Research article

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Existence, multiplicity and nonexistence results for Kirchhoff type equations

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Abstract: In this paper, we study following Kirchhoff type equation:

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(u) + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We consider first the case that $\Omega \subset \mathbb{R}^3$ is a bounded domain. Existence of at least one or two positive solutions for above equation is obtained by using the monotonicity trick. Nonexistence criterion is also established by virtue of the corresponding Pohožaev identity. In particular, we show nonexistence properties for the 3-sublinear case as well as the critical case. Under general assumption on the nonlinearity, existence result is also established for the whole space case that $\Omega = \mathbb{R}^3$ by using property of the Pohožaev identity and some delicate analysis.

Keywords: Kirchhoff type equations; Subcritical or critical growth; Positive solutions; Nonexistence results**MSC:** 35J60; 35J25

1 Introduction and main results

This paper is concerned with following Kirchhoff type equation:

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(u) + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain or $\Omega = \mathbb{R}^3$, $0 \leq h \in L^2(\Omega)$ and $f \in C(\mathbb{R}, \mathbb{R})$. Problem like (1.1) is associated with the stationary analogue of the wave equation arising in the study of string or membrane vibrations:

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u). \quad (1.2)$$

Such type equation proposed first by Kirchhoff [1] is used to describe the transversal oscillations of a stretched string. The nonlocal term is also used to model suspension bridges [2] and to describe the growth and movement of a particular species in biological systems [3]. More mathematical and physical background and applications of such problems can be found in [1, 4–6] and the references therein.

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Using $f(x, u)$ instead of $f(u) + h$ in (1.1), we have the stationary version of equation (1.2)

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain ($N \geq 1$). After the pioneering work of Lions [7] introducing the abstract framework, problem like or similar to (1.3) has been widely studied by many researchers via variational methods. Depending on various assumptions on f , there are many results concerning the solutions to (1.3), such as, the existence of positive solutions [6, 8–10], ground state solutions [11–14], sign-changing solutions [14–18], multiple or infinitely many solutions [10, 11, 13, 15, 19–22] and some other related results [3, 5, 23–27].

When f is critical growth at infinity, particularly, behaving like following form:

$$f(x, t) = t^5 + \lambda g(x, t), \quad \text{and} \quad 0 < \rho \int_0^t g(x, s) ds \leq g(x, t)t, \quad (1.4)$$

mountain pass solution for (1.3) was established by Alves et al. in [23] when $4 < \rho < 6$ and the parameter $\lambda > 0$ is sufficiently large. This result was improved by Figueiredo [9] by extending the range of ρ to $(2, 6)$, moreover, existence of positive solution was also obtained by them using truncation technique. In [22], two distinct solutions of (1.3) were obtained by Xie et al. for case that $5 < \rho < 6$ and $\lambda = 1$ in (1.4). For the case that $\rho \in (1, 2)$, two positive solutions of (1.3) were constructed by Lei et al. [11] via the concentration compactness argument provided that $g(x, t) \equiv t^q$ and the parameters b, λ are sufficiently small. See also [10] where multiple results were obtained via the perturbation method. In recent paper [13], the case $f(x, t) = Q(x)|t|^4 t + \lambda|t|^{q-1}t$ ($3 < q < 5$) was considered by Qin et al, and existence of positive ground state solutions was established there based on energy estimates and the Nehari manifold method, moreover, by minimizing the corresponding energy functional on some proper subsets of the Nehari manifold associated with the barycenter map, the relation between the number of positive solutions for (1.3) and the number of maxima of Q was also studied there [13].

When f has subcritical growth at infinity, i.e.,

$$\limsup_{t \rightarrow +\infty} \frac{f(x, t)}{t^5} = 0 \quad \text{uniformly in } x,$$

existence and uniqueness properties for parabolic problem (1.2) and elliptic equation (1.3) were studied by Chipot et al. [5]. Corrêa [6] found positive solutions to (1.3) for all $N \geq 1$ with the aid of fixed point theorems. Assuming the well known Ambrosetti-Rabinowitz type condition or following monotonic condition,

(AR) there exist $\eta > 4$ and $r > 0$ such that

$$0 < \eta F(x, u) \leq t f(x, t), \quad \forall |t| \geq r,$$

(F1) $t \mapsto \frac{f(x, t)}{|t|^3}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$,

He and Zou [20] obtained infinitely many large energy solutions for (1.1) via the fountain theorems. See [21] for multiplicity results concerning an oscillating behavior of f . In [16], following eigenvalue problem was studied by Perera and Zhang, and the corresponding eigenvalues are $0 < \mu_1 \leq \mu_2 \leq \dots$,

$$\begin{cases} -(\int_{\Omega} |\nabla u|^2 dx) \Delta u = \mu u^3, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.5)$$

Particularly, μ_1 is simple and isolated, and it can be expressed by (see [16, Section 3] and [18])

$$\mu_1 := \inf_{v \in S} \psi(v) > 0, \quad (1.6)$$

where

$$\psi(v) = \|v\|^4, \quad v \in \mathcal{S} := \left\{ v \in H_0^1(\Omega), \int_{\Omega} |v|^4 dx = 1 \right\}.$$

They [16] mainly concerned with the asymptotically 3-linear case, precisely, f satisfies

$$\lim_{t \rightarrow 0} \frac{f(x, t)}{at} = \lambda, \quad \lim_{t \rightarrow \infty} \frac{f(x, t)}{bt^3} = \mu, \quad \text{uniformly in } x. \quad (1.7)$$

Existence of nontrivial solution to (1.3) was established by them via critical group and Yang index when $\lambda \in (\lambda_l, \lambda_{l+1})$ and $\mu \in (\mu_m, \mu_{m+1})$ with $l \neq m$, where

$$0 < \lambda_1 < \lambda_2 \leq \dots \text{ is the sequence of all eigenvalues of } -\Delta \text{ in } H_0^1(\Omega). \quad (1.8)$$

Taking advantage of the invariant sets of descent flow, the 3-sublinear and 3-superlinear cases (AR) was assumed there) were also studied by Zhang and Perera [18] (see also [15]), moreover, existence of sign-changing solutions was constructed by them.

By developing the Nehari manifold method, the asymptotically 3-linear case was also considered by Qin et al. [12] where existence of ground state solution to (1.3) was established under following weak version of the monotonic condition (F1) (see [12, Lemma 3.9]).

(F2) for all $\theta \geq 0$ and $t \in \mathbb{R}$,

$$\frac{1 - \theta^4}{4} f(t)t + F(\theta t) - F(t) \geq 0.$$

In [8], Cheng and Wu studied (1.3) by assuming (F1) and following condition on f ,

$$\lim_{t \rightarrow 0} \frac{f(x, t)}{at} = p(x), \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{f(x, t)}{t^3} = q(x) \neq 0 \text{ uniformly for a.e. } x \in \Omega, \quad (1.9)$$

where $0 \leq p, q \in L^\infty(\Omega)$ and $\|p\|_\infty < \lambda_1$. Positive solutions were obtained for the 3-linear case and 3-superlinear case, respectively. In particular, nonexistence result was showed there by virtue of the monotonic condition (F1). See also [19]. In recent paper [3], Chen et al. considered (1.3) with concave-convex nonlinearity having the form:

$$f(x, t) = \lambda h(x)|t|^{q-2}t + g(x)|t|^{p-2}t, \quad 1 < q < 2 < p < 2^*,$$

where $2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$, and the continuous functions h, g satisfy $h^+ := \max\{h, 0\} \neq 0$ and $g^+ := \max\{g, 0\} \neq 0$. Taking advantage of the Nehari manifold approach, they got some existence results for the case $p > 4$, $p = 4$ and $p < 4$, respectively. Recently, Shuai [17] studied (1.3) under the monotonic condition (F1), moreover, they succeeded in finding a least energy sign-changing solution via the degree theory and quantitative deformation lemma. Such result was improved later by Tang et al. [14] based on the Non-Nehari manifold method.

