Research article

Ky Ho and Yun-Ho Kim*

The concentration-compactness principles for $W^{S,p(\cdot,\cdot)}(\mathbb{R}^N)$ and application

https://doi.org/10.1515/anona-2020-0160
Received January 19, 2020; accepted October 16, 2020.

Abstract: We obtain a critical imbedding and then, concentration-compactness principles for fractional Sobolev spaces with variable exponents. As an application of these results, we obtain the existence of many solutions for a class of critical nonlocal problems with variable exponents, which is even new for constant exponent case.

Keywords: Fractional $p$-Laplacian; $p(\cdot)$-Laplacian; fractional Sobolev spaces with variable exponents; critical imbedding; concentration-compactness principles; variational methods

MSC: 35B33, 35D30, 35J20, 35R11, 46E35, 49J35

1 Introduction

Nonlocal equations have been modeled for various problems in real fields, for instance, phase transitions, thin obstacle problem, soft thin films, crystal dislocation, stratified materials, anomalous diffusion, semipermeable membranes and flame propagation, material science, ultra-relativistic limits of quantum mechanics, multiple scattering, minimal surfaces, water waves, etc. After the seminal papers by Caffarelli et al. [1–3], problems involving fractional $p$-Laplacian have been intensively studied. On the other hand, various other real fields such as electrorheological fluids and image processing, etc. require partial differential equations with variable exponents (see e.g., [4, 5]). Natural solution spaces for those problems are Sobolev spaces with fractional order or variable exponents, which were comprehensively investigated in [6] and [7].

Recently, many authors have been studied the fractional Sobolev spaces with variable exponents and the corresponding nonlocal equations with variable exponents (see e.g., [8–11]). To the authors’ best knowledge, though most properties of the classical fractional Sobolev spaces have been extended to the fractional Sobolev spaces with variable exponents, there have no results for the critical Sobolev type imbedding for these spaces. Consequently, there have no results on nonlocal equations with variable critical growth because the critical Sobolev type imbedding is essential in the study of such critical equations. The critical problem was initially studied in the seminal paper by Brezis-Nirenberg [12], which treated for Laplace equations. Since then there have been extensions of [12] in many directions. Elliptic equations involving critical growth are delicate due to the lack of compactness arising in connection with the variational approach. For such problems, the concentration-compactness principles (the CCPs, for short) introduced by P.L. Lions [13, 14] and its variant at infinity [15–17] have played a decisive role in showing a minimizing sequence or a Palais-Smale sequence is precompact. By using these CCPs or extending them to the Sobolev spaces with fractional order or vari-
able exponents, many authors have been successful to deal with critical problems involving \( p \)-Laplacian or \( p(\cdot) \)-Laplacian or fractional \( p \)-Laplacian, see e.g., [18–27] and references therein.

As we mentioned above, there have no results for the critical Sobolev type imbedding for the fractional Sobolev spaces with variable exponents. Although the usual critical Sobolev immersion theorem holds in the fractional order or variable exponents setting, we do not know this assertion even in fractional Sobolev spaces with variable exponents defined in bounded domain; see [8–11]. Because of this, our first aim of the present paper is to obtain a critical imbedding from fractional Sobolev spaces with variable exponents into spaces with variable exponents defined in bounded domain; see [8–11]. Thanks to this result, we then establish two Lions type concentration-compactness principles for fractional Sobolev spaces with variable exponents, which are our second aim (Theorems 3.3). As we mentioned above, there have no results for the critical Sobolev type imbedding for the fractional \(-\)Laplacian or variable exponents setting, we do not know this assertion even in fractional Sobolev spaces with variable exponents. Although the usual critical Sobolev immersion theorem holds in the case of constant exponents.

In this section, we briefly review the Lebesgue spaces with variable exponents and the classical fractional Sobolev spaces. In Section 2, we briefly review some properties of the Sobolev spaces with fractional order or variable exponents. In Section 3, we establish a critical Sobolev type imbedding for the fractional Sobolev spaces with variable exponents, which is a key to our arguments. In Section 4 we establish Lions type concentration-compactness principles for fractional Sobolev spaces with variable exponents. In Section 5, we show the existence of many solutions for a superlinear nonlocal problem with variable exponents using genus theory. In Appendix, we give an auxiliary result, which is used to prove our CCPs.

