Research article

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Blowing-up solutions of the time-fractional dispersive equations

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Abstract: This paper is devoted to the study of initial-boundary value problems for time-fractional analogues of Korteweg-de Vries, Benjamin-Bona-Mahony, Burgers, Rosenau, Camassa-Holm, Degasperis-Procesi, Ostrovsky and time-fractional modified Korteweg-de Vries-Burgers equations on a bounded domain. Sufficient conditions for the blowing-up of solutions in finite time of aforementioned equations are presented. We also discuss the maximum principle and influence of gradient non-linearity on the global solvability of initial-boundary value problems for the time-fractional Burgers equation. The main tool of our study is the Pohozaev nonlinear capacity method. We also provide some illustrative examples.

Keywords: Caputo derivative; Burgers equation; Korteweg-de Vries equation; Benjamin-Bona-Mahony equation; Camassa-Holm equation, Rosenau equation, Ostrovsky equation; blow-up

MSC: Primary 35B50; Secondary 26A33, 35K55, 35J60

1 Introduction

Nonlinear wave phenomenon is one of the important areas of scientific investigation. Among the mathematical models describing the dynamics of wave equations include Korteweg-de Vries equation, Burgers equation, Benjamin-Bona-Mahony equation, Rosenau equation and Ostrovsky equation.

Bateman-Burgers equation or Burgers equation [6, 9]

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0, \quad (1.1)$$

is a fundamental partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, traffic flow.

The Korteweg-de Vries equation [16] is well known in different fields of science and technology, it reads

$$u_t + uu_x + u_{xxx} = 0. \quad (1.2)$$

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In [7], Benjamin, Bona and Mahony proposed the following equation to describe long waves on the water surface

$$u_t - u_{txx} + uu_x = 0. \quad (1.3)$$

In [47] Rosenau suggested the following equation to describe waves on “shallow” water:

$$u_t + u_{txxxx} + u_x + uu_x = 0. \quad (1.4)$$

In [37] Ostrovsky derived an equation for weakly nonlinear surface and internal waves in a rotating ocean

$$u_{tx} + u_{xx} + u_{xxx} + (uu_x)_x = 0. \quad (1.5)$$

The Camassa-Holm equation

$$u_t - u_{txx} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad \kappa > 0, \quad (1.6)$$

was introduced by Camassa and Holm [10] as a bi-Hamiltonian model for waves in shallow water and the following Degasperis-Procesi equation, one of the important model of mathematical physics

$$u_t - u_{txx} + 2\kappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad \kappa > 0. \quad (1.7)$$

The Korteweg-de Vries-Burgers equation

$$u_t + uu_x + u_{xxx} = \nu u_{xx}, \quad \nu > 0, \quad (1.8)$$

modified Korteweg-de Vries-Burgers equation

$$u_t + u^2 u_x + u_{xxx} = \nu u_{xx}, \quad \nu > 0, \quad (1.9)$$

Benjamin-Bona-Mahony-Burgers equation

$$u_t - u_{txx} + uu_x = \nu u_{xx}, \quad (1.10)$$

Korteweg-de Vries-Benjamin-Bona-Mahony equation

$$u_t - u_{txx} + u_{xxx} + uu_x = 0, \quad (1.11)$$

Rosenau-Burgers equation

$$u_t + u_{txxxx} + u_x + uu_x = \nu u_{xx}, \quad (1.12)$$

Rosenau-Korteweg-de Vries equation

$$u_t + u_{txxxx} + u_{xxx} + u_x + uu_x = 0, \quad (1.13)$$

Rosenau-Benjamin-Bona-Mahony equation

$$u_t - u_{txx} + u_{txxxx} + u_x + uu_x = 0, \quad (1.14)$$

have important applications in different physical situations such as waves on shallow water, and processes in semiconductors with differential conductivity [8, 13, 48, 52, 53, 58].

This paper is devoted to blowing-up solutions of time-fractional analogues of the above equations. The approach to the problem is based on the Pohozaev nonlinear capacity method [33–35]; more precisely, on the choice of test functions according to initial and boundary conditions under consideration.

Here, we give a simple case of the analysis of a rough blow-up, i.e., the case where the solution tends to infinity as $t \rightarrow T^*$ on $[0, L]$; more exactly, when the integral

$$\int_0^L u(x, t) \phi(x) dx$$

tends to infinity as $t \rightarrow T^*$ for the given function ϕ .

A small difference in our approach, we will study nonlinear problems without boundary conditions. We assume the boundary conditions are such that, there exists a smooth function ϕ , for which the functional $B(u, \phi)$ containing u and ϕ and their k -derivatives is lower bounds by a certain functional of ϕ . This allows us to consider some families of boundary conditions.

In [19–23] Korpusev et al. obtained sufficient conditions for the finite time blow-up of solutions of initial-boundary problems for Burgers, Korteweg-de Vries, Benjamin-Bona-Mahony and Rosenau type equations. We also note that the blow-up of solutions of the initial problems for the Korteweg-de Vries and critical Korteweg-de Vries equations are investigated in [30–32, 39–45]. Blow-up of solutions of the initial problems for the Ostrovsky equation is proved in [27].

Descriptions of some physical applications and numerical simulations of the time-fractional dispersive equations are given in [12, 15, 24, 29, 46, 49, 50, 55, 57].

Recently, the study of blowing-up solutions of time-fractional nonlinear partial differential equations received great attention. For example, the authors of this paper obtained results on the blow-up of the solutions of time-fractional Burgers equation [4, 54] and fractional reaction-diffusion equation [5]. We note that the blow-up of the solution of various nonlinear fractional problems was investigated in [1, 3, 11, 18, 25, 38, 56].

Let us briefly describe the problems investigated in this paper:

- Blowing-up solutions of the time-fractional Rosenau-KdV-BBM-Burgers equation with initial conditions described as follows:

$$\begin{aligned} \partial_{+0,t}^\alpha (u - au_{xx} + bu_{xxxx}) + cu_{xxx} - du_{xx} + u_x + uu_x &= 0, \quad x \in (0, L), \quad t > 0, \quad \alpha \in (0, 1], \\ u(x, 0) &= u_0(x), \quad x \in [0, L], \end{aligned}$$

where $a, b, c, d \in \mathbb{R}$ and u_0 is a given function.

- Blowing-up solutions of the initial-boundary problem for the time-fractional Camassa-Holm–Degasperis–Procesi equation

$$\begin{aligned} \partial_{+0,t}^\alpha (u - u_{xx}) + au_x + buu_x - cu_x u_{xx} - duu_{xxx} &= 0, \quad x \in (0, L), \quad t > 0, \quad \alpha \in (0, 1], \\ u(x, 0) &= u_0(x), \quad x \in [0, L], \end{aligned}$$

where $a, b, c, d \in \mathbb{R}$ and u_0 is a given function.

