Research article

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Combined effects of Choquard and singular nonlinearities in fractional Kirchhoff problems

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Abstract: The aim of this paper is to study the existence and multiplicity of solutions for a class of fractional Kirchhoff problems involving Choquard type nonlinearity and singular nonlinearity. Under suitable assumptions, two nonnegative and nontrivial solutions are obtained by using the Nehari manifold approach combined with the Hardy-Littlewood-Sobolev inequality.

Keywords: Fractional Kirchhoff problem; Choquard nonlinearity; Singular nonlinearity; Nehari manifold method; Multiple solutions

MSC 2020: 35A15; 35B38; 35D30

1 Introduction

In this paper, we study the following Choquard-Kirchhoff problem involving singular nonlinearity:

$$\begin{cases} \mathcal{L}(u) = \lambda \frac{f(x)}{u^\beta} + \left(\int_{\mathbb{R}^N} \frac{g(y)|u(y)|^q}{|x-y|^\mu} dy \right) g(x)u^{q-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $N \geq 2$, $1 < p < N/s$ with $s \in (0, 1)$, $1 < q < p_{\mu,s}^*$ and

$$\mathcal{L}(u) = \left(a + b[u]_{s,p}^{(\theta-1)p} \right) (-\Delta)_p^s u$$

with $a > 0$, $b \geq 0$, $\theta \in [1, 2q]$ and

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{1/p}.$$

Here $\mu \in (0, N)$, $p_{\mu,s}^* = \frac{p}{2} \cdot \frac{2N-\mu}{N-ps}$ is the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, $\lambda, \mu > 0$ are two parameters, $0 < \beta < 1$, and $(-\Delta)_p^s$ is the fractional p -Laplacian which, up to a

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normalization constant, is defined for any $x \in \mathbb{R}^N$ as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dy$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Here $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ centered at x with radius $\varepsilon > 0$. For further details about the fractional Laplacian and its applications, we refer to [31]. Throughout our paper, the following assumptions will be satisfied:

(f₁) $f : \mathbb{R}^N \rightarrow \mathbb{R}^+$ such that $f \in L^q(\mathbb{R}^N)$, where $q = \frac{p_s^*}{p_s^* - 1 + \beta}$ with $p_s^* = \frac{Np}{N - ps}$ is the fractional critical Sobolev exponent;

(g₁) $g \in L^{\frac{2_\mu^* p_s^*}{p_s^* - q 2_\mu^*}}(\mathbb{R}^N)$, where $2_\mu^* = \frac{2N}{2N - \mu}$.

When $s = 1$ and $b = 0$, the equation (1.1) covers the Choquard-Pekar equation which has come forth in quantum physics of a polaron at rest [39] and which describes the modeling of an electron ensnared in its own hole [26], see also [32]. Equation (1.1) is also related with the fractional Kirchhoff model which was first proposed by Fiscella and Valdinoci [11]. Indeed, the study of Kirchhoff-type problems, which arise in various models of physical and biological systems, have received more and more attention in recent years. Precisely, Kirchhoff in [24] extended the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations and established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - M \left(\int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where $M \left(\int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) = \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right)$ and ρ, p_0, h, E, L are constants. For recent results about fractional Kirchhoff problems, we refer to [10, 19–23, 28–30, 49, 50] and the references cited there.

In recent years, much attention has been focused on the existence and properties of nontrivial solutions for fractional Choquard equation involving fractional p -Laplacian, see for example [37, 41, 47, 51]. In [37], Mukherjee and Sreenadh studied the following subcritical Choquard system involving fractional p -Laplacian and perturbations

$$\begin{cases} (-\Delta)_p^s u + a_1(x)u|u|^{p-2} = \alpha(|x|^{-\mu} * |u|^q)|u|^{q-2}u + \beta(|x|^{-\mu} * |v|^q)|u|^{q-2}u + f_1(x) & \text{in } \mathbb{R}^n, \\ (-\Delta)_p^s v + a_2(x)v|v|^{p-2} = \gamma(|x|^{-\mu} * |v|^q)|v|^{q-2}v + \beta(|x|^{-\mu} * |u|^q)|v|^{q-2}v + f_2(x) & \text{in } \mathbb{R}^n. \end{cases}$$

The authors proved that the system admits at least two solutions by means of Nehari manifold and mini-max methods. Pucci, Xiang and Zhang in [41] discussed the following Schrödinger-Choquard-Kirchhoff type fractional p -Laplacian equations with upper critical exponents

$$M(\|u\|_s^p)[(-\Delta)_p^s u + V(x)|u|^{p-2}u] = \lambda f(x, u) + \left(\int_{\mathbb{R}^N} \frac{|u|^{p_{\mu,s}^*}}{|x - y|^\mu} dy \right) |u|^{p_{\mu,s}^* - 2}u \quad \text{in } \mathbb{R}^N,$$

where $p_{\mu,s}^* = (pN - p\mu/2)/(N - ps)$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality and

$$\|u\|_s = \left([u]_{s,p}^p + \int_{\mathbb{R}^N} V(x)|u|^p dx \right)^{1/p}.$$

The authors established the existence and asymptotic behavior of solutions for above problem in the cases of f satisfies superlinear and sublinear nonlinearities, respectively. The main techniques used in the paper are the mountain pass lemma and Ekeland's variational principle. Recently, Yang et al. [51] considered the

following problem

$$\begin{cases} (-\Delta)_p^s(u) = \lambda \frac{|u|^{r-2}u}{|x|^a} + y \left(\int_{\Omega} \frac{|u|^q}{|x-y|^\mu} dy \right) |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.2)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $1 < p < \infty$, $0 < s < 1$, $0 < \mu < N$, $N > sp$, $0 \leq a \leq sp$, $p \leq r \leq p_a^*$, and $p \leq 2q \leq p_{\mu,s}^*$ with $p_a^* = \frac{p(N-a)}{N-ps}$. The authors analyzed the minimizer of energy functional associated to the problem (1.2) on positive Nehari and sign-changing Nehari sets, and obtained the existence of positive and sign-changing solutions for the problem (1.2). Further discussions on Choquard equation can be found in the survey papers [33, 36] and the references cited there.

On the other hand, critical and subcritical fractional problems involving singular nonlinearity have received more and more attention. In [2], Barrios et al. considered the existence of solutions for the problem

$$\begin{cases} (-\Delta)^s u = \lambda \frac{f(x)}{u^y} + Mu^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $y > 0$, $p > 1$ and $M \in \{0, 1\}$. The existence of solutions were obtained by using the approximation method by Boccardo and Orsina. Canino et al. [6] extended the above problem to the fractional p -Laplacian and considered the following problem

$$\begin{cases} (-\Delta)_p^s u = \frac{f(x)}{u^y} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $y > 0$. The authors considered two cases: $y \in (0, 1)$ and $y > 1$. The approximation method by Boccardo and Orsina was used to get the existence of solutions. Moreover, the uniqueness and symmetry of solutions were also investigated. In [12], Fiscella and Mishra studied the following Kirchhoff problem involving singular and critical nonlinearities

$$\begin{cases} \mathcal{L}(u) = \lambda \frac{f(x)}{u^q} + g(x)u^{2_s^*-1} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \quad (1.3)$$

where $0 < q < 1$, $\mathcal{L}(u)$ is defined as in (1.1) with $p = 2$ and $\theta \in [1, 2_s^*/2)$, $\lambda > 0$, and g is a sign-changing function. The authors analyzed the fibering map and gave the compactness property of the energy functional corresponding to the problem (1.3). The authors required b small enough in order to get some key estimates of the energy functional on the Nehari manifold. When λ is small enough, the authors obtained the existence of two positive solutions by using Nehari manifold method. For fractional Kirchhoff problems with singular nonlinearity, we also refer the interested readers to [44]. Very recently, Goel and Sreenadh [16] used the similar method as in [12] to discuss the following critical Choquard-Kirchhoff type problem

$$\begin{cases} \mathcal{K}u = \lambda f(x)|u|^{q-2}u + \left(\int_{\Omega} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy \right) |u|^{2_\mu^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where $\mathcal{K}(u) = -\left(a + \epsilon^p \left(\int_{\Omega} |\nabla u|^2 dx\right)^{\theta-1}\right) \Delta u$ with $a > 0$, $p > N - 2(N \geq 3)$ and $\theta \in [1, 2_\mu^*)$. Here $0 < \mu < N$, $1 < q \leq 2$ and λ is a positive parameter. The authors established the existence of two positive solutions for (1.4). They applied minimization argument on the Nehari sub-manifolds to obtain the first solution. To

get the second solution, the authors divided the proof into two cases: $\mu < \min\{4, N\}$ and $\mu \geq \min\{4, N\}$. Furthermore, do Ó, Giacomoni and Mishra [9] studied fractional Kirchhoff system with critical and concave-convex nonlinearities.

In present paper, we are interested in the multiplicity of solutions for fractional Kirchhoff equations with Choquard and singular nonlinearities. Since the energy functional associated to (1.1) in general is not differentiable on $D^{s,p}(\mathbb{R}^N)$, the usual critical point theory is not available. Inspired by [9, 45, 46], we shall use the Nehari manifold approach to get the existence of two solutions for (1.1). Clearly, equation (1.1) is different from the problems considered in the literature, since (1.1) deals with fractional p -Kirchhoff equations with Choquard type and singular nonlinearities. Thus, our equation and result are new. Definitely, we encounter some difficulties in analyzing the fiberling map and discussing the existence of local minimizes on Nehari manifold. Our discussions are more elaborate than the papers in the literature.

To introduce the main result of this paper more precisely, we first give the definition of weak solutions.

Definition 1.1. We say that $u \in D^{s,p}(\mathbb{R}^N)$ is a (weak) solution of (1.1), if $f(x)u^{-\beta}\phi \in L^1(\mathbb{R}^N)$ and

$$(a + b[u]_{s,p}^{p(\theta-1)})\langle u, \phi \rangle_{s,p} = \lambda \int_{\mathbb{R}^N} f(x)u^{-\beta}\phi dx + \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u(y))^q}{|x-y|^\mu} (u(x))^{q-1}\phi(x) dx dy$$

for all $\phi \in D^{s,p}(\mathbb{R}^N)$, where

$$\langle u, \phi \rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+ps}} dx dy.$$

Let

$$\Lambda_0 := \left[\frac{p-1+\beta}{2q-p} \left(\frac{a(2q-p)}{2q-1+\beta} \right)^{\frac{2q-1+\beta}{p-1+\beta}} S^{\frac{2q-1+\beta}{p-1+\beta}} (C_g(N, \mu))^{-1} \right]^{\frac{p}{2q-p}}, \quad (1.5)$$

$$\Lambda_{00} := \left(\frac{a(p-1+\beta)S^{\frac{2q}{p}}}{C_g(N, \mu)(2q-1+\beta)} \right)^{\frac{p-1+\beta}{2q-p}} \frac{S^{\frac{1-\beta}{p}} a(2q-p)}{(2q-1+\beta)\|f\|_{\frac{p_s^*}{p_s^*-1+\beta}}} \quad (1.6)$$

and

$$\Lambda_* := \min\{\Lambda_0, \Lambda_{00}\}. \quad (1.7)$$

Here $S > 0$ denotes the best constant of embedding from $D^{s,p}(\mathbb{R}^N)$ to $L^{p_s^*}(\mathbb{R}^N)$ and $C_g(N, \mu) > 0$ will be given by (2.2).

