Research article

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Exponential stability of the nonlinear Schrödinger equation with locally distributed damping on compact Riemannian manifold

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Abstract: In this paper, we consider the following nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta_g u + ia(x)u - |u|^{p-1}u = 0 & (x, t) \in \mathcal{M} \times (0, +\infty), \\ u(x, 0) = u_0(x) & x \in \mathcal{M}, \end{cases} \quad (0.1)$$

where $(\mathcal{M}, g)$ is a smooth complete compact Riemannian manifold of dimension n ($n = 2, 3$) without boundary. For the damping terms $-a(x)(1 - \Delta)^{-1}a(x)u_t$ and $ia(x)(-\Delta)^{\frac{1}{2}}a(x)u$, the exponential stability results of system (0.1) have been proved by Dehman et al. (Math Z 254(4): 729-749, 2006), Laurent. (SIAM J. Math. Anal. 42(2): 785-832, 2010) and Cavalcanti et al. (Math Phys 69(4): 100, 2018). However, from the physical point of view, it would be more important to consider the stability of system (0.1) with the damping term $ia(x)u$, which is still an open problem. In this paper, we obtain the exponential stability of system (0.1) by Morawetz multipliers in non Euclidean geometries and compactness-uniqueness arguments.

Keywords: nonlinear Schrödinger equation, Morawetz multipliers in non Euclidean geometries, exponential stability

MSC: 58J45, 93D20

1 Introduction

1.1 Notations

Suppose that $(\mathcal{M}, g)$ is a smooth complete compact Riemannian manifold of dimension n ($n = 2, 3$) without boundary.

Denote

$$\langle X, Y \rangle_g = g(X, Y), \quad |X|_g^2 = \langle X, X \rangle_g, \quad X, Y \in \mathcal{M}_x, x \in \mathcal{M}. \quad (1.1)$$

Let D be the Levi-Civita connection of the metric g and H be a vector field. The covariant differential DH of the vector field H is a tensor field of order 2 as follow:

$$DH(X, Y)(x) = \langle D_Y H, X \rangle_g(x) \quad X, Y \in \mathcal{M}_x, x \in \mathcal{M}. \quad (1.2)$$

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Finally, we set div_g , ∇_g and Δ_g as the divergence operator, the gradient operator and the Laplace–Beltrami operator of $(\mathcal{M}, g)$, respectively.

1.2 Nonlinear Schrödinger equation

We consider the following system:

$$\begin{cases} iu_t + \Delta_g u + ia(x)u - |u|^{p-1}u = 0 & (x, t) \in \mathcal{M} \times (0, +\infty), \\ u(x, 0) = u_0(x) & x \in \mathcal{M}, \end{cases} \quad (1.3)$$

where $1 < p < +\infty$ for dimension $n = 2$, $1 < p \leq 3$ for dimension $n = 3$. And $a(x) \in C^1(\mathcal{M})$ is a nonnegative real function.

Define the energy of system (1.3) by

$$E(t) = \frac{1}{2} \int_{\mathcal{M}} (|u|^2 + |\nabla_g u|_g^2) dx_g + \frac{1}{p+1} \int_{\mathcal{M}} |u|^{p+1} dx_g, \quad (1.4)$$

where dx_g denotes the volume element of $(\mathcal{M}, g)$ and

$$|u|^2 = u\bar{u}, \quad |\nabla_g u|_g^2 = \langle \nabla_g u, \nabla_g \bar{u} \rangle_g. \quad (1.5)$$

There are extensive available literatures on the estimates of solutions of the Schrödinger equation on Riemannian manifold. See [6, 9, 18, 20, 30] for Strichartz estimates, [1–5, 7, 10, 21, 27–29, 32] for local energy decay and [11–16, 19, 24] for stability results on compact manifold or Euclidean space.

When dimension $n = 2$, the exponential stability of the following system

$$\begin{cases} iu_t + \Delta u - a(x)(1 - \Delta)^{-1}a(x)u_t = P'(|u|^2)u & (x, t) \in \mathcal{M} \times [0, T], \\ u(x, 0) = u_0(x) & x \in \mathcal{M} \end{cases} \quad (1.6)$$

has been proved by [19] and the exponential stability of the following system

$$\begin{cases} iu_t + \Delta u - f(|u|^2)u + ia(x)(-\Delta)^{\frac{1}{2}}a(x)u = 0 & (x, t) \in \mathcal{M} \times (0, +\infty), \\ u(x, 0) = u_0(x) & x \in \mathcal{M} \end{cases} \quad (1.7)$$

has been obtained in [13].

When dimension $n = 3$, the exponential stability of the following system:

$$\begin{cases} iu_t + \Delta_g u - a(x)(1 - \Delta_g)^{-1}a(x)u_t = (1 + |u|^2)u & (x, t) \in \mathcal{M} \times (0, T), \\ u(x, 0) = u_0(x) & x \in \mathcal{M}, \end{cases} \quad (1.8)$$

has been established in [24].

In fact, from the physical point of view, it would be more important to consider the damping term like $ia(x)u$. When dimension $n = 2$, asymptotic stability of the system (1.3) has been proved by [13]. It is said that the energy of system (1.3) goes to zero as time goes to infinity. However, the exact stability (especially the exponential stability) of system (1.3) is still an open problem. In this paper, under suitable geometric assumptions, we obtain the exponential stability of system (1.3) by Morawetz multipliers in non Euclidean geometries and compactness-uniqueness arguments.

