Research article

Peng Chen* and Xianhua Tang

Periodic solutions for a differential inclusion problem involving the $p(t)$ -Laplacian

<https://doi.org/10.1515/anona-2020-0156>

Received June 27, 2020; accepted September 30, 2020.

Abstract: In the present paper, we consider the nonlinear periodic systems involving variable exponent driven by $p(t)$ -Laplacian with a locally Lipschitz nonlinearity. Our arguments combine the variational principle for locally Lipschitz functions with the properties of the generalized Lebesgue-Sobolev space. Applying the nonsmooth critical point theory, we establish some new existence results.

Keywords: Locally Lipschitz; Periodic solution; $p(t)$ -Laplacian; Nonsmooth critical point theory

MSC: 34C25; 58E30; 47H04

1 Introduction

In recent years, the study on $p(t)$ -Laplacian problems has attracted more and more attention. The $p(t)$ -Laplacian possesses more complicated nonlinearities than the p -Laplacian. For example, it is inhomogeneous, this causes many troubles, and some classical theories and methods, such as the theory of Sobolev spaces, are not applicable. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to problems of this type is the model of motion of electro-rheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field [44]. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the filtration processes of an ideal barotropic gas through a porous medium [1,2]. Another field of application of equations with variable exponent growth conditions is image processing [4]. We refer the reader to [35,40-44] for an overview of and references on this subject, and to [6-9,13,14,20,21,33,34, 37,39, 40-42] for the study of the $p(t)$ -Laplacian equations and the corresponding variational problems.

Recently, Wang and Yuan [37] obtained the existence of periodic solutions for $p(t)$ -Laplacian system:

$$\begin{cases} -(|u'(t)|^{p(t)-2}u'(t))' = \nabla j(t, u(t)), & a.e. t \in [0, T], \\ u(0) = u(T), & u'(0) = u'(T), \end{cases} \quad (1.1)$$

where $j(t, u)$ is measurable in $t \in [0, T]$, continuously differentiable in $u \in \mathbb{R}^N$. More precisely, they were able to prove that, under suitable conditions, the system might have at least one solution, or have infinite number of solutions. Since many free boundary problems and obstacle problems may be reduced to partial

*Corresponding Author: Peng Chen, College of Science, China Three Gorges University, Yichang, Hubei 443002, P.R. China
Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, Hubei 443002, P.R. China, E-mail:
pengchen729@sina.com

Xianhua Tang, School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, P.R. China, E-mail:
tangxh@mail.csu.edu.cn

differential equations with discontinuous nonlinearities, now a question arises: whether there exist solutions for system (1.1) in the case where the potential function $j(t, x)$ is nonsmooth in $x \in \mathbb{R}^N$. We require that $j(t, \cdot)$ is only locally Lipschitz. That is the main problem which we want to solve in the present paper.

The operator $(|u'(t)|^{p(t)-2}u'(t))'$ is said to be $p(t)$ -Laplacian, which becomes p -Laplacian when $p(t) \equiv p$ (a constant) and the problem (1.1) reduces to the following

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' \in \partial j(t, u), & \text{a.e. } t \in [0, T], \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \quad (1.2)$$

where $j(t, u)$ is locally Lipschitz in $u \in \mathbb{R}^N$.

Periodic problems involving the scalar p -Laplacian were studied by many authors. We mention the works by Dang and Oppenheimer [10], Del Pino, Manasevich and Murua [11], Gasinski and Papageorgiou [16-18], Papageorgiou and Rădulescu [30], Yang [38] and the references [15, 22, 28, 30-32, 38].

The goal of this paper is to discuss the existence of solutions of the following differential equation with $p(t)$ -Laplacian and a nonsmooth potential

$$\begin{cases} -(|u'(t)|^{p(t)-2}u'(t))' \in \partial j(t, u(t)), & \text{a.e. } t \in [0, T] \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \quad (1.3)$$

where $p(t) > 1$, $u \in \mathbb{R}^N$, $j(t, s)$ is locally Lipschitz function in the s -variable integrand (in general it can be nonsmooth), and $\partial j(t, s)$ is the subdifferential with respect to the s -variable in the sense of Clarke [3, 5].

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on generalized gradient of the locally Lipschitz function and variable exponent Sobolev spaces. In Section 3, we give the main results and their proofs.

2 Preliminaries

The nonsmooth critical point theory for locally Lipschitz functionals is based on the subdifferential theory of Clarke [5].

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*) . A function $\phi : X \rightarrow \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, we can find a neighbourhood U of x and a constant $k > 0$ (depending on U), such that $|\phi(y) - \phi(z)| \leq k\|y - z\|$, $\forall y, z \in U$.

For a locally Lipschitz function $\phi : X \rightarrow \mathbb{R}$ we define

$$\phi^0(x; h) = \lim_{x' \rightarrow x; \lambda \downarrow 0} \sup \frac{\phi(x' + \lambda h) - \phi(x')}{\lambda}.$$

It is obvious that the function $h \mapsto \phi^0(x; h)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^* -compact set $\partial\phi(x) \subseteq X^*$, defined by

$$\partial\phi(x) = \{x^* \in X^*; \langle x^*, h \rangle \leq \phi^0(x; h), \forall h \in X\}.$$

The multifunction $x \mapsto \partial\phi(x)$ is known as the generalized (or Clarke) subdifferential of ϕ . If $\phi, \psi : X \rightarrow \mathbb{R}$ are locally Lipschitz functions, then $\partial(\phi + \psi)(x) \subseteq \partial\phi(x) + \partial\psi(x)$ and for every $\lambda \in \mathbb{R}$, $\partial(\lambda\phi)(x) = \lambda\partial\phi(x)$.

Let $\phi : X \rightarrow \mathbb{R}$ be a locally Lipschitz function. A point $x \in X$ is said to be a critical point of ϕ if $0 \in \partial\phi(x)$. If $x \in X$ is a critical point of ϕ , then $c = \phi(x)$ is a critical value of ϕ . It is easy to see that, if $x \in X$ is a local extremum of ϕ , then $0 \in \partial\phi(x)$. Moreover, the multifunction $x \mapsto \partial\phi(x)$ is upper semicontinuous from X into X^* equipped with the w^* topology, i.e., for any $U \subseteq X^*$ w^* -open, the set $\{x \in X : \partial\phi(x) \subseteq U\}$ is open in X . For more details we refer to Clarke [5]. The critical point theory for smooth functions uses a compactness condition known as the Palais-Smale condition (PS). In the present nonsmooth setting this condition takes the following form:

The locally Lipschitz function $\phi : X \rightarrow \mathbb{R}$ satisfies the nonsmooth PS condition if any sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\{\phi(x_n)\}_{n \geq 1}$ is bounded and $m(x_n) = \min\{\|x^\| : x^* \in \partial\phi(x_n)\} \rightarrow 0$ as $n \rightarrow \infty$, has a strongly convergent subsequence.*

If $\phi \in C^1(X, \mathbb{R})$, then as we already mentioned $\partial\phi(x) = \{\phi'(x)\}$ and so the above definition of the P.S. condition coincides with the classical (smooth) one. In the context of the smooth theory, Cerami introduced a weaker compactness condition which in our nonsmooth setting has the following form:

The locally Lipschitz function $\phi : X \rightarrow \mathbb{R}$ satisfies the nonsmooth Cerami condition (nonsmooth C-condition for short), if any sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\{\phi(x_n)\}_{n \geq 1}$ is bounded and $(1 + \|x_n\|)m(x_n) \rightarrow 0$ as $n \rightarrow \infty$, has a strongly convergent subsequence.