Our result is the following theorem.

Theorem 1.1. Let $N \geq 2$, $1 < p < N/s$ with $s \in (0, 1)$, $1 < q < p_{\mu,s}^*$ and $\theta \in [1, 2q)$. Assume that $a > 0$, $b \geq 0$, (f_1) and (g_1) hold. Then equation (1.1) has at least two nonnegative and nontrivial solutions for all $\lambda \in (0, \Lambda_*)$.

The rest of our paper is organized as follows. In Section 2, we recall some definitions and preliminaries which will be used in our discussion. In Section 3, the properties of fiberling maps are analyzed. Furthermore, a compactness result is also given. In Section 4, two nontrivial and nonnegative solutions are obtained by applying the Nehari manifold approach.

2 Preliminaries

In this section, we recall some basic results on fractional Sobolev spaces and the Hardy-Littlewood-Sobolev inequality. For the details, we refer to [8, 38, 42, 43].

Firstly, we denote by $D^{s,p}(\mathbb{R}^N)$ the usual fractional Sobolev space

$$D^{s,p}(\mathbb{R}^N) := \left\{ u \in L^{p_s^*}(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

endowed with the norm

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{1/p}.$$

Theorem 2.1. (Hardy-Littlewood-Sobolev inequality, see [27]) Assume that $1 < r, t < \infty$, $0 < \mu < N$ and

$$\frac{1}{r} + \frac{1}{t} + \frac{\mu}{N} = 2.$$

Then there exists $C(N, \mu, r, t) > 0$ such that

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)| \cdot |v(y)|}{|x - y|^\mu} dx dy \leq C(N, \mu, r, t) \|u\|_r \|v\|_t \quad (2.1)$$

for all $u \in L^r(\mathbb{R}^N)$ and $v \in L^t(\mathbb{R}^N)$. If $r = t = 2N/(2N - \mu)$, then

$$C(N, \mu, r, t) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N-\mu}{2})}{\Gamma(\frac{2N-\mu}{2})} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right)^{-1 + \frac{\mu}{N}}.$$

In this case, the equality in (2.1) holds if and only if $u = ch$ and

$$h(x) = A(y^2 + |x - x_0|^2)^{-\frac{(2N-\mu)}{2}}$$

for some $A \in \mathbb{R}$, $0 \neq y \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$.

Note that, by Theorem 2.1, we know

$$\iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)|u(x)|^q|u(y)|^q}{|x - y|^\mu} dx dy$$

is finite if $g|u|^q \in L^{2_\mu^*}(\mathbb{R}^N)$ for some $2_\mu^* > 1$ defined as

$$2_\mu^* = \frac{2N}{2N - \mu}.$$

Hence, by the fractional Sobolev embedding theorem, if $u \in D^{s,p}(\mathbb{R}^N)$ this occurs provided that $1 < 2_\mu^* q < p_s^*$

and $g \in L^{\frac{p_s^* 2_\mu^*}{p_s^* - 2_\mu^* q}}(\mathbb{R}^N)$. Thus, q has to satisfy

$$q < \frac{(N - \mu/2)p}{N - sp} = p_{\mu,s}^*.$$

Hence, $p_{\mu,s}^*$ is said to be the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

By Theorem 2.1 and $g \in L^{\frac{p_s^* 2_\mu^*}{p_s^* - 2_\mu^* q}}(\mathbb{R}^N)$, we have

$$\begin{aligned} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)|u(x)|^q|u(y)|^q}{|x - y|^\mu} dx dy &\leq C(N, \mu) \|g(x)|u|^q\|_{2_\mu^*}^2 \\ &\leq C(N, \mu) \|g\|_{\frac{p_s^* 2_\mu^*}{p_s^* - 2_\mu^* q}}^2 \|u\|_{p_s^*}^{2q} \text{ for all } u \in D^{s,p}(\mathbb{R}^N), \end{aligned}$$

where $C(N, \mu) > 0$ is a suitable constant. Here and from now on, we shortly denote by $\|\cdot\|_v$ for the norm of Lebesgue space $L^v(\mathbb{R}^N)$. Further, by the fractional Sobolev inequality, we get

$$\iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)|u(x)|^q|u(y)|^q}{|x-y|^\mu} dx dy \leq C_g(N, \mu) S^{-\frac{2q}{p}} [u]_{s,p}^{2q} \quad \text{for all } u \in D^{s,p}(\mathbb{R}^N), \quad (2.2)$$

where $C_g(N, \mu) = C(N, \mu) \|g\|_{\frac{p_s^* 2\mu}{p_s^* - 2\mu q}}^2$ and $S > 0$ is given by

$$S = \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{s,p}^p}{\|u\|_{p_s^*}^p}.$$

3 Fibering map analysis

First, we define the energy functional $I_\lambda : D^{s,p}(\mathbb{R}^N) \rightarrow \mathbb{R}$ corresponding to equation (1.1) as

$$I_\lambda(u) = \frac{a}{p} [u]_{s,p}^p + \frac{b}{p\theta} [u]_{s,p}^{p\theta} - \frac{\lambda}{1-\beta} \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx - \frac{1}{2q} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy.$$

Here $u^+ = \max\{u, 0\}$. For each $u \in D^{s,p}(\mathbb{R}^N)$, we define the fibering map $\Phi_{\lambda,u} : \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$\Phi_{\lambda,u}(t) = I_\lambda(tu)$$

for all $t > 0$. A simple calculation gives that

$$\Phi'_{\lambda,u}(t) = at^{p-1} [u]_{s,p}^p + bt^{\theta p-1} [u]_{s,p}^{p\theta} - \lambda t^{-\beta} \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx - t^{2q-1} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy$$

and

$$\begin{aligned} \Phi''_{\lambda,u}(t) &= a(p-1)t^{p-2} [u]_{s,p}^p + b(\theta p-1)t^{\theta p-2} [u]_{s,p}^{p\theta} + \lambda \beta t^{-\beta-1} \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx \\ &\quad - (2q-1)t^{2q-2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy. \end{aligned}$$

Define the Nehari manifold

$$\begin{aligned} \mathcal{N}_\lambda &= \{u \in D^{s,p}(\mathbb{R}^N) : a[u]_{s,p}^p + b[u]_{s,p}^{p\theta} - \lambda \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx \\ &\quad - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy = 0\}. \end{aligned} \quad (3.1)$$

Then $u \in \mathcal{N}_\lambda$ if and only if $\Phi'_{\lambda,u}(1) = 0$. Thus, it is natural to divide $\mathcal{N}_\lambda$ into three parts as

$$\mathcal{N}_\lambda^+ = \{u \in \mathcal{N}_\lambda : \Phi''_{\lambda,u}(1) > 0\}, \quad (3.2)$$

$$\mathcal{N}_\lambda^0 = \{u \in \mathcal{N}_\lambda : \Phi''_{\lambda,u}(1) = 0\} \quad (3.3)$$

and

$$\mathcal{N}_\lambda^- = \{u \in \mathcal{N}_\lambda : \Phi''_{\lambda,u}(1) < 0\}. \quad (3.4)$$

Lemma 3.1. For $u \in D^{s,p}(\mathbb{R}^N)$ and $\lambda \in (0, \Lambda_0)$. Then there exist unique $t_{\max} = t_{\max}(u) > 0$, $t^+ = t^+(u) > 0$, $t^- = t^-(u) > 0$ with $t^+ < t_{\max} < t^-$ such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$ and $I_\lambda(t^+u) = \min_{0 \leq t \leq t^-} I_\lambda(tu)$, $I_\lambda(t^-u) = \max_{t \geq t_{\max}} I_\lambda(tu)$.

Proof. For fixed $u \in D^{s,p}(\mathbb{R}^N)$, define $\psi_{\lambda,u} : \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$\psi_{\lambda,u}(t) = at^{p-2q}[u]_{s,p}^p + bt^{p\theta-2q}[u]_{s,p}^{p\theta} - \lambda t^{-\beta-2q+1} \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy.$$

A simple calculation gives that

$$\psi'_{\lambda,u}(t) = a(p-2q)t^{p-2q-1}[u]_{s,p}^p + b(p\theta-2q)t^{p\theta-2q-1}[u]_{s,p}^{p\theta} + (\beta+2q-1)t^{-\beta-2q}\lambda \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx. \quad (3.5)$$

This means that $\psi'_{\lambda,u}(t) = t^{p\theta-2q-1}h_{\lambda,u}(t)$, where

$$h_{\lambda,u}(t) = a(p-2q)t^{p-p\theta}[u]_{s,p}^p + b(p\theta-2q)[u]_{s,p}^{p\theta} + (\beta+2q-1)t^{-p\theta+1-\beta} \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx.$$

Let $h'_{\lambda,u}(t) = 0$. Then we have

$$t = \left(\frac{(\beta+2q-1)(p\theta+1-\beta)\lambda \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx}{a(2q-p)(p\theta-p)[u]_{s,p}^p} \right)^{\frac{1}{p-1+\beta}}.$$

Observe that

$$\lim_{t \rightarrow 0^+} h_{\lambda,u}(t) = -\infty \text{ and } \lim_{t \rightarrow +\infty} h_{\lambda,u}(t) = b(p\theta-2q)[u]_{s,p}^{p\theta} < 0,$$

Then there exists a unique $t_{\max} > 0$ such that $h_{\lambda,u}(t_{\max}) = 0$. Thus, we have $\psi'_{\lambda,u}(t_{\max}) = 0$, $\psi_{\lambda,u}(t)$ is increasing on $(0, t_{\max})$, decreasing on $(t_{\max}, +\infty)$.