2 Variable exponent Lebesgue spaces and fractional Sobolev spaces

In this section, we briefly review the Lebesgue spaces with variable exponents and the classical fractional Sobolev spaces.

Let \( \Omega \) be a Lipschitz domain in \( \mathbb{R}^N \). Denote

\[
C_+ (\Omega) = \left\{ h \in C(\overline{\Omega}) : 1 < \inf_{x \in \Omega} h(x) \leq \sup_{x \in \Omega} h(x) < \infty \right\},
\]

and for \( h \in C_+ (\Omega) \), denote

\[
h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).
\]
For $p \in C_0(\Omega)$ and a $\sigma$-finite, complete measure $\mu$ in $\Omega$, define the variable exponent Lebesgue space $L^p_\mu(\Omega)$ as

$$L^p_\mu(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ is } \mu \text{-measurable, } \int_\Omega |u(x)|^p(x) \, d\mu < \infty \right\}$$

endowed with the Luxemburg norm

$$|u|_{L^p_\mu(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \frac{|u(x)|^p(x)}{\lambda} \, d\mu \leq 1 \right\}.$$

When $\mu$ is the Lebesgue measure, we write $d\mu$, $L^p(\Omega)$ and $|u|_{L^p(\Omega)}$ instead of $d\mu$, $L^p_\mu(\Omega)$ and $|u|_{L^p_\mu(\Omega)}$, respectively. Set $L^\infty_\mu(\Omega) := \left\{ u \in L^\infty(\Omega) : u > 0 \text{ a.e. in } \Omega \right\}$ and for a Lebesgue measurable and positive a.e. function $w : \Omega \to \mathbb{R}$, set $L^p(\Omega) := L^p(\omega)$ with $d\mu = w(x) \, dx$. Some basic properties of $L^p_\mu(\Omega)$ are listed in the next three propositions.

**Proposition 2.1** ([7, Corollary 3.3.4]). Let $\alpha, \beta \in C_0(\Omega)$ such that $\alpha(x) \leq \beta(x)$ for all $x \in \Omega$. Then, we have

$$|u|_{L^p_\mu(\Omega)} \leq 2 \left[ 1 + \mu(\Omega) \right] |u|_{L^\infty_\mu(\Omega)}, \forall u \in L^p_\mu(\Omega) \cap L^\infty_\mu(\Omega).$$

**Proposition 2.2** ([30]). Define the modular $\rho : L^p_\mu(\Omega) \to \mathbb{R}$ as

$$\rho(u) := \int_\Omega |u|^p(x) \, d\mu, \forall u \in L^p(\Omega).$$

Then, we have the following relations between the norm and modular.

(i) For $u \in L^p_\mu(\Omega) \setminus \{0\}$, $\lambda = |u|_{L^p_\mu(\Omega)}$ if and only if $\rho(\frac{u}{\lambda}) = 1$.
(ii) $\rho(u) > 1$ ($= 1; < 1$) if and only if $|u|_{L^p_\mu(\Omega)} > 1$ ($= 1; < 1$), respectively.
(iii) If $|u|_{L^p_\mu(\Omega)} > 1$, then $|u|_{L^p_\mu(\Omega)} \leq \rho(u) \leq |u|_{L^p_\mu(\Omega)}^\prime$.
(iv) If $|u|_{L^p_\mu(\Omega)} < 1$, then $|u|_{L^p_\mu(\Omega)} \leq \rho(u) \leq |u|_{L^p_\mu(\Omega)}^\prime$.