Our paper is organized as follows. In Section 2, we will state our main results. Then, multiplier identities and key lemmas are presented in Section 3. Finally, we prove the exponential stability of the nonlinear Schrödinger equation in Section 4.

2 Main results

When $n = 2$, the well-posedness of system (1.3) has been proved by Theorem 1.4 in [13]. When $n = 3$, the unique global weak solution of system (1.3) has been given by [8] and the well-posedness of system (1.3) in Bourgain spaces has been proved by [3]. Throughout the paper, we assume that the system (1.3) is well-posed such that

$$u \in C\left([0, +\infty), H^1(\mathcal{M})\right). \quad (2.1)$$

The followings are the main assumptions of this paper.

Assumption (A) There exists a C^2 vector field H on $\mathcal{M}$ such that

$$DH(X, X) \geq \delta |X|_g^2, \quad X \in \mathcal{M}_x, x \in \overline{\Omega}, \quad (2.2)$$

where $\delta > 0$ is a constant and $\Omega \subset \mathcal{M}$ is an open set with smooth boundary.

Moreover, $a(x)$ satisfies

$$a(x) \geq a_0 > 0, \quad x \in \mathcal{M} \setminus \Omega, \quad (2.3)$$

where $a_0 > 0$ is a constant.

Remark 2.1. The vector field given by assumption (A) is called escape vector field and it was introduced by Yao [33] for the controllability of the wave equation with variable coefficients, which is also a useful condition for the controllability and the stabilization of the quasilinear wave equation (see [17, 34, 36]). Existence of escape vector field depends on the sectional curvature of the Riemannian manifold $(\mathcal{M}, g)$. There are a number of methods and examples in [35] to find out escape vector field. The explicit expression of $D\left(r \frac{\partial}{\partial r}\right)$ in Riemannian manifold $(\mathbb{R}^n, g)$ is given by [25, 26].

Assumption (B) (Unique continuation) Let $\Omega \subset \mathcal{M}$ be an open set with smooth boundary and $\omega \subset \Omega$ be an open subset. Assume that ω satisfies the geometric control condition:

(GCC) There exists constant $T_0 > 0$ such that for any $x \in \Omega$ and any unit-speed geodesic $\gamma(t)$ of $(\mathcal{M}, g)$ starting at x , there exists $t < T_0$ such that $\gamma(t) \subset \omega$.

As a consequence, for every $T > 0$, the only solution in $C([0, T], H^1(\Omega))$ to the system

$$\begin{cases} iu_t + \Delta_g u + b_1(x, t)u + b_2(x, t)\bar{u} = 0 & (x, t) \in \Omega \times (0, T), \\ u = 0 & (x, t) \in \omega \times (0, T), \end{cases} \quad (2.4)$$

is the trivial one $u \equiv 0$, where $b_1(x, t)$ and $b_2(x, t) \in L^\infty([0, T], L^3(\Omega))$.

Remark 2.2. Let $H = D\varphi$, where H is given by (2.2) and φ is a strictly convex function, then assumption (B) follows from Proposition B.3 in [24]. It can also be proved in particular cases by Carleman estimates in Euclidean space, see [22, 23, 31].

Theorem 2.1. Let assumption (A) and assumption (B) hold true. Given constant $E_0 > 0$. Assume that $\|u_0\|_{L^2(\mathcal{M})} \leq E_0$. Then there exist positive constants C_1 and C_2 , which are only dependent on E_0 , such that

$$E(t) \leq C_1 e^{-C_2 t} E(0), \quad \forall t > 0. \quad (2.5)$$

3 Multiplier Identities and Key Lemmas

We need to establish several multiplier identities, which are useful for our problem.

Lemma 3.1. Suppose that $u(x, t)$ solves the following equation:

$$iu_t + \Delta_g u + ia(x)u - |u|^{p-1}u = 0 \quad (x, t) \in \mathcal{M} \times (0, +\infty). \quad (3.1)$$

Let $\mathcal{H}$ be a C^1 vector field defined on $\mathcal{M}$. Then

$$\begin{aligned} 0 &= \frac{1}{2} \int_{\mathcal{M}} \operatorname{Im} (u \mathcal{H}(\bar{u})) dx_g \Big|_0^T + \int_0^T \int_{\mathcal{M}} \operatorname{Re} D\mathcal{H}(\nabla_g \bar{u}, \nabla_g u) dx_g dt \\ &\quad + \int_0^T \int_{\mathcal{M}} \operatorname{Im} (a(x)u \mathcal{H}(\bar{u})) dx_g dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\mathcal{M}} \left(\operatorname{Im} (u \bar{u}_t) - |\nabla_g u|_g^2 - \frac{2}{p+1} |u|^{p+1} \right) \operatorname{div}_g \mathcal{H} dx_g dt. \end{aligned} \quad (3.2)$$

Moreover, assume that the real function $P \in C^1(\mathcal{M})$. Then

$$\int_0^T \int_{\mathcal{M}} \left(\operatorname{Im} (u \bar{u}_t) - |\nabla_g u|_g^2 - |u|^{p+1} \right) P dx_g dt = \frac{1}{2} \int_0^T \int_{\mathcal{M}} \nabla_g P (|u|^2) dx_g dt. \quad (3.3)$$