In the following, we estimate $\psi_{\lambda,u}(t_{\max})$ as follows

$$\begin{aligned} \psi_{\lambda,u}(t_{\max}) &= \max_{t>0} \psi_{\lambda,u}(t) \\ &\geq \max_{t>0} \left\{ at^{p-2q}[u]_{s,p}^p - t^{-\beta-2q+1}\lambda \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy \right\} \\ &= \frac{p+\beta-1}{2q-p} \left(\frac{2q-p}{2q-1+\beta} \right)^{\frac{2q+\beta-1}{p+\beta-1}} \frac{(a[u]_{s,p}^p)^{\frac{2q-1+\beta}{p+\beta-1}}}{(\lambda \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx)^{\frac{2q-p}{p-1+\beta}}} - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy. \end{aligned}$$

It follows from the condition (f_1) and Hölder's inequality that

$$\lambda \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx \leq S^{-\frac{1-\beta}{p}} \lambda \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}} [u]_{s,p}^{1-\beta}. \quad (3.6)$$

By (3.6) and (2.2), we deduce that

$$\psi_{\lambda,u}(t_{\max}) \geq \left[\frac{\frac{p+\beta-1}{2q-p} \left(\frac{a(2q-p)}{2q-1+\beta} \right)^{\frac{2q+\beta-1}{p+\beta-1}}}{\left(S^{-\frac{1-\beta}{p}} \lambda \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}} \right)^{\frac{2q-p}{p-1+\beta}}} - C_g(N, \mu) S^{-\frac{2q}{p}} \right] [u]_{s,p}^{2q} := C(\lambda) [u]_{s,p}^{2q},$$

and $C(\lambda) = 0$ if and only if

$$\lambda = \Lambda_0 := \left[\frac{p-1+\beta}{2q-p} \left(\frac{a(2q-p)}{2q-1+\beta} \right)^{\frac{2q-1+\beta}{p-1+\beta}} S^{\frac{2q-1+\beta}{p-1+\beta}} (C_g(N, \mu))^{-1} \right]^{\frac{p}{2q-p}}.$$

When $\lambda \in (0, \Lambda_0)$, we have $C(\lambda) > 0$ and $\psi_{\lambda, u}(t_{\max}) > 0$. Thus we can find two zero points t^+ and t^- of $\psi_{\lambda, u}(t)$ with $t^+ = t^+(u) < t_{\max}$ and $t^- = t^-(u) > t_{\max}$. That is $t^+u, t^-u \in \mathcal{N}_\lambda$. By (3.5), we have that $\psi'_{\lambda, u}(t^+) > 0$ and $\psi'_{\lambda, u}(t^-) < 0$. Therefore $t^+u \in \mathcal{N}_\lambda^+$ and $t^-u \in \mathcal{N}_\lambda^-$.

In view of $1 - \beta < p\theta < 2q$, we derive that $\lim_{t \rightarrow 0^+} \psi_{\lambda, u}(t) = -\infty$ and $\lim_{t \rightarrow +\infty} \psi_{\lambda, u}(t) < 0$, which yield that $\psi_{\lambda, u}(t) < 0$ for all $t \in [0, t^+)$ and $t \in (t^-, +\infty)$, $\psi_{\lambda, u}(t) > 0$ for all $t \in (t^+, t^-)$. Consequently, $I_\lambda(t^+u) = \min_{0 \leq t \leq t^+} I_\lambda(tu)$ and $I_\lambda(t^-u) = \max_{t \geq t_{\max}} I_\lambda(tu)$. $\square$

Lemma 3.2. For all $\lambda \in (0, \Lambda_{00})$, we have $\mathcal{N}_\lambda^0 = \{0\}$.

Proof. Arguing by contradiction, we assume that there exists $u \in \mathcal{N}_\lambda^0 \setminus \{0\}$. Then we have

$$a(p-1+\beta)[u]_{s,p}^p + b(p\theta-1+\beta)[u]_{s,p}^{p\theta} = (2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy, \quad (3.7)$$

and

$$a(2q-p)[u]_{s,p}^p + b(2q-p\theta)[u]_{s,p}^{p\theta} = (2q-1+\beta)\lambda \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx. \quad (3.8)$$

Consider functional $T_\lambda : \mathcal{N}_\lambda \rightarrow \mathbb{R}$ defined as

$$T_\lambda(v) = \frac{a(p-1+\beta)[v]_{s,p}^p + b(p\theta-1+\beta)[v]_{s,p}^{p\theta}}{2q-1+\beta} - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy$$

for all $v \in \mathcal{N}_\lambda$. By (3.7), we obtain that $T_\lambda(u) = 0$ for all $u \in \mathcal{N}_\lambda^0 \setminus \{0\}$. It follows from the definition of T and (2.2) that

$$0 = T_\lambda(u) \geq \frac{a(p-1+\beta)}{2q-1+\beta} [u]_{s,p}^p - C_g(N, \mu) S^{\frac{2q}{p}} [u]_{s,p}^{2q},$$

which implies that

$$[u]_{s,p} \geq \left(\frac{a(p-1+\beta)S^{\frac{-2q}{p}}}{C_g(N, \mu)(2q-1+\beta)} \right)^{\frac{1}{2q-p}}. \quad (3.9)$$

On the other hand, we deduce from (3.8) and (3.6) that

$$\begin{aligned} a(2q-p)[u]_{s,p}^p &< a(2q-p)[u]_{s,p}^p + b(2q-p\theta)[u]_{s,p}^{p\theta} \\ &\leq \lambda(2q-1+\beta) \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx \\ &\leq \lambda S^{-\frac{1-\beta}{p}} (2q-1+\beta) \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}} [u]_{s,p}^{1-\beta}, \end{aligned}$$

which yields that

$$[u]_{s,p} < \left(\frac{S^{-\frac{1-\beta}{p}} \lambda(2q-1+\beta) \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}}}{a(2q-p)} \right)^{\frac{1}{p-1+\beta}}. \quad (3.10)$$

Combining (3.9) with (3.10), we obtain

$$\lambda > \left(\frac{a(p-1+\beta)S^{\frac{2q}{p}}}{C_g(N, \mu)(2q-1+\beta)} \right)^{\frac{p-1+\beta}{2q-p}} \frac{S^{\frac{1-\beta}{p}} a(2q-p)}{(2q-1+\beta) \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}}} := \Lambda_{00}.$$

This contradicts the assumption $\lambda < \Lambda_{00}$. In conclusion, we prove that $\mathcal{N}_\lambda^0 = \{0\}$ for all $\lambda \in (0, \Lambda_{00})$. $\square$

Lemma 3.3. *Let $\lambda \in (0, \Lambda_{00})$. Then*

$$[U]_{s,p} > A_0 > A_\lambda > [u]_{s,p} \text{ for all } u \in \mathcal{N}_\lambda^+ \text{ and } U \in \mathcal{N}_\lambda^-,$$

where

$$A_0 := \left[\frac{a(p-1+\beta)S^{\frac{2q}{p}}}{(2q-1+\beta)C_g(N, \mu)} \right]^{\frac{1}{2q-p}}, \quad A_\lambda := \left(\frac{(2q-1+\beta)S^{-\frac{1-\beta}{p}} \lambda \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}}}{a(2q-p)} \right)^{\frac{1}{p+\beta-1}}.$$

Proof. For $u \in \mathcal{N}_\lambda^+$. By (f_1) and (3.6), we get

$$\begin{aligned} a(2q-p)[u]_{s,p}^p + b(2q-p\theta)[u]_{s,p}^{p\theta} &< (2q-1+\beta)\lambda \int_{\Omega} f(x)(u^+)^{1-\beta} dx \\ &\leq (2q-1+\beta)S^{-\frac{1-\beta}{p}} \lambda \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}} [u]_{s,p}^{1-\beta}, \end{aligned}$$

which implies

$$[u]_{s,p} < \left(\frac{(2q-1+\beta)S^{-\frac{1-\beta}{p}} \lambda \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}}}{a(2q-p)} \right)^{\frac{1}{p+\beta-1}} := A_\lambda.$$

For $U \in \mathcal{N}_\lambda^-$. It follows from (2.2) that

$$\begin{aligned} a(p-1+\beta)[U]_{s,p}^p + b(p\theta-1+\beta)[U]_{s,p}^{p\theta} &< (2q-1+\beta) \int_{\Omega} \int_{\Omega} \frac{g(x)g(y)(U^+(x))^q (U^+(y))^q}{|x-y|^\mu} dx dy \\ &\leq (2q-1+\beta)C_g(N, \mu)S^{-\frac{2q}{p}} [U]_{s,p}^{2q}, \end{aligned}$$

which leads to

$$[U]_{s,p} > \left[\frac{a(p-1+\beta)S^{\frac{2q}{p}}}{(2q-1+\beta)C_g(N, \mu)} \right]^{\frac{1}{2q-p}} := A_0.$$

Clearly, $A_0 > A_\lambda$ since $\lambda \in (0, \Lambda_{00})$. Thus, the proof is complete. $\square$

Lemma 3.4. *Let $\lambda \in (0, \Lambda_{00})$. Then $\mathcal{N}_\lambda^-$ is a closed subset of $\mathcal{N}_\lambda$.*

Proof. The desired result follows by combining Lemma 3.2 and Lemma 3.3. $\square$

Lemma 3.5. *For $\lambda > 0$ and $u \in \mathcal{N}_\lambda^+$. There exist $R > 0$ and a continuous function $\zeta : B_R(0) \rightarrow \mathbb{R}^+$ such that*

$$\zeta(v) > 0, \quad \zeta(0) = 1, \quad \zeta(v)(u+v) \in \mathcal{N}_\lambda^+ \text{ for all } v \in B_R(0),$$

where $B_R(0) = \{v \in D^{s,p}(\mathbb{R}^N) : [v]_{s,p} < R\}$.