- Blowing-up solutions of the time-fractional Ostrovsky equation with initial conditions:

$$\begin{aligned} \partial_{+0,t}^\alpha u_x + au_{xx} + bu_{xxxx} + (uu_x)_x &= 0, \quad x \in (0, L), \quad t > 0, \quad \alpha \in (0, 1], \\ u(x, 0) &= u_0(x), \quad x \in [0, L], \end{aligned}$$

where $a, b \in \mathbb{R}$ and u_0 is a given function.

- Blowing-up solutions of the initial problem for the time-fractional analogue of the modified Korteweg-de Vries-Burgers equation with dissipation:

$$\begin{aligned} \partial_{+0,t}^\alpha u + u^2 u_x + au_{xxx} - bu_{xx} &= 0, \quad x \in (0, L), \quad t > 0, \quad \alpha \in (0, 1], \\ u(x, 0) &= u_0(x), \quad x \in [0, L], \end{aligned}$$

where $a, b \in \mathbb{R}$ and u_0 is a sufficiently smooth function.

- Maximum principle and gradient blow-up in time-fractional Burgers equation

$$\partial_{+0,t}^\alpha u + uu_x = vu_{xx}, \quad x \in (0, L), \quad t > 0, \quad \alpha \in (0, 1],$$

with an initial condition

$$u(x, 0) = u_0(x), \quad x \in [0, L],$$

where $v > 0$ and u_0 is a sufficiently smooth function.

1.1 Preliminaries

1.1.1 Fractional operators

Here, we recall definitions and properties of fractional order integral and differential operators [17, 36, 51].

Definition 1.1. [17] (Riemann-Liouville integral). Let f be a locally integrable real-valued function on $-\infty < a < t < b < +\infty$. The Riemann-Liouville fractional integral I_{+a}^α of order $\alpha \in \mathbb{R}$ ($\alpha > 0$) is defined as

$$I_{+a}^\alpha f(t) = (f * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

where $K_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, Γ denotes the Euler gamma function.

The convolution here will be understood in the sense of the above definition.

Definition 1.2. [17] (Riemann-Liouville derivative). Let $f \in C([a, b])$, $-\infty < a < t < b < +\infty$ and $f * K_{m-\alpha}(t) \in C^m([a, b])$, $m = [\alpha] + 1$, $\alpha > 0$. The Riemann-Liouville fractional derivative D_{+a}^α of order $\alpha > 0$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as

$$D_{+a}^\alpha f(t) = \frac{d^m}{dt^m} I_{+a}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} f(s) ds.$$

Definition 1.3. [17] (Caputo derivative). Let $f \in C^m([a, b])$, $-\infty < a < t < b < +\infty$ and $f^{(m)} * K_{m-\alpha}(t) \in L^1([a, b])$, $m = [\alpha] + 1$, $\alpha > 0$. The Caputo fractional derivative ∂_{+a}^α of order $\alpha \in \mathbb{R}$ ($m-1 < \alpha < m$, $m \in \mathbb{N}$) is defined as

$$\partial_{+a}^\alpha [f](t) = I_{+a}^{m-\alpha} f^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} f^{(m)}(s) ds.$$

Property 1.4. [2] Let $0 < \alpha \leq 1$ and f absolutely continuous on $[0, T]$. Then

$$2f(t)\partial_{+0}^\alpha f(t) \geq \partial_{+0}^\alpha f^2(t). \quad (1.15)$$

Property 1.5. [28] Let $f \in C^1((0, T)) \cap C([0, T])$ attain its maximum over the interval $[0, T]$ at $t_0 \in (0, T]$. Then $\partial_{+0,t}^\alpha f(t_0) \geq 0$.

Let $f \in C^1((0, T)) \cap C([0, T])$ attain its minimum over the interval $[0, T]$ at $t_0 \in (0, T]$. Then $\partial_{+0,t}^\alpha f(t_0) \leq 0$.

1.1.2 Finite time blow-up of solutions of a fractional differential equation

We consider the fractional differential equation

$$\begin{aligned} \partial_{0+}^\alpha u(t) &= u^2(t), \quad t > 0, \quad 0 < \alpha < 1, \\ u(0) &= u_0 \in \mathbb{R}. \end{aligned} \quad (1.16)$$

The blow-up of solutions to (1.16) is assured by the following theorem.

Theorem 1.6. [14] If $u_0 > 0$, then the solution of problem (1.16) blows-up in a finite time

$$\left(\frac{\Gamma(\alpha+1)}{4u_0} \right)^{\frac{1}{\alpha}} \leq T^* \leq \left(\frac{\Gamma(\alpha+1)}{u_0} \right)^{\frac{1}{\alpha}}, \quad (1.17)$$

that is $\lim_{t \rightarrow T^*} u(t) = +\infty$.

2 Blowing-up solutions of the time-fractional Rosenau-KdV-BBM-Burgers equation

In this section we consider the time-fractional Rosenau-KdV-BBM-Burgers equation:

$$\begin{aligned} \partial_{+0,t}^\alpha (u - au_{xx} + bu_{xxxx}) + cu_{xxx} - du_{xx} + eu_x + uu_x &= 0, \quad 0 < x < L, \quad t > 0, \\ u(x, 0) &= u_0(x), \quad x \in [0, L], \end{aligned} \quad (2.1)$$

where $a, b, c, d, e \in \mathbb{R}$ and u_0 is a given function.

The equation (2.1) is called the Rosenau-KdV-BBM-Burgers equation with time-fractional derivative as it is a generalization of the following well-known equations:

- If $\alpha = 1$ and $a = b = c = e = 0$, $d > 0$, then the equation (2.1) coincides with the classical Burgers equation (1.1);
- If $\alpha = 1$ and $a = b = d = e = 0$, $c = 1$, then the equation (2.1) coincides with the classical KdV equation (1.2);
- If $\alpha = 1$ and $b = c = d = e = 0$, $a = 1$, then the equation (2.1) coincides with the classical BBM equation (1.3);
- If $\alpha = 1$ and $a = b = e = 0$, $c = 1$, $d > 0$, then the equation (2.1) coincides with the classical KdV-Burgers equation (1.8);
- If $\alpha = 1$ and $b = c = e = 0$, $a = 1$, $d > 0$, then the equation (2.1) coincides with the classical BBM-Burgers equation (1.10);
- If $\alpha = 1$ and $b = d = e = 0$, $a = c = 1$, then the equation (2.1) coincides with the classical KdV-BBM equation (1.11);
- If $\alpha = 1$ and $a = c = d = 0$, $b = e = 1$, then equation (2.1) coincides with the classical Rosenau equation (1.4);
- If $\alpha = 1$ and $a = c = 0$, $b = e = 1$, then equation (2.1) coincides with the classical Rosenau-Burgers equation (1.12);
- If $\alpha = 1$ and $c = d = 0$, $a = b = e = 1$, then equation (2.1) coincides with the classical Rosenau-BBM equation (1.14);
- If $\alpha = 1$ and $a = d = 0$, $b = c = e = 1$, then equation (2.1) coincides with the classical Rosenau-KdV equation (1.13).