Proof. Multiplying the Schrödinger equation in (3.1) by $\mathcal{H}(\bar{u})$ and then integrating over $\mathcal{M} \times (0, T)$, we deduce that

$$\begin{aligned} \operatorname{Re} (iu_t \mathcal{H}(\bar{u})) &= -\operatorname{Im} (u_t \mathcal{H}(\bar{u})) = -\frac{1}{2} \operatorname{Im} (u_t \mathcal{H}(\bar{u}) - \bar{u}_t \mathcal{H}(u)) \\ &= -\frac{1}{2} \operatorname{Im} ((u \mathcal{H}(\bar{u}))_t - \mathcal{H}(u \bar{u}_t)) = -\frac{1}{2} \operatorname{Im} (u \mathcal{H}(\bar{u}))_t + \frac{1}{2} \operatorname{Im} \mathcal{H}(u \bar{u}_t) \\ &= -\frac{1}{2} \operatorname{Im} (u \mathcal{H}(\bar{u}))_t + \frac{1}{2} \operatorname{Im} \operatorname{div}_g (u \bar{u}_t \mathcal{H}) - \frac{1}{2} \operatorname{Im} (u \bar{u}_t \operatorname{div}_g \mathcal{H}), \end{aligned} \quad (3.4)$$

$$\begin{aligned} \operatorname{Re} (\mathcal{H}(\bar{u}) \Delta_g u) &= \operatorname{Re} \left(\operatorname{div}_g \mathcal{H}(\bar{u}) \nabla_g u - \nabla_g u \langle \mathcal{H}, \nabla_g \bar{u} \rangle_g \right) \\ &= \operatorname{Re} \operatorname{div}_g \mathcal{H}(\bar{u}) \nabla_g u - \operatorname{Re} \nabla_g u \langle \mathcal{H}, \nabla_g \bar{u} \rangle_g \\ &= \operatorname{Re} \operatorname{div}_g \mathcal{H}(\bar{u}) \nabla_g u - \operatorname{Re} D\mathcal{H}(\nabla_g \bar{u}, \nabla_g u) - \operatorname{Re} D^2 \bar{u}(\mathcal{H}, \nabla_g u) \\ &= \operatorname{Re} \operatorname{div}_g \mathcal{H}(\bar{u}) \nabla_g u - \operatorname{Re} D\mathcal{H}(\nabla_g \bar{u}, \nabla_g u) - \operatorname{Re} D^2 \bar{u}(\nabla_g u, \mathcal{H}) \\ &= \operatorname{Re} \operatorname{div}_g \mathcal{H}(\bar{u}) \nabla_g u - \operatorname{Re} D\mathcal{H}(\nabla_g \bar{u}, \nabla_g u) - \frac{1}{2} \mathcal{H}(|\nabla_g u|_g^2) \\ &= \operatorname{Re} \operatorname{div}_g \mathcal{H}(\bar{u}) \nabla_g u - \operatorname{Re} D\mathcal{H}(\nabla_g \bar{u}, \nabla_g u) - \frac{1}{2} \operatorname{div}_g (|\nabla_g u|_g^2 \mathcal{H}) \\ &\quad + \frac{1}{2} |\nabla_g u|_g^2 \operatorname{div}_g \mathcal{H}, \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \operatorname{Re} (ia(x)u - |u|^{p-1}u) \mathcal{H}(\bar{u}) &= -\operatorname{Im} (a(x)u \mathcal{H}(\bar{u})) - \frac{1}{p+1} \operatorname{div}_g (|u|^{p+1} \mathcal{H}) \\ &\quad + \frac{|u|^{p+1}}{p+1} \operatorname{div}_g \mathcal{H}. \end{aligned} \quad (3.6)$$

The equality (3.2) follows from Green's formula.

In addition, we multiply the Schrödinger equation in (3.1) by $P\bar{u}$ and integrate over $\mathcal{M} \times (0, T)$. It gives that

$$\operatorname{Re} (iPu_t \bar{u}) = -\operatorname{Im} (Pu_t \bar{u}) = \operatorname{Im} (Pu \bar{u}_t), \quad (3.7)$$

$$\operatorname{Re} (P\bar{u} \Delta_g u) = \operatorname{Re} (\operatorname{div}_g P\bar{u} \nabla_g u - \nabla_g u (P\bar{u}))$$

$$= \operatorname{Re} \operatorname{div}_g P \bar{u} \nabla_g u - P |\nabla_g u|_g^2 - \frac{1}{2} \nabla_g P (|u|^2), \quad (3.8)$$

and

$$\begin{aligned} \operatorname{Re} \left(i a(x) u - |u|^{p-1} u \right) P \bar{u} &= \operatorname{Re} \left(i a(x) P |u|^2 \right) - P |u|^{p+1} \\ &= -P |u|^{p+1}. \end{aligned} \quad (3.9)$$

The equality (3.3) follows from Green's formula. $\square$

The following lemma shows the relationship between the metric g and the geometric control condition.