Proof. We only show the proof for the case $u \in \mathcal{N}_\lambda^+$ while the proof of the case $u \in \mathcal{N}_\lambda^-$ is similar. Define $F : D^{s,p}(\mathbb{R}^N) \times \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$F(v, t) = at^{p-1+\beta}[u+v]_{s,p}^p + bt^{p\theta-1+\beta}[u+v]^{p\theta} - \lambda \int_{\mathbb{R}^N} f(x)((u+v)^+)^{1-\beta} dx \\ - t^{2q-1+\beta} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u+v)^+(x))^q((u+v)^+(y))^q}{|x-y|^\mu} dx dy.$$

Since $u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, it follows that

$$F(0, 1) = a[u]_{s,p}^p + b[u]_{s,p}^{p\theta} - \lambda \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy = 0. \quad (3.11)$$

and

$$\left. \frac{\partial F}{\partial t} \right|_{(0,1)} = a(p-1+\beta)[u]_{s,p}^p + b(p\theta-1+\beta)[u]_{s,p}^{p\theta} - (2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy > 0. \quad (3.12)$$

Applying the implicit function theorem at the point $(0, 1)$, there exists $\hat{R} > 0$ such that for every $v \in D^{s,p}(\mathbb{R}^N)$ with $[v]_{s,p} < \hat{R}$, the equation $F(v, t) = 0$ has a unique solution $t = \zeta(v) > 0$. By (3.11), we have $\zeta(0) = 1$. This together with $F(v, \zeta(v)) = 0$ for every $v \in D^{s,p}(\mathbb{R}^N)$ with $[v]_{s,p} < \hat{R}$ yields that

$$0 = a\zeta^{p-1+\beta}(v)[u+v]_{s,p}^p + b\zeta^{p\theta-1+\beta}(v)[u+v]_{s,p}^{p\theta} - \lambda \int_{\mathbb{R}^N} f(x)((u+v)^+)^{1-\beta} dx \\ - \zeta^{2q-1+\beta} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u+v)^+(x))^q((u+v)^+(y))^q}{|x-y|^\mu} dx dy \\ = \frac{1}{\zeta^{1-\beta}(v)} \left(a[\zeta(v)(u+v)]_{s,p}^p + b[\zeta(v)(u+v)]_{s,p}^{p\theta} - \lambda \int_{\mathbb{R}^N} f(x)(\zeta(v)(u+v)^+)^{1-\beta} dx \right. \\ \left. - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(\zeta(v)(u+v)^+(x))^q(\zeta(v)(u+v)^+(y))^q}{|x-y|^\mu} dx dy \right),$$

which means that

$$\zeta(v)(u+v) \in \mathcal{N}_\lambda \text{ for all } v \in D^{s,p}(\mathbb{R}^N) \text{ with } [v]_{s,p} < \hat{R}.$$

Observe by (3.12) that

$$\left. \frac{\partial F}{\partial t} \right|_{(v, \zeta(v))} = a(p-1+\beta)\zeta^{p-2+\beta}(v)[u+v]_{s,p}^p + b(p\theta-1+\beta)\zeta^{p\theta-2+\beta}(v)[u+v]_{s,p}^{p\theta} \\ - (2q-1+\beta)\zeta^{p\theta-2+\beta}(v) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u+v)^+(x))^q((u+v)^+(y))^q}{|x-y|^\mu} dx dy \\ = \frac{1}{\zeta^{2-\beta}(v)} \left[a(p-1+\beta)[\zeta(v)(u+v)]_{s,p}^p + b(p\theta-1+\beta)[\zeta(v)(u+v)]_{s,p}^{p\theta} \right. \\ \left. - (2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(\zeta(v)(u+v)^+(x))^q(\zeta(v)(u+v)^+(y))^q}{|x-y|^\mu} dx dy \right].$$

Then we can choose sufficiently small $R \in (0, \hat{R})$ such that

$$a(p-1+\beta)[\zeta(v)(u+v)]_{s,p}^p + b(p\theta-1+\beta)[\zeta(v)(u+v)]_{s,p}^{p\theta}$$

$$-(2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(\zeta(v)(u+v)^+(x))^q(\zeta(v)(u+v)^+(y))^q}{|x-y|^\mu} dx dy > 0,$$

for all $v \in D^{s,p}(\mathbb{R}^N)$ with $[v]_{s,p} < R$. Thus,

$$\zeta(v)(u+v) \in \mathcal{N}_\lambda^+ \text{ for all } v \in B_R(0).$$

The proof is completed. $\square$

Lemma 3.6. *The functional I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. For $u \in \mathcal{N}_\lambda$, it follows from $\theta p < 2q$ and (3.6) that

$$\begin{aligned} I_\lambda(u) &= \frac{a(2q-p)}{2pq} [u]_{s,p}^p + \frac{b(2q-p\theta)}{2p\theta q} [u]_{s,p}^{p\theta} - \left(\frac{1}{1-\beta} - \frac{1}{2q} \right) \lambda \int_{\mathbb{R}^N} f(x)(u^+)^{1-\beta} dx \\ &\geq \frac{a(2q-p)}{2pq} [u]_{s,p}^p - \left(\frac{1}{1-\beta} - \frac{1}{2q} \right) S^{\frac{\beta-1}{p}} \lambda \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}} [u]_{s,p}^{1-\beta}. \end{aligned}$$

Since $p > 1-\beta$, we know that I_λ is coercive on $\mathcal{N}_\lambda$. Define

$$G(t) = \frac{a(2q-p)}{2pq} t^p - \left(\frac{1}{1-\beta} - \frac{1}{2q} \right) S^{\frac{\beta-1}{p}} \lambda \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}} t^{1-\beta},$$

then $G(t)$ attains its minimum at

$$t_{\min} = \left(\frac{2q-1+\beta}{a(2q-p)} S^{\frac{\beta-1}{p}} \lambda \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}} \right)^{\frac{1}{p-1+\beta}}.$$

Thus

$$I_\lambda(u) \geq \left(\frac{a(2q-p)}{2pq} \right)^{\frac{\beta-1}{p-1+\beta}} \left(\frac{\left(1 - \frac{1-\beta}{2q}\right) S^{\frac{\beta-1}{p}} \lambda \|f\|_{\frac{p_s^*}{p_s^*-1+\beta}}}{p} \right)^{\frac{p}{p-1+\beta}} \frac{1-\beta-p}{1-\beta}.$$

Therefore, we get that I_λ is bounded from below on $\mathcal{N}_\lambda$. $\square$

Set

$$c_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+ \cup \{0\}} I_\lambda(u) \text{ and } c_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u).$$

By Lemma 3.2 and 3.4, we know that $\mathcal{N}_\lambda^+ \cup \{0\}$ and $\mathcal{N}_\lambda^-$ are two closed sets in E . Using Ekeland's variational principle [1], we can extract a minimizing sequence $(u_n)_n \subset \mathcal{N}_\lambda^+ \cup \{0\}(\mathcal{N}_\lambda^-)$ with

$$c_\lambda^+(c_\lambda^-) < I_\lambda(u_n) < c_\lambda^+(c_\lambda^-) + \frac{1}{n}, \quad I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n}[v - u_n]_{s,p} \quad (3.13)$$

for all $v \in \mathcal{N}_\lambda^+ \cup \{0\}(\mathcal{N}_\lambda^-)$. In view of Lemma 3.6, we derive that the sequence $(u_n)_n$ is bounded in $\mathcal{N}_\lambda$ with $[u_n]_{s,p} \leq C_1$ for some $C_1 > 0$. Thus, up to a subsequence still denoted by $(u_n)_n$ we may assume that there exists $u_0 \in D^{s,p}(\mathbb{R}^N)$ such that

$$\begin{cases} u_n \rightharpoonup u_0 \text{ weakly in } D^{s,p}(\mathbb{R}^N) \\ u_n \rightarrow u_0 \text{ a.e. in } \mathbb{R}^N. \end{cases} \quad (3.14)$$

Lemma 3.7. *For $\lambda \in (0, \Lambda_0)$. Then there exists a constant $C_2 > 0$ such that the following conclusions hold:*

(i) if $(u_n)_n \subset \mathcal{N}_\lambda^+$, for each $n \in \mathbb{N}$, we have

$$a(p-1+\beta)[u_n]_{s,p}^p + b(p\theta-1+\beta)[u_n]_{s,p}^{p\theta} - (2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy \geq C_2;$$

(ii) if $(u_n)_n \subset \mathcal{N}_\lambda^-$, for each $n \in \mathbb{N}$, we have

$$a(p-1+\beta)[u_n]_{s,p}^p + b(p\theta-1+\beta)[u_n]_{s,p}^{p\theta} - (2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy \leq -C_2.$$

Proof. We only prove the case (i), while the proof of (ii) follows similarly. Since $(u_n)_n \subset \mathcal{N}_\lambda^+$, it suffices to prove that

$$\liminf_{n \rightarrow \infty} [a(2q-p)[u_n]_{s,p}^p + b(2q-p\theta)[u_n]_{s,p}^{p\theta}] < (2q-1+\beta)\lambda \int_{\mathbb{R}^N} f(x)(u_0^+)^{1-\beta} dx. \quad (3.15)$$

Arguing by contradiction, we assume that

$$\liminf_{n \rightarrow \infty} [a(2q-p)[u_n]_{s,p}^p + b(2q-p\theta)[u_n]_{s,p}^{p\theta}] = (2q-1+\beta)\lambda \int_{\mathbb{R}^N} f(x)(u_0^+)^{1-\beta} dx.$$

Since $(u_n)_n \subset \mathcal{N}_\lambda^+$, we have

$$a(2q-p)[u_n]_{s,p}^p + b(2q-p\theta)[u_n]_{s,p}^{p\theta} < (2q-1+\beta)\lambda \int_{\mathbb{R}^N} f(x)(u_n^+)^{1-\beta} dx.$$

By $f \in L^{\frac{p_s^*}{p_s^*-1+\beta}}(\mathbb{R}^N)$ and Vitali's convergence theorem, one can prove that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x)(u_n^+)^{1-\beta} dx = \int_{\mathbb{R}^N} f(x)(u_0^+)^{1-\beta} dx$$

Then

$$\begin{aligned} \liminf_{n \rightarrow \infty} [a(2q-p)[u_n]_{s,p}^p + b(2q-p\theta)[u_n]_{s,p}^{p\theta}] &< \limsup_{n \rightarrow \infty} [a(2q-p)[u_n]_{s,p}^p + b(2q-p\theta)[u_n]_{s,p}^{p\theta}] \\ &\leq (2q-1+\beta)\lambda \int_{\mathbb{R}^N} f(x)(u_0^+)^{1-\beta} dx, \end{aligned}$$

which yields that

$$\lim_{n \rightarrow \infty} [a(2q-p)[u_n]_{s,p}^p + b(2q-p\theta)[u_n]_{s,p}^{p\theta}] = (2q-1+\beta)\lambda \int_{\mathbb{R}^N} f(x)(u_0^+)^{1-\beta} dx.$$

Hence, there exists $A > 0$ such that $[u_n]_{s,p}^p \rightarrow A$ as $n \rightarrow \infty$. Consequently, we get

$$a(2q-p)A + b(2q-p\theta)A^\theta = (2q-1+\beta)\lambda \int_{\mathbb{R}^N} f(x)(u_0^+)^{1-\beta} dx. \quad (3.16)$$

By Lemma 3.1, for $\lambda \in (0, \Lambda_0)$, we have

$$0 < \frac{p-1+\beta}{2q-p} \left(\frac{2q-p}{2q-1+\beta} \right)^{\frac{2q-1+\beta}{p-1+\beta}} \frac{(a[u_n]_{s,p}^p)^{\frac{2q-1+\beta}{p-1+\beta}}}{(\lambda \int_{\mathbb{R}^N} f(x)(u_n^+)^{1-\beta} dx)^{\frac{2q-p}{p-1+\beta}}} - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy.$$