Lemma 2.1. [27] Assume that φ is a locally Lipschitz functional on a Banach space X and $\varphi : X \rightarrow \mathbb{R}$ satisfies:

- (i) φ is weakly lower semicontinuous;
- (ii) φ is coercive.

Then there exists $x^* \in X$ such that $\varphi(x^*) = \min_{x \in X} \varphi(x)$.

Lemma 2.2. [22] Let X be a Banach space and $\varphi : X \rightarrow \mathbb{R}$ a locally Lipschitz functional satisfying the (C) condition. If $X = Y \oplus V$ with Y a finite-dimensional subspace of X , φ satisfies the nonsmooth C-condition,

$$c = \inf_{y \in \Gamma} \max_{x \in E} \varphi(y(x))$$

and there exists an $r > 0$ such that

$$\max\{\varphi(x) : x \in Y, \|x\| = r\} \leq \inf\{\varphi(x) : x \in V\},$$

where $\Gamma = \{y \in C(E, X) : y|_{\partial E} = \text{id}\}$, $E = \{x \in Y : \|x\| \leq r\}$ and $\partial E = \{x \in Y : \|x\| = r\}$, then

$$c \geq \inf_V \varphi$$

and c is a critical value of φ . Moreover, if $c = \inf_V \varphi$, then $V \cap K_c \neq \emptyset$, where $K_c = \{x \in X : \varphi(x) = c, \lambda(x) = 0\}$.

Lemma 2.3. [22] Let X be a reflexive Banach space, $\phi : X \rightarrow \mathbb{R}$ a locally Lipschitz functional satisfying the PS-condition. Assume that there exist $x_0, x_1 \in X$, $c_0 \in \mathbb{R}$ and $\varrho > 0$ such that $\|x_1 - x_0\| > \varrho$ and

$$\max\{\phi(x_0), \phi(x_1)\} < c_0 = \inf\{\phi(y) : \|y - x_0\| = \varrho\}.$$

Then, ϕ has a critical point $x \in X$ with $c = \phi(x) \geq c_0$ given by

$$c = \inf_{y \in \Gamma_1} \max_{t \in T} \phi(y(t)),$$

where $\Gamma_1 = \{y \in C([0, 1], X) : y(0) = x_0, y(1) = x_1\}$.

In order to discuss (1.3), we recall some known results from critical point theory and the properties of space $W^{1,p(t)}$ for the convenience of the readers.

Let $p(t) \in C(0, T; \mathbb{R})$ and $1 < p_- := \inf_{t \in [0, T]} p(t) \leq \sup_{t \in [0, T]} p(t) := p^+ < \infty$. Define

$$L^{p(t)}(0, T; \mathbb{R}^N) = \left\{ u \in L^1(0, T; \mathbb{R}^N) \mid \int_0^T |u(t)|^{p(t)} dt < \infty \right\}$$

with the norm

$$\|u\|_{p(t)} = \inf \left\{ \lambda > 0 \mid \int_0^T \left| \frac{u}{\lambda} \right|^{p(t)} dt \leq 1 \right\}.$$

For $u \in L^1_{\text{loc}}(0, T; \mathbb{R}^N)$, let u' denote the weak derivative of u , i.e., $u' \in L^1_{\text{loc}}(0, T; \mathbb{R}^N)$ and satisfy

$$\int_0^T u' \varphi dt = - \int_0^T u \varphi' dt, \quad \forall \varphi \in C_0^\infty(0, T; \mathbb{R}^N).$$

Define

$$W^{1,p(t)}(0, T; \mathbb{R}^N) = \left\{ u \in L^{p(t)}(0, T; \mathbb{R}^N) \mid u' \in L^{p(t)}(0, T; \mathbb{R}^N) \right\}$$

with the norm

$$\|u\| = \inf \left\{ \lambda > 0 \mid \int_0^T \left(\left| \frac{u'}{\lambda} \right|^{p(t)} + \left| \frac{u}{\lambda} \right|^{p(t)} \right) dt \leq 1 \right\}.$$

We call the space $L^{p(t)}$ a generalized Lebesgue space, it is a special kind of generalized Orlicz spaces. The space $W^{1,p(t)}$ is called a generalized Sobolev space, it is a special kind of generalized Orlicz-Sobolev spaces. For the general theory of generalized Orlicz spaces and generalized Orlicz-Sobolev spaces, see [13,14].

Proposition 2.4. [14] *Let*

$$\rho(u) = \int_0^T |u'|^{p(t)} dt, \quad \forall u \in L^{p(t)},$$

then

- (i) $|u|_{p(t)} < 1$ ($= 1$; > 1) $\iff \rho(u) < 1$ ($= 1$; > 1);
- (ii) $|u|_{p(t)} > 1 \Rightarrow |u|_{p(t)}^{p^-} \leq \rho(u) \leq |u|_{p(t)}^{p^+}$,
 $|u|_{p(t)} < 1 \Rightarrow |u|_{p(t)}^{p^+} \leq \rho(u) \leq |u|_{p(t)}^{p^-}$;
- (iii) $|u|_{p(t)} \rightarrow 0 \iff \rho(u) \rightarrow 0$; $|u|_{p(t)} \rightarrow \infty \iff \rho(u) \rightarrow \infty$.
- (iv) *Let* $u \in L^{p(t)} \setminus \{0\}$, *then* $\|u\|_{p(t)} = \lambda$ *if and only if* $\rho\left(\frac{u}{\lambda}\right) = 1$.

Proposition 2.5. [13] $L^{p(t)}$ and $W^{1,p(t)}$ are Banach spaces with the norms defined above. When $p^- > 1$, they are reflexive.

Let $C_T^\infty = C_T^\infty(\mathbb{R}, \mathbb{R}^N) = \{u \in C^\infty(\mathbb{R}, \mathbb{R}^N) : u \text{ is } T\text{-periodic}\}$.

Definition 2.6. [13] *Let* $u, v \in L^1(0, T; \mathbb{R}^N)$. *If*

$$\int_0^T v \varphi dt = - \int_0^T u \dot{\varphi} dt, \quad \forall \varphi \in C_T^\infty(\mathbb{R}, \mathbb{R}^N),$$

then v *is called a* T -*weak derivative of* u *and is denoted by* $\dot{u}$.

Definition 2.7. [24] *Define*

$$W_T^{1,p}(0, T; \mathbb{R}^N) = \left\{ u \in L^p(0, T; \mathbb{R}^N) : \dot{u} \in L^p(0, T; \mathbb{R}^N) \right\}$$

with the norm $\|u\|_{W_T^{1,p}} = (|u|_p^p + |\dot{u}|_p^p)^{1/p}$.

Definition 2.8. [13, 14] *Define*

$$W_T^{1,p(t)}(0, T; \mathbb{R}^N) = \left\{ u \in L^{p(t)}(0, T; \mathbb{R}^N) : \dot{u} \in L^{p(t)}(0, T; \mathbb{R}^N) \right\}$$

and $H_T^{1,p(t)}(0, T; \mathbb{R}^N)$ *to be the closure of* C_T^∞ *in* $W_T^{1,p(t)}(0, T; \mathbb{R}^N)$.