Lemma 3.2. *Let $\Omega \subset \mathcal{M}$ be a bounded domain. Assume that there exists a C^1 vector field H on $\mathcal{M}$ such that*

$$DH(X, X) \geq \delta |X|_g^2, \quad X \in \mathcal{M}_x, x \in \overline{\Omega}, \quad (3.10)$$

where $\delta > 0$ is a constant.

Then, for any $x \in \Omega$ and any unit-speed geodesic $y(t)$ starting at x , if

$$y(t) \in \Omega, \quad 0 \leq t \leq t_0, \quad (3.11)$$

we have

$$t_0 \leq \frac{2}{\delta} \sup \left\{ |H|_g(x) \mid x \in \overline{\Omega} \right\}. \quad (3.12)$$

Proof. Note that

$$|y'(t)|_g = 1, \quad D_{y'(t)} y'(t) = 0. \quad (3.13)$$

Then

$$\langle H, y'(t) \rangle_g \Big|_0^{t_0} = \int_0^{t_0} y'(t) \langle H, y'(t) \rangle_g dt = \int_0^{t_0} DH(y'(t), y'(t)) dt \geq \delta t_0. \quad (3.14)$$

Hence

$$t_0 \leq \frac{2}{\delta} \sup \left\{ |H|_g(x) \mid x \in \overline{\Omega} \right\}. \quad (3.15)$$

$\square$

Lemma 3.3. *Let $u(x, t)$ solve the system (1.3). Then*

$$\int_{\mathcal{M}} |u|^2 dx_g \Big|_0^T = -2 \int_0^T \int_{\mathcal{M}} a(x) |u|^2 dx_g dt, \quad (3.16)$$

$$\begin{aligned} \int_{\mathcal{M}} \left(|\nabla_g u|_g^2 + \frac{2}{p+1} |u|^{p+1} \right) dx_g \Big|_0^T &= -2 \int_0^T \int_{\mathcal{M}} a(x) \left(|\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\ &\quad - \int_0^T \int_{\mathcal{M}} \nabla_g a(x) (|u|^2) dx_g dt, \end{aligned} \quad (3.17)$$

for any $T > 0$.

Proof. After multiplying the Schrödinger equation in (1.3) by $2\bar{u}$ and integrating over $\mathcal{M} \times (0, T)$, the equality (3.16) holds.

Multiplying the Schrödinger equation in (1.3) by $2\bar{u}_t$ and integrating over $\mathcal{M} \times (0, T)$, we obtain

$$\int_{\mathcal{M}} \left(|\nabla_g u|_g^2 + \frac{2}{p+1} |u|^{p+1} \right) dx_g \Big|_0^T = -2 \int_0^T \int_{\mathcal{M}} \operatorname{Im} (a(x) u \bar{u}_t) dx_g dt. \quad (3.18)$$

Let $P = a(x)$ in (3.3). Substituting (3.3) into (3.18), we have

$$\begin{aligned} \int_{\mathcal{M}} \left(|\nabla_g u|_g^2 + \frac{2}{p+1} |u|^{p+1} \right) dx_g \Big|_0^T &= -2 \int_0^T \int_{\mathcal{M}} a(x) \left(|\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\ &\quad - \int_0^T \int_{\mathcal{M}} \nabla_g a(x) (|u|^2) dx_g dt. \end{aligned} \quad (3.19)$$

□

4 Exponential stability of the nonlinear Schrödinger equation

From Lemma 3.2, the following lemma holds true.

Lemma 4.1. *Let assumption (A) hold true. Then, there exists $t_0 > 0$, for any $x \in \Omega$ and any unit-speed geodesic $y(t)$ starting at x , there exists $t < t_0$ such that*

$$y(t) \in \partial\Omega. \quad (4.1)$$

The following lemmas follow from assumption (B).

Lemma 4.2. (Unique continuation) *Let assumption (B) hold true. Let $\Omega \subset \mathcal{M}$ be an open set with smooth boundary and $\omega \subset \Omega$ be an open subset. Assume that ω satisfies the geometric control condition:*

(GCC) *There exists constant $T_0 > 0$ such that for any $x \in \Omega$ and any unit-speed geodesic $y(t)$ of $(\mathcal{M}, g)$ starting at x , there exists $t < T_0$ such that $y(t) \subset \omega$.*

Accordingly, for every $T > 0$, the only solution in $C([0, T], H^1(\Omega))$ to the system

$$\begin{cases} iu_t + \Delta_g u = 0 & (x, t) \in \Omega \times (0, T), \\ u = 0 & (x, t) \in \omega \times (0, T), \end{cases} \quad (4.2)$$

is the trivial one $u \equiv 0$.

Lemma 4.3. (Unique continuation) *Let assumption (B) hold true. Let $\Omega \subset \mathcal{M}$ be an open set with smooth boundary and $\omega \subset \Omega$ be an open subset. Assume that ω satisfies the geometric control condition:*

(GCC) *There exists constant $T_0 > 0$ such that for any $x \in \Omega$ and any unit-speed geodesic $y(t)$ of $(\mathcal{M}, g)$ starting at x , there exists $t < T_0$ such that $y(t) \subset \omega$.*

Therefore, for every $T > 0$, the only solution in $C([0, T], H^1(\Omega))$ to the system

$$\begin{cases} iu_t + \Delta_g u - |u|^{p-1}u = 0 & (x, t) \in \Omega \times (0, T), \\ u = 0 & (x, t) \in \omega \times (0, T), \end{cases} \quad (4.3)$$

is the trivial one $u \equiv 0$.