Hence

$$0 \leq \frac{p-1+\beta}{2q-p} \left(\frac{2q-p}{2q-1+\beta} \right)^{\frac{2q-1+\beta}{p-1+\beta}} \frac{(aA)^{\frac{2q-p}{p-1+\beta}}}{(\lambda \int_{\mathbb{R}^N} f(x)(u_0^+)^{1-\beta} dx)^{\frac{2q-p}{p-1+\beta}}} - \lim_{n \rightarrow \infty} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy. \quad (3.17)$$

Since $(u_n)_n \subset \mathcal{N}_\lambda^+$, it follows from (3.16) that

$$\lim_{n \rightarrow \infty} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy = \frac{aA(p-1+\beta)}{2q-1+\beta} + \frac{bA^\theta(p\theta-1+\beta)}{2q-1+\beta}. \quad (3.18)$$

Substituting (3.16) and (3.18) into (3.17), we arrive at

$$\begin{aligned} & \frac{p-1+\beta}{2q-p} \left(\frac{2q-p}{2q-1+\beta} \right)^{\frac{2q-1+\beta}{p-1+\beta}} (aA)^{\frac{2q-1+\beta}{p-1+\beta}} \left[\frac{aA(2q-p)}{2q-1+\beta} \right. \\ & \left. + \frac{bA^\theta(2q-p\theta)}{2q-1+\beta} \right]^{\frac{p-2q}{p-1+\beta}} - \frac{aA(p-1+\beta)}{2q-1+\beta} - \frac{bA^\theta(p\theta-1+\beta)}{2q-1+\beta} \geq 0, \end{aligned}$$

this together with $2q > p$ implies that

$$\frac{p-1+\beta}{2q-p} \left(\frac{2q-p}{2q-1+\beta} \right)^{\frac{2q-1+\beta}{p-1+\beta}} (aA)^{\frac{2q-1+\beta}{p-1+\beta}} \left[\frac{aA(2q-p)}{2q-1+\beta} \right]^{\frac{p-2q}{p-1+\beta}} - \frac{aA(p-1+\beta)}{2q-1+\beta} - \frac{bA^\theta(p\theta-1+\beta)}{2q-1+\beta} \geq 0.$$

It follows that

$$\begin{aligned} -\frac{bA^\theta(p\theta-1+\beta)}{2q-1+\beta} & \geq \frac{aA(p-1+\beta)}{2q-1+\beta} - \frac{p-1+\beta}{2q-p} \left(\frac{2q-p}{2q-1+\beta} \right)^{\frac{2q-1+\beta}{p-1+\beta}} (aA)^{\frac{2q-1+\beta}{p-1+\beta}} \left[\frac{aA(2q-p)}{2q-1+\beta} \right]^{\frac{p-2q}{p-1+\beta}} \\ & = 0, \end{aligned}$$

which is impossible. Therefore, the proof is complete. $\square$

For $\varphi \in D^{s,p}(\mathbb{R}^N)$ with $\varphi \geq 0$. Recalling the constants $C_1 > 0$ with $[u_n]_{s,p} \leq C_1$ and $C_2 > 0$ given in Lemma 3.7. Thus, for $n \in \mathbb{N}$ sufficiently large so that

$$\frac{1-\beta}{n} C_1 \leq C_2. \quad (3.19)$$

Lemma 3.5 guarantees to extract a sequence of functions $(\zeta_n)_n$ satisfying $\zeta_n(0) = 1$ and $\zeta_n(t\varphi)(u_n + t\varphi) \in \mathcal{N}_\lambda^+$ for $t > 0$ sufficiently small. By $(u_n)_n \subset \mathcal{N}_\lambda$ and $\zeta_n(t\varphi)(u_n + t\varphi) \in \mathcal{N}_\lambda$, we get

$$a[u_n]_{s,p}^p + b[u_n]_{s,p}^{p\theta} - \lambda \int_{\mathbb{R}^N} f(x)(u_n^+)^{1-\beta} dx - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy = 0 \quad (3.20)$$

and

$$\begin{aligned} & a\zeta_n^p(t\varphi)[u_n + t\varphi]_{s,p}^p + b\zeta_n^{p\theta}(t\varphi)[u_n + t\varphi]_{s,p}^{p\theta} - \lambda \zeta_n^{1-\beta}(t\varphi) \int_{\mathbb{R}^N} f(x)((u_n + t\varphi)^+)^{1-\beta} dx \\ & - \zeta_n^{2q}(t\varphi) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_n + t\varphi)^+(x))^q((u_n + t\varphi)^+(y))^q}{|x-y|^\mu} dx dy = 0, \end{aligned} \quad (3.21)$$

where $\zeta_n'(0)$ denotes the derivative of ζ_n' at the point 0 with $(\zeta_n'(0), \varphi) \in \mathbb{R}$ for every $\varphi \in D^{s,p}(\mathbb{R}^N)$. If it does not exist, $\zeta_n'(0)$ should be replaced by $\lim_{k \rightarrow \infty} \frac{\zeta_n(t_n \varphi) - 1}{t_n}$ for some sequence $(t_n)_{n=1}^\infty$ with $\lim_{n \rightarrow \infty} t_n = 0$ and $t_n > 0$.

Lemma 3.8. Let $\lambda \in (0, \Lambda_0)$ and $(u_n)_n \subset \mathcal{N}_\lambda^+$. Then $(\zeta_n'(0), \varphi)$ is uniformly bounded for every $\varphi \in D^{s,p}(\mathbb{R}^N)$ with $\varphi \geq 0$.

Proof. We only prove the case that $(u_n)_n \subset \mathcal{N}_\lambda^+$. In view of (3.20) and (3.21), we have

$$\begin{aligned} 0 & = a[\zeta_n^p(t\varphi) - 1][u_n + t\varphi]_{s,p}^p + a([u_n + t\varphi]_{s,p}^p - [u_n]_{s,p}^p) + b[\zeta_n^{p\theta}(t\varphi) - 1][u_n + t\varphi]_{s,p}^{p\theta} + b([u_n + t\varphi]_{s,p}^{p\theta} - [u_n]_{s,p}^{p\theta}) \\ & \quad - [\zeta_n^{1-\beta}(t\varphi) - 1] \lambda \int_{\mathbb{R}^N} f(x)((u_n + t\varphi)^+)^{1-\beta} dx - \lambda \int_{\mathbb{R}^N} f(x)[((u_n + t\varphi)^+)^{1-\beta} - (u_n^+)^{1-\beta}] dx \end{aligned}$$

$$\begin{aligned}
& - [\zeta_n^{2q}(t\varphi) - 1] \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_n + t\varphi)^+(x))^q((u_n + t\varphi)^+(y))^q}{|x - y|^\mu} dx dy \\
& - \iint_{\mathbb{R}^{2N}} g(x)g(y) \left[\frac{((u_n + t\varphi)^+(x))^q((u_n + t\varphi)^+(y))^q}{|x - y|^\mu} - \frac{(u_n^+(x))^q(u_n^+(y))^q}{|x - y|^\mu} \right] dx dy.
\end{aligned}$$

Dividing the above estimate by $t > 0$ and letting $t \rightarrow 0^+$, we obtain

$$\begin{aligned}
0 & \leq pa(\zeta'_n(0), \varphi)[u_n]_{s,p}^p + pa\langle u_n, \varphi \rangle_{s,p} + pb\theta(\zeta'_n(0), \varphi)[u_n]_{s,p}^{p\theta} + pb\theta[u_n]_{s,p}^{p(\theta-1)}\langle u_n, \varphi \rangle_{s,p} \\
& - (1 - \beta)(\zeta'_n(0), \varphi)\lambda \int_{\mathbb{R}^N} f(x)(u_n^+)^{1-\beta} dx - 2q(\zeta'_n(0), \varphi) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x - y|^\mu} dx dy \\
& - q \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}\varphi(x)(u_n^+(y))^q}{|x - y|^\mu} dx dy - q \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q\varphi(y)(u_n^+(y))^{q-1}}{|x - y|^\mu} dx dy \\
& = (\zeta'_n(0), \varphi) \left(pa[u_n]_{s,p}^p + pb\theta[u_n]_{s,p}^{p\theta} - (1 - \beta)\lambda \int_{\mathbb{R}^N} f(x)(u_n^+)^{1-\beta} dx \right. \\
& \quad \left. - 2q \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x - y|^\mu} dx dy \right) + (pa + pb\theta[u_n]_{s,p}^{p\theta-1})\langle u_n, \varphi \rangle_{s,p} \\
& \quad - 2q \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}\varphi(x)(u_n^+(y))^q}{|x - y|^\mu} dx dy.
\end{aligned}$$

By (3.20), we have

$$\begin{aligned}
0 & \leq (\zeta'_n(0), \varphi) \left(a(p - 1 + \beta)[u_n]_{s,p}^p + b(p\theta - 1 + \beta)[u_n]_{s,p}^{p\theta} - (2q - 1 + \beta) \iint_{\mathbb{R}^{2N}} \frac{(u_n^+(x))^q(u_n^+(y))^q}{|x - y|^\mu} dx dy \right) \\
& + (pa + pb\theta[u_n]_{s,p}^{p(\theta-1)})\langle u_n, \varphi \rangle_{s,p} - 2q \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}\varphi(x)(u_n^+(y))^q}{|x - y|^\mu} dx dy.
\end{aligned}$$

Then by Lemma 3.7-(i) and the boundedness of the sequence $(u_n)_n$, we obtain that $(\zeta'_n(0), \varphi)$ is bounded from below for every $\varphi \in D^{s,p}(\mathbb{R}^N)$ with $\varphi \geq 0$.