From Definition 2.7 and 2.8 we see that, for $u \in L^1(0, T; \mathbb{R}^N)$, the weak derivative u' and the T -weak $\dot{u}$ derivative are two different conceptions (see [13] for details). Although the two derivatives are distinct, we have

Proposition 2.9. [13, 14]

- (1) $C_T^\infty(0, T; \mathbb{R}^N)$ is dense in $W_T^{1,p(t)}(0, T; \mathbb{R}^N)$;
- (2) $W_T^{1,p(t)}(0, T; \mathbb{R}^N) = H_T^{1,p(t)}(0, T; \mathbb{R}^N) = \{u \in W_T^{1,p(t)}(0, T; \mathbb{R}^N) : u(0) = u(T)\}$;
- (3) *If* $u \in H_T^{1,1}$, *the weak derivative* $\dot{u}$ *is also the* T -*weak derivative* $\dot{u}$, *i.e.,* $u' = \dot{u}$.

Proposition 2.10. [13] *Let* $u \in W_T^{1,1}$, *then*

- (1) $\int_0^T \dot{u} dt = 0$;
- (2) u has its continuous representation, which is still denoted by u ,

$$u(t) = \int_0^t \dot{u}(s) ds + u(0), \quad u(0) = u(T).$$

- (3) $\dot{u}$ is the classical derivative of u if $\dot{u} \in C(0, T; \mathbb{R}^N)$.

Proposition 2.11. [13]

- (1) $H_T^1(0, T; \mathbb{R}^N)$ is a reflexive Banach space if $p^- > 1$;
- (2) There is a continuous embedding $W_T^{1,p(t)}(H_T^{1,p(t)}) \hookrightarrow C(0, T; \mathbb{R}^N)$. When $p^- > 1$, the embedding is compact.

Proposition 2.12. [14] If

$$\frac{1}{p(t)} + \frac{1}{q(t)} = 1,$$

then

- (i) $(L^{p(t)})^* = L^{q(t)}$, where $(L^{p(t)})^*$ is the dual space of $L^{p(t)}$;
- (ii) $\forall u \in L^{p(t)}, v \in L^{q(t)}$, we have

$$\left| \int_{\Omega} u(t)v(t) dt \right| \leq 2|u|_{p(t)}|v|_{q(t)}.$$

Let

$$W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) = \{u \in L^{p(t)}(0, T; \mathbb{R}^N) : u(0) = u(T), u' \in L^{p(t)}(0, T; \mathbb{R}^N)\}.$$

Obviously, $W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ is a reflexive Banach space with the norm

$$\|u\| = \inf \left\{ \lambda > 0 \mid \int_0^T \left(\left| \frac{u'}{\lambda} \right|^{p(t)} + \left| \frac{u}{\lambda} \right|^{p(t)} \right) dt \leq 1 \right\}.$$

Consider the following functional:

$$J(u) = \int_0^T \frac{1}{p(t)} |u'|^{p(t)} dt, \quad u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N).$$

We know that (see [2]) $J \in C^1(W_{\text{per}}^{1,p(t)}, \mathbb{R})$ and $p(t)$ -Laplacian operator $(|u'|^{p(t)-2}u')'$ is the derivative operator of J in the weak sense. Denote $A = J' : W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \rightarrow (W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N))^*$, then

$$\langle A(u), v \rangle = \int_0^T (|u'(t)|^{p(t)-2}u'(t), v'(t))_{\mathbb{R}^N} dt, \quad \forall u, v \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N).$$

Proposition 2.13. [14] J' is a mapping of $(S)_+$, i.e., if

$$u_n \rightharpoonup u \quad \text{and} \quad \lim_{n \rightarrow \infty} (J'(u_n) - J'(u), u_n - u) \leq 0,$$

then u_n has a convergent subsequence in $W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$.

For every $u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$, set

$$\bar{u} = \frac{1}{T} \int_0^T u(t) dt, \quad \tilde{u}(t) = u(t) - \bar{u}.$$

By virtue of [35], there exists $a > 0$ such that

$$\|\tilde{u}\|_{\infty} \leq a \|u'\|_{p(t)}, \quad \forall u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N). \quad (2.1)$$

The corresponding functional $\varphi : W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \rightarrow \mathbb{R}$ for (1.3) is defined by:

$$\varphi(u) = \int_0^T \frac{1}{p(t)} |u'|^{p(t)} dt - \int_0^T j(t, u) dt, \quad \forall u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N). \quad (2.2)$$

3 Main results and their Proofs

Our hypotheses on the function $p(t)$ and $j(t, u)$ are the following:

(P) $p(t) \in C(0, T; \mathbb{R})$, $p(t+T) = p(t)$ and $1 < p^- := \inf_{t \in [0, T]} p(t) \leq \sup_{t \in [0, T]} p(t) := p^+ < \infty$, $\forall t \in \mathbb{R}$;

$H(j)_1$ $j : [0, T] \times \mathbb{R}^N$ is a function such that:

- (i) for all $u \in \mathbb{R}^N$, $t \mapsto j(t, u)$ is measurable;
- (ii) for almost all $t \in [0, T]$, $u \mapsto j(t, u)$ is locally Lipschitz;
- (iii) for every $w \in \partial j(t, u)$, there exists $\alpha \in L^1[0, T]$ such that for almost all $t \in [0, T]$, we have $|w| \leq \alpha(t)$;
- (iv) $j(t, u) \rightarrow -\infty$ as $|u| \rightarrow \infty$ uniformly for almost all $t \in [0, T]$;
- (v) there exists $u_0 \in \mathbb{R}^N \setminus \{0\}$ such that $\int_0^T j(t, u_0) dt > 0$ and $\int_0^T j(t, 0) dt \leq 0$.

Theorem 3.1 *If hypotheses (P), $H(j)_1$ hold, then problem (1.3) has at least one nontrivial periodic solution.*

We can weaken hypothesis $H(j)_1$ (iv) at the expense of introducing an extra unilateral growth condition. More precisely, the new hypotheses on $j(t, u)$ are the following:

$H(j)_2$ $j : [0, T] \times \mathbb{R}^N$ is a function such that:

- (i) for all $u \in \mathbb{R}^N$, $t \mapsto j(t, u)$ is measurable;
- (ii) for almost all $t \in [0, T]$, $u \mapsto j(t, u)$ is locally Lipschitz;
- (iii) for every $w \in \partial j(t, u)$, there exists $\alpha \in L^1[0, T]$ such that for almost all $t \in [0, T]$, we have $|w| \leq \alpha(t)$;
- (iv) $j(t, u) \rightarrow -\infty$ as $|u| \rightarrow \infty$ for almost all $t \in E \subset [0, T]$ with $\text{meas} E > 0$;
- (v) there exists $\beta \in L^1[0, T]$ such that for almost all $t \in [0, T]$, $u \in \mathbb{R}^N$, we have

$$j(t, u) \leq \beta(t);$$

- (vi) there exists $u_0 \in \mathbb{R}^N \setminus \{0\}$ such that $\int_0^T j(t, u_0) dt > 0$ and $\int_0^T j(t, 0) dt \leq 0$.

Theorem 3.2 *If hypotheses (P), $H(j)_2$ hold, then problem (1.3) has at least one nontrivial periodic solution.*

In the previous existence theorems the energy functional φ defined in (2.2) was coercive and so the solution was obtained by an application of the least action principle. In the next existence theorem the energy functional φ is bounded below but not necessarily coercive. In this case the hypotheses on the nonsmooth potential $j(t, u)$ are the following:

$H(j)_3$ $j : [0, T] \times \mathbb{R}^N$ is a function such that $j(\cdot, 0) \in L^1[0, T]$ and:

- (i) for all $u \in \mathbb{R}^N$, $t \mapsto j(t, u)$ is measurable;

- (ii) for almost all $t \in [0, T]$, $u \mapsto j(t, u)$ is locally Lipschitz;
- (iii) for every $r > 0$, there exists $\alpha_r \in L^1[0, T]$ such that for almost all $t \in [0, T]$, $|u| \leq r$ and all $w \in \partial j(t, u)$, we have $|w| \leq \alpha_r(t)$;
- (iv) there exist $0 < \mu < p^-$ and $M > 0$ such that for almost all $t \in [0, T]$ and all $|u| \geq M$, we have

$$j^0(t, u; u) < \mu j(t, u);$$

- (v) $j(t, u) \rightarrow +\infty$ uniformly for almost all $t \in [0, T]$ as $|u| \rightarrow \infty$.