**Proposition 2.3** ([30, 31]). The space $L^p(\Omega)$ is a separable, uniformly convex Banach space, and its dual space is $L^{p'}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, we have

$$\left| \int_\Omega uv \, dx \right| \leq 2 |u|_{L^p(\Omega)} |v|_{L^{p'}(\Omega)}.$$

Let $s \in (0, 1)$ and $p \in (1, \infty)$ be constants. Define the fractional Sobolev space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty \right\}$$

endowed with norm

$$\|u\|_{s,p,\Omega} := \left( \int_\Omega |u(x)|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{1/p}.$$

We recall the following crucial imbeddings.

**Proposition 2.4** ([6]). Let $s \in (0, 1)$ and $p \in (1, \infty)$ be such that $sp < N$. It holds that

(i) $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $\Omega$ is bounded and $1 \leq q < \frac{Np}{N-sp} := p_1^*$;
(ii) $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $p \leq q \leq p_1^*$. 
3 The Sobolev spaces \( W^{s,p(\cdot,\cdot)}(\Omega) \)

In this section, we recall the fractional Sobolev spaces with variable exponents that was first introduced in [11], and was then refined in [10]. Furthermore, we will obtain a critical Sobolev type imbedding on these spaces.

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \) or \( \Omega = \mathbb{R}^N \). Throughout this article, we assume that

\[(p_1) \quad s \in (0, 1); \ p = C(\overline{\Omega} \times \overline{\Omega}) \]

is uniformly continuous and symmetric such that

\[ 1 < p^+ := \inf_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x, y) \leq p^* := \sup_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x, y) < \frac{N}{s}. \]

In the following, we write \( p(x) \) instead of \( p(x, x) \) and with this notation, \( p \in C(\overline{\Omega}) \). Define

\[ W^{s,p(\cdot,\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{Ns p(x,y)}} \, dx \, dy < +\infty \right\} \]

endowed with the norm

\[ \|u\|_{s,p,\Omega} := \inf \left\{ \lambda > 0 : M_{\Omega} \left( \frac{u}{\lambda} \right) < 1 \right\}, \]

where \( M_{\Omega}(u) := \int_{\Omega} |u|^{p(x)} \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{Ns p(x,y)}} \, dx \, dy \). Then, \( W^{s,p(\cdot,\cdot)}(\Omega) \) is a separable reflexive Banach space (see [8, 9, 11]). On \( W^{s,p(\cdot,\cdot)}(\Omega) \), we also make use of the following norm

\[ \|u\|_{s,p,\Omega} := \|u\|_{L^{p(\cdot)}(\Omega)} + [u]_{s,p,\Omega}, \]

where

\[ [u]_{s,p,\Omega} := \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{Ns p(x,y)}} \, dx \, dy < 1 \right\}. \]

Note that \( \| \cdot \|_{s,p,\Omega} \) and \( [ \cdot ]_{s,p,\Omega} \) are equivalent norms on \( W^{s,p(\cdot,\cdot)}(\Omega) \) with the relation

\[ \frac{1}{2} \|u\|_{s,p,\Omega} \leq [u]_{s,p,\Omega} \leq 2 \|u\|_{s,p,\Omega} \quad \forall u \in W^{s,p(\cdot,\cdot)}(\Omega). \] (3.1)

In what follows, when \( \Omega \) is understood, we just write \( \| \cdot \|_{s,p} \), \( [ \cdot ]_{s,p} \) and \( [ \cdot ]_{s,p,\Omega} \) instead of \( \| \cdot \|_{s,p,\Omega} \), \( [ \cdot ]_{s,p,\Omega} \) and \( [ \cdot ]_{s,p,\Omega} \), respectively. We also denote the ball in \( \mathbb{R}^N \) centered at \( z \) with radius \( \varepsilon \) by \( B_{\varepsilon}(z) \) and denote the Lebesgue measure of a set \( E \subset \mathbb{R}^N \) by \( |E| \). For brevity, we write \( B_{\varepsilon} \) and \( B_{\varepsilon}^c \) instead of \( B_{\varepsilon}(0) \) and \( \mathbb{R}^N \setminus B_{\varepsilon}(0) \), respectively.