Proof.

Let

$$b_1(x, t) = -|u|^{p-1}u, \quad b_2(x, t) = 0, \quad (x, t) \in \Omega \times (0, T). \quad (4.4)$$

Note that

$$E(t) = E(0), \text{ for all } t \in [0, T], \quad (4.5)$$

and

$$H^1(\Omega) \hookrightarrow L^{3(p-1)}(\Omega). \quad (4.6)$$

Hence

$$b_1(x, t), b_2(x, t) \in L^\infty([0, T], L^{3(p-1)}(\Omega)). \quad (4.7)$$

It follows from assumption **(B)** that $u \equiv 0$. $\square$

Lemma 4.4. *For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$|\nabla_g a(x)|_g^2 \leq C_\epsilon a(x) + \epsilon, \text{ for all } x \in \mathcal{M}, \quad (4.8)$$

where $a(x)$ is given by (1.3).

Proof. We prove (4.8) by contradiction. If (4.8) doesn't hold true, then there exist constant $\epsilon_0 > 0$ and $\{x_k\}_{k=1}^\infty \subset \mathcal{M}$ such that

$$|\nabla_g a(x)|_g^2 \geq ka(x_k) + \epsilon_0. \quad (4.9)$$

Therefore, there exists x_0 and a subset of $\{x_k\}_{k=1}^\infty$, still denoted by $\{x_k\}_{k=1}^\infty$, such that

$$\lim_{k \rightarrow +\infty} x_k = x_0. \quad (4.10)$$

Note that

$$a(x) \geq 0, \quad x \in \mathcal{M}, \quad (4.11)$$

and

$$\sup_{x \in \mathcal{M}} |\nabla_g a(x)|_g^2 < +\infty. \quad (4.12)$$

It follows from (4.9) that

$$a(x_0) = 0, \quad (4.13)$$

and

$$|\nabla a(x_0)| \geq \sqrt{\epsilon_0}. \quad (4.14)$$

With (4.11) and (4.13), we obtain

$$\nabla a(x_0) = 0, \quad (4.15)$$

which contradicts (4.14). $\square$

Lemma 4.5. *Let assumption **(A)** hold true. Let $u(x, t)$ solve the system (1.3). Then*

$$\begin{aligned} E(0) + \int_0^T E(t) dt &\leq C \int_0^T \int_{\mathcal{M}} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\ &\quad + C \int_0^T \int_{\mathcal{M}} |u|^2 dx_g dt, \end{aligned} \quad (4.16)$$

for sufficiently large T .

Proof.

Note that $a(x) \in C^1(\mathcal{M})$, then there exists an open set $\Omega_0 \in \mathcal{M}$, such that

$$\overline{\Omega_0} \subset \Omega, \quad (4.17)$$

$$a(x) \geq \frac{a_0}{2}, \quad x \in \mathcal{M} \setminus \Omega_0. \quad (4.18)$$

Let $b(x) \in C^\infty(\mathcal{M})$ be a nonnegative function satisfying

$$b(x) = 1, \quad x \in \Omega_0 \quad \text{and} \quad b(x) = 0, \quad x \in \mathcal{M} \setminus \Omega. \quad (4.19)$$

Let

$$\mathcal{H}(x) = b(x)H, \quad x \in \mathcal{M}. \quad (4.20)$$

Note that

$$D\mathcal{H}(X, X) \geq \delta |X|_g^2 \quad \text{for all } X \in \mathcal{M}_x, x \in \Omega_0, \quad (4.21)$$

$$\operatorname{div}_g \mathcal{H} = \operatorname{tr} D\mathcal{H} \geq n\delta \quad \text{for all } x \in \Omega_0. \quad (4.22)$$

It follows from (3.2) that

$$\begin{aligned} 0 &\geq \frac{1}{2} \int_{\Omega} \operatorname{Im} (uH(\bar{u})) \, dx_g \Big|_0^T + \delta \int_0^T \int_{\Omega_0} |\nabla_g u|_g^2 \, dx_g \, dt \\ &\quad - C \int_0^T \int_{\Omega \setminus \Omega_0} |\nabla_g u|_g^2 \, dx_g \, dt + \int_0^T \int_{\Omega} \operatorname{Im} (a(x)uH(\bar{u})) \, dx_g \, dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\Omega} \left(\operatorname{Im} (u\bar{u}_t) - |\nabla_g u|_g^2 - \frac{2}{p+1} |u|^{p+1} \right) \operatorname{div}_g H \, dx_g \, dt \\ &= \frac{1}{2} \int_{\Omega} \operatorname{Im} (uH(\bar{u})) \, dx_g \Big|_0^T + \delta \int_0^T \int_{\Omega_0} |\nabla_g u|_g^2 \, dx_g \, dt \\ &\quad - C \int_0^T \int_{\Omega \setminus \Omega_0} |\nabla_g u|_g^2 \, dx_g \, dt + \int_0^T \int_{\Omega} \operatorname{Im} (a(x)uH(\bar{u})) \, dx_g \, dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\Omega} \left(\operatorname{Im} (u\bar{u}_t) - |\nabla_g u|_g^2 - |u|^{p+1} \right) \operatorname{div}_g H \, dx_g \, dt \\ &\quad + \int_0^T \int_{\Omega} \frac{(p-1) \operatorname{div}_g H}{2(p+1)} |u|^{p+1} \, dx_g \, dt. \end{aligned} \quad (4.23)$$