Inspired by above works [8, 12, 19, 28], we consider first the bounded domain case and focus our attention on the subcritical case, precisely, the 3-sublinear, 3-linear growth and 3-superlinear are considered separately in this paper. Moreover, existence of at least one or two positive solutions for the equation (1.1) are obtained correspondingly. Based on the Pohožaev identity, we also establish a nonexistence criterion, in particular, we show a nonexistence property for the 3-sublinear case. In order to find one solution at the mountain pass level, we have to overcome the difficulty caused by the lack of a priori bounds on the Palais-Smale sequences. By combining the monotonicity trick used in [29] and some techniques, as well as the strategy used in [12], we manage to obtain the boundedness of the sequence. By the compactness of the Sobolev embedding, then we show the convergence of the Palais-Smale sequence. The hypotheses imposed on f and h are generic and natural.

For the case that $\Omega \subset \mathbb{R}^3$ is a bounded domain, recall that the working space to (1.1) is $H_0^1(\Omega)$, which is a Hilbert space equipped with following inner product

$$(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx, \quad \forall u, v \in H_0^1(\Omega), \quad (1.10)$$

and the corresponding norm is $\|u\| = (u, u)^{1/2}$. As well known, the embedding from $H_0^1(\Omega)$ to $L^s(\Omega)$ is continuous for $s \in [1, 2^*]$, and is compact for $s \in [1, 2^*)$ (see [30, Theorem 1.9]), here $2^* = 6$ since $N = 3$. Then we find the existence of $y_s > 0$ satisfying

$$\|u\|_s \leq y_s \|u\| = y_s \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega), \quad (1.11)$$

where $\|\cdot\|_s$ stands for the usual $L^s(\Omega)$ norm.

Let $u \in H_0^1(\Omega)$ be a solution of (1.1), then we have following useful Pohožaev identity (see [30]):

$$\begin{aligned} P(u) &:= \frac{1}{2} \left(a + b \|\nabla u\|_2^2 \right) \left[\|\nabla u\|_2^2 + \int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{v}) ds \right] \\ &\quad - 3 \int_{\Omega} F(u) dx - 3 \int_{\Omega} h u dx - \int_{\Omega} (\nabla h \cdot x) u dx = 0, \end{aligned} \quad (1.12)$$

where $\vec{v}$ is the unit outward normal to $\partial\Omega$. Multiplying (1.1) by u and calculating each side of the equation, one has

$$a \|\nabla u\|_2^2 + b \|\nabla u\|_2^4 - \int_{\Omega} f(u) u dx - \int_{\Omega} h u dx = 0.$$

By above two equalities, we obtain following non-existence criterion.

Theorem 1.1. *Let $F \in L^1(\Omega)$ and $h, \nabla h \cdot x \in L^2(\Omega)$. If $u \in H_{\text{loc}}^2(\overline{\Omega})$ is a solution of (1.1), then following identities hold:*

$$\left(a + b \|\nabla u\|_2^2 \right) \left[\|\nabla u\|_2^2 - \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{v}) ds \right] = \frac{3}{2} \int_{\Omega} \left[f(u) u - 2F(u) - \left(h + \frac{2}{3} \nabla h \cdot x \right) u \right] dx,$$

and

$$\left(a + b \|\nabla u\|_2^2 \right) \int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{v}) ds + \int_{\Omega} (f(u) u - 6F(u)) dx = \int_{\Omega} (5h + 2 \nabla h \cdot x) u dx.$$

In particular, this solution is trivial, i.e., $u \equiv 0$ if

$$3 \int_{\Omega} \left[f(u) u - 2F(u) - \left(h + \frac{2}{3} \nabla h \cdot x \right) u \right] dx \leq - \left(a + b \|\nabla u\|_2^2 \right) \int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{v}) ds.$$

Consider first the 3-sublinear case. Assume that $f(t) = v g(t)$ with $v > 0$ and that $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying following assumptions:

(g1) $\lim_{t \rightarrow +\infty} \frac{g(t)}{t^3} = 0$;

(g2) there exists $t_0 > 0$ such that $G(t_0) := \int_0^{t_0} g(s) ds > 0$;

(g3) there exists $C_0 > 0$ such that $|g(t)| \leq C_0 |t|^3$ for all $t \in \mathbb{R}$.

Since we focus our attention only on non-negative solutions of (1.1), we assume, without loss of generality, that $g(t) = 0$, $\forall t \leq 0$. Our results for the 3-sublinear case read as follows.

Theorem 1.2. *Let (g1) be satisfied, then for problem (1.1) with $f = v g(t)$, we have following properties.*

(a) *If $h \not\equiv 0$, then (1.1) has a positive solution for all $v > 0$.*

(b) If $h \equiv 0$ and (g2) is satisfied, then

i) there exists $v_1 > 0$ such that (1.1) has a positive solution for any $v > v_1$;

ii) there exists $v_0 := \frac{b\mu_1}{c_0} > 0$ such that (1.1) has no nontrivial solution if $0 < v < v_0$ and (g3) is satisfied in addition.

Consider now the 3-linear and 3-superlinear cases. For the first eigenvalues $\lambda_1, \mu_1 > 0$ given by (1.6) and (1.8), respectively, we make use of following assumptions on the continuous function $f : [0, \infty) \rightarrow \mathbb{R}$:

(f1) $\limsup_{t \rightarrow 0} \frac{f(t)}{t} \leq a\lambda_1$;

(f2) there exists $p \in (3, 5)$ such that $\limsup_{t \rightarrow +\infty} \frac{f(t)}{t^p} = 0$;

(f3) $\liminf_{t \rightarrow +\infty} \frac{f(t)}{t^3} > b\mu_1$.

(f4) $l := \lim_{t \rightarrow +\infty} \frac{f(t)}{t^3} < +\infty$.

(f5) $\lim_{t \rightarrow +\infty} \frac{f(t)}{t^3} = +\infty$, and

$$\liminf_{t \rightarrow +\infty} \frac{f(t)t - 4F(t)}{|t|^\tau} \geq -d, \text{ for some constants } d > 0, \tau \in [0, 2].$$

Now we are ready to state our results for these two cases.

Theorem 1.3. Let (f1), (f2) be satisfied and $\|h\|_2$ be sufficiently small, then we have following properties for problem (1.1).

(a) There exists at least one non-negative solution for (1.1), and this solution is positive when $h \not\equiv 0$.

(b) Let (f3) be satisfied. Then there exist at least two positive solutions for (1.1) if $h \not\equiv 0$, and at least a positive solution if $h \equiv 0$, provided that any one of following conditions is satisfied:

(i) (F2) is satisfied;

(ii) (f4) is satisfied with $\frac{l}{b} \neq \mu_m, m = 2, 3, \dots$;

(iii) (f5) is satisfied.

Next, we give a result for the critical growth case, precisely, the nonlinearity has the form $f(t) = |t|^4 t + v|t|^{q-1}t$.

Theorem 1.4. For problem (1.1) with $f(t) = |t|^4 t + v|t|^{q-1}t, 0 < q < 5$, we have following properties.

(a) Problem (1.1) has no nontrivial solution in $H_0^1(\Omega) \cap H_{loc}^2(\overline{\Omega})$ if

$$\left(a + b\|\nabla u\|_2^2\right) \int_{\partial\Omega} |\nabla u|^2 (x \cdot \vec{\nu}) ds + v \frac{q-5}{q+1} |u|^{q+1} \neq \int_{\Omega} (5h + 2\nabla h \cdot x) u dx.$$

In particular, when Ω is a smooth starshaped bounded domain, problem (1.1) has no nontrivial solution if $h \equiv 0$ and no positive solution if $5h + 2\nabla h \cdot x \leq 0$ for all $x \in \Omega$.

(b) If $h \equiv 0$ and $3 < q < 5$, then (1.1) has a positive ground state solution for all $v > 0$.