Theorem 3.3 If hypotheses (P), $H(j)_3$ hold, then problem (1.3) has at least one nontrivial periodic solution.

Next, we will establish some existence results by using nonsmooth Mountain Pass theorem. Our hypotheses on the nonsmooth potential function $j(t, u)$ are the following:

$H(j)_4$ $j : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a function such that $j(t, 0) = 0$ for every $t \in [0, T]$;

- (i) for all $u \in \mathbb{R}^N$, $t \mapsto j(t, u)$ is measurable;
- (ii) for almost all $t \in [0, T]$, $u \mapsto j(t, u)$ is locally Lipschitz;
- (iii) for every $w \in \partial j(t, u)$, there exists $\alpha \in C([0, T], \mathbb{R})$ such that for almost all $t \in [0, T]$, we have

$$|w| \leq b(t)|u|^{\alpha(t)-1},$$

where $b \in L^{\alpha(t)}[0, T] \cap L_+^\infty[0, T]$ and $p^+ < \alpha^- \leq \alpha(t) \leq \alpha^+$;

- (iv) there exists constants $M > 0$, $\alpha, \beta > 0$ and $\nu < p^-$ such that for almost all $t \in [0, T]$ and all $u \in \mathbb{R}^N \setminus \{0\}$, we have

$$0 \leq \left(p^+ + \frac{1}{\alpha + \beta|u|^\nu} \right) j(t, u) \leq -j^0(t, u; -u);$$

- (v) for every $u \in \mathbb{R}^N$, $w \in \partial j(t, u)$ we have

$$\lim_{|u| \rightarrow 0} \frac{w}{|u|^{p(t)-1}} = 0;$$

- (vi) there exists a function $q(t) > 0$ ($p^+ < q^- \leq q(t) \leq q^+$) such that for almost all $t \in [0, T]$ and all $u \in \mathbb{R}^N$, we have

$$\lim_{|u| \rightarrow +\infty} \inf \frac{j(t, u)}{|u|^{q(t)}} > 0.$$

Theorem 3.4 If hypotheses (P), $H(j)_4$ hold, then problem (1.3) has at least one nontrivial periodic solution.

Theorem 3.5 If hypotheses (P), $H(j)_4$ (j), (i)-(iii), (v) and the following conditions hold:

- (vi') there exist $\mu > p^+$ and $M > 0$ such that

$$\mu j(t, u) \leq -j^0(t, u; -u), \quad \forall t \in [0, T], \quad |u| \geq M,$$

and $\int_0^T j(t, u_0) dt > 0$ as $u_0 \in \mathbb{R}^N$, $|u_0| \geq M$.

Then problem (1.3) has at least one nontrivial periodic solution.

Proof of Theorem 3.1. We consider the locally Lipschitz energy functional $\varphi : W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\varphi(u) = \int_0^T \frac{1}{p(t)} |u'|^{p(t)} dt - \int_0^T j(t, u) dt.$$

It is easy to verify that φ is locally Lipschitz by $H(j)_1$ (iii) and (2.1).

Because of hypothesis $H(j)_1$ (iv) and Lemma 3 [35], for almost all $t \in [0, T]$, $u \in \mathbb{R}^N$, we have

$$j(t, u) \leq -G(u) + y(t), \quad (3.1)$$

where $y \in L^1[0, T]$, $G \in C(\mathbb{R}^N, \mathbb{R})$ is subadditive, i.e., $G(x + y) \leq G(x) + G(y)$ for all $x, y \in \mathbb{R}^N$ and is coercive, i.e., $G(u) \rightarrow +\infty$ as $|u| \rightarrow \infty$.

Consider the direct sum decomposition

$$W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) = \mathbb{R}^N \oplus V,$$

where

$$V = \left\{ v \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) : \int_0^T v(t) dt = 0 \right\}.$$

For every $u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ we have $u = \bar{u} + \tilde{u}$, where $\bar{u} \in \mathbb{R}^N$, $\tilde{u} \in V$. By (3.1), we have

$$\begin{aligned} \varphi(u) &= \int_0^T \frac{1}{p(t)} |u'|^{p(t)} dt - \int_0^T j(t, u) dt \\ &\geq \frac{1}{p^+} \int_0^T |u'|^{p(t)} dt + \int_0^T G(u(t)) dt - \|y\|_1. \end{aligned} \quad (3.2)$$

From the properties of the continuous function G mentioned above, we have

$$\begin{aligned} G(\bar{u}) &= G(\bar{u} + \tilde{u}(t) - \tilde{u}(t)) \leq G(\bar{u} + \tilde{u}(t)) + G(-\tilde{u}(t)) \\ &\Rightarrow G(\bar{u}) - G(-\tilde{u}(t)) \leq G(u(t)) \\ &\Rightarrow G(\bar{u}) - \|\tilde{u}(t)\| - 4 \leq G(u(t)), \quad t \in [0, T]. \end{aligned} \quad (3.3)$$

Using (3.3) in (3.2), we obtain

$$\varphi(u) \geq \frac{1}{p^+} \int_0^T |\tilde{u}'|^{p(t)} dt + G(\bar{u})T - c_1 \|\tilde{u}'\|_{p(t)} - c_2, \quad (3.4)$$

where $c_1, c_2 > 0$.

Noting that $\|u\| \leq \|\bar{u}\| + \|\tilde{u}\|$, by virtue of (2.1), (3.4) and the coercivity on G , we infer that φ is coercive.

Owing to the fact the compact embedding of $W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \hookrightarrow C(0, T; \mathbb{R}^N)$ and the weak lower semicontinuity of the norm functional in a Banach space, we infer that φ is weakly lower semicontinuous. So by hypothesis $H(j)_1$ (v) and the Weierstrass theorem (Lemma 2.1) we can find u_0 such that

$$\begin{aligned} -\infty < m = \inf \varphi = \varphi(u) \leq \varphi(u_0) < 0 \leq \varphi(0) \\ \Rightarrow u \neq 0, \quad 0 \in \partial\varphi(u). \end{aligned}$$

Next, we will show that u is the solution of (1.3).

In fact, let $u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ be such that $0 \in \partial\varphi(u)$. Define the nonlinear operator $A : W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \rightarrow (W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N))^*$ as follows

$$\langle A(u), v \rangle = \int_0^T |u'|^{p(t)-2} (u'(t), v'(t))_{\mathbb{R}^N} dt, \quad \forall u, v \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N).$$

Thus,

$$A(u) = w,$$

where $w \in L^{q(t)}(0, T; \mathbb{R}^N)$ and $w \in \partial j(t, u(t))$. Here $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$.