We study the question of the blow-up of a classical solution $u \in C_{t,x}^{1,4}([0, T] \times [0, L])$ of problem (2.1).

Let us consider a function $\varphi \in C^4([0, L])$ and suppose that the solution $u \in C_{t,x}^{1,4}([0, T] \times [0, L])$ of problem (2.1) exists. Multiplying equation (2.1) by φ and integrating by parts, we obtain

$$\begin{aligned} &\partial_{+0,t}^\alpha \int_0^L u(x, t)(\varphi(x) - a\varphi''(x) + b\varphi''''(x))dx \\ &= \int_0^L u(x, t)(c\varphi'''(x) + d\varphi''(x) + e\varphi'(x))dx + \frac{1}{2} \int_0^L u^2(x, t)\varphi'(x)dx \\ &\quad + \mathcal{B}(u(L, t), \varphi(L)) - \mathcal{B}(u(0, t), \varphi(0)), \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} \mathcal{B}(u(x, t), \varphi(x)) &= a\partial_{+0,t}^\alpha u_x(x, t)\varphi(x) - a\partial_{+0,t}^\alpha u(x, t)\varphi'(x) \\ &\quad - b\partial_{+0,t}^\alpha u_{xxx}(x, t)\varphi(x) + b\partial_{+0,t}^\alpha u_{xx}(x, t)\varphi'(x) \\ &\quad - b\partial_{+0,t}^\alpha u_x(x, t)\varphi''(x) + b\partial_{+0,t}^\alpha u(x, t)\varphi'''(x) \\ &\quad - cu_{xx}(x, t)\varphi(x) + cu_x(x, t)\varphi'(x) - cu(x, t)\varphi''(x) \\ &\quad + du_x(x, t)\varphi(x) - du(x, t)\varphi'(x) \end{aligned}$$

$$- eu(x, t)\varphi(x) - \frac{1}{2}u^2(x, t)\varphi(x).$$

Let the function $\varphi(x)$ be monotonically nondecreasing:

$$\varphi'(x) \geq 0, \quad \varphi'(x) \not\equiv 0 \quad \text{for } x \in [0, L] \quad (2.3)$$

and satisfy the following properties

$$\begin{cases} \theta_1 := \frac{1}{2} \int_0^L \frac{(c\varphi'''(x) + d\varphi''(x) + e\varphi'(x))^2}{\varphi'(x)} dx < \infty; \\ \theta_2 := 2 \int_0^L \frac{(\varphi(x) - a\varphi''(x) + b\varphi''''(x))^2}{\varphi'(x)} dx < \infty. \end{cases} \quad (2.4)$$

Then we have

$$\begin{aligned} & 2 \int_0^L u(x, t)(c\varphi'''(x) + d\varphi''(x) + e\varphi'(x))dx + \int_0^L u^2(x, t)\varphi'(x)dx \\ &= \int_0^L v^2(x, t)\varphi'(x)dx - \int_0^L \frac{(c\varphi'''(x) + d\varphi''(x) + e\varphi'(x))^2}{\varphi'(x)} dx, \end{aligned}$$

where

$$v(x, t) = u(x, t) + \frac{c\varphi'''(x) + d\varphi''(x) + e\varphi'(x)}{\varphi'(x)}.$$

Using the Hölder inequality, we obtain the following estimate

$$\begin{aligned} & \left(\int_0^L v(x, t)(\varphi(x) - a\varphi''(x) + b\varphi''''(x))dx \right)^2 \\ & \leq \int_0^L v^2(x, t)\varphi'(x)dx \int_0^L \frac{(\varphi(x) - a\varphi''(x) + b\varphi''''(x))^2}{\varphi'(x)} dx. \end{aligned}$$

Then, expression (2.2) takes the form

$$\partial_{+0,t}^\alpha F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1, \quad (2.5)$$

where

$$F(t) = \int_0^L v(x, t) (\varphi(x) - a\varphi''(x) + b\varphi''''(x)) dx$$

and

$$\Phi(t) = \mathcal{B}(u(L, t), \varphi(L)) - \mathcal{B}(u(0, t), \varphi(0)).$$

Then the following theorem holds.

Theorem 2.1. Let $u_0(x) \in L^1([0, L])$ and the solution u of the equation (2.1) is such that $u \in C_{t,x}^{1,4}((0, L) \times (0, T))$ and let the function φ satisfy conditions (2.3), (2.4). If $\Phi(t) - \theta_1 \geq 0$, for all $t > 0$, and $F(0) > 0$, then

$$F(t) \rightarrow +\infty \quad \text{for } t \rightarrow T^*,$$

where T^* satisfies estimate (1.17).

Proof. Obviously

$$\partial_{+0,t}^\alpha \tilde{F}(t) \geq \tilde{F}^2(t),$$

where $\tilde{F}(t) = \theta_2^{-1} F(t)$.

Since the function $\tilde{F}(t)$ is an upper solution of equation (1.16), therefore $\tilde{F}(t) \rightarrow +\infty$ for $t \rightarrow T^*$, where T^* satisfies estimate (1.17). Whereupon by comparison principle (see [26]) we have $F(t) \rightarrow +\infty$ for $t \rightarrow T^*$. $\square$

Note that the trial function method has great practical convenience.