Finally, we consider the whole space case, i.e., following problem:

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u = f(u), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \quad (1.13)$$

where $V \in C^1(\mathbb{R}^3, (0, +\infty))$ and $f \in C([0, \infty), \mathbb{R})$ satisfying following assumptions:

(f₁) $\lim_{t \rightarrow 0} \frac{f(t)}{t} = 0$;

(f₃) $\lim_{t \rightarrow +\infty} \frac{f(t)}{t} = +\infty$.

$$(f'_3) \lim_{t \rightarrow +\infty} \frac{f(t)}{t^3} = +\infty.$$

$$(f_6) \quad tf(t) - 2F(t) \geq 0 \text{ for all } t \geq 0.$$

$$(V1) \text{ there exists } \theta \in [0, 1) \text{ such that } \nabla V(x) \cdot x \leq \frac{\theta a}{2|x|^2}, \forall x \in \mathbb{R}^3 \setminus \{0\}.$$

Problem (1.13) has been widely investigated in the literature, see, for example, [31] where positive ground state solution was obtained via the Nehari-Pohožaev manifold when $f(t) = |t|^{p-2}t$, $3 < p < 6$, and [32] where the monotonicity trick [29] is well applied in order to find the ground states of (1.13) and the range of p is extended to $(2, 6)$ under a stronger assumption of (V1) and some smooth restrictions on f . Later, these results were partially improved by Tang and Chen in recent papers [33, 34]. For other results and related Kirchhoff problem, we refer the readers to [28, 35–41] and the references therein.

For convenience of the verification of compactness, as in [39], we consider following working space for problem (1.13),

$$E := \begin{cases} H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}, & \text{if } V \equiv \text{const}, \\ \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 dx < +\infty\}, & \text{if } V \not\equiv \text{const}, \end{cases} \quad (1.14)$$

and assume that

$$(V2) \quad E \subset H^1(\mathbb{R}^3) \text{ such that the following embedding is compact,}$$

$$E \hookrightarrow L^s(\mathbb{R}^3) \text{ for } 2 < s < 6.$$

Define the inner product by

$$(u, v) = \int_{\mathbb{R}^3} (a \nabla u \cdot \nabla v + V(x)uv) dx, \quad \forall u, v \in E, \quad (1.15)$$

and the norm $\|u\| = (u, u)^{1/2}$. Condition (V2) holds obviously if $V \equiv \text{const}$ since the embedding $E = H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ ($2 < s < 6$) is compact. For case that $V \not\equiv \text{const}$, by assuming that $V(x)$ is coercive or that $(V3) \inf_{x \in \mathbb{R}^N} V(x) \geq a_1 > 0$, and for each $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$.

we also get the compact embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ ($2 \leq s < 6$) (see [42, Lemma 3.4]).

By using property of the Pohožaev identity associated to problem (1.13) and some delicate analysis, we establish following existence result.

Theorem 1.5. *Let (V1), (V2) (f₁), (f₂) and (f₆) be satisfied. Then problem (1.13) has a positive solution if one of following conditions holds:*

- (i) $V_\infty := \lim_{|x| \rightarrow \infty} V(x) < +\infty$, and (f₃) is satisfied;
- (ii) (f'_3) is satisfied.

Remark 1.6. *Note that $\mu_1 \leq \lambda_1^2 |\Omega|$, thus (f₃) holds if following condition is satisfied.*

$$(f_3') \quad \liminf_{t \rightarrow +\infty} \frac{f(t)}{t^3} > b \lambda_1^2 |\Omega|.$$

Indeed, taking the first eigenfunction $\varphi_1 > 0$ of the operator $-\Delta$, by (1.6) and Hölder inequality we have

$$\mu_1 \|\varphi_1\|_4^4 \leq \|\nabla \varphi_1\|_2^4 = \lambda_1^2 \|\varphi_1\|_2^4 \leq \lambda_1^2 |\Omega| \|\varphi_1\|_4^4,$$

which implies the conclusion.

Remark 1.7. *Compared with [16, Theorem 1.1] and [18, Theorem 1.1 (iii)], the μ in (1.7) allows to be an eigenvalue of problem (1.5) in above Theorem 1.3 when (F2) holds and the restriction $\mu > \mu_2$ is got rid of. Following*

assumption used in [18, Theorem 1.1 (i)] is also removed in Theorem 1.2 for the 3-sublinear case.

$$\exists \lambda > \lambda_2 \text{ such that } F(x, t) \geq \frac{a\lambda}{2} t^2, \quad \forall |t| > 0 \text{ small.}$$

For the 3-superlinear case, the (AR) condition assumed in [18, Theorem 1.1 (iv)] is not needed in Theorem 1.3 provided that (F2) or (f5) holds in addition. Compared with [8, Corollary 2(2), Theorem 3], $p(x)$ in (1.9) allows to equal λ_1 here and the monotonic condition (F1) is also weakened to (F2) (see [12, Lemma 3.9]). Note that following condition used in [19, Theorem 3.1] is also weakened to (f5) in Theorem 1.3.

$$\liminf_{t \rightarrow +\infty} \frac{f(t)t - 4F(t)}{|t|^\tau} > -\alpha, \quad \text{uniformly in } x \in \Omega, \quad \tau \in [0, 2] \text{ and } 0 < \alpha < a\lambda_1.$$

Moreover, existence of two positive solutions is established, and nonexistence properties for the 3-sublinear case and the critical case are obtained. Thus, Theorems 1.2–1.5 improve and extend the related results of [8, 9, 11, 16, 18, 19, 22, 23, 31, 32].

Now, we give some nonlinear examples to illustrate our assumptions. Assume that $\alpha > 0$, $l_0 > b\mu_1$ and $3 \leq \beta < 6$, following functions g, f satisfy all assumptions of Theorems 1.2 and 1.3.

Example 1.8.

$$g(t) = \frac{|t|^3}{1 + |t|^\alpha}; \quad f(t) = l_0 \left[1 - \frac{1}{1 + |t|^\alpha} \right] t^\beta.$$

Example 1.9.

$$g(t) = \alpha t^2 \log(1 + |t|); \quad f(t) = l_0 \left[1 - \frac{1}{\ln(e + |t|)} \right] t^\beta.$$

Example 1.10.

$$g(t) = t^2 \sin(\alpha t); \quad f(t) = l_0 |t|^\beta + |t|^\alpha, \quad \text{with } 3 \leq \alpha \leq \beta.$$

Example 1.11.

$$g(t) = \min \left\{ |t|^\alpha, |t|^3 \right\}, \quad \text{with } 0 < \alpha < 3; \quad f(t) = l_0 \min \left\{ |t|^\alpha, |t|^\beta \right\}, \quad \text{with } 3 \leq \alpha \leq \beta.$$

Remaining of the paper is organized as follows. In Section 2, some preliminary lemmas are presented. We consider the bounded domain case in Section 3 and study the 3-sublinear case and the rest cases respectively, then we show Theorems 1.2 and 1.3 by using some energy estimates and the monotonicity trick. In the last section, we consider the whole space case and give the proof of Theorem 1.5. In this paper, we use $C, C_1, C_2, C_3 \dots$ to denote different positive constants in different places. $L^s(\Omega)$ ($1 \leq s < \infty$) denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_\Omega |u|^s dx \right)^{1/s}$.

2 Preliminaries

In this section, we give some preliminaries. Define first $u_+ = \max\{u, 0\}$ and $u_- = \max\{-u, 0\}$. Let us consider following auxiliary problem:

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^2 dx) \Delta u = f(u_+) + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

The corresponding functional is defined as follows

$$\Phi(u) := \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 - \int_{\Omega} F(u_+) dx - \int_{\Omega} h u dx, \quad (2.2)$$

where $F(t) := \int_0^t f(s) ds$. Under (g1) or (f2), the functional Φ is of class C^1 and

$$\langle \Phi'(u), v \rangle = a \int_{\Omega} \nabla u \nabla v dx + b \|\nabla u\|_2^2 \int_{\Omega} \nabla u \nabla v dx - \int_{\Omega} f(u_+) v dx - \int_{\Omega} h v dx. \quad (2.3)$$

Lemma 2.1. *Let $h \geq 0$. Then any solution $u \in H_0^1(\Omega)$ of problem (2.1) is non-negative, and it is a solution to (1.1). In particular, such solution is positive if $h \not\equiv 0$.*

Proof. If $u \in H_0^1(\Omega)$ is a solution of problem (2.1), then, multiplying (2.1) by u_- , we have

$$a \int_{\Omega} \nabla u \nabla u_- dx + b \|\nabla u\|_2^2 \int_{\Omega} \nabla u \nabla u_- dx = \int_{\Omega} f(u_+) u_- dx + \int_{\Omega} h u_- dx.$$

From the fact that $\int_{\Omega} f(u_+) u_- dx = 0$ and $\int_{\Omega} h u_- dx \geq 0$, we deduce that

$$-a \int_{\Omega} |\nabla u_-|^2 dx - b \|\nabla u\|_2^2 \int_{\Omega} |\nabla u_-|^2 dx \geq 0,$$

which implies that $u \geq 0$, hence it is a solution of (1.1). When $h \not\equiv 0$, we get that $u \geq 0$ is nontrivial, thus by the strong maximum principle, $u > 0$ for $x \in \Omega$. $\square$

Lemma 2.2. *Let (f2) be satisfied, then*

- (i) Φ is weakly sequentially lower semicontinuous;
- (ii) Φ is coercive under (g1).