Let $P = \frac{\operatorname{div}_g H}{2}$ in (3.3). Substituting (3.3) into (4.23), we obtain

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} \operatorname{Im} (uH(\bar{u})) \, dx_g \Big|_0^T + \frac{1}{4} \int_0^T \int_{\Omega} \nabla_g P(|u|^2) \, dx_g \, dt \\ &\quad + \int_0^T \int_{\Omega} \operatorname{Im} (a(x)uH(\bar{u})) \, dx_g \, dt \\ &\quad + \delta \int_0^T \int_{\Omega_0} |\nabla_g u|_g^2 \, dx_g \, dt + \int_0^T \int_{\Omega_0} \frac{\delta n(p-1)}{2(p+1)} |u|^{p+1} \, dx_g \, dt \\ &\leq C \int_0^T \int_{\Omega \setminus \Omega_0} \left(|\nabla_g u|_g^2 + |u|^{p+1} \right) \, dx_g \, dt. \end{aligned} \quad (4.24)$$

Then

$$\int_0^T \int_{\Omega_0} \left(|\nabla_g u|_g^2 + |u|^{p+1} \right) \, dx_g \, dt$$

$$\begin{aligned}
&\leq C(E(0) + E(T)) + C \int_0^T \int_{\Omega} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\
&\quad + \int_0^T \int_{\Omega} \left(C_{\epsilon} |u|^2 + \epsilon |\nabla_g u|_g^2 \right) dx_g dt.
\end{aligned} \tag{4.25}$$

Therefore

$$\begin{aligned}
&\int_0^T \int_{\Omega_0} \left(|\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\
&\leq C(E(0) + E(T)) + C \int_0^T \int_{\Omega} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\
&\quad + C \int_0^T \int_{\Omega} |u|^2 dx_g dt.
\end{aligned} \tag{4.26}$$

Hence

$$\begin{aligned}
\int_0^T E(t) dt &\leq C(E(0) + E(T)) \\
&\quad + C \int_0^T \int_{\mathcal{M}} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\
&\quad + C \int_0^T \int_{\mathcal{M}} |u|^2 dx_g dt.
\end{aligned} \tag{4.27}$$

With (3.16) and (3.17), we deduce that

$$\begin{aligned}
CE(T) &= CE(0) - C \int_0^T \int_{\mathcal{M}} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\
&\quad - \frac{C}{2} \int_0^T \int_{\mathcal{M}} \nabla_g a(x) (|u|^2) dx_g dt,
\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}
4CE(0) &= \int_0^{4C} E(t) dt - \int_0^{4C} (E(t) - E(0)) dt \\
&\leq \int_0^{4C} E(t) dt + 4C \int_0^{4C} \int_{\mathcal{M}} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\
&\quad + 4C \int_0^{4C} \int_{\mathcal{M}} \left(|\nabla_g a(x)|_g |u| \right) |\nabla_g u|_g dx_g dt.
\end{aligned} \tag{4.29}$$

Substituting (4.28) and (4.29) into (4.27), for $T > 5C$, we have

$$E(0) + \int_0^T E(t) dt \leq C \int_0^T \int_{\mathcal{M}} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt$$

$$+ \int_0^T \int_{\Omega} \left(C_{\epsilon} |u|^2 + \epsilon |\nabla_g u|_g^2 \right) dx_g dt. \quad (4.30)$$

Therefore

$$\begin{aligned} E(0) + \int_0^T E(t) dt &\leq C \int_0^T \int_{\mathcal{M}} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\ &\quad + C \int_0^T \int_{\Omega} |u|^2 dx_g dt. \end{aligned} \quad (4.31)$$

The estimate (4.16) holds true. $\square$

Lemma 4.6. *Let assumption (A) and assumption (B) hold true. Let T be sufficiently large. Then for any $\|u_0\|_{L^2(\mathcal{M})} \leq E_0$, there exists positive constant $C(E_0, T)$ such that*

$$E(0) + \int_0^T E(t) dt \leq C(E_0, T) \int_0^T \int_{\mathcal{M}} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt. \quad (4.32)$$

Proof. We apply compactness-uniqueness arguments to prove the conclusion. It follows from (4.16) that

$$\begin{aligned} E(0) + \int_0^T E(t) dt &\leq C \int_0^T \int_{\mathcal{M}} a(x) \left(|u|^2 + |\nabla_g u|_g^2 + |u|^{p+1} \right) dx_g dt \\ &\quad + C \int_0^T \int_{\Omega} |u|^2 dx_g dt. \end{aligned} \quad (4.33)$$

Then, if the estimate (4.32) doesn't hold true, there exist $\{u_k\}_{k=1}^{\infty}$ satisfying (1.3) and

$$\|u_{k0}\|_{L^2(\mathcal{M})} \leq E_0, \quad (4.34)$$

$$\int_0^T \int_{\Omega_0} |u_k|^2 dx_g dt \geq k \int_0^T \int_{\mathcal{M}} a(x) \left(|u_k|^2 + |\nabla_g u_k|_g^2 + |u_k|^{p+1} \right) dx_g dt. \quad (4.35)$$