Example 2.2. (Fractional Korteweg-de Vries equation). Consider the problem (2.1) with $a = b = d = e = 0$, $c = 1$ on the interval $[0, 1]$ equipped with the boundary conditions:

$$\begin{aligned} u_x(0, t) &= 0, \quad t \geq 0, \\ u(1, t) &= 0, \quad t \geq 0, \\ u_{xx}(1, t) &= u_x(1, t), \quad t \geq 0. \end{aligned}$$

Letting $\varphi(x) = x$, we obtain

$$\theta_1 := 0, \quad \theta_2 := \frac{2}{3}$$

and

$$\Phi(t) = \theta_1 = 0, \quad \text{for all } t > 0;$$

hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition

$$\int_0^1 u_0(x) x dx > 0.$$

Example 2.3. (Fractional Burgers equation). Let $a = b = c = e = 0$, $d > 0$ in problem (2.1) on the interval $[0, 1]$ and let the solution of problem (2.1) satisfy the Robin type nonlinear boundary conditions:

$$\begin{aligned} u(0, t) &= 0, \quad t \geq 0, \\ u_x(1, t) - u(1, t) - \frac{1}{2d} u^2(1, t) &= 0, \quad t \geq 0. \end{aligned}$$

Then, if $\varphi(x) = x$, we obtain

$$\theta_1 := 0, \quad \theta_2 := \frac{2}{3}$$

and

$$\Phi(t) = \theta_1 = 0, \quad \text{for all } t > 0;$$

hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition

$$\int_0^1 u_0(x) x dx > 0.$$

Example 2.4. (Fractional Benjamin-Bona-Mahony equation). With $b = c = d = e = 0$, $a = 1$, consider the problem (2.1) on the interval $[0, 1]$ supplemented with final boundary conditions:

$$\begin{aligned} u(1, t) &= 0, \quad t \geq 0, \\ u_x(1, t) &= 0, \quad t \geq 0. \end{aligned}$$

Taking $\varphi(x) = x^4$, we obtain

$$\theta_1 := 0, \quad \theta_2 := \frac{397}{3}$$

and

$$\Phi(t) = 0, \text{ for all } t > 0;$$

hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition

$$\int_0^1 u_0(x)x^2(x^2 - 12)dx > 0.$$

Example 2.5. (Fractional Rosenau equation). Let $a = c = d = 0$, $b = e = 1$ and consider problem (2.1) on the interval $[0, 1]$ with boundary conditions

$$\begin{aligned} u(0, t) &= 0, \quad t \geq 0, \\ u_{xx}(0, t) &= 0, \quad t \geq 0, \\ u_{xx}(1, t) &= 0, \quad t \geq 0, \\ -\partial_{+0,t}^\alpha u_{xxx}(0, t) &= f(t), \quad t \geq 0. \end{aligned}$$

Suppose that $f(t) \geq \frac{1}{2}$, for all $t > 0$. Then, if $\varphi(x) = x - 1$, we obtain

$$\theta_1 := \frac{1}{2}, \quad \theta_2 := \frac{2}{3}$$

and

$$\Phi(t) - \theta_1 = f(t) - \frac{1}{2} \geq 0, \text{ for all } t > 0;$$

hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition

$$\int_0^1 u_0(x)(x - 1)dx > \frac{1}{2}.$$

Example 2.6. (Fractional Rosenau-Burgers equation). Let $a = c = 0$, $b = e = 1$ and consider problem (2.1) on the interval $[0, 1]$ with nonlocal dynamical boundary conditions

$$\begin{aligned} u(1, t) &= 0, \quad t \geq 0, \\ u_x(1, t) + u(0, t) &= 0, \quad t \geq 0, \\ u_{xx}(0, t) &= 0, \quad t \geq 0, \\ -\partial_{+0,t}^\alpha u_{xxx}(1, t) + \partial_{+0,t}^\alpha u_{xx}(1, t) &= \frac{1}{2}, \quad t \geq 0. \end{aligned}$$

Letting $\varphi(x) = x$, we obtain

$$\theta_1 := \frac{1}{2}, \quad \theta_2 := \frac{2}{3}$$

and

$$\Phi(t) - \theta_1 = 0, \text{ for all } t > 0;$$

hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition

$$\int_0^1 u_0(x)xdx > -\frac{1}{2}.$$

3 Blowing-up solutions of the time-fractional Camassa-Holm–Degasperis-Procesi equation

In this section we consider the time-fractional Camassa-Holm–Degasperis-Procesi equation:

$$\begin{aligned} \partial_{+0,t}^\alpha (u - u_{xx}) + au_x + buu_x - cu_x u_{xx} - duu_{xxx} &= 0, \quad 0 < x < L, \quad t > 0, \\ u(x, 0) &= u_0(x), \quad x \in [0, L], \end{aligned} \quad (3.1)$$

where $0 < \alpha \leq 1$, $a, b, c, d \in \mathbb{R}$ and u_0 is a given function.

The equation (3.1) is called the Camassa-Holm–Degasperis-Procesi equation with time-fractional derivative as it is a generalization of the following well-known equations:

- If $\alpha = 1$ and $a = 2\kappa > 0$, $b = 3$, $c = 2$, $d = 1$, then the equation (3.1) coincides with the classical Camassa-Holm equation (1.6);
- If $\alpha = 1$ and $a = 2\kappa > 0$, $b = 4$, $c = 3$, $d = 1$, then the equation (3.1) coincides with the classical Degasperis-Procesi equation (1.7).

We study the question of the blow-up of a classical solution $u \in C_{t,x}^{1,3}([0, T] \times [0, L])$ of problem (3.1). Let us consider a function $\varphi \in C^3([0, L])$ and suppose that the solution $u \in C_{t,x}^{1,3}([0, T] \times [0, L])$ of problem (3.1) exists. Using the equality

$$(u^2)_{xxx} = 6u_x u_{xx} + 2uu_{xxx},$$

we reduce the equation (3.1) to the equation

$$\partial_{+0,t}^\alpha (u - u_{xx}) + au_x + buu_x + (3d - c)u_x u_{xx} - \frac{d}{2}(u^2)_{xxx} = 0, \quad 0 < x < L, \quad t > 0. \quad (3.2)$$

Multiplying equation (3.1) by φ and integrating by parts, we obtain

$$\begin{aligned} &\partial_{+0,t}^\alpha \int_0^L u(x, t)(\varphi(x) - \varphi''(x))dx \\ &= a \int_0^L u(x, t)\varphi'(x)dx + \frac{3d - c}{2} \int_0^L u_x^2(x, t)\varphi'(x)dx \\ &\quad + \frac{1}{2} \int_0^L u^2(x, t)(b\varphi'(x) - d\varphi'''(x))dx \\ &\quad + \mathcal{B}(u(L, t), \varphi(L)) - \mathcal{B}(u(0, t), \varphi(0)), \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} \mathcal{B}(u(x, t), \varphi(x)) &= \partial_{+0,t}^\alpha u_x(x, t)\varphi(x) - \partial_{+0,t}^\alpha u(x, t)\varphi'(x) \\ &\quad - au(x, t)\varphi(x) - \frac{b}{2}u^2(x, t)\varphi(x) \\ &\quad - \frac{d - c}{2}u_x^2(x, t)\varphi(x) + du(x, t)u_{xx}(x, t)\varphi(x) \\ &\quad - du(x, t)u_x(x, t)\varphi'(x) + \frac{d}{2}u^2(x, t)\varphi''(x). \end{aligned}$$