Proof. Since the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is compact for $s \in [1, 2^*)$, by virtue of the weakly sequentially lower semicontinuous property of the norm, we can certify the first part (i).

In view of (g1), we have

$$G(t) \leq \frac{b}{4y_4} t^4 + C_1, \quad \forall t \in \mathbb{R}.$$

Then by (1.11) and (4.2), we have

$$\begin{aligned} \Phi(u) &\geq \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 - \frac{b}{4y_4} \|u\|_4^4 - C_1 |\Omega| - \|h\|_2 \|u\|_2 \\ &\geq \frac{a}{2} \|\nabla u\|_2^2 - C_1 |\Omega| - y_2 \|h\|_2 \|\nabla u\|_2, \end{aligned}$$

which shows that Φ is coercive and bounded from below. $\square$

In order to find a Palais-Smale sequence at the mountain pass level, we introduce following Proposition established by Jeanjean [29, Theorem 1.1], which helps us to find a second solution.

Proposition 2.3. *Assume that X is a Banach space with the norm $\|\cdot\|$ and $J \subset \mathbb{R}_+$ is an interval. Let*

$$I_{\rho}(u) = A(u) - \rho B(u), \quad \forall \rho \in J,$$

be a family of C^1 -functions on X such that $B(u) \geq 0$, $\forall u \in X$, and either $A(u) \rightarrow +\infty$ or $B(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$, moreover, there are two points (v_1, v_2) in X satisfying

$$c_{\rho} := \inf_{\sigma \in \Sigma} \max_{t \in [0,1]} I_{\rho}(\sigma(t)) > \max \{I_{\rho}(v_1), I_{\rho}(v_2)\},$$

where

$$\Sigma := \{\sigma \in C([0, 1], X) : \sigma(0) = v_1, \sigma(1) = v_2\}.$$

Then, for almost every $\rho \in J$, there exists a sequence $\{v_n\} \subset X$ such that

- (i) $\{v_n\}$ is bounded;
- (ii) $I_\rho(v_n) \rightarrow c_\rho$ as $n \rightarrow \infty$;
- (iii) $I'_\rho(v_n) \rightarrow 0$ in the dual X^{-1} of X .

3 Proofs of main results for the bounded domain case

In this section, we consider the 3-sublinear, 3-linear and 3-superlinear cases, respectively, and show Theorems 1.2 and 1.3 by using the monotonicity trick and some analytical techniques.

3.1 3-sublinear case

Lemma 3.1. *Let $h \equiv 0$ and (g2) be satisfied. Then there exists $v_1 > 0$ such that for all $v > v_1$,*

$$\inf_{u \in H_0^1(\Omega)} \Phi(u) < 0.$$

Proof. For any fixed compact set $M \subset \Omega$, using the Tietze's extension theorem, we find the existence of map $v \in H_0^1(\Omega) \cap C(\overline{\Omega})$ such that $v \equiv t_0$ in M , and $|v| \leq t_0$ in $\Omega \setminus M$. By (g2), one has

$$\int_{\Omega} G(v) dx = G(t_0)|M| + \int_{\Omega \setminus M} G(v) dx \geq G(t_0)|M| - \max_{s \in [0, t_0]} |G(s)| |\Omega \setminus M|,$$

which implies that $\int_{\Omega} G(v) dx > 0$ when $|M|$ approaches $|\Omega|$ close enough. Thus for v sufficiently large,

$$\Phi(v) = \frac{a}{2} \|\nabla v\|_2^2 + \frac{b}{4} \|\nabla v\|_2^4 - \int_{\Omega} vG(v) dx < 0.$$

The proof is completed. $\square$

Lemma 3.2. *Let $h \equiv 0$ and (g2) be satisfied. Then problem (1.1) has no nontrivial solution for $v > 0$ small.*

Proof. Assume $v > 0$ such that (1.1) has a nontrivial solution $u \in H_0^1(\Omega)$. Then, multiplying (1.1) by u and calculating each side of the equation, we deduce from (g2) and (1.6) that

$$b\mu_1 \|u\|_4^4 \leq a \|\nabla u\|_2^2 + b \|\nabla u\|_2^4 = \int_{\Omega} v g(u) u dx \leq v C_0 \|u\|_4^4.$$

Thus choosing $0 < v < \frac{b\mu_1}{C_0} := v_0$, we get the desired result. $\square$

Proof of Theorem 1.2

(a). By Lemma 2.2, the functional Φ has a global minimum $u \in H_0^1(\Omega)$, and it is a solution of (2.1), which is a non-negative solution to (1.1) by Lemma 2.1. Particularly, this solution is positive if $h \not\equiv 0$.

(b). By Lemmas 2.2 and 3.1, the global minimum $u \in H_0^1(\Omega)$ of Φ is nontrivial if $v > 0$ sufficiently large. Thus it is positive by Lemma 2.1 and this shows i), and ii) follows from Lemma 3.2. $\square$

3.2 3-linear and 3-superlinear cases

Lemma 3.3. *Let (f1)-(f2) be satisfied. If $\|h\|_2$ is sufficiently small, then we find the existence of $\kappa > 0$ and $r > 0$ such that $\Phi(u) \geq \kappa$ for all $u \in H_0^1(\Omega)$ with $\|u\| = r$.*

Proof. By (f1) and (f2), there exists $C > 0$ such that

$$F(u_+) \leq \frac{a\lambda_1}{2} u_+^2 + C u_+^{p+1}.$$

Then by (1.11) and (4.2),

$$\begin{aligned} \Phi(u) &= \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 - \int_{\Omega} F(u_+) dx - \int_{\Omega} h u dx \\ &\geq \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 - \frac{a\lambda_1}{2} \|u_+\|_2^2 - C \|u_+\|_{p+1}^{p+1} - \|h\|_2 \|u\|_2 \\ &\geq \frac{b}{4} \|u\|_2^4 - C \gamma_{p+1}^{p+1} \|u_+\|^{p+1} - \|h\|_2 \|u\|_2. \end{aligned}$$

Taking $\|h\|_2$ small enough and noting that $p > 3$, we obtain the conclusion. $\square$

Lemma 3.4. For $\|h\|_2$ given by Lemma 3.3, problem (1.1) has a solution which is a local minimizer of Φ . Moreover, it is nontrivial if $h \not\equiv 0$.