Thus,

$$E_k(0) + \int_0^T E_k(t) dt \leq CE_0, \quad (4.36)$$

where

$$E_k(t) = \frac{1}{2} \int_{\mathcal{M}} \left(|u_k|^2 + |\nabla_g u_k|_g^2 \right) dx_g + \frac{1}{p+1} \int_{\mathcal{M}} |u_k|^{p+1} dx_g. \quad (4.37)$$

It follows from (3.16), (3.17) and (4.36) that there exists $C(T) > 0$ such that

$$E_k(t) \leq C(T)E_0, \quad \forall t \in [0, T], \quad \forall 1 \leq k < +\infty, \quad (4.38)$$

which implies

$$\{u_k\} \text{ are bounded in } L^{\infty}([0, T], H^1(\mathcal{M})). \quad (4.39)$$

Therefore, there exists $\hat{u}$ and a subset of $\{u_k\}_{k=1}^{\infty}$, still denoted by $\{u_k\}_{k=1}^{\infty}$, such that

$$u_k \rightharpoonup \hat{u} \text{ weakly in } L^2([0, T], H^1(\mathcal{M})), \quad (4.40)$$

and

$$u_k \rightarrow \hat{u} \text{ strongly in } L^2(\mathcal{M}) \text{ for arbitrarily fixed } t \in [0, T]. \quad (4.41)$$

Note that

$$\|u_k - \hat{u}\|_{L^2(\mathcal{M})}^2 \leq \widehat{C}(T)E_0, \quad \forall t \in [0, T], \quad \forall 1 \leq k < +\infty. \quad (4.42)$$

Lebesgue's dominated convergence theorem yields

$$u_k \rightarrow \hat{u} \text{ strongly in } L^2(\mathcal{M} \times (0, T)). \quad (4.43)$$

Case a:

$$\int_0^T \int_{\mathcal{M}} |\hat{u}|^2 dx_g dt > 0. \quad (4.44)$$

Note that

$$H^1(\mathcal{M}) \hookrightarrow L^{2p}(\mathcal{M}). \quad (4.45)$$

Therefore, it follows from (4.39) that

$$\{|u_k|^{p-1}u_k\} \text{ are bounded in } L^2(\mathcal{M} \times (0, T)). \quad (4.46)$$

Hence, there exists a subset of $\{u_k\}_{k=1}^\infty$, still denoted by $\{u_k\}_{k=1}^\infty$, such that

$$|u_k|^{p-1}u_k \rightarrow |\hat{u}|^{p-1}\hat{u} \text{ weakly in } L^2(\mathcal{M} \times (0, T)). \quad (4.47)$$

It follows from (4.34) and (4.35) that

$$a(x)\hat{u} = 0 \quad (x, t) \in \mathcal{M} \times (0, T). \quad (4.48)$$

Therefore, with (4.40) and (4.47), we obtain

$$\begin{cases} i\hat{u}_{0t} + \Delta_g \hat{u} - |\hat{u}|^{p-1}\hat{u} = 0 & (x, t) \in \mathcal{M} \times (0, T), \\ a(x)\hat{u} = 0 & (x, t) \in \mathcal{M} \times (0, T). \end{cases} \quad (4.49)$$

With (4.1) and Lemma 4.3, we have

$$\hat{u} \equiv 0 \quad \text{on } \mathcal{M} \times (0, T), \quad (4.50)$$

which contradicts (4.44).

Case b:

$$\hat{u} \equiv 0 \quad \text{on } \mathcal{M} \times (0, T). \quad (4.51)$$

Denote

$$v_k = \frac{u_k}{\sqrt{c_k}} \quad \text{for } k \geq 1, \quad (4.52)$$

where

$$c_k = \int_0^T \int_{\mathcal{M}} |u_k|^2 dx_g dt. \quad (4.53)$$

Then v_k satisfies

$$iv_{kt} + \Delta_g v_k + ia(x)v_k - |u_k|^{p-1}v_k = 0 \quad (x, t) \in \mathcal{M} \times (0, T), \quad (4.54)$$

and

$$\int_0^T \int_{\mathcal{M}} |v_k|^2 dx_g dt = 1. \quad (4.55)$$

It follows from (4.35) that

$$1 \geq k \int_0^T \int_{\mathcal{M}} a(x) \left(|v_k|^2 + |\nabla_g v_k|_g^2 + |u_k|^{p-1} |v_k|^2 \right) dx_g dt. \quad (4.56)$$

Therefore, it follows from (4.33) that

$$\widehat{E}_k(0) + \int_0^T \widehat{E}_k(t) dt \leq 1 + \frac{1}{k} \leq 2, \quad (4.57)$$

where

$$\widehat{E}_k(t) = \frac{1}{2} \int_{\mathcal{M}} \left(|v_k|^2 + |\nabla_g v_k|_g^2 + \frac{2}{p+1} |u_k|^{p-1} |v_k|^2 \right) dx_g. \quad (4.58)$$

With (4.38), (4.52) and (4.57), we obtain

$$\widehat{E}_k(t) \leq C(T) \widehat{E}_k(0) \leq 2C(T), \quad \forall t \in (0, T), \quad \forall 1 \leq k < +\infty, \quad (4.59)$$

which implies

$$\{v_k\} \text{ are bounded in } L^\infty([0, T], H^1(\mathcal{M})). \quad (4.60)$$