**Proposition 3.1 ([10]).** On \( W^{s,p(\cdot,\cdot)}(\Omega) \) it holds that

(i) for \( u \in W^{s,p(\cdot,\cdot)}(\Omega) \setminus \{0\} \), \( \lambda = \|u\|_{s,p} \) if and only if \( M_{\Omega}(\frac{u}{\lambda}) = 1 \);

(ii) \( M_{\Omega}(u) > 1 (= 1; < 1) \) if and only if \( \|u\|_{s,p} > 1 (= 1; < 1) \), respectively;

(iii) if \( \|u\|_{s,p} \geq 1 \), then \( u \|_{s,p} \leq M_{\Omega}(u) \leq [u]_{s,p}^{p^*} ; \\
(iv) \) if \( \|u\|_{s,p} < 1 \), then \( u \|_{s,p}^{p^*} \leq M_{\Omega}(u) \leq [u]_{s,p}^{p^*} . \)

**Theorem 3.2** (Subcritical imbeddings, [10]). It holds that

(i) \( W^{s,p(\cdot,\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \), if \( \Omega \) is a bounded Lipschitz domain and \( r \in C_{+}(\Omega) \) such that \( r(x) < \frac{N(p(x))}{Ns p(x)} =: p_0^{s}(x) \) for all \( x \in \overline{\Omega} \);

(ii) \( W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow L^{r(\cdot)}(\mathbb{R}^N) \) for any uniformly continuous function \( r \in C_{+}(\mathbb{R}^N) \) satisfying \( p(x) \leq r(x) \) for all \( x \in \mathbb{R}^N \) and \( \inf_{x \in \mathbb{R}^N} (p_0^{s}(x) - r(x)) > 0 ; \\
(iii) \) \( W^{s,p(\cdot,\cdot)}(\mathbb{R}^N) \hookrightarrow L^{4/_{\Omega}}_{\text{loc}}(\mathbb{R}^N) \) for any \( r \in C_{+}(\mathbb{R}^N) \) satisfying \( r(x) \leq p_0^{s}(x) \) for all \( x \in \mathbb{R}^N \).

The next critical imbedding is our first main result.
Theorem 3.3 (Critical imbedding). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \) or \( \Omega = \mathbb{R}^N \). Let \((p_1)\) hold. Furthermore, let the variable exponent \( p \) satisfy the following log-Hölder type continuity condition

\[
\inf_{\varepsilon > 0} \sup_{(x, y) \in \Omega \times \Omega_{\varepsilon}} \left| p(x, y) - p_{\Omega_{\varepsilon}} \right| \log \frac{1}{|x - y|} < \infty, \tag{3.2}
\]

where \( \Omega_{\varepsilon} := B_\varepsilon(z) \cap \Omega \) for \( z \in \Omega \) and \( \varepsilon > 0 \), and \( p_{\Omega_{\varepsilon}} := \inf_{(x', y') \in \Omega_{\varepsilon} \times \Omega_{\varepsilon}} p(x', y') \). Let \( q : \mathbb{R} \to \mathbb{R} \) be a function satisfying

\[
(\Omega_1) \quad q \in C_c(\overline{\Omega}) \text{ such that for any } x \in \Omega, \text{ there exists } \varepsilon = \varepsilon(x) > 0 \text{ such that the following locally critical growth condition holds:}
\]

\[
\sup_{y \in \Omega_{\varepsilon}} q(y) \leq N \inf_{(y, z) \in \Omega_{\varepsilon} \times \Omega_{\varepsilon}} p(y, z), \tag{3.3}
\]

In addition, when \( \Omega = \mathbb{R}^N \), \( q \) is uniformly continuous and \( p(x) < q(x) \) for all \( x \in \mathbb{R}^N \).