Proof. Define

$$m := \inf_{u \in \overline{B}(0, r)} \Phi(u),$$

where r is given in Lemma 3.3. Let $\{u_n\} \subset \overline{B}(0, r)$ be a minimizing sequence, then it is bounded and there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. By Lemma 2.2-(i), we have $\Phi(u) = m$. Note that $m \leq \Phi(0) < \kappa$, then u is a critical point of Φ . Moreover, by Lemma 2.1, u is positive provided that $h \not\equiv 0$. $\square$

Lemma 3.5. Suppose that (f2) and (f3) (or (f3')) are satisfied. Let $\varphi_1 > 0$ and $\psi_1 > 0$ be the first eigenfunction of the operator $-\Delta$ and the eigenvalue problem (1.5), respectively. Then there exists $t_1 > 0$ such that

$$\Phi(t\varphi_1) < 0 \text{ and } \Phi(t\psi_1) < 0 \text{ for all } t \geq t_1.$$

Proof. For φ_1, ψ_1 , we have

$$\|\nabla \varphi_1\|_2^2 = \lambda_1 \int_{\Omega} |\varphi_1|^2 dx \leq \lambda_1 \|\varphi_1\|_4^2 |\Omega|^{\frac{1}{2}}, \quad (3.1)$$

and

$$\|\nabla \psi_1\|_2^4 = \mu_1 \|\psi_1\|_4^4. \quad (3.2)$$

By (f3), there exist $\delta > 0$ and $C = C(\delta) > 0$ such that

$$F(s) \geq \frac{(\delta + b)\lambda_1^2 |\Omega|}{4} s^4 - C \text{ or } F(s) \geq \frac{(\delta + b)\mu_1}{4} s^4 - C, \quad \forall s \geq 0. \quad (3.3)$$

Then by (1.11), (4.2), (3.1) and (3.2), one has

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\Phi(t\varphi_1)}{t^4} &= \limsup_{t \rightarrow \infty} \left[\frac{a}{2t^2} \|\nabla \varphi_1\|_2^2 + \frac{b}{4} \|\nabla \varphi_1\|_2^4 - \frac{1}{t^4} \int_{\Omega} F(t\varphi_1) dx - \frac{1}{t^3} \int_{\Omega} h \varphi_1 dx \right] \\ &\leq \limsup_{t \rightarrow \infty} \left[\frac{a}{2t^2} \|\nabla \varphi_1\|_2^2 + \frac{b}{4} \|\nabla \varphi_1\|_2^4 \right] \\ &\quad - \liminf_{t \rightarrow \infty} \left[\frac{(\delta + b)\lambda_1^2 |\Omega|}{4} \|\varphi_1\|_4^4 + \frac{C|\Omega|}{t^4} + \frac{1}{t^3} \int_{\Omega} h \varphi_1 dx \right] \\ &\leq \frac{b}{4} \|\nabla \varphi_1\|_2^4 - \frac{(\delta + b)}{4} \|\nabla \varphi_1\|_2^4 \end{aligned}$$

$$= -\frac{\delta}{4}\|\nabla\varphi_1\|_2^4,$$

and

$$\begin{aligned}\limsup_{t \rightarrow \infty} \frac{\Phi(t\psi_1)}{t^4} &= \limsup_{t \rightarrow \infty} \left[\frac{a}{2t^2} \|\nabla\psi_1\|_2^2 + \frac{b}{4} \|\nabla\psi_1\|_2^4 - \frac{1}{t^4} \int_{\Omega} F(t\psi_1) dx - \frac{1}{t^3} \int_{\Omega} h\psi_1 dx \right] \\ &\leq \limsup_{t \rightarrow \infty} \left[\frac{a}{2t^2} \|\nabla\psi_1\|_2^2 + \frac{b}{4} \|\nabla\psi_1\|_2^4 \right] \\ &\quad - \liminf_{t \rightarrow \infty} \left[\frac{(\delta+b)\mu_1}{4} \|\psi_1\|_4^4 + \frac{C|\Omega|}{t^4} + \frac{1}{t^3} \int_{\Omega} h\psi_1 dx \right] \\ &= \frac{b}{4} \|\nabla\psi_1\|_2^4 - \frac{(\delta+b)}{4} \mu_1 \|\varphi_1\|_4^4 \\ &= -\frac{\delta}{4} \|\nabla\varphi_1\|_2^4.\end{aligned}$$

Both inequalities above prove the lemma. $\square$

To apply Proposition 2.3, define first the continuous functions $f_i : [0, \infty) \rightarrow \mathbb{R}$, $i = 1, 2$ as follows,

$$f_1(t) = \max\{f(t), 0\}, \text{ and } f_2(t) = f_1(t) - f(t).$$

Clearly, $f_1(t), f_2(t) \geq 0$ for all $t \geq 0$, and (F3) implies that $f(t) > 0$ for t large, this yields $f_2(t) = 0$ for t large. Set

$$F_1(t) = \int_0^t f_1(s) ds, \text{ and } F_2(t) = \int_0^t f_2(s) ds.$$

Then

$$F_1(t) \geq 0, \text{ and } 0 \leq F_2(t) \leq C_2 \text{ for some } C_2 > 0.$$

Define $J := (1 - \alpha, 1 + \alpha)$ for some $\alpha > 0$ small. Consider the family of C^1 -functionals:

$$\Phi_\rho(u) = \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 - \rho \int_{\Omega} F_1(u_+) dx + \int_{\Omega} F_2(u_+) dx - \int_{\Omega} h u dx.$$

If we let

$$A(u) = \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 + \int_{\Omega} F_2(u_+) dx - \int_{\Omega} h u dx, \quad B(u) = \int_{\Omega} F_1(u_+) dx,$$

then functional $\Phi_\rho(u)$ can be written as

$$\Phi_\rho(u) = A(u) - \rho B(u), \quad \forall u \in H_0^1(\Omega).$$

Note that $B(u) \geq 0$ for all $u \in H_0^1(\Omega)$, and

$$A(u) \geq \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 - C_2 |\Omega| - \|h\|_2 \|u\|_2, \quad \forall u \in H_0^1(\Omega),$$

then we see that $A(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$. Since Lemmas 3.3 and 3.5 hold also for Φ_ρ with $\rho \in J = (1 - \alpha, 1 + \alpha)$ for some $\alpha > 0$ sufficiently small. Thus all the assumptions of Proposition 2.3 are satisfied, and following result holds.

Lemma 3.6. *There exists an increasing sequence $\{\rho_n\}$ such that $\rho_n \rightarrow 1$ and for any $n \in \mathbb{N}$, Φ_{ρ_n} has a bounded Palais-Smale sequence at the mountain pass level c_{ρ_n} .*

Lemma 3.7. Every bounded Palais-Smale sequence $\{u_n\} \subset H_0^1(\Omega)$ for Φ_ρ with $\rho \in J$ has a convergent subsequence.

Proof. We assume, passing to a subsequence, that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_n \rightarrow u$ in $L^q(\Omega)$ for all $q \in [1, 6)$. By (f2) and Lebesgue dominated convergence theorem, we have

$$\int_{\Omega} F_1(u_{n+}) dx \rightarrow \int_{\Omega} F_1(u_+) dx, \quad \int_{\Omega} F_2(u_{n+}) dx \rightarrow \int_{\Omega} F_2(u_+) dx, \quad \text{and} \quad \int_{\Omega} hu_n dx \rightarrow \int_{\Omega} h u dx.$$

Then by $\langle \Phi'_\rho(u_n), u_n - u \rangle \rightarrow 0$ and $\langle \Phi'_\rho(u), u_n - u \rangle \rightarrow 0$, we get

$$\begin{aligned} o(1) &= \langle \Phi'_\rho(u_n) - \Phi'_\rho(u), u_n - u \rangle \\ &= a \|\nabla u_n - \nabla u\|_2^2 + b \|\nabla u_n\|_2^2 \int_{\Omega} \nabla u_n (\nabla u_n - \nabla u) dx \\ &\quad - b \|\nabla u\|_2^2 \int_{\Omega} \nabla u (\nabla u_n - \nabla u) dx + o(1) \\ &= a \|\nabla u_n - \nabla u\|_2^2 + b \|\nabla u_n\|_2^2 \|\nabla u_n - \nabla u\|_2^2 \\ &\quad + b (\|\nabla u_n\|_2^2 - \|\nabla u\|_2^2) \int_{\Omega} \nabla u (\nabla u_n - \nabla u) dx + o(1) \\ &= (a + b \|\nabla u_n\|_2^2) \|\nabla u_n - \nabla u\|_2^2 + o(1). \end{aligned}$$

Thus $u_n \rightarrow u$ in $H_0^1(\Omega)$, we complete the proof. $\square$

In view of Lemmas 3.3, 3.6 and 3.7, there exist sequences $\{\rho_n\}$ and $\{u_n\} \subset H_0^1(\Omega)$ such that

- (i) $\rho_n \rightarrow 1$ as $n \rightarrow \infty$ and $\{\rho_n\}$ is increasing;
- (ii) $u_n > 0$ for $x \in \Omega$, $\Phi_{\rho_n}(u_n) = c_{\rho_n} \geq \kappa$, and $\Phi'_{\rho_n}(u_n) = 0$ in the dual space $H_0^{-1}(\Omega)$.

By the proof of [29, Lemma 2.3], we see that the map $\rho \mapsto c_\rho$ is continuous from the left. Thus $c_{\rho_n} \rightarrow c_1$ as $n \rightarrow \infty$.