Next, we show that $(\zeta'_n(0), \varphi)$ is bounded from above. Arguing by contradiction, we assume that $(\zeta'_n(0), \varphi) = \infty$. Since

$$\begin{aligned}
|\zeta_n(t\varphi) - 1|[u_n]_{s,p} + \zeta_n(t\varphi)[t\varphi]_{s,p} & \geq [(\zeta_n(t\varphi) - 1)u_n + \zeta_n(t\varphi)t\varphi]_{s,p} \\
& = [\zeta_n(t\varphi)(u_n + t\varphi) - u_n]_{s,p}
\end{aligned} \tag{3.22}$$

and $\zeta_n(t\varphi) > \zeta_n(0) = 1$ for sufficiently large n . By the definition of $\zeta'_n(0)$ and (3.13), we get

$$\begin{aligned}
& [\zeta_n(t\varphi) - 1] \frac{[u_n]_{s,p}}{n} + \zeta_n(t\varphi) \frac{[t\varphi]_{s,p}}{n} \\
& \geq \frac{1}{n} [\zeta_n(t\varphi)(u_n + t\varphi) - u_n]_{s,p} \\
& \geq I_\lambda(u_n) - I_\lambda[\zeta_n(t\varphi)(u_n + t\varphi)] \\
& = a \left(\frac{1}{1 - \beta} - \frac{1}{p} \right) ([u_n + t\varphi]_{s,p}^p - [u_n]_{s,p}^p) + b \left(\frac{1}{1 - \beta} - \frac{1}{p\theta} \right) ([u_n + t\varphi]_{s,p}^{p\theta} - [u_n]_{s,p}^{p\theta}) \\
& + a \left(\frac{1}{1 - \beta} - \frac{1}{p} \right) (\zeta_n^{p\theta}(t\varphi) - 1)[u_n + t\varphi]_{s,p}^p + b \left(\frac{1}{1 - \beta} - \frac{1}{p\theta} \right) [\zeta_n^{p\theta}(t\varphi) - 1][u_n + t\varphi]_{s,p}^{p\theta} \\
& - \left(\frac{1}{1 - \beta} - \frac{1}{2q} \right) \zeta_n^{2q}(t\varphi) \iint_{\mathbb{R}^{2N}} g(x)g(y) \left[\frac{((u_n + t\varphi)^+(x))^q((u_n + t\varphi)^+(y))^q}{|x - y|^\mu} \right. \\
& \quad \left. - \frac{(u_n^+(x))^q(u_n^+(y))^q}{|x - y|^\mu} \right] dx dy
\end{aligned}$$

$$- \left(\frac{1}{1-\beta} - \frac{1}{2q} \right) (\zeta_n^{2q}(t\varphi) - 1) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy.$$

Dividing above inequality by $t > 0$ and letting $t \rightarrow 0^+$, we get

$$\begin{aligned} (\zeta'_n(0), \varphi) \frac{[u_n]_{s,p}}{n} + \frac{[\varphi]_{s,p}}{n} &\geq \left(a \frac{p-1+\beta}{1-\beta} + b \frac{p\theta-1+\beta}{1-\beta} [u_n]_{s,p}^{p(\theta-1)} \right) \langle u_n, \varphi \rangle_{s,p} \\ &\quad + a \frac{p-1+\beta}{1-\beta} (\zeta'_n(0), \varphi) [u_n]_{s,p}^p + b \frac{p\theta-1+\beta}{1-\beta} (\zeta'_n(0), \varphi) [u_n]_{s,p}^{p\theta} \\ &\quad - \frac{2q-1+\beta}{1-\beta} \iint_{\mathbb{R}^{2N}} g(x)g(y) \frac{(u_n^+(x))^{q-1} \varphi(x)(u_n^+(y))^q}{|x-y|^\mu} dx dy \\ &\quad - \frac{2q-1+\beta}{1-\beta} (\zeta'_n(0), \varphi) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy \\ &= \frac{(\zeta'_n(0), \varphi)}{1-\beta} \left[a(p-1+\beta) [u_n]_{s,p}^p + b(p\theta-1+\beta) [u_n]_{s,p}^{p\theta} \right. \\ &\quad \left. - (2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy \right] \\ &\quad + \left(a \frac{p-1+\beta}{1-\beta} + b \frac{p\theta-1+\beta}{1-\beta} [u_n]_{s,p}^{p(\theta-1)} \right) \langle u_n, \varphi \rangle_{s,p} \\ &\quad - \frac{2q-1+\beta}{1-\beta} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1} \varphi(x)(u_n^+(y))^q}{|x-y|^\mu} dx dy. \end{aligned}$$

That is

$$\begin{aligned} \frac{[\varphi]_{s,p}}{n} &\geq \frac{(\zeta'_n(0), \varphi)}{1-\beta} \left[a(p-1+\beta) [u_n]_{s,p}^p + b(p\theta-1+\beta) [u_n]_{s,p}^{p\theta} \right. \\ &\quad \left. - (2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy - \frac{1-\beta}{n} [u_n]_{s,p} \right] \\ &\quad + \left(a \frac{p-1+\beta}{1-\beta} + b \frac{p\theta-1+\beta}{1-\beta} [u_n]_{s,p}^{p(\theta-1)} \right) \langle u_n, \varphi \rangle_{s,p} \\ &\quad - \frac{2q-1+\beta}{1-\beta} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1} \varphi(x)(u_n^+(y))^q}{|x-y|^\mu} dx dy, \end{aligned}$$

which contradicts with our assumption that $(\zeta'_n(0), \varphi) = \infty$. From Lemma 3.7 and (3.19), we have

$$\begin{aligned} a(p-1+\beta) [u_n]_{s,p}^p + b(p\theta-1+\beta) [u_n]_{s,p}^{p\theta} - (2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy - \frac{1-\beta}{n} [u_n]_{s,p} \\ \geq C_2 - \frac{(1-\beta)}{n} C_1 > 0. \end{aligned}$$

This ends the proof. $\square$

Lemma 3.9. For $\lambda \in (0, \Lambda_0)$ and $(u_n)_n \subset \mathcal{N}_\lambda^+$. Then, for every $\varphi \in D^{s,p}(\mathbb{R}^N)$ and $n \in \mathbb{N}$, we have

$$f(x)(u_n^+)^{-\beta} \varphi \in L^1(\mathbb{R}^N),$$

and

$$(a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \varphi \rangle_{s,p} - \lambda \int_{\mathbb{R}^N} f(x)(u_n^+)^{-\beta} \varphi dx - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(y))^q(u_n^+(x))^{q-1} \varphi(x)}{|x-y|^\mu} dx dy$$

$$:= H(u_n, \varphi) = o_n(1), \quad (3.23)$$

as $n \rightarrow \infty$.

Proof. Let $\varphi \in D^{s,p}(\mathbb{R}^N)$ with $\varphi \geq 0$. By (3.13) and (3.22), we have

$$\begin{aligned} (\zeta_n(t\varphi) - 1) \frac{[u_n]_{s,p}}{n} + \zeta_n(t\varphi) \frac{[t\varphi]_{s,p}}{n} &\geq I_\lambda(u_n) - I_\lambda(\zeta_n(t\varphi)(u_n + t\varphi)) \\ &= -\frac{\zeta_n^p(t\varphi) - 1}{p} a[u_n]_{s,p}^{p\theta} - \frac{\zeta_n^p(t\varphi) - 1}{p} b[u_n]_{s,p}^{p\theta} \\ &\quad - \frac{\zeta_n^p(t\varphi)}{p} a([u_n + t\varphi]_{s,p}^p - [u_n]_{s,p}^p) - \frac{\zeta_n^{p\theta}(t\varphi)}{p\theta} b([u_n + t\varphi]_{s,p}^{p\theta} - [u_n]_{s,p}^{p\theta}) \\ &\quad + \frac{\zeta_n^{1-\beta}(t\varphi) - 1}{1-\beta} \lambda \int_{\mathbb{R}^N} f(x) ((u_n + t\varphi)^+)^{1-\beta} dx \\ &\quad + \frac{\lambda}{1-\beta} \int_{\mathbb{R}^N} f(x) [(u_n + t\varphi)^+)^{1-\beta} - (u_n^+)^{1-\beta}] dx \\ &\quad + \frac{\zeta_n^{2q}(t\varphi) - 1}{2q} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_n + t\varphi)^+(x))^q((u_n + t\varphi)^+(y))^q}{|x-y|^\mu} dx dy \\ &\quad + \frac{1}{2q} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_n + t\varphi)^+(x))^q((u_n + t\varphi)^+(y))^q - (u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy. \end{aligned}$$

Dividing by $t > 0$ and letting $t \rightarrow 0^+$, we obtain

$$\begin{aligned} |(\zeta'_n(0), \varphi)| \frac{[u_n]_{s,p}}{n} + \frac{[\varphi]_{s,p}}{n} &\geq -a(\zeta'_n(0), \varphi)[u_n]_{s,p}^p - b(\zeta'_n(0), \varphi)[u_n]_{s,p}^{p\theta} \\ &\quad - (a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \varphi \rangle_{s,p} \\ &\quad + (\zeta'_n(0), \varphi) \lambda \int_{\mathbb{R}^N} f(x) (u_n^+)^{1-\beta} dx \\ &\quad + \liminf_{t \rightarrow 0^+} \frac{\lambda}{1-\beta} \int_{\mathbb{R}^N} f(x) \frac{((u_n + t\varphi)^+)^{1-\beta} - (u_n^+)^{1-\beta}}{t} dx \\ &\quad + (\zeta'_n(0), \varphi) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+(y))^q \varphi(x)}{|x-y|^\mu} dx dy. \end{aligned}$$

Since $u_n \in \mathcal{N}_\lambda$, it follows that

$$\begin{aligned} |(\zeta'_n(0), \varphi)| \frac{[u_n]_{s,p}}{n} + \frac{[\varphi]_{s,p}}{n} &\geq - (a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \varphi \rangle_{s,p} \\ &\quad + \liminf_{t \rightarrow 0^+} \frac{\lambda}{1-\beta} \int_{\mathbb{R}^N} f(x) \frac{((u_n + t\varphi)^+)^{1-\beta} - (u_n^+)^{1-\beta}}{t} dx \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+(y))^q \varphi(x)}{|x-y|^\mu} dx dy \end{aligned}$$

Note that

$$f(x) \frac{((u_n + t\varphi)^+)^{1-\beta} - (u_n^+)^{1-\beta}}{t} \geq 0$$

in $\mathbb{R}^N$ and $(u_n)_n$ is bounded in $D^{s,p}(\mathbb{R}^N)$. Using Fatou's lemma and Lemma 3.8, we know that

$$\liminf_{t \rightarrow 0^+} \int_{\mathbb{R}^N} f(x) \frac{((u_n + t\varphi)^+)^{1-\beta} - (u_n^+)^{1-\beta}}{t} dx$$

is finite, and

$$\begin{aligned} \lambda \int_{\mathbb{R}^N} f(x) (u_n^+)^{-\beta} \varphi dx &\leq \liminf_{t \rightarrow 0^+} \frac{\lambda}{1-\beta} \int_{\mathbb{R}^N} f(x) \frac{((u_n + t\varphi)^+)^{1-\beta} - (u_n^+)^{1-\beta}}{t} dx \\ &\leq \frac{1}{n} \left(|(\zeta'_n(0), \varphi)| [u_n]_{s,p} + [\varphi]_{s,p} \right) + (a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \varphi \rangle_{s,p} \\ &\quad - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+(y))^q \varphi(x)}{|x-y|^\mu} dx dy \\ &\leq \frac{1}{n} (C_1 C_3 + [\varphi]_{s,p}) + (a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \varphi \rangle_{s,p} \\ &\quad - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+(y))^q \varphi(x)}{|x-y|^\mu} dx dy \end{aligned}$$

with $C_3 > 0$ given by the boundedness of $(\zeta'_n(0), \varphi)$, which implies that

$$(a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \varphi \rangle_{s,p} - \lambda \int_{\mathbb{R}^N} f(x) (u_n^+)^{-\beta} \varphi dx - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+(y))^q \varphi(x)}{|x-y|^\mu} dx dy \geq o_n(1). \quad (3.24)$$