Then, it holds that

\[
W^{s, p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega). \tag{3.4}
\]

Proof. By the closed graph theorem, to prove (3.4) it suffices to show that \( W^{s, p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega) \). Let \( u \in W^{s, p(\cdot)}(\Omega) \setminus \{0\} \) be arbitrary and fixed. We will show that \( u \in L^{q(\cdot)}(\Omega) \), namely,

\[
\int_{\Omega} |u|^{q(x)} \, dx < \infty. \tag{3.5}
\]

To this end, we first note that by (3.2), there exists constant \( \varepsilon_0 \in (0, 1) \) such that

\[
\sup_{(x, y) \in \Omega \times \Omega_{\varepsilon_0}} \left| p(x, y) - p_{\Omega_{\varepsilon_0}} \right| \log \frac{1}{|x - y|} < C. \tag{3.6}
\]

Here and in the remainder of the proof, \( C \) denotes a positive constant independent of \( u \) and may vary from line to line. We consider the following two cases.

Case 1: \( \Omega \) is a bounded Lipschitz domain.

We cover \( \Omega \) by \( \{B_{\varepsilon_i}(x_i)\}_{i=1}^m \) with \( x_i \in \Omega \) and \( \varepsilon_i \in (0, \varepsilon_0) \) such that \( \Omega_i := B_{\varepsilon_i}(x_i) \cap \Omega \) being Lipschitz domains and the locally critical growth condition (3.3) being satisfied for all \( i \in \{1, \cdots, m\} \). Fix \( i \in \{1, \cdots, m\} \) and denote \( p_i := \inf_{(y, z) \in \Omega_i \times \Omega_i} p(y, z) \) and \( q_i := \sup_{x \in \Omega_i} q(x) \). By (3.3) and the choice of \( \varepsilon_i \), we have

\[
q_i \leq \frac{Np_i}{N - sp_i} =: p_i^*. \tag{3.7}
\]

From this and Proposition 2.4, we have

\[
\int_{\Omega_i} |u|^{q_i} \, dx \leq C \left[ \int_{\Omega_i} |u|^{p_i} \, dx + \int_{\Omega_i} \int_{\Omega_i} \frac{|u(x) - u(y)|^{p_i}}{|x - y|^{N + sp_i}} \, dy \, dx \right]^{\frac{q_i}{p_i^*}}, \tag{3.8}
\]

and hence,

\[
\int_{\Omega_i} |u|^{q_i} \, dx - |\Omega_i| \leq C \left[ |\Omega_i| + \int_{\Omega_i} |u|^{p_i} \, dx + \int_{\Omega_i} \int_{\Omega_i} \frac{|u(x) - u(y)|^{p_i}}{|x - y|^{N + sp_i}} \, dy \, dx \right]^{\frac{q_i}{p_i^*}}. \tag{3.9}
\]

On the other hand, we have

\[
\int_{\Omega_i} \int_{\Omega_i} \frac{|u(x) - u(y)|^{p(x, y)}}{|x - y|^{N + sp(x, y)}} \, dy \, dx = \int_{\Omega_i} \int_{\Omega_i} \frac{|u(x) - u(y)|^{p(x, y)}}{|x - y|^{2s}} \frac{1}{|x - y|^{N - sp_i}} \frac{1}{|x - y|^{p(x, y) - p_i} s} \, dy \, dx. \tag{3.10}
\]
Note that by (3.6), we have
\[ |x - y|^{-\delta(p(x,y) - p_1)} = e^{-\delta(p(x,y) - p_1) \log |x - y|} \leq C, \quad \forall x, y \in \Omega, x \neq y. \] (3.9)

Thus, (3.8) yields
\[ \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy \geq C \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{2s}} \, dx \, dy \]
\[ \geq C \int_\Omega \int_\Omega \left( \frac{|u(x) - u(y)|^{p_1}}{|x - y|^{N + sp_1}} - 1 \right) \frac{1}{|x - y|^{N - sp_1}} \, dx \, dy. \] (3.10)

Hence,
\[ \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy \]
\[ \geq C \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p_1}}{|x - y|^{N + sp_1}} \, dx \, dy - C \int_\Omega \int_\Omega \frac{1}{|x - y|^{N - sp_1}} \, dx \, dy. \] (3.11)

We have
\[ \int_\Omega \int_\Omega \frac{1}{|x - y|^{N - sp_1}} \, dx \, dy