Lemma 3.8. The sequence $\{u_n\} \subset H_0^1(\Omega)$ is bounded when one of following conditions is satisfied:

- (i) (F2) is satisfied;
- (ii) (f4) is satisfied with $\frac{1}{b} \neq \mu_m$, $m = 2, 3, \dots$;
- (iii) (f5) is satisfied.

Proof. By (ii) above, for n large we have

$$\begin{aligned} c_1 + 1 &\geq c_{\rho_n} = \Phi_{\rho_n}(u_n) \\ &= \frac{a}{2} \|\nabla u_n\|_2^2 + \frac{b}{4} \|\nabla u_n\|_2^4 - \rho_n \int_{\Omega} F_1(u_n) dx + \int_{\Omega} F_2(u_n) dx - \int_{\Omega} hu_n dx \geq \kappa, \end{aligned} \quad (3.4)$$

and u_n is solution of the problem

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \rho_n f_1(u_+) - f_2(u_+) + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.5)$$

Then there holds

$$\begin{aligned} 0 &= \langle \Phi'_{\rho_n}(u_n), u_n \rangle \\ &= a \|\nabla u_n\|_2^2 + b \|\nabla u_n\|_2^4 - \rho_n \int_{\Omega} f_1(u_n) u_n dx + \int_{\Omega} f_2(u_n) u_n dx - \int_{\Omega} hu_n dx. \end{aligned} \quad (3.6)$$

Arguing by contradiction, suppose that $\|u_n\| \rightarrow +\infty$ as $n \rightarrow \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = \|\nabla v_n\|_2 = 1$. Up to a subsequence, we may assume that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \rightarrow v$ in $L^q(\Omega)$ for all $q \in [1, 6)$, and $v_n \rightarrow v \geq 0$ a.e. on Ω .

If (f5) is satisfied, then there exists $t_0 > 1$, $C > 0$ such that

$$f(t)t - 4F(t) \geq \begin{cases} -2d|t|^\tau, & \forall t \geq t_0 \\ -2d|t|^\tau - C, & \forall t \geq 0. \end{cases}$$

Note that for n large enough, modifying t_0 if necessary, there holds

$$\rho_n (f_1(t)t - 4F_1(t)) - (f_2(t)t - 4F_2(t)) \geq \begin{cases} -3d|t|^\tau, & \forall t \geq t_0 \\ -3d|t|^\tau - C, & \forall t \geq 0. \end{cases}$$

By (3.4) and (3.6), we have

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{c_1 + 1}{\|\nabla u_n\|_2^2} \geq \lim_{n \rightarrow \infty} \frac{1}{\|\nabla u_n\|_2^2} \left[\Phi_{\rho_n}(u_n) - \frac{1}{4} \langle \Phi'_{\rho_n}(u_n), u_n \rangle \right] \\ &= \frac{1}{4} \lim_{n \rightarrow \infty} \frac{1}{\|\nabla u_n\|_2^2} \left[a \|\nabla u_n\|_2^2 + \rho_n \int_{\Omega} f_1(u_n)u_n - 4F_1(u_n)dx \right. \\ &\quad \left. - \int_{\Omega} f_2(u_n)u_n - 4F_2(u_n)dx + \int_{\Omega} (-3h)u_n dx \right] \\ &= \frac{1}{4} \left[a + \lim_{n \rightarrow \infty} \frac{1}{\|\nabla u_n\|_2^2} \left(\rho_n \int_{\Omega} f_1(u_n)u_n - 4F_1(u_n)dx \right. \right. \\ &\quad \left. \left. - \int_{\Omega} f_2(u_n)u_n - 4F_2(u_n)dx \right) \right] \\ &\geq \lim_{n \rightarrow \infty} \frac{\kappa}{\|\nabla u_n\|_2^2} = 0. \end{aligned}$$

If $v = 0$, then $v_n \rightarrow 0$ in $L^q(\Omega)$, $\forall q \in [1, 6)$. Set $\Omega_n := \{x \in \Omega : |u_n| \geq t_0\}$, we get

$$\begin{aligned} -a + o(1) &= \frac{1}{\|\nabla u_n\|_2^2} \left(\rho_n \int_{\Omega} f_1(u_n)u_n - 4F_1(u_n)dx - \int_{\Omega} f_2(u_n)u_n - 4F_2(u_n)dx \right) \\ &\geq \frac{1}{\|\nabla u_n\|_2^2} \left[-3d \int_{\Omega_n} |u_n|^\tau dx - C|\Omega| \right] \\ &\geq \frac{1}{\|\nabla u_n\|_2^2} \left[-3d \int_{\Omega_n} |u_n|^2 dx - C|\Omega| \right] \\ &= -3d \int_{\Omega_n} |v_n|^2 dx + o(1) \\ &\geq -3d \|v_n\|_2^2 + o(1) = o(1), \end{aligned}$$

this contradiction implies that $v \neq 0$. By (f5), for any $M > 0$, there exists $t_1 = t_1(M) > 1$ such that

$$F(t) \geq \begin{cases} Mt^4, & \forall t \geq t_1 \\ Mt^4 - C, & \forall t \geq 0. \end{cases}$$

Choosing $t_1 = t_1(M) > 1$ large enough, we have $F_2(u_n) = 0$ on $\Omega'_n := \{x \in \Omega : |u_n| \geq t_1\}$, thus by (3.4), there holds

$$o(1) = \kappa \frac{1}{\|\nabla u_n\|_2^4} \leq \frac{1}{\|\nabla u_n\|_2^4} \Phi_{\rho_n}(u_n)$$

$$\begin{aligned}
&= \frac{1}{\|\nabla u_n\|_2^4} \left[\frac{a}{2} \|\nabla u_n\|_2^2 + \frac{b}{4} \|\nabla u_n\|_2^4 - \rho_n \int_{\Omega} F_1(u_n) dx + \int_{\Omega} F_2(u_n) dx - \int_{\Omega} h u_n dx \right] \\
&= \frac{b}{4} + \frac{1}{\|\nabla u_n\|_2^4} \left[-\rho_n \int_{\Omega'_n} F_1(u_n) dx + \int_{\Omega \setminus \Omega'_n} F_2(u_n) - \rho_n F_1(u_n) dx - \int_{\Omega} h u_n dx \right] \\
&\leq \frac{b}{4} + \frac{1}{\|\nabla u_n\|_2^4} \left[-\rho_n \int_{\Omega'_n} M |u_n|^4 dx + C |\Omega| + \|h\|_2 \|u_n\|_2 \right] \\
&\leq \frac{b}{4} - \frac{M}{2} \int_{\Omega'_n} |v_n|^4 dx + o(1) \\
&= \frac{b}{4} - \frac{M}{2} \int_{\Omega'_n} |v|^4 dx + o(1) \\
&< 0 \text{ for } M \text{ large.}
\end{aligned}$$

This contradiction shows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Consider the case that (f4) is satisfied with $\frac{l}{b} \neq \mu_m, m = 2, 3, \dots$. By (3.4) we have

$$\begin{aligned}
\frac{b}{4} &= \lim_{n \rightarrow \infty} \left[\frac{1}{2\|\nabla u_n\|_2^2} + \frac{b}{4} \right] \\
&= \lim_{n \rightarrow \infty} \left[\frac{1}{\|\nabla u_n\|_2^4} \left(\rho_n \int_{\Omega} F_1(u_n) dx - \int_{\Omega} F_2(u_n) dx + \int_{\Omega} h u_n dx \right) \right] \\
&= \frac{l}{4} \lim_{n \rightarrow \infty} \|v_n\|_4^4.
\end{aligned} \tag{3.7}$$

If $v = 0$, then $v_n \rightarrow 0$ in $L^q(\Omega), \forall q \in [1, 6)$. From above equality, we deduce that $b = l \lim_{n \rightarrow \infty} \|v_n\|_4^4 = 0$, which is a contradiction. Thus $v \neq 0$. It follows from (3.5) that

$$\left(a + b \|\nabla u_n\|_2^2 \right) \int_{\Omega} \nabla u_n \nabla \phi dx = \int_{\Omega} (\rho_n f_1(u_n) - f_2(u_n) + h) \phi dx, \quad \forall \phi \in C_0^\infty(\Omega). \tag{3.8}$$