Let $3d - c \geq 0$ and the function $\varphi(x)$ be monotonically nondecreasing:

$$\varphi'(x) \geq 0 \quad \text{for } x \in [0, L], \quad (3.4)$$

then from (3.3) we have

$$\begin{aligned} \partial_{+0,t}^\alpha \int_0^L u(x,t)(\varphi(x) - \varphi''(x))dx &\geq a \int_0^L u(x,t)\varphi'(x)dx \\ &+ \frac{1}{2} \int_0^L u^2(x,t)(b\varphi'(x) - d\varphi'''(x))dx \\ &+ \mathcal{B}(u(L,t), \varphi(L)) - \mathcal{B}(u(0,t), \varphi(0)). \end{aligned} \quad (3.5)$$

Let φ satisfy the following properties

$$\begin{cases} b\varphi'(x) - d\varphi'''(x) \geq 0, \quad x \in [0, L]; \\ \theta_1 := \frac{1}{2} \int_0^L \frac{a^2 \varphi'^2(x)}{b\varphi'(x) - d\varphi'''(x)} dx < \infty; \\ \theta_2 := 2 \int_0^L \frac{(\varphi(x) - \varphi''(x))^2}{b\varphi'(x) - d\varphi'''(x)} dx < \infty. \end{cases} \quad (3.6)$$

Then we have

$$\begin{aligned} 2a \int_0^L u(x,t)\varphi'(x)dx + \int_0^L u^2(x,t)(b\varphi'(x) - d\varphi'''(x))dx \\ = \int_0^L v^2(x,t)(b\varphi'(x) - d\varphi'''(x))dx - a^2 \int_0^L \frac{\varphi'^2(x)}{b\varphi'(x) - d\varphi'''(x)} dx, \end{aligned}$$

where

$$v(x,t) = u(x,t) + a \frac{\varphi'(x)}{b\varphi'(x) - d\varphi'''(x)}.$$

Using the Hölder inequality, we obtain the following estimate

$$\begin{aligned} \left(\int_0^L v(x,t)(\varphi(x) - \varphi''(x))dx \right)^2 \\ \leq \int_0^L v^2(x,t)(b\varphi'(x) - d\varphi'''(x))dx \int_0^L \frac{(\varphi(x) - \varphi''(x))^2}{b\varphi'(x) - d\varphi'''(x)} dx. \end{aligned}$$

Then, expression (3.3) takes the form

$$\partial_{+0,t}^\alpha F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1, \quad (3.7)$$

where

$$F(t) = \int_0^L v(x,t) (\varphi(x) - \varphi''(x)) dx$$

and

$$\Phi(t) = \mathcal{B}(u(L,t), \varphi(L)) - \mathcal{B}(u(0,t), \varphi(0)).$$

Then the following theorem holds.

Theorem 3.1. Let $u_0(x) \in L^1([0, L])$ and the solution u of the equation (3.1) is such that $u \in C_{t,x}^{1,3}((0, L) \times (0, T))$ and let the function φ satisfy conditions (3.4), (3.6). If $\Phi(t) - \theta_1 \geq 0$, for all $t > 0$, and $F(0) > 0$, then

$$F(t) \rightarrow +\infty \text{ for } t \rightarrow T^*,$$

where T^* satisfies estimate (1.17).

The Theorem 3.1 can be proved as Theorem 2.1.

Below we give some examples.

Example 3.2. (Fractional Camassa-Holm equation). Consider the problem (3.1) with $a = 2\kappa > 0$, $b = 3$, $c = 2$, $d = 1$, on the interval $[0, 1]$ equipped with the dynamical boundary conditions:

$$\begin{aligned} u(0, t) &= 0, \quad t \geq 0, \\ u(1, t) &= 0, \quad t \geq 0, \\ \partial_{+0,t}^\alpha u_x(1, t) + \frac{1}{2} u_x^2(1, t) &= f(t) \geq \frac{2\kappa^2}{3}, \quad t \geq 0. \end{aligned}$$

Letting $\varphi(x) = x$, we obtain

$$\theta_1 := \frac{2\kappa^2}{3}, \quad \theta_2 := \frac{2}{9}$$

and

$$\Phi(t) - \theta_1 \geq 0, \quad \text{for all } t > 0;$$

hence it follows from Theorem 3.1 that the solution of problem (3.1) blows up in finite time under the condition

$$\int_0^1 u_0(x)x dx > -\frac{\kappa}{3}.$$

Example 3.3. (Fractional Degasperis-Procesi). Let $a = 2\kappa > 0$, $b = 4$, $c = 3$, $d = 1$, in problem (3.1) on the interval $[0, 1]$ and let the solution of problem (3.1) satisfy the nonlinear nonlocal boundary conditions:

$$\begin{aligned} u(1, t) &= 0, \quad t \geq 0, \\ u_x(0, t) &= 0, \quad t \geq 0, \\ \partial_{+0,t}^\alpha u_x(1, t) + \partial_{+0,t}^\alpha u(0, t) + u_x^2(1, t) &= g(t) \geq \frac{\kappa^2}{2}, \quad t \geq 0. \end{aligned}$$

Then, if $\varphi(x) = x$, we obtain

$$\theta_1 := \frac{\kappa^2}{2}, \quad \theta_2 := \frac{1}{6}$$

and

$$\Phi(t) - \theta_1 \geq 0, \quad \text{for all } t > 0;$$

hence it follows from Theorem 3.1 that the solution of problem (3.1) blows up in finite time under the condition

$$\int_0^1 u_0(x)x dx > -\frac{\kappa}{4}.$$

4 Blowing-up solutions of the time-fractional Ostrovsky equation

We consider the equation

$$\partial_{+0,t}^\alpha u_x + au_{xx} + bu_{xxx} + (uu_x)_x = 0, \quad x \in (0, L), \quad t > 0, \quad (4.1)$$

with Cauchy data

$$u(x, 0) = u_0(x), \quad x \in [0, L], \quad (4.2)$$

where $0 < \alpha \leq 1$, $a, b \in \mathbb{R}$ and u_0 is a sufficiently smooth function.