For every $x, y \in \mathbb{R}^N$, the following inequalities hold [23]:

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq (p-1)|x - y|^2(1 + |x|^2 + |y|^2)^{\frac{p-2}{2}}, \quad \forall p \in [1, 2]. \quad (3.5)$$

and

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \leq (p-1)(|x|^{\frac{p-2}{2}} + |y|^{\frac{p-2}{2}})|x|^{\frac{p-2}{2}}x - |y|^{\frac{p-2}{2}}y, \quad \forall p \in [2, +\infty). \quad (3.6)$$

An argument similar to the one used in [37] shows that for every $v \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$, thus

$$\langle A(u), v \rangle = \int_0^T |u'(t)|^{p(t)-2}(u'(t), v'(t))dt = \int_0^T (w(t), v(t))dt,$$

which implies that $(|u'(t)|^{p(t)-2}u'(t))'$ has T -weak derivative and satisfy

$$-(|u'(t)|^{p(t)-2}u'(t))' = w.$$

It follows from $C([0, T], \mathbb{R}^N) \subseteq W_{\text{per}}^{1,p(t)}([0, T], \mathbb{R}^N)$ and $W_{\text{per}}^{1,p(t)}([0, T], \mathbb{R}^N) \hookrightarrow C([0, T], \mathbb{R}^N)$ that $u(t)$ is continuous for every $t \in [0, T]$. It is obviously that $(|u'(t)|^{p(t)-2}u'(t))' \in L^{q(t)}([0, T], \mathbb{R}^N)$. Since $L^{q(t)} \hookrightarrow L^1$, then we have $(|u'(t)|^{p(t)-2}u'(t))' \in L^1([0, T], \mathbb{R}^N)$, which implies that $(|u'(t)|^{p(t)-2}u'(t))' \in W_{\text{per}}^{1,1}([0, T], \mathbb{R}^N)$. By Proposition 2.4 (2), we have

$$|u'(0)|^{p(0)-2}u'(0) = |u'(T)|^{p(T)-2}u'(T).$$

Noting that $p(0) = p(T)$, we can obtain $u'(0) = u'(T)$. Since $u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$, then $u(0) = u(T)$. So u is the solution of (1.3).

Proof of Theorem 3.2. Given $\delta > 0$, by Lemma 2 [35] and $H(j)_2$ (iv), we can find $E_\delta \subseteq E$ such that $\text{meas}(E \setminus E_\delta) < \delta$ as $|u| \rightarrow \infty$, $j(t, u) \rightarrow -\infty$ uniformly for all $t \in E_\delta$, $u \in \mathbb{R}^N$, so for almost all $t \in E_\delta$, $u \in \mathbb{R}^N$ we have

$$j(t, u) \leq -G(u) + y(t), \quad (3.7)$$

with $G \in C(\mathbb{R}^N, \mathbb{R})$, $y \in L^1(E_\delta)$ as in the proof of Theorem 3.1. Then for every $u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$, by (3.4), (3.7) and $H(j)_2$ (v) we have

$$\begin{aligned} \varphi(u) &= \int_0^T \frac{1}{p(t)} |u'|^{p(t)} dt - \int_0^T j(t, u) dt \\ &\geq \frac{1}{p^+} \int_0^T |u'|^{p(t)} dt - \int_{E_\delta} j(t, u) dt - \int_{[0, T] \setminus E_\delta} j(t, u) dt \\ &\geq \frac{1}{p^+} \int_0^T |u'|^{p(t)} dt + \int_{E_\delta} G(u(t)) dt - \|y\|_1 - \|\beta\|_1 \\ &\geq \frac{1}{p^+} \int_0^T |\tilde{u}'|^{p(t)} dt + G(\bar{u}) \text{meas}(E_\delta) - c_3 |\tilde{u}'|_{p(t)} - c_4. \end{aligned}$$

where $c_3, c_4 > 0$. Therefore φ is coercive. The rest of the proof goes as that of Theorem 3.1.

Proof of Theorem 3.3. Consider the locally Lipschitz energy functional $\varphi : W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\varphi(u) = \int_0^T \frac{1}{p(t)} |u'(t)|^{p(t)} dt - \int_0^T j(t, u(t)) dt.$$

Step 1: φ satisfies the nonsmooth C-condition.

Let $\{u_n\}_{n \geq 1}$ be a (C)-sequence of φ , i.e., there exists $M_1 > 0$ such that

$$|\varphi(u_n)| \leq M_1, \quad (1 + \|u_n\|)m(u_n) \rightarrow 0.$$

Since $\partial\varphi(u_n) \subseteq W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)^*$ is weakly compact and the norm functional in a Banach space is weakly compact, from the Weierstrass theorem (Lemma 2.1), we can find $u_n^* \in \partial\varphi(u_n)$ such that $m(u_n) = \|u_n^*\|$.

Define the nonlinear operator $A : W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \rightarrow (W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N))^*$

$$\langle A(u), v \rangle = \int_0^T |u'(t)|^{p(t)-2} (u'(t), v'(t))_{\mathbb{R}^N} dt, \quad \forall u, v \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N).$$

Then

$$u_n^* = A(u_n) - w_n,$$

where $w_n \in \partial j(t, u_n)$.

From the choice of the sequence $\{u_n\}_{n \geq 1} \subseteq W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ we have

$$\langle u_n^*, u_n \rangle = \int_0^T |u_n'(t)|^{p(t)} dt - \int_0^T (w_n(t), u_n(t)) dt \leq \epsilon_n, \quad \epsilon_n \downarrow 0.$$

Thus,

$$\int_0^T |u_n'(t)|^{p(t)} dt - \int_0^T j^0(t, u_n(t); u_n(t)) dt \leq \epsilon_n, \quad (3.8)$$

and also

$$\frac{\mu}{p^+} \int_0^T |u_n'|^{p(t)} + \int_0^T \mu j(t, u_n(t)) dt \leq \mu M_1. \quad (3.9)$$

It follows from (3.8) and (3.9) that

$$\begin{aligned} & \left(1 - \frac{\mu}{p^+}\right) \int_0^T |u_n'|^{p(t)} + \int_0^T [\mu j(t, u_n(t)) - j^0(t, u_n(t); u_n(t))] dt \\ & \leq \epsilon_n + \mu M_1. \end{aligned} \quad (3.10)$$

By H(j)₃ (iv), we have

$$\begin{aligned} & \int_0^T [\mu j(t, u_n(t)) - j^0(t, u_n(t); u_n(t))] dt \\ & = \int_{|u_n| \leq M} [\mu j(t, u_n(t)) - j^0(t, u_n(t); u_n(t))] dt \\ & \quad + \int_{|u_n| > M} [\mu j(t, u_n(t)) - j^0(t, u_n(t); u_n(t))] dt \geq -c_5, \end{aligned}$$

where $c_5 > 0$ is independent of n .

Therefore, from (3.10) we have

$$\left(1 - \frac{\mu}{p^+}\right) \int_0^T |u_n'|^{p(t)} \leq \epsilon_n + \mu M_1 + c_6, \quad n \geq 1.$$

By the Poincaré-Wirtinger inequality (2.1), $\{\tilde{u}_n\}$ is bounded in $W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$.

It follows from $H(j)_3$ and Lemma 3 [35] that for almost all $t \in [0, T]$ and all $u \in \mathbb{R}^N$, we have

$$j(t, u) \geq G(u) + y(t), \quad (3.11)$$

with $y \in L^1[0, T]$, $G \in C(\mathbb{R}^N, \mathbb{R})$ is subadditive, coercive and satisfies $G(u) \leq |u| + 4$ for all $u \in \mathbb{R}^N$.