Taking $\phi = \frac{v}{\|\nabla u_n\|_2^3}$, we deduce from (3.7) that

$$\begin{aligned}
b \|\nabla v\|_2^2 &= b \int_{\Omega} \nabla v_n \nabla v dx + o(1) = \frac{1}{\|\nabla u_n\|_2^2} \left(a + b \|\nabla u_n\|_2^2 \right) \int_{\Omega} \nabla v_n \nabla v dx \\
&= \frac{1}{\|\nabla u_n\|_2^3} \int_{\Omega} (\rho_n f_1(u_n) - f_2(u_n) + h) v dx \\
&= l \int_{\Omega} v_n^3 v dx + o(1) \\
&= l \|v\|_4^4 = b.
\end{aligned}$$

This shows that $\|\nabla v\|_2 = 1 = \|\nabla v_n\|_2$, which together with $v_n \rightharpoonup v$ implies that $v_n \rightarrow v$ in $H_0^1(\Omega)$. Then by (3.8),

$$b \|\nabla v\|_2^2 \int_{\Omega} \nabla v \nabla \phi dx = l \int_{\Omega} v^3 \phi dx, \quad \forall \phi \in C_0^\infty(\Omega).$$

Thus $v \neq 0$ is an eigenfunction of the problem (1.5) and the corresponding eigenvalue is $\frac{l}{b}$. But $\frac{l}{b} \neq \mu_m, m = 2, 3, \dots$, then we get the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$.

If (F2) is satisfied, then similar to the argument of [12, Lemma 3.1], we have for any $u \in E$,

$$\begin{aligned} \Phi_\rho(u) &\geq \Phi_\rho(tu) + \frac{(1-t^2)^2}{4} a \|u\|^2 + \frac{1-t^4}{4} \langle \Phi'_\rho(u), u \rangle + \frac{(t-1)(3-t)}{4} \int_\Omega h u dx \\ &\quad - \int_\Omega \left[\frac{1-t^4}{4} f_2(u)u + F_2(tu) - F_2(u) \right] dx, \quad \forall t \geq 0, \end{aligned}$$

which, together with the fact that $\Phi'_{\rho_n}(u_n) = 0$, implies that

$$\begin{aligned} \Phi_{\rho_n}(u_n) &\geq \Phi_{\rho_n}(t_n u_n) + \frac{(1-t_n^2)^2}{4} a \|u_n\|^2 + \frac{(t_n-1)(3-t_n)}{4} \int_\Omega h u_n dx \\ &\quad - \int_\Omega \left[\frac{1-t_n^4}{4} f_2(u_n)u_n + F_2(t_n u_n) - F_2(u_n) \right] dx, \quad \forall t_n \geq 0. \end{aligned}$$

For r given in Lemma 3.3, we set $t_n = \frac{r}{\|u_n\|}$. Then for n large, $t_n < 1$ and by (3.4) and Lemma 3.3,

$$\begin{aligned} c_1 + 1 &\geq \Phi_{\rho_n}(u_n) \\ &\geq \Phi_{\rho_n}(t_n u_n) + \frac{(1-t_n^2)^2}{4} a \|u_n\|^2 + \frac{(t_n-1)(3-t_n)}{4} \int_\Omega h u_n dx \\ &\quad - \int_\Omega \left[\frac{1-t_n^4}{4} f_2(u_n)u_n + F_2(t_n u_n) - F_2(u_n) \right] dx \\ &\geq \kappa + \frac{(1-t_n^2)^2}{4} a \|u_n\|^2 - \frac{(1-t_n)(3-t_n)}{4} \|h\|_2 \|u_n\|_2 - C|\Omega| \\ &\rightarrow +\infty, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This contradiction shows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. $\square$

Lemma 3.9. *The sequence $\{u_n\}$ is a bounded Palais-Smale sequence for Φ at the level c_1 .*

Proof. Noting that as $n \rightarrow \infty$,

$$\Phi(u_n) = \Phi_{\rho_n}(u_n) + (\rho_n - 1)B(u_n) = c_{\rho_n} + o(1) \rightarrow c_1,$$

and

$$\Phi'(u_n) = \Phi'_{\rho_n}(u_n) + (\rho_n - 1)B'(u_n) \rightarrow 0 \text{ in } H_0^{-1}(\Omega),$$

thus, $\{u_n\}$ is a bounded Palais-Smale sequence for Φ at the mountain pass level c_1 . $\square$

Proof of Theorem 1.3

(a) follows directly from Lemmas 3.4 and 2.1. In view of Lemmas 3.9 and 3.7, the bounded sequence $\{u_n\}$ has a convergent subsequence, i.e., $u_{n_k} \rightarrow u$ as $k \rightarrow \infty$, thus $\Phi'(u) = 0$ and $\Phi(u) = c_1 \geq \kappa > 0$ by Lemma 3.3. Then, according to Lemmas 3.4 and 2.1, we show (b). $\square$

Proof of Theorem 1.4

By Theorem 1.1 and in view of the concise form $f(t) = |t|^4 t + \nu |t|^{q-1} t$, we get (a). (b) follows directly from [13, Theorem 1.1]. $\square$

4 Proof of main result for the whole space $\mathbb{R}^3$

In this section, we consider problem (1.13) and show Theorem 1.5 by virtue of the Pohožaev identity and some delicate analysis.

As in section 2, we consider following auxiliary problem:

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u = f(u_+), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \quad (4.1)$$

Note that any solution $u \in E$ of problem (4.1) is non-negative, and it is a solution to (1.13). The corresponding functional of problem (4.1) is

$$\Phi(u) := \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 dx - \int_{\mathbb{R}^3} F(u_+) dx, \quad u \in E. \quad (4.2)$$

By (f_1) and (f_3) , there exists $t_0 > 1$ large such that $f(t_0) = t_0$ and $f(t) > t$ for $t > t_0$. Then similar to section 3, we can define continuous functions $f_i : [0, \infty) \rightarrow \mathbb{R}$, $i = 1, 2$ as follows,

$$f_1(t) = \begin{cases} \max\{f(t), t\}, & \forall 0 \leq t \leq t_0 \\ f(t), & \forall t > t_0, \end{cases} \quad \text{and } f_2(t) = f_1(t) - f(t).$$

Note that, $f_1(t), f_2(t) \geq 0$ for all $t \geq 0$, and $f_2(t) = 0$ for $t \geq t_0$. Set

$$F_1(t) = \int_0^t f_1(s) ds, \quad \text{and } F_2(t) = \int_0^t f_2(s) ds.$$

Then

$$F_1(t) \geq 0, \quad \text{and } 0 \leq F_2(t) \leq \frac{t^2}{2} \text{ for all } t \geq 0.$$

Similarly, we consider the family of C^1 -functionals:

$$\Phi_\rho(u) = \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 dx - \rho \int_{\mathbb{R}^3} F_1(u_+) dx + \int_{\mathbb{R}^3} F_2(u_+) dx, \quad u \in E.$$

where $\rho \in J := (1 - \alpha, 1 + \alpha)$ and $\alpha > 0$ is small.