Multiplying equation (4.1) by a function $\varphi(x) \in C^4([0, L])$ and integrating by parts, we obtain

$$\begin{aligned} \partial_{+0,t}^\alpha \int_0^L u(x, t) \varphi'(x) dx &= \frac{1}{2} \int_0^L u^2(x, t) \varphi''(x) dx \\ &+ \int_0^L u(x, t) (a \varphi''(x) + b \varphi''''(x)) dx + \mathcal{B}(u(x, t), \varphi(x)) \Big|_0^L, \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} \mathcal{B}(u(x, t), \varphi(x)) &= \partial_{+0,t}^\alpha u(x, t) \varphi(x) + a u_x(x, t) \varphi(x) - a u(x, t) \varphi'(x) \\ &+ b u_{xxx}(x, t) \varphi(x) - b u_{xx}(x, t) \varphi'(x) + b u_x(x, t) \varphi''(x) \\ &- b u(x, t) \varphi'''(x) + u(x, t) u_x(x, t) \varphi(x) - \frac{1}{2} u^2(x, t) \varphi'(x). \end{aligned}$$

Let the function $\varphi(x)$ satisfy the properties:

$$\varphi''(x) \geq 0 \quad \text{for } x \in [0, L] \quad (4.4)$$

and

$$\begin{cases} \theta_1 := \frac{1}{2} \int_0^L \frac{(a \varphi''(x) + b \varphi''''(x))^2}{\varphi''(x)} dx < \infty; \\ \theta_2 := 2 \int_0^L \frac{\varphi'^2(x)}{\varphi''(x)} dx < \infty. \end{cases} \quad (4.5)$$

Then we have

$$\begin{aligned} &2 \int_0^L u(x, t) (a \varphi''(x) + b \varphi''''(x)) dx + \int_0^L u^2(x, t) \varphi''(x) dx \\ &= \int_0^L v^2(x, t) \varphi''(x) dx - \int_0^L \frac{(a \varphi''(x) + b \varphi''''(x))^2}{\varphi''(x)} dx, \end{aligned}$$

where

$$v(x, t) = u(x, t) + \frac{a \varphi''(x) + b \varphi''''(x)}{\varphi''(x)}.$$

Using the Hölder inequality, we obtain the following estimate

$$\left(\int_0^L v(x, t) \varphi'(x) dx \right)^2 \leq \int_0^L v^2(x, t) \varphi''(x) dx \int_0^L \frac{\varphi'^2(x)}{\varphi''(x)} dx.$$

Then, the expression (4.3) can be rewritten as

$$\partial_{+0,t}^\alpha F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1, \quad (4.6)$$

where

$$F(t) = \int_0^L v(x, t) \varphi'(x) dx$$

and

$$\Phi(t) = \mathcal{B}(u(L, t), \varphi(L)) - \mathcal{B}(u(0, t), \varphi(0)).$$

Then the following theorem holds.

Theorem 4.1. Let $u_0(x) \in L^1([0, L])$ and the solution u of the problem (4.1), (4.2) is such that $u \in C_{t,x}^{1,4}((0, L) \times (0, T))$ and let the function φ satisfy conditions (4.4), (4.5). If $\Phi(t) - \theta_1 \geq 0$, for all $t > 0$, and $F(0) > 0$, then

$$F(t) \rightarrow +\infty \text{ for } t \rightarrow T^*,$$

where T^* satisfies estimate (1.17).

The Theorem 4.1 can be proved as Theorem 2.1.

Example 4.2. With $a = 1$ and $b = -1$, consider problem (4.1), (4.2) on the interval $[0, 1]$ subject to boundary conditions

$$\begin{aligned} u(1, t) &= 0, \quad t \geq 0, \\ u_x(1, t) &= 0, \quad t \geq 0, \\ u_x(0, t) &= 0, \quad t \geq 0, \\ -u_{xxx}(1, t) + 2u_{xx}(1, t) &= f(t), \quad t \geq 0. \end{aligned}$$

Suppose that $f(t) \geq 1$, for all $t > 0$. Then, if $\varphi(x) = x^2$, we obtain

$$\theta_1 := 1, \quad \theta_2 := \frac{4}{3}$$

and

$$\Phi(t) - \theta_1 = f(t) - 1 \geq 0, \text{ for all } t > 0;$$

hence it follows by Theorem 4.1 that the solution of problem (4.1), (4.2) blows up in finite time under the condition

$$\int_0^1 u_0(x)x dx > -\frac{1}{2}.$$

5 Blowing-up solutions of the time-fractional modified KdV-Burgers equation

Consider the initial value problem for the time-fractional analogue of the well-known modified Korteweg-de Vries-Burgers equation with dissipation:

$$\partial_{+0,t}^\alpha u + u^2 u_x + a u_{xxx} - b u_{xx} = 0, \quad x \in (0, L), \quad t > 0, \quad (5.1)$$

$$u(x, 0) = u_0(x), \quad x \in [0, L], \quad (5.2)$$

where $0 < \alpha \leq 1$, $a, b \in \mathbb{R}$ and u_0 is a sufficiently smooth function.

Let a function $\varphi \in C^3([0, L])$ satisfy the properties:

$$\varphi(x) \leq 0, \quad \varphi'(x) \geq 0 \text{ for } x \in [0, L], \quad (5.3)$$

$$3a\varphi'(x) + 2b\varphi(x) \leq 0 \text{ for } x \in [0, L], \quad (5.4)$$

and

$$\begin{cases} \theta_1 := 2 \int_0^L \frac{(a\varphi'''(x) + b\varphi''(x))^2}{\varphi'(x)} dx < \infty; \\ \theta_2 := \frac{1}{2} \int_0^L \frac{\varphi^2(x)}{\varphi'(x)} dx < \infty. \end{cases} \quad (5.5)$$

Multiplying equation (5.1) by $u(x, t)\varphi(x)$ and integrating by parts, we obtain

$$\begin{aligned} & \int_0^L \partial_{+0,t}^\alpha u(x, t) u(x, t) \varphi(x) dx \\ &= \frac{1}{4} \int_0^L u^4(x, t) \varphi'(x) dx + \frac{1}{2} \int_0^L u^2(x, t) (a\varphi'''(x) + b\varphi''(x)) dx \\ & - \int_0^L u_x^2(x, t) \left(\frac{3a}{2} \varphi'(x) + b\varphi(x) \right) dx + \mathcal{B}(u(x, t), \varphi(x)) \Big|_0^L, \end{aligned} \quad (5.6)$$

where

$$\begin{aligned} \mathcal{B}(u(x, t), \varphi(x)) = & -\frac{1}{4} u^4(x, t) \varphi(x) + u(x, t) u_x(x, t) (a\varphi'(x) + b\varphi(x)) \\ & - \frac{u^2(x, t)}{2} (a\varphi''(x) + b\varphi'(x)) + \frac{a}{2} u_x^2(x, t) \varphi(x) \\ & - au(x, t) u_{xx}(x, t) \varphi(x). \end{aligned}$$