From the choice of the sequence $\{u_n\}_{n \geq 1} \subseteq W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ we have

$$\left| \frac{1}{p^+} \int_0^T |\tilde{u}'_n|^{p(t)} - \int_0^T j(t, u_n(t)) dt \right| \leq M_1, \quad n \geq 1.$$

Since $\{\tilde{u}_n\}$ is bounded, there exists $M_2 > 0$, $n \geq 1$ such that

$$\int_0^T j(t, u_n(t)) dt \leq M_2.$$

By (3.11), we get

$$\int_0^T G(u_n(t)) dt + \|y\|_1 \leq M_2,$$

thus, there exists $M_3 > 0$ such that

$$G(\tilde{u}_n) b \leq M_3, \quad n \geq 1.$$

Due to the coercivity of G , we infer that $\{\tilde{u}_n\}_{n \geq 1} \subseteq \mathbb{R}^N$ is bounded. Therefore $\{u_n\}_{n \geq 1} \subseteq W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ is bounded and so by passing to a subsequence if necessary, we may assume that

$$u_n \rightharpoonup u \text{ in } W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N), \quad u_n \rightarrow u \text{ in } C_{\text{per}}(0, T; \mathbb{R}^N).$$

Next, we will prove that $u_n \rightarrow u$ in $W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$. By Proposition 2.13, it suffices to prove that the following inequality hold:

$$\overline{\lim}_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle \leq 0, \quad \varepsilon_n \downarrow 0.$$

In fact, from the choice of the sequence $\{u_n\}_{n \geq 1}$ we have

$$|\langle u_n^*, u_n \rangle| \leq \varepsilon_n \downarrow 0.$$

Recall that $u_n^* = A(u_n) - w_n$, then we have

$$\langle A(u_n), u_n - u \rangle - \int_0^T (w_n(t), (u_n(t) - u(t)))_{\mathbb{R}^N} dt \leq \varepsilon_n, \quad \forall n \geq 1.$$

By $H(j)_3$ (iii), $\{w_n\} \subseteq L^1[0, T]$ is bounded and

$$\int_0^T (w_n(t), (u_n(t) - u(t)))_{\mathbb{R}^N} dt \rightarrow 0 \quad (n \rightarrow \infty).$$

Then

$$\overline{\lim}_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0.$$

So $\overline{\lim}_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle \leq 0$, $\varepsilon_n \downarrow 0$.

Step 2: Similar to the proof in [30, 31], we have $j(t, sx) \leq s^\mu j(t, x)$ for every $s \geq 1$, we omit its proof course.

Step 3: $\varphi|_V$ is coercive, i.e., for every $v \in V$, $\varphi(v) \rightarrow +\infty$ as $\|v\| \rightarrow \infty$.

Let $v \in V$ be such that $|\{t \in T : \|v(t) > M\}|_1 > 0$, we have

$$\begin{aligned}\varphi(v) &= \int_0^T \frac{1}{p(t)} |v'|^{p(t)} dt - \int_0^T j(t, v(t)) dt \\ &= \int_0^T \frac{1}{p(t)} |v'|^{p(t)} dt - \int_{|v| \geq M} j(t, v(t)) dt - \int_{|v| < M} j(t, v(t)) dt.\end{aligned}$$

Because of hypothesis $H(j)_3$ (iii) and the mean value theorem for locally Lipschitz functions, we see that there exists $c_7 > 0$ such that

$$\int_{|v| < M} j(t, v(t)) dt \leq c_7. \quad (3.12)$$

Also using Step 2, we have

$$\begin{aligned}\int_{|v| \geq M} j(t, v(t)) dt &= \int_{|v| \geq M} j\left(t, \frac{|v(t)|}{M} \frac{Mv(t)}{|v(t)|}\right) dt \\ &\leq \int_{|v| \geq M} \frac{|v(t)|^\mu}{M^\mu} j\left(t, \frac{Mv(t)}{|v(t)|}\right) dt \\ &\leq \frac{\|v(t)\|_\infty^\mu}{M^\mu} \int_{|v| \geq M} j\left(t, \frac{Mv(t)}{|v(t)|}\right) dt.\end{aligned} \quad (3.13)$$

Noting that for $\|u\| = M$, from the subdifferential chain rule, for almost all $t \in T$ we have

$$\begin{aligned}j(t, x) &= j(t, 0) + \int_0^1 (\partial_x j(t, rx), x) dr \\ \Rightarrow j(t, x) &\leq \beta_M(t), \quad \beta_M \in L^1[0, T].\end{aligned} \quad (3.14)$$

By (3.13) and (3.14), we have

$$\int_{|v| \geq M} j(t, v(t)) dt \leq \frac{\|v\|_\infty^\mu}{M^\mu} c_8, \quad c_8 > 0. \quad (3.15)$$

Using (3.15) and Poincaré-Wirtinger inequality (2.1) we have

$$\begin{aligned}\varphi(v) &\geq \frac{1}{p^-} |v'|_{p(t)}^{p^-} - \frac{c_8}{M^\mu} \|v\|_\infty^\mu - c_9 \\ &\geq \frac{1}{p^-} |v'|_{p(t)}^{p^-} - c_{10} |v'|_{p(t)}^\mu - c_{11},\end{aligned}$$

which yields that $\varphi|_V$ is coercive since $\mu < p^-$.

Step 4: $\varphi|_{\mathbb{R}^N}$ is antioercive, i.e., $\varphi(y) \rightarrow -\infty$ as $\|y\| \rightarrow \infty$, $y \in \mathbb{R}^N$, this claim is a direct consequence of hypothesis $H(j)_3$ (v).

Step 1, 3 and 4 permit the application of the nonsmooth saddle point theorem, so we can find $u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ such that

$$0 \in \partial\varphi(u).$$

As before, we can show that $u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ solves (1.3).

Proof of Theorem 3.4. Consider the locally Lipschitz energy functional $\varphi : W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\varphi(u) = \int_0^T \frac{1}{p(t)} |u'|^{p(t)} dt - \int_0^T j(t, u(t)) dt.$$

We divide our proof into two steps.

Step 1: φ satisfies nonsmooth (C)-condition.

Let $\{u_n\}_{n \geq 1} \subseteq W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ be a (C)-sequence of φ , i.e., there exists $M_1 > 0$ such that

$$|\varphi(u_n)| \leq M_1, \quad (1 + \|u_n\|)m(u_n) \rightarrow 0 \quad (n \rightarrow \infty).$$

It is apparent that $m(u_n) = \|u_n^*\|$, $n \geq 1$ for some $u_n^* \in \partial\varphi(u_n)$. Taking into account the fact that the set $\partial\varphi(u_n) \subseteq (W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N))^*$ is weakly compact, we define the nonlinear operator $A : W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \rightarrow W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)^*$ as follows

$$\langle A(u), v \rangle = \int_0^T |u'(t)|^{p(t)-2} u'(t) v'(t) dt, \quad \forall u, v \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N).$$

From the definition of and Proposition 2.13 we know that A is maximal monotone (see [16]), hence we have

$$u_n^* = A(u_n) - w_n,$$

where $w_n \in L^{q(t)}(0, T; \mathbb{R}^N)$, $w_n(t) \in \partial j(t, u_n(t))$, $\forall n \geq 1$, with $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$. Then, we have

$$\begin{aligned} C_1 &\geq p^+ \varphi(u_n) - \langle u_n^*, u_n \rangle \\ &\geq \int_0^T \frac{p^+}{p(t)} |u_n'|^{p(t)} dt - p^+ \int_{\mathbb{R}} j(t, u_n) dt \\ &\quad - \int_0^T |u_n'|^{p(t)} dt + \int_{\mathbb{R}} w_n u_n \\ &\geq \int_0^T [-p^+ j(t, u_n) dt - j^0(t, u_n; -u_n)] dt \end{aligned} \quad (3.16)$$

for every $(t, u) \in [0, T] \times (\mathbb{R}^N \setminus \{0\})$. By $H(j)_4$ (iv) we have

$$-p^+ j(t, u_n) - j^0(t, u_n; -u_n) \geq 0$$

and

$$j(t, u) \leq (\alpha + \beta |u|^\nu) [-p^+ j(t, u) - j^0(t, u; -u)]. \quad (3.17)$$