Hence, there exist $\hat{v}$ and a subset of $\{v_k\}_{k=1}^\infty$, still denoted by $\{v_k\}_{k=1}^\infty$, such that

$$v_k \rightharpoonup \hat{v} \text{ weakly in } L^2([0, T], H^1(\mathcal{M})), \quad (4.61)$$

and

$$v_k \rightarrow \hat{v} \text{ strongly in } L^2(\mathcal{M}) \text{ for any fixed } t \in [0, T]. \quad (4.62)$$

Then by Lebesgue's dominated convergence theorem, we have

$$v_k \rightarrow \hat{v} \text{ strongly in } L^2(\mathcal{M} \times (0, T)). \quad (4.63)$$

Note that

$$H^1(\mathcal{M}) \hookrightarrow L^{2p}(\mathcal{M}). \quad (4.64)$$

Therefore, it follows from (4.60) that

$$\{|v_k|^{p-1} v_k\} \text{ are bounded in } L^\infty([0, T], L^2(\mathcal{M})). \quad (4.65)$$

Hence

$$\int_{\mathcal{M}} \left(|u_k|^{p-1} |v_k| \right)^2 dx_g = c_k^{p-1} \int_{\mathcal{M}} |v_k|^2 dx_g \leq c_k^{p-1} C(T). \quad (4.66)$$

With (4.51) and (4.53), we obtain

$$|u_k|^{p-1} |v_k| \rightarrow 0 \text{ strongly in } L^\infty([0, T], L^2(\mathcal{M})). \quad (4.67)$$

It follows from (4.55), (4.56) and (4.63) that

$$\int_0^T \int_{\mathcal{M}} \hat{v}^2 dx_g dt = 1, \quad (4.68)$$

and

$$a(x) \hat{v} = 0 \quad (x, t) \in \mathcal{M} \times (0, T). \quad (4.69)$$

Therefore, it follows from (4.54), (4.61) and (4.67) that

$$\begin{cases} i\hat{v}_t + \Delta_g \hat{v} = 0 & (x, t) \in \mathcal{M} \times (0, T), \\ a(x)\hat{v} = 0 & (x, t) \in \mathcal{M} \times (0, T). \end{cases} \quad (4.70)$$

With (4.1) and Lemma 4.2, we have

$$\hat{v} \equiv 0 \quad \text{on } \mathcal{M} \times (0, T), \quad (4.71)$$

which contradicts (4.68). $\square$

Proof of Theorem 2.1 Let T be sufficiently large. It follows from (3.16) that $\|u\|_{L^2(\mathcal{M})}$ is non-increasing. It follows from (4.8), (3.16), (3.17) and (4.32) that

$$\begin{aligned} & E(0) + \int_0^T E(t) dt \\ & \leq C(E_0, T) \left(E(0) - E(T) + \int_0^T \int_{\mathcal{M}} \left(|\nabla_g a(x)|_g |u| \right) |\nabla_g u|_g dx_g dt \right) \\ & \leq C(E_0, T) \left(E(0) - E(T) + C_\epsilon \int_0^T \int_{\mathcal{M}} a(x) |u|^2 dx_g dt + \epsilon \int_0^T \int_{\mathcal{M}} (|u|^2 + |\nabla_g u|_g^2) dx_g dt \right) \\ & = C(E_0, T) \left(E(0) - E(T) - \frac{C_\epsilon}{2} \int_0^T |u|^2 dx_g \Big|_0^T + \epsilon \int_0^T \int_{\mathcal{M}} (|u|^2 + |\nabla_g u|_g^2) dx_g dt \right). \end{aligned} \quad (4.72)$$

Therefore

$$E(0) \leq C(E_0, T) \left(E(0) - E(T) - \widehat{C}(E_0, T) \int_0^T |u|^2 dx_g \Big|_0^T \right). \quad (4.73)$$

Denote

$$\widetilde{E}(t) = E(t) + \widehat{C}(E_0, T) \int_0^t |u|^2 dx_g. \quad (4.74)$$

From (4.73), we obtain

$$\widetilde{E}(0) \leq \widetilde{C}(E_0, T) (\widetilde{E}(0) - \widetilde{E}(T)). \quad (4.75)$$

Then

$$\widetilde{E}(T) \leq \frac{\widetilde{C}(E_0, T) - 1}{\widetilde{C}(E_0, T)} \widetilde{E}(0). \quad (4.76)$$

It follows from (3.16), (3.17) and (4.74) that there exists $\widetilde{C}(T) > 0$ such that

$$\widetilde{E}(t) \leq \widetilde{C}(E_0, T) \widetilde{E}(0), \quad \forall 0 \leq t \leq T. \quad (4.77)$$

Note that $\|u\|_{L^2(\mathcal{M})}$ is non-increasing. Therefore, it follows from (4.76) that $\widetilde{E}(t)$ is of exponential decay. Hence, there exist $C_1(E_0), C_2(E_0) > 0$ such that

$$E(t) \leq C_1(E_0) e^{-C_2(E_0)t} E(0), \quad \forall t > 0. \quad (4.78)$$

$\square$

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