Then we have

$$\begin{aligned} & \frac{1}{2} \int_0^L u^2(x, t) (a\varphi'''(x) + b\varphi''(x)) dx + \frac{1}{4} \int_0^L u^4(x, t) \varphi'(x) dx \\ &= \frac{1}{4} \int_0^L v^4(x, t) \varphi'(x) dx - \frac{1}{4} \int_0^L \frac{(a\varphi'''(x) + b\varphi''(x))^2}{\varphi'(x)} dx, \end{aligned}$$

where

$$v^2(x, t) = u^2(x, t) + \frac{a\varphi'''(x) + b\varphi''(x)}{\varphi'(x)}.$$

Using the Hölder inequality and inequality (1.15), we obtain

$$\begin{aligned} \left(\int_0^L v^2(x, t) \varphi(x) dx \right)^2 &\leq \int_0^L v^4(x, t) \varphi'(x) dx \int_0^L \frac{\varphi^2(x)}{\varphi'(x)} dx, \\ \partial_{+0,t}^\alpha u(x, t) u(x, t) \varphi(x) &\geq \frac{1}{2} \partial_{+0,t}^\alpha \left(u^2(x, t) \varphi(x) \right). \end{aligned}$$

From (5.4), we also get

$$- \int_0^L u_x^2(x, t) \left(\frac{3a}{2} \varphi'(x) + b\varphi(x) \right) dx \geq 0.$$

Then, expression (5.6) takes the form

$$\partial_{+0,t}^\alpha F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1, \quad (5.7)$$

where

$$F(t) = \int_0^L v^2(x, t) \varphi(x) dx$$

and

$$\Phi(t) = 2\mathcal{B}(u(L, t), \varphi(L)) - 2\mathcal{B}(u(0, t), \varphi(0)).$$

Then the following theorem holds.

Theorem 5.1. Let $u_0(x) \in L^1([0, L])$ and the solution u of the equation (5.1) is such that $u \in C_{t,x}^{1,3}((0, L) \times (0, T))$ and let the function φ satisfy conditions (5.3), (5.4) and (5.5). If $\Phi(t) - \theta_1 \geq 0$, for all $t > 0$, and $F(0) > 0$, then

$$F(t) \rightarrow +\infty \text{ for } t \rightarrow T^*,$$

where T^* satisfies estimate (1.17).

Proof. Since $\Phi(t) - \theta_1 \geq 0$, for all $t > 0$, it follows from (5.7) that

$$\partial_{+0,t}^\alpha \tilde{F}(t) \geq \tilde{F}^2(t),$$

where $\tilde{F}(t) = \theta_2^{-1} F(t)$.

As the function $\tilde{F}(t)$ is an upper solution of equation (1.16), therefore $\tilde{F}(t) \rightarrow +\infty$ for $t \rightarrow T^*$, where T^* satisfies estimate (1.17). Whereupon $F(t) \rightarrow +\infty$ for $t \rightarrow T^*$. $\square$

Example 5.2. Consider problem (5.1) with $a = 2$ and $b = 3$ on the interval $[0, 1]$, supplemented with Dirichlet type nonlinear boundary conditions

$$\begin{aligned} u(0, t) &= 0, \quad t \geq 0, \\ u(1, t) &= 0, \quad t \geq 0, \\ u_x^2(0, t) - e^{-1} u_x^2(1, t) &= f(t), \quad t \geq 0, \end{aligned}$$

where $e = \exp(1)$ is the Euler's number. Suppose that $f(t) \geq 2(1 - e^{-1})$ for all $t > 0$. Then, if

$$\varphi(x) = -\exp(-x),$$

we obtain

$$\theta_1 := 2(1 - e^{-1}), \quad \theta_2 := \frac{1 - e^{-1}}{2}$$

and

$$\Phi(t) - \theta_1 \geq 0, \quad \text{for all } t > 0;$$

hence it follows from Theorem 5.1 that the solution of problem (5.1) blows up in finite time under the condition

$$\int_0^1 u_0^2(x) \exp(-x) dx < 1 - e^{-1}.$$

Example 5.3. Let $a = 0$ and $b > 0$. Let in problem (5.1) on the interval $[0, 1]$ be given Dirichlet boundary conditions

$$\begin{aligned} u(0, t) &= 0, \quad t \geq 0, \\ u(1, t) &= 0, \quad t \geq 0. \end{aligned}$$

Then, if

$$\varphi(x) = x - 1,$$

we obtain

$$\theta_1 := 0, \quad \theta_2 := \frac{1}{6}$$

and

$$\Phi(t) = \theta_1 = 0, \quad \text{for all } t > 0;$$

hence it follows from Theorem 5.1 that the solution of problem (5.1) blows up in finite time under the condition

$$\int_0^1 u_0^2(x)(x - 1) dx > 0,$$

.

6 Maximum principle and gradient blow-up in time-fractional Burgers equation

The purpose of this section is to study time-fractional Burgers equation

$$\partial_{+0,t}^\alpha u + uu_x = \nu u_{xx}, \quad x \in (0, L), \quad t > 0, \quad (6.1)$$

with the initial condition

$$u(x, 0) = u_0(x), \quad x \in [0, L], \quad (6.2)$$

where $\nu > 0$ and u_0 is a sufficiently smooth function.

6.1 Maximum principle

In this subsection, we present a maximum principle for the time-fractional Burgers equation (6.1).