It follows from (2.1), (3.16) and (3.17) that

$$\begin{aligned} \frac{1}{p^+} |u_n'|_{p(t)}^{p^+} &\leq \int_0^T \frac{1}{p(t)} |u_n'|^{p^+} dt \\ &= I(u_n) + \int_0^T j(t, u_n(t)) dt \\ &\leq I(u_n) + \int_0^T (\alpha + \beta |u_n(t)|^\nu) [-p^+ j(t, u_n) - j^0(t, u_n; -u_n)] dt \end{aligned}$$

$$\begin{aligned}
&\leq C_2 + \int_0^T (\alpha + \beta |u_n(t)|^v) [-p^+ j(t, u_n) - j^0(t, u_n; -u_n)] dt \\
&\leq C_2 + (\alpha + \beta \|u_n\|_\infty^v) \int_0^T [-p^+ j(t, u_n) - j^0(t, u_n; -u_n)] dt \\
&\leq C_2 + C_1 (\alpha + \beta \|u_n\|_\infty^v) \\
&\leq C_2 + C_1 (\alpha + \beta \|u_n\|_\infty^v) \\
&\leq C_2 + C_1 (\alpha + C^v \beta |u'_n|_{p(t)}^v), \tag{3.18}
\end{aligned}$$

which yields that $|u'_n|_{p(t)}$ is bounded since $v < p^-$.

We claim that $\|u_n\|$ is bounded. If not, suppose that the sequence $\|u_n\|$ is not bounded. Note that $\|u_n\| = (|u_n|_{p(t)} + |u'_n|_{p(t)}) \rightarrow \infty$ and $|u'_n|_{p(t)}$ is bounded, we have $|u_n|_{p(t)} \rightarrow \infty$. From Proposition 3.1 of [35], we have $|u_n(t)| \rightarrow \infty$ as $n \rightarrow \infty$ uniformly for $t \in [0, T]$. By virtue of $H(j)_4$ (vi), there exists constants $c_1, M_2 > 0$ such that

$$\frac{j(t, u(t))}{|u|^{q(t)}} \geq c_1 > 0$$

as $|u| \geq M_2$. Namely,

$$j(t, u(t)) \geq c_1 |u|^{q(t)}. \tag{3.19}$$

Then

$$\lim_{n \rightarrow \infty} \int_0^T j(t, u_n(t)) dt = +\infty. \tag{3.20}$$

However

$$\left| \int_0^T j(t, u_n(t)) dt \right| = \left| \int_0^T \frac{1}{p(t)} |u_n(t)|^{p(t)} dt - \varphi(u_n(t)) \right| \leq \frac{1}{p^-} \rho(u'_n) + |\varphi(u_n(t))|$$

is bounded, which is contrary to (3.20). Thus $\{u_n\}$ is a bounded sequence in $W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ and so we may assume that $u_n \rightharpoonup u$ in $W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$.

Next, we prove that $u_n \rightarrow u$ in $W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$. From Proposition 2.13, it suffices to prove that

$$\overline{\lim}_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle \leq 0, \quad \varepsilon_n \downarrow 0.$$

Indeed, from the choice of the sequence $\{u_n\}_{n \geq 1}$, we have

$$|\langle u_n^*, u_n \rangle| \leq \varepsilon_n \downarrow 0.$$

Using $u_n^* = A(u_n) - w_n$, we have

$$\langle A(u_n), u_n - u \rangle - \int_0^T (w_n(t), (u_n(t) - u(t)))_{\mathbb{R}^N} dt \leq \varepsilon_n, \quad \forall n \geq 1.$$

Since $\{w_n\}_{n \geq 1} \subseteq L^1[0, T]$ is bounded and

$$\int_0^T (w_n(t), (u_n(t) - u(t))) dt \rightarrow 0 (n \rightarrow \infty).$$

Then

$$\overline{\lim}_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle \leq 0.$$

Step 2: φ satisfies nonsmooth Mountain Pass Theorem.

Let $\varepsilon > 0$ be small enough, by virtue of hypotheses $H(j)_4$ (iii), (v) we have

$$j(t, u) \leq \varepsilon |u|^{p(t)} + c(\varepsilon) |u|^{\alpha(t)}, \quad \forall (t, u) \in [0, T] \times \mathbb{R}^N.$$

Let $|u|_{p(t)} = \rho$ be small enough, then

$$\begin{aligned} \varphi(u) &\geq \frac{1}{p^+} \int_0^T |u'(t)|^{p(t)} dt \\ &\quad - \varepsilon \int_0^T |u(t)|^{p(t)} dt - c(\varepsilon) \int_0^T |u(t)|^{\alpha(t)} dt \\ &\geq \frac{1}{p^+} |u|_{p(t)}^{p^+} - \varepsilon \int_0^T |u'|^{p(t)} dt - c(\varepsilon) \int_0^T |u|^{\alpha(t)} dt \\ &\geq \frac{1}{p^+} |u|_{p(t)}^{p^+} - \varepsilon |u|_{p(t)}^{p^+} - c(\varepsilon) |u|_{p(t)}^{\alpha^-} \end{aligned}$$

For the arbitrariness of ε , we may choose ε small enough such that $\frac{\varepsilon}{a_0} < \frac{1}{2p^+}$, then

$$\varphi(u) \geq \frac{1}{2p^+} |u|_{p(t)}^{p^+} - c(\varepsilon) |u|_{p(t)}^{\alpha^-}.$$

Since $p^+ < \alpha^-$, there exist a constant $r > 0$ such that $\varphi(u) \geq r$ as $\|u\| = \rho$ small enough.

By virtue of $H(j)_4$ (vi), there exists constants $c_1, M_2 > 0$ such that

$$\frac{j(t, u(t))}{|u|^{q(t)}} \geq c_1 > 0$$

as $|u| \geq M_2$. Namely,

$$j(t, u(t)) \geq c_1 |u|^{q(t)}. \quad (3.21)$$

By $H(j)_4$ (iii), we have

$$|j(t, u(t))| \leq c_2 b(t), \quad \forall t \in [0, T], \quad |u| \leq M_2. \quad (3.22)$$

Then, for every $u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N) \setminus \{0\}$ and $\sigma > 1$ we have

$$\begin{aligned} \varphi(\sigma u) &= \int_0^T \frac{1}{p(t)} |\sigma u'|^{p(t)} dt - \int_0^T j(t, \sigma u) dt \\ &\leq \frac{\sigma^{p^+}}{p^-} \int_0^T |u'|^{p(t)} dt - c_1 \sigma^{q^-} \int_0^T |u|^{q(t)} dt + c_2 \int_0^T b(t) dt, \end{aligned}$$

which implies that $\varphi(\sigma u) \rightarrow -\infty$ ($\sigma \rightarrow +\infty$) since $q^- > p^+$.

As before, we can show that $u \in W_{\text{per}}^{1,p(t)}(0, T; \mathbb{R}^N)$ solves (1.3), we omit its proof process.