In the following, we prove that (3.24) holds for any $\varphi \in D^{s,p}(\mathbb{R}^N)$. In (3.24), we choose $\varphi = \omega_\epsilon^+$ with $\omega_\epsilon = u_n^+ + \epsilon \varphi$ and $\varphi \in D^{s,p}(\mathbb{R}^N)$ as test function. Then

$$\begin{aligned} o_n(1) &\leq (a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \omega_\epsilon + \omega_\epsilon^- \rangle_{s,p} - \lambda \int_{\mathbb{R}^N} f(x) (u_n^+)^{1-\beta} (\omega_\epsilon + \omega_\epsilon^-) dx \\ &\quad - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+(y))^q (\omega_\epsilon + \omega_\epsilon^-)(x)}{|x-y|^\mu} dx dy. \end{aligned} \quad (3.25)$$

Observe that

$$\langle u_n, \omega_\epsilon + \omega_\epsilon^- \rangle_{s,p} \leq [u_n]_{s,p}^p + \epsilon \langle u_n, \varphi \rangle_{s,p} + \langle u_n, \omega_\epsilon^- \rangle_{s,p}.$$

Hence, denote $\Omega_\epsilon = \{x \in \mathbb{R}^N : \omega_\epsilon(x) \leq 0\}$. In view of (3.25) and (3.3), we deduce $n \rightarrow \infty$

$$\begin{aligned} o_n(1) &\leq (a + b[u_n]_{s,p}^{p(\theta-1)}) [u_n]_{s,p}^p - \lambda \int_{\mathbb{R}^N} f(x) (u_n^+)^{1-\beta} dx \\ &\quad - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(x,y)(u_n^+(x))^{p_{\mu,s}^*}(u_n^+(y))^{p_{\mu,s}^*}}{|x-y|^\mu} dx dy \\ &\quad + \epsilon \left[(a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \varphi \rangle_{s,p} - \lambda \int_{\mathbb{R}^N} f(x) (u_n^+)^{-\beta} \varphi dx \right. \\ &\quad \left. - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+(y))^q \varphi(x)}{|x-y|^\mu} dx dy \right] + (a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \omega_\epsilon^- \rangle_{s,p} \\ &\quad + \iint_{\mathbb{R}^N \setminus \Omega_\epsilon} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+ + \epsilon \varphi)(x)(u_n^+(y))^q}{|x-y|^\mu} dx dy \\ &= \epsilon \left[(a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \varphi \rangle_{s,p} - \lambda \int_{\mathbb{R}^N} f(x) (u_n^+)^{-\beta} \varphi dx \right. \end{aligned}$$

$$\begin{aligned}
& - \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+(y))^q \varphi(x)}{|x-y|^\mu} dx dy \Big] + (a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, \omega_\epsilon^- \rangle_{s,p} \\
& + \int_{\mathbb{R}^N} \int_{\Omega_\epsilon} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+ + \epsilon \varphi)(x)(u_n^+(y))^q}{|x-y|^\mu} dx dy.
\end{aligned} \quad (3.26)$$

Furthermore,

$$\begin{aligned}
\langle u_n, \omega_\epsilon^- \rangle_{s,p} &= \iint_{\Omega_\epsilon \times \Omega_\epsilon} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\omega_\epsilon^-(x) - \omega_\epsilon^-(y))}{|x-y|^{N+ps}} dx dy \\
&+ 2 \iint_{\Omega_\epsilon \times (\mathbb{R}^N \setminus \Omega_\epsilon)} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\omega_\epsilon^-(x) - \omega_\epsilon^-(y))}{|x-y|^{N+ps}} dx dy \\
&\leq 2\epsilon \iint_{\Omega_\epsilon \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2} |(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))|}{|x-y|^{N+ps}} dx dy \\
&\leq 2C\epsilon \left(\iint_{\Omega_\epsilon \times \mathbb{R}^N} \left| \frac{\varphi(x) - \varphi(y)}{|x-y|^{\frac{N+ps}{p}}} \right|^p dx dy \right)^{\frac{1}{p}}.
\end{aligned}$$

For any $\delta > 0$, there exists R_δ large enough such that

$$\iint_{(\text{supp } \varphi) \times (\mathbb{R}^N \setminus B_{R_\delta})} \left| \frac{\varphi(x) - \varphi(y)}{|x-y|^{\frac{N+ps}{p}}} \right|^p dx dy < \frac{\delta}{2}.$$

In view of the definition of $\Omega_{1,\epsilon}$, we have $\Omega_{1,\epsilon} \subset \text{supp } \varphi$ and $|\Omega_{1,\epsilon} \times B_{R_\delta}| \rightarrow 0$ as $\epsilon \rightarrow 0^+$. Hence, there exist $\kappa_\delta > 0$ and $\epsilon_\delta > 0$ such that for every $\epsilon \in (0, \epsilon_\delta]$,

$$|\Omega_{1,\epsilon} \times B_{R_\delta}| < \kappa_\delta, \text{ and } \iint_{\Omega_\epsilon \times B_{R_\delta}} \left| \frac{\varphi(x) - \varphi(y)}{|x-y|^{\frac{N+ps}{p}}} \right|^p dx dy < \frac{\delta}{2}.$$

Consequently, for every $\epsilon \in (0, \epsilon_\delta]$,

$$\iint_{\Omega_\epsilon \times \mathbb{R}^N} \left| \frac{\varphi(x) - \varphi(y)}{|x-y|^{\frac{N+ps}{p}}} \right|^p dx dy < \delta,$$

it follows that

$$\lim_{\epsilon \rightarrow 0^+} \iint_{\Omega_\epsilon \times \mathbb{R}^N} \left| \frac{\varphi(x) - \varphi(y)}{|x-y|^{\frac{N+ps}{p}}} \right|^p dx dy = 0. \quad (3.27)$$

Now, we show that

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \iint_{\mathbb{R}^N \setminus \Omega_\epsilon} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+ + \epsilon \varphi)(x)(u_n^+(y))^q}{|x-y|^\mu} dx dy = 0. \quad (3.28)$$

In fact,

$$\begin{aligned}
& \int_{\mathbb{R}^N} \int_{\Omega_\epsilon} \frac{g(x)g(y)(u_n^+(x))^{q-1}(u_n^+ + \epsilon \varphi)(x)(u_n^+(y))^q}{|x-y|^\mu} dx dy \\
& \leq \int_{\mathbb{R}^N} \int_{\Omega_\epsilon} \frac{g(x)g(y)(u_n^+(x))^q(u_n^+(y))^q}{|x-y|^\mu} dx dy + \epsilon \iint_{\mathbb{R}^N \setminus \Omega_\epsilon} \frac{g(x)g(y)(u_n^+(x))^{q-1}\varphi(x)(u_n^+(y))^q}{|x-y|^\mu} dx dy
\end{aligned}$$

$$\begin{aligned}
& + \epsilon \left(\int_{\Omega_\epsilon} \int_{\Omega_\epsilon} \frac{g(x)g(y)(u_n^+(x))^{q-1} \varphi(x)(u_n^+(y))^{q-1}}{\varphi} (y)|x-y|^\mu dx dy \right)^{\frac{1}{2}} \\
& \cdot \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)g(y)(u_n^+(x))^q (u_n^+(y))^q}{|x-y|^\mu} dx dy \right)^{\frac{1}{2}} \\
& \leq CC_g(N, \mu) \left(\int_{\Omega_\epsilon} (u_n^+(x))^{p_s^*} dx \right)^{\frac{q}{p_s^*}} + C\epsilon C_g(N, \mu) \left(\int_{\Omega_\epsilon} ((u_n^+(x))^{q-1} \varphi(x))^{\frac{p_s^*}{q}} dx \right)^{\frac{q}{p_s^*}} \\
& \leq CC_g(N, \mu) \left(\int_{\Omega_\epsilon} (u_n^+(x))^{p_s^*} dx \right)^{\frac{q}{p_s^*}} \\
& + C\epsilon C_g(N, \mu) \left(\int_{\Omega_\epsilon} (u_n^+(x))^{p_s^*} dx \right)^{\frac{q-1}{p_s^*}} \left(\int_{\Omega_\epsilon} |\varphi(x)|^{p_s^*} dx \right)^{\frac{1}{p_s^*}} \\
& \leq CC_g(N, \mu) \epsilon^q \left(\int_{\Omega_{1,\epsilon}} |\varphi(x)|^{p_s^*} dx \right)^{\frac{q}{p_s^*}} + \tilde{C}\epsilon C_g(N, \mu) \left(\int_{\Omega_\epsilon} |\varphi(x)|^{p_s^*} dx \right)^{\frac{1}{p_s^*}}
\end{aligned}$$

Dividing by ϵ the above estimate and using that $|\Omega_\epsilon| \rightarrow 0$ as $\epsilon \rightarrow 0^+$, we can get (3.28).

Dividing (3.26) by ϵ , together with (3.27)-(3.28), and letting $\epsilon \rightarrow 0^+$, we deduce that (3.24) holds. According to the arbitrariness of φ , we show that (3.23) holds. $\square$

Lemma 3.10. For $\lambda \in (0, \lambda_0)$, $(u_n)_n \subset \mathcal{N}_\lambda^+$, and $I_\lambda(u_n) \rightarrow c$ as $n \rightarrow \infty$. Then, the sequence $(u_n)_n$ contains a subsequence strongly convergent to u_0 in $D^{s,p}(\mathbb{R}^N)$.