Theorem 6.1. *Let $u(x, t)$ satisfy the time-fractional Burgers equation (6.1) with Cauchy data (6.2). Then*

$$u(x, t) \geq \min_{(x,t)} \{u(0, t), u(L, t), u_0(x)\} \text{ for } (x, t) \in [0, L] \times [0, T].$$

Proof. Let

$$m = \min_{(x,t)} \{u(0, t), u(L, t), u_0(x)\}$$

and

$$\tilde{u}(x, t) = u(x, t) - m.$$

Then, we have

$$\tilde{u}(0, t) = u(0, t) - m \geq 0, \quad t \in [0, T],$$

$$\tilde{u}(L, t) = u(L, t) - m \geq 0, \quad t \in [0, T],$$

and

$$\tilde{u}(x, 0) = u_0(x) - m \geq 0, \quad x \in [0, L].$$

Since

$$\partial_{+0,t}^\alpha \tilde{u}(x, t) = \partial_{+0,t}^\alpha u(x, t)$$

and

$$\tilde{u}_{xx}(x, t) = u_{xx}(x, t),$$

it follows that $\tilde{u}(x, t)$ satisfies:

$$\partial_{+0,t}^\alpha \tilde{u}(x, t) + \tilde{u}(x, t)\tilde{u}_x(x, t) + m\tilde{u}_x(x, t) = \nu \tilde{u}_{xx}(x, t),$$

and the initial condition

$$\tilde{u}(x, 0) = u_0(x) - m \geq 0, \quad x \in [0, L].$$

Suppose that there exists some $(x, t) \in [0, L] \times [0, T]$ such that $\tilde{u}(x, t)$ is negative. Since

$$\tilde{u}(x, t) \geq 0, \quad (x, t) \in \{0\} \times [0, T] \cup \{L\} \times [0, T] \cup [0, L] \times \{0\},$$

there is $(x_0, t_0) \in (0, L) \times (0, T]$ such that $\tilde{u}(x_0, t_0)$ is the negative minimum of $\tilde{u}$ over $(0, L) \times (0, T]$. It follows from Property 1.5 that $\partial_{+0,t}^\alpha u(x_0, t_0) \leq 0$.

Therefore at (x_0, t_0) , we get

$$\partial_{+0,t}^\alpha \tilde{u}(x_0, t_0) \leq 0, \quad \tilde{u}_x(x_0, t_0) = 0 \quad \text{and} \quad \nu \tilde{u}_{xx}(x_0, t_0) > 0.$$

This contradiction shows that $\tilde{u}(x, t) \geq 0$, whereupon $u(x, t) \geq m$ on $[0, L] \times [0, T]$. □

A similar result can be obtained for a nonpositive solution $u(x, t)$ by considering $-u(x, t)$.

Theorem 6.2. *Suppose that $u(x, t)$ satisfies (6.1), (6.2). Then*

$$u(x, t) \leq \max_{(x,t)} \{u(L, t), u(0, t), u(x, 0)\}, \quad (x, t) \in [0, L] \times [0, T).$$

6.2 Gradient blow-up

Now suppose that the boundary conditions are set in such a way that the global in time solution of equation (6.1) is bounded. Let there exist a smooth bounded solution u such that $|u(x, t)| \leq M$. Differentiating equation (6.1) with respect to x , we obtain

$$\partial_{+0,t}^\alpha u_x + uu_{xx} + u_x^2 - vu_{xxx} = 0, \quad x \in (0, L), \quad t > 0. \quad (6.3)$$

Substituting the expression for u_{xx} from (6.1) into (6.3), we obtain

$$\partial_{+0,t}^\alpha u_x + u_x^2 + \frac{1}{v} u \partial_{+0,t}^\alpha u + \frac{1}{v} u^2 u_x = vu_{xxx}, \quad x \in (0, L), \quad t > 0. \quad (6.4)$$

Multiply the equation (6.4) by the function $0 \leq \varphi(x) \in C^3([0, L])$ and integrate by parts over the domain $[0, L]$ to get

$$\begin{aligned} & \int_0^L \left(\partial_{+0,t}^\alpha u_x(x, t) + \frac{1}{v} u(x, t) \partial_{+0,t}^\alpha u(x, t) \right) \varphi(x) dx \\ &= - \int_0^L \left(u_x^2(x, t) + \frac{1}{v} u^2(x, t) u_x(x, t) \right) \varphi(x) dx - v \int_0^L u(x, t) \varphi'''(x) dx \\ &+ v \left(u_{xx}(x, t) \varphi(x) - u_x(x, t) \varphi'(x) + u(x, t) \varphi''(x) \right) \Big|_0^L. \end{aligned} \quad (6.5)$$

Denote $v = -u_x - \frac{1}{2v} u^2$. Then, using $|u(x, t)| \leq M$ and inequality (1.15), we obtain

$$\begin{aligned} \partial_{+0,t}^\alpha \int_0^L v(x, t) \varphi(x) dx &\geq \int_0^L v^2(x, t) \varphi(x) dx \\ &- \frac{M^4}{4v^2} \int_0^L \varphi(x) dx - Mv \int_0^L |\varphi'''(x)| dx \\ &- v \left(u_{xx}(x, t) \varphi(x) - u_x(x, t) \varphi'(x) + u(x, t) \varphi''(x) \right) \Big|_0^L. \end{aligned} \quad (6.6)$$

Let

$$\theta_1 := \frac{M^4}{4v^2} \int_0^L \varphi(x) dx + Mv \int_0^L |\varphi'''(x)| dx < \infty, \quad (6.7)$$

$$\theta_2 := \int_0^L \varphi(x) dx < \infty. \quad (6.8)$$

Using Hölder inequality, we can rewrite the expression (6.6) in the form

$$\partial_{+0,t}^\alpha F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1, \quad (6.9)$$

where

$$F(t) = \int_0^L v(x, t) \varphi(x) dx$$

and

$$\Phi(t) = -v \left(u_{xx}(x, t)\varphi(x) - u_x(x, t)\varphi'(x) + u(x, t)\varphi''(x) \right) \Big|_0^L.$$

Then the following theorem holds.

Theorem 6.3. Let $u_0(x) \in C^1([0, L])$ and the solution u of the equation (6.1) be such that $u \in C_{t,x}^{1,3}((0, L) \times (0, T))$ and let the function φ satisfy conditions (6.7) and (6.8). If $\Phi(t) - \theta_1 \geq 0$, for all $t > 0$, and

$$F(0) = - \int_0^L \left(u'_0(x) + \frac{1}{2v} u_0^2(x) \right) \varphi(x) dx > 0,$$

then

$$F(t) \rightarrow +\infty \text{ for } t \rightarrow T^*,$$

where T^* satisfies estimate (1.17).

We do not provide the proof of this theorem as it runs parallel to that of Theorem 5.1.

Conclusion

In this article, we have studied blowing-up solutions to some time-fractional nonlinear partial differential equations. In precise terms, we have obtained the following results:

- Blowing-up solutions of the time-fractional Rosenau-KdV-BBM-Burgers equation with initial conditions;
- Blowing-up solutions of the time-fractional Camassa-Holm–Degasperis-Procesi equation with initial conditions;
- Blowing-up solutions of the time-fractional Ostrovsky equation with Cauchy data;
- Blowing-up solutions of the initial problem for the time-fractional analogue of the modified Korteweg-de Vries-Burgers equation with dissipation;
- Maximum principle and gradient blow-up in time-fractional Burgers equation.

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