Acknowledgement: This work is partially supported by the NSFC (No:11871305) and supported by Foundation of Hubei Educational Committee (D20161206). The authors would like to express their gratitude to the referees for valuable comments and suggestions.

References

- [1] S. N. Antontsev, S. I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: Existence, uniqueness and localization properties of solutions, *Nonlinear Anal.* **60**(2005), 515-545.

- [2] S. N. Antontsev, J. F. Rodrigues, On stationary thermo-rheological viscous flows, *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **52**(2006), 19-36.
- [3] K. C. Chang, Variational methods for nondifferentiable functions and their applications to partial differential equations, *J. Math. Anal. Appl.* **80**(1981), no. 1, 102-129.
- [4] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* **66**(2006), no. 4, 1383-1406.
- [5] F. H. Clarke, *Optimization and Nonsmooth Analysis*. Wiley, New York, 1993.
- [6] G. Dai, Infinitely many solutions for a Neumann-type differential inclusion problem involving the $p(x)$ -Laplacian, *Nonlinear Anal.* **70**(2009), 2297-2305.
- [7] G. Dai, Infinitely many solutions for a hemivariational inequality involving the $p(x)$ -Laplacian, *Nonlinear Anal.* **71**(2009), 186-195.
- [8] G. Dai, Three solutions for a Neumann-type differential inclusion problem involving the $p(x)$ -Laplacian, *Nonlinear Anal.* **70**(2009), 3755-3760.
- [9] G. Dai, Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$ involving the $p(x)$ -Laplacian, *Nonlinear Anal.* **71**(2009), 1116-1123.
- [10] H. Dang, S. F. Oppenheimer, Existence and uniqueness results for some nonlinear boundary value problems, *J. Math. Anal. Appl.* **198**(1996), 35-48.
- [11] M. Del Pino, R. Manasevich, A. Murua, Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE, *Nonlinear Anal.* **18**(1992), 79-92.
- [12] T. L. Dinu, Entire solutions of multivalued nonlinear Schrödinger equations in Sobolev spaces with variable exponent, *Nonlinear Anal.* **65**(2006), 1414-1424.
- [13] X. L. Fan, X. Fan, A Knobloch-type result for $p(t)$ -Laplacian systems, *J. Math. Anal. Appl.* **282**(2003), 453-464.
- [14] X. L. Fan, D. Zhao, On the space $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263**(2001), 424-446.
- [15] M. E. Filippakis, Existence and multiplicity results for nonlinear nonautonomous second-order systems, *Nonlinear Anal.* **68**(2008), 1611-1626.
- [16] M. Filippakis, L. Gasinski, N. S. Papageorgiou, On the existence of positive solutions for Hemivariational inequalities driven by the p -Laplacian, *J. Global Optimization*. **31**(2005), 173-189.
- [17] L. Gasinski, N. S. Papageorgiou, On the existence of multiple periodic solutions for equations driven by the p -Laplacian and a nonsmooth potential, *Proc. Edinb. Math. Soc.* **46**(2003), 229-249.
- [18] L. Gasinski, N. S. Papageorgiou, Existence and multiplicity solutions for second order periodic systems with the p -Laplacian and a nonsmooth potential, *Monatsheft für Mathematik*, **158** (2009), no. 2, 121-150.
- [19] S. Hu, N. S. Papageorgiou, Solutions and multiple solutions for problems with the p -Laplacian, *Monatshefte für Mathematik*. **150**(2007), no. 4, 309-326.
- [20] A. Iannizzoto, N. S. Papageorgiou, Existence of three nontrivial solutions for nonlinear Neumann Hemivariational inequalities. *Nonlinear Anal.* **70**(2009), no. 9, 3285-3297.
- [21] N. C. Kourageorou, N. S. Papageorgiou, Nonsmooth Critical Point Theory and Nonlinear Elliptic Equations at Resonance, *Journal of the Australian Mathematical Society (Series A)*, **69** (2000), no. 2, 245-271.
- [22] N. C. Kouragenis, N. S. Papageorgiou, Nonsmooth critical point theory and nonlinear elliptic equations at resonance, *J. Aust. Math. Soc.* **69**(2000), 245-271.
- [23] P. Lindqvist, Notes on the p -Laplacian equation, 2017.
- [24] J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, in: *Appl. Math. Sci.*, vol. 74, Springer, New York, 1989.
- [25] S. Marano, D. Motreanu, Infinitely many critical points of non-differentiable functions and applications to Neumann-type problem involving the p -Laplacian, *J. Differential Equations*, **182**(2002), 108-135.
- [26] E. Papageorgiou, N. S. Papageorgiou, Two nontrivial solutions for quasilinear periodic problems, *Proc. Amer. Math. Soc.* **132**(2004), 429-434.
- [27] N. S. Papageorgiou, S. K. Yiallourou, *Handbook of Applied Analysis*. New York: Springer Verlag, 2009.
- [28] E. H. Papageorgiou, N. S. Papageorgiou, Existence of Solutions and of Multiple Solutions for Nonlinear Nonsmooth Periodic Systems, *Czechoslovak Mathematical Journal*. **54**(2004), no. 129, 347-371.
- [29] N. S. Papageorgiou, V. Staicu, Three nontrivial solutions for the p -Laplacian with a nonsmooth potential, *Nonlinear Anal.* **68**(2008), no. 12, 3812-3827.
- [30] N. S. Papageorgiou, V. D. Rădulescu, D. D. Repovš, Periodic solutions for a class of evolution inclusions, *Comput. Math. Appl.* **75**(2018), 3047-3065.
- [31] E. H. Papageorgiou, N. S. Papageorgiou, Nonlinear Second-order Periodic Systems with non-smooth Potential, *Proc. Indian Acad. Sci. (Math. Sci.)* **114**(2004), no. 3, 269-298.
- [32] C. Y. Qian, Z. F. Shen, M. B. Yang, Existence of Solutions for $p(x)$ -Laplacian Nonhomogeneous Neumann Problems with Indefinite Weight, *Nonlinear Anal.: RWA*. **11**(2010), no. 1, 446-458.
- [33] V. D. Rădulescu, Nonlinear elliptic equations with variable exponent: Old and new, *Nonlinear Anal.* **121**(2015), 336-369.
- [34] S. Samko, On a progress in the theory of Lebesgue spaces with variable exponent maximal and singular operators, *Integral Transforms Spec. Funct.* **16**(2005), 461-482.

- [35] C. L. Tang, X. P. Wu, Periodic Solutions for Second Order Systems with Not Uniformly Coercive Potential, *J. Math. Anal. Appl.*, **259**(2011), no. 2, 386-397.
- [36] K. Teng, X. Wu, Existence and subharmonicity of solutions for nonlinear non-smooth periodic systems with a p -Laplacian, *Nonlinear Anal.* **68**(2008), 3742-3756.
- [37] X. Wang, R. Yuan, Existence of periodic solutions for $p(t)$ -Laplacian systems, *Nonlinear Anal.* **70**(2009), 866-880.
- [38] X. Yang, Multiple periodic solutions of a class of p -Laplacian, *J. Math. Anal. Appl.* **314**(2006), 17-29.
- [39] Q. H. Zhang, Existence of radial solutions for $p(x)$ -Laplacian equations in $\mathbb{R}^N$, *J. Math. Anal. Appl.* **315**(2006), 506-516.
- [40] V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994; translated from the Russian by G.A. Yosifian.
- [41] V. V. Zhikov, On some variational problems, *Russ. J. Math. Phys.* **5**(1997), 105-116.
- [42] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.* **9**(1987), 33-66.