Proof. By (3.14), $(u_n)_n$ and $(u_n^-)_n$ are both bounded in $D^{s,p}(\mathbb{R}^N)$. Using (3.23) with $\varphi = u_n^-$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} (a + b[u_n]_{s,p}^{(\theta-1)}) \langle u_n, u_n^- \rangle_{s,p} = 0,$$

which together with $a > 0$ yields that $[u_n]_{s,p} \rightarrow 0$ as $n \rightarrow \infty$. Consequently, we can assume that $(u_n)_n$ is a sequence of nonnegative functions. Furthermore, we can extract a subsequence, still denoted by $(u_n)_n$ such that

$$\begin{cases} u_n \rightharpoonup u_0 \geq 0 \text{ in } L^{p_s^*}(\mathbb{R}^N), [u_n]_{s,p} \rightarrow \chi, \\ u_n \rightarrow u_0 \text{ in } L_{loc}^v(\mathbb{R}^N), \text{ for each } v \in [1, p_s^*), \\ u_n \rightarrow u_0 \text{ a.e. in } \mathbb{R}^N, \end{cases} \quad (3.29)$$

as $n \rightarrow \infty$. If $\chi = 0$, then $u_n \rightarrow 0$ in $D^{s,p}(\mathbb{R}^N)$ as $n \rightarrow \infty$. Hence, we assume that $\chi > 0$. By [4, Lemma 3.2], [40, Lemma 2.4], and [34, Lemma 2.3] we obtain

$$[u_n]_{s,p}^p = [u_n - u_0]_{s,p}^p + [u_0]_{s,p}^p + o(1), \quad (3.30)$$

and

$$\begin{aligned} \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_n^+(x))^q (u_n^+(y))^q}{|x-y|^\mu} dx dy &= \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_n(x) - u_0(x))^+)^q ((u_n(y) - u_0(y))^+)^q}{|x-y|^\mu} dx dy \\ &+ \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_0^+(x))^q (u_0^+(y))^q}{|x-y|^\mu} dx dy + o(1). \end{aligned} \quad (3.31)$$

It follows from (3.29), (3.30) and (3.31) that

$$\begin{aligned}
 o(1) &= (a + b[u_n]_{s,p}^{p(\theta-1)}) \langle u_n, u_n - u_0 \rangle_{s,p} - \lambda \int_{\mathbb{R}^N} f(x)(u_n^+)^{-\beta} (u_n - u_0) dx \\
 &\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)g(y)((u_n(x))^+)^{q-1}((u_n(y))^+)^q}{|x-y|^\mu} (u_n(x) - u_0(x)) dx dy \\
 &= (a + b\chi^{p(\theta-1)})(\chi^p - [u_0]_{s,p}^p) - \lambda \int_{\Omega} f(x)(u_n^+)^{-\beta} (u_n - u_0) dx \\
 &\quad - \int_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_n(x))^+)^q((u_n(y))^+)^q}{|x-y|^\mu} dx dy \\
 &\quad - \int_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_0(x))^+)^q((u_0(y))^+)^q}{|x-y|^\mu} dx dy + o(1) \\
 &= (a + b\chi^{p(\theta-1)})[u_n - u_0]_{s,p}^p - \lambda \int_{\mathbb{R}^N} f(x)(u_n^+)^{-\beta} (u_n - u_0) dx \\
 &\quad - \int_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_n(x) - u_0(x))^+)^q((u_n(y) - u_0(y))^+)^q}{|x-y|^\mu} dx dy + o(1).
 \end{aligned}$$

Hence

$$\begin{aligned}
 (a + b\chi^{p(\theta-1)}) \lim_{n \rightarrow \infty} [u_n - u_0]_{s,p}^p &= \lambda \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x)(u_n^+)^{-\beta} (u_n - u_0) dx \\
 &\quad + \lim_{n \rightarrow \infty} \int_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_n(x) - u_0(x))^+)^q((u_n(y) - u_0(y))^+)^q}{|x-y|^\mu} dx dy. \quad (3.32)
 \end{aligned}$$

Since $\beta \in (0, 1)$ and $f \in L^{\frac{p_s^*}{p_s^*-1+\beta}}(\mathbb{R}^N)$, we deduce from the Vitali's convergence theorem that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x)(u_n^+)^{1-\beta} dx = \int_{\mathbb{R}^N} f(x)(u_0^+)^{1-\beta} dx. \quad (3.33)$$

By virtue of Lemma 3.9, we obtain

$$f(x)u_n^{-\beta}u_0 \in L^1(\mathbb{R}^N)$$

for each $n \in \mathbb{N}$. It follows from Fatou's lemma that

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \lambda f(x)(u_n^+)^{-\beta} u_0 dx \geq \lambda \int_{\mathbb{R}^N} f(x)(u_0^+)^{1-\beta} dx. \quad (3.34)$$

Similarly, by $g \in L^{\frac{p_s^* 2_\mu^*}{p_s^* - 2_\mu^* q}}(\mathbb{R}^N)$ and Vitali's convergence theorem, one can deduce that

$$g(x)|u_n - u_0|^q \rightarrow 0 \text{ strongly in } L^{2_\mu^*}(\mathbb{R}^N).$$

Furthermore, it follows from (2.2) that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^{2N}} \frac{g(x)g(y)((u_n(x) - u_0(x))^+)^q((u_n(y) - u_0(y))^+)^q}{|x-y|^\mu} dy dx = 0. \quad (3.35)$$

From (3.32)-(3.35), we get

$$(a + b\chi^{p(\theta-1)}) \lim_{n \rightarrow \infty} [u_n - u_0]_{s,p}^p = 0, \quad (3.36)$$

which together with $a > 0$ yields that $u_n \rightarrow u_0$ strongly in $D^{s,p}(\mathbb{R}^N)$. $\square$

4 Proof of the main result

Theorem 4.1. *Let $0 < \lambda < \Lambda^* = \min\{\Lambda_0, \Lambda_{00}\}$. Assume f satisfies (f_1) and g satisfies (g_1) . Then equation (1.1) has a nontrivial and nonnegative solution in $\mathcal{N}_\lambda^+$*

Proof. Firstly, we show that $c_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u) < 0$. In fact, for $\lambda > 0$ and $u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we obtain

$$I_\lambda(u) = a \left(\frac{1}{p} - \frac{1}{1-\beta} \right) [u_n]_{s,p}^p + b \left(\frac{1}{p\theta} - \frac{1}{1-\beta} \right) [u_n]_{s,p}^{p\theta} - \left(\frac{1}{2q} - \frac{1}{1-\beta} \right) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy.$$

Since $2q > \theta p$ and $I_\lambda(0) = 0$, we obtain

$$I_\lambda(u) \leq -\frac{1}{pq(1-\beta)} \left[a(p-1+\beta)[u_n]_{s,p}^p + b(p\theta-1+\beta)[u_n]_{s,p}^{p\theta} - p(2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u^+(x))^q(u^+(y))^q}{|x-y|^\mu} dx dy \right] < 0.$$

Now, fix $\lambda < \Lambda^* = \min\{\Lambda_0, \Lambda_{00}\}$. According to Ekeland's variational principle and Lemma 3.2, there exists a minimizing sequence $(u_n)_n \subset \mathcal{N}_\lambda^+ \cup \{0\}$ satisfying (3.11) and (3.12). Hence $I_\lambda(u_n) \rightarrow c_\lambda^+ < 0$ as $n \rightarrow \infty$, which yields that $(u_n)_n \subset \mathcal{N}_\lambda^+$. As a consequence, using Lemma 3.10 with $c = c_\lambda^+$, we know that $u_n \rightarrow u_0$ in $D^{s,p}(\mathbb{R}^N)$, up to a subsequence. Furthermore, according to Lemma 3.7 and (3.12), we get

$$a(p-1+\beta)[u_0]_{s,p}^p + b(p\theta-1+\beta)[u_0]_{s,p}^{p\theta} - p(2q-1+\beta) \iint_{\mathbb{R}^{2N}} \frac{g(x)g(y)(u_0^+(x))^q(u_0^+(y))^q}{|x-y|^\mu} dx dy > 0,$$

which means that $u_0 \in \mathcal{N}_\lambda^+$, and c_λ^+ is attained at u_0 by I_λ is continuous on $D^{s,p}(\mathbb{R}^N)$.

Letting $n \rightarrow \infty$ in (3.23), together with Fatou's lemma, we obtain the inequality $H(u_0, \varphi) \geq 0$ for every test function $\varphi \in D^{s,p}(\mathbb{R}^N)$ with $\varphi \geq 0$. Taking a test function Φ_ϵ^+ in above inequality with $\omega_\epsilon = u_0^+ + \epsilon\varphi$, $\epsilon > 0$ and $\varphi \in D^{s,p}(\mathbb{R}^N)$. Repeating the same discussion as (3.23)-(3.28) with u_0 instead of u_n , we know that $H(u_0, \varphi) \geq 0$ for any $\varphi \in D^{s,p}(\mathbb{R}^N)$, which yields that $\lambda f(x)(u_0^+)^{-\beta}\varphi \in L^1(\mathbb{R}^N)$ for any $\varphi \in D^{s,p}(\mathbb{R}^N)$ and u_0 satisfies (3.2). It follows from Lemma 3.2 that $u_0 \neq 0$. Moreover, Using (3.2) with test function $\phi_1 = u_0^-$, we deduce that $(a + b[u_0]_{s,p}^{p(\theta-1)})[u_0^-]_{s,p}^p = 0$, which means that u_0 is a nontrivial and nonnegative solution of equation (1.1). $\square$

Theorem 4.2. *Let $0 < \lambda < \Lambda^* = \min\{\Lambda_0, \Lambda_{00}\}$. Assume f satisfies (f_1) and g satisfies (g_1) . Then equation (1.1) has a nontrivial and nonnegative solution in $\mathcal{N}_\lambda^-$*

Proof. Since $\mathcal{N}_\lambda^-$ is a closed set in $D^{s,p}(\mathbb{R}^N)$, we can extract the minimizing sequence $(U_n)_n \subset \mathcal{N}_\lambda^-$ satisfying the Ekeland variational principle for $\inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u)$. Since $(U_n)_n$ is bounded in $D^{s,p}(\mathbb{R}^N)$, after taking a subsequence, we suppose that $(U_n)_n \rightharpoonup U_0$ in $D^{s,p}(\mathbb{R}^N)$. By Lemma 3.10, we know that $U_n \rightarrow U_0$ in $D^{s,p}(\mathbb{R}^N)$, up to a subsequence, if necessary, so that $U_n \in \mathcal{N}_\lambda^-$ and $I_\lambda(U_0) = c_\lambda^-$. Repeating the same argument as in the proof of Theorem 4.1, U_0 satisfies $H(U_0, \varphi) \geq 0$, so that $\lambda f(x)(U_0^+)^{-\beta}\varphi \in L^1(\mathbb{R}^N)$ for any $\varphi \in D^{s,p}(\mathbb{R}^N)$ and U_0 satisfies (3.2). Combining this with Lemma 3.2, we deduce that U_0 is a nontrivial solution of equation (1.1). The proof is complete. $\square$

Proof of Theorem 1.1. Combining Theorem 4.1 and Theorem 4.2, we know that equation (1.1) admits at least two nontrivial and nonnegative solutions u_0 and U_0 . Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, we know that u_0 and U_0 are distinct. $\square$

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