Lemma 4.1. *Let (f_3) be satisfied. Then for any $u \in E \setminus \{0\}$, there exists $t_1 = t_1(u) > 0$ such that*

(i) *if $V_\infty := \lim_{|x| \rightarrow \infty} V(x) < +\infty$, then*

$$\Phi_\rho(tu(t^{-2}x)) < 0 \quad \text{for all } t \geq t_1.$$

(ii) *if (f'_3) is satisfied, then*

$$\Phi_\rho(tu) < 0 \quad \text{for all } t \geq t_1.$$

Proof. By (f_3) , for $\rho \in J = (1 - \alpha, 1 + \alpha)$ with $\alpha > 0$ small, we have

$$\lim_{t \rightarrow +\infty} \frac{\rho f_1(t) - f_2(t)}{t} = +\infty.$$

If $V_\infty < +\infty$, then for any $u \in E \setminus \{0\}$,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{\Phi_\rho(tu(t^{-2}x))}{t^8} \\ &= \limsup_{t \rightarrow \infty} \left[\frac{a}{2t^4} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 + \frac{1}{2} \int_{\mathbb{R}^3} V(t^2x)u^2 dx \right. \\ & \quad \left. - \frac{1}{t^2} \int_{\mathbb{R}^3} \rho F_1(tu) - F_2(tu) dx \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{b}{4} \|\nabla u\|_2^4 + \frac{1}{2} \int_{\mathbb{R}^3} V_\infty u^2 dx - \liminf_{t \rightarrow \infty} \left[\frac{1}{t^2} \int_{\mathbb{R}^3} \rho F_1(tu) - F_2(tu) dx \right] \\
&\leq \frac{b}{4} \|\nabla u\|_2^4 + \frac{1}{2} \int_{\mathbb{R}^3} V_\infty u^2 dx - \int_{\mathbb{R}^3} \liminf_{t \rightarrow \infty} \frac{1}{t^2} [\rho F_1(tu) - F_2(tu)] dx \\
&\rightarrow -\infty,
\end{aligned}$$

and the result (i) holds.

If (f'_3) is satisfied, then for $\rho \in J$,

$$\lim_{t \rightarrow +\infty} \frac{\rho f_1(t) - f_2(t)}{t^3} = +\infty.$$

and

$$\begin{aligned}
&\limsup_{t \rightarrow \infty} \frac{\Phi_\rho(tu)}{t^4} \\
&= \limsup_{t \rightarrow \infty} \left[\frac{a}{2t^2} \|\nabla u\|_2^2 + \frac{b}{4} \|\nabla u\|_2^4 + \frac{1}{2t^2} \int_{\mathbb{R}^3} V(x) u^2 dx \right. \\
&\quad \left. - \frac{1}{t^4} \int_{\mathbb{R}^3} \rho F_1(tu) - F_2(tu) dx \right] \\
&\leq \frac{b}{4} \|\nabla u\|_2^4 - \liminf_{t \rightarrow \infty} \left[\frac{1}{t^4} \int_{\mathbb{R}^3} \rho F_1(tu) - F_2(tu) dx \right] \\
&\leq \frac{b}{4} \|\nabla u\|_2^4 - \int_{\mathbb{R}^3} \liminf_{t \rightarrow \infty} \frac{1}{t^4} [\rho F_1(tu) - F_2(tu)] dx \\
&\rightarrow -\infty,
\end{aligned}$$

which implies (ii), and the proof is completed. $\square$

By virtue of (f_1) and (f_2) , it is easy to verify that Lemma 3.3 holds for Φ_ρ . Then applying Lemma 4.1 and Proposition 2.3, we get Lemma 3.6. For $\rho \in J = (1 - \alpha, 1 + \alpha)$ with $\alpha > 0$ sufficiently small, by (V2) and arguing similarly as in the proof of [43, Lemma 4], we show Lemma 3.7. Then we find sequences $\{\rho_n\}$ and $\{u_n\} \subset E$ such that

- (i) $\rho_n \rightarrow 1$ as $n \rightarrow \infty$ and $\{\rho_n\}$ is increasing;
- (ii) $u_n > 0$ for $x \in \mathbb{R}^3$, $\Phi_{\rho_n}(u_n) = c_{\rho_n} \geq \kappa$, and $\Phi'_{\rho_n}(u_n) = 0$ in the dual space E^* .

Moreover, boundedness of $\{u_n\}$ can be derived from following lemma.

Lemma 4.2. *Let (V1) and (f_6) be satisfied. Then the sequence $\{u_n\} \subset E$ is bounded.*

Proof. By (ii) above, for n large we have

$$\begin{aligned}
c_1 + 1 &\geq c_{\rho_n} = \Phi_{\rho_n}(u_n) \\
&= \frac{a}{2} \|\nabla u_n\|_2^2 + \frac{b}{4} \|\nabla u_n\|_2^4 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |u_n|^2 dx - \rho_n \int_{\mathbb{R}^3} F_1(u_n) dx + \int_{\mathbb{R}^3} F_2(u_n) dx \\
&\geq \kappa,
\end{aligned} \tag{4.3}$$

and u_n is solution of the problem

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u = \rho_n f_1(u_+) - f_2(u_+), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{4.4}$$

Then the corresponding Pohožaev identity is

$$\begin{aligned} P_{\rho_n}(u_n) &:= \frac{1}{2} \left(a + b \|\nabla u_n\|_2^2 \right) \|\nabla u_n\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} (3V(x) + \nabla V \cdot x) |u_n|^2 dx \\ &\quad - 3\rho_n \int_{\mathbb{R}^3} F_1(u_n) dx + 3 \int_{\mathbb{R}^3} F_2(u_n) dx = 0, \end{aligned} \quad (4.5)$$

and there holds

$$\begin{aligned} 0 &= \langle \Phi'_{\rho_n}(u_n), u_n \rangle \\ &= a \|\nabla u_n\|_2^2 + b \|\nabla u_n\|_2^4 + \int_{\mathbb{R}^3} V(x) |u_n|^2 dx - \rho_n \int_{\mathbb{R}^3} f_1(u_n) u_n dx + \int_{\mathbb{R}^3} f_2(u_n) u_n dx. \end{aligned} \quad (4.6)$$

By (f6), for $\rho \in J = (1 - \alpha, 1 + \alpha)$ with $\alpha > 0$ sufficiently small, we have

$$\rho (tf_1(t) - 2F_1(t)) - (tf_2(t) - 2F_2(t)) \geq 0 \quad \text{for all } t \geq 0.$$

First we show $\{\|\nabla u_n\|_2\}$ is bounded. Arguing by contradiction, suppose that $\|\nabla u_n\|_2 \rightarrow +\infty$ as $n \rightarrow \infty$. It follows from Hardy inequality that

$$\|\nabla u\|_2^2 \geq \frac{1}{4} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2} dx, \quad \forall u \in H^1(\mathbb{R}^3). \quad (4.7)$$

Then by (4.3), (4.5) and (4.6), we have for n large,

$$c_1 + 1 \geq \Phi_{\rho_n}(u_n) = \Phi_{\rho_n}(u_n) - \frac{1}{8} \langle \Phi'_{\rho_n}(u_n), u_n \rangle - \frac{1}{4} P_{\rho_n}(u_n) \geq \kappa > 0, \quad (4.8)$$

and by (V1) and (4.7),

$$\begin{aligned} &\Phi_{\rho_n}(u_n) - \frac{1}{8} \langle \Phi'_{\rho_n}(u_n), u_n \rangle - \frac{1}{4} P_{\rho_n}(u_n) \\ &= \frac{a}{4} \|\nabla u_n\|_2^2 - \frac{1}{8} \int_{\mathbb{R}^3} (\nabla V \cdot x) |u_n|^2 dx + \rho_n \int_{\mathbb{R}^3} f_1(u_n) u_n - 2F_1(u_n) dx \\ &\quad - \int_{\mathbb{R}^3} f_2(u_n) u_n - 2F_2(u_n) dx \\ &\geq \frac{a}{4} (1 - \theta) \|\nabla u_n\|_2^2 + \rho_n \int_{\mathbb{R}^3} f_1(u_n) u_n - 2F_1(u_n) dx - \int_{\mathbb{R}^3} f_2(u_n) u_n - 2F_2(u_n) dx \\ &\geq \frac{a}{4} (1 - \theta) \|\nabla u_n\|_2^2. \end{aligned}$$

Then we get a contradiction from above inequality and (4.8). Thus $\{\|\nabla u_n\|_2\}$ is bounded, then by (4.3), (f₁) and (f₂), we show that $\{\|u_n\|_2\}$ is bounded, so we get the boundedness of $\{u_n\}$. $\square$

By the same proof of Lemma 3.9, we show that $\{u_n\}$ is a bounded Palais-Smale sequence for Φ at the level c_1 , then by Lemma 3.7 we get a convergence subsequence of $\{u_n\}$, i.e., $u_{n_k} \rightarrow u$ in E as $k \rightarrow \infty$, thus $\Phi'(u) = 0$ and $\Phi(u) = c_1 \geq \kappa > 0$ by Lemma 3.3. This implies that u is a positive solution of (1.13), and the proof of Theorem 1.5 is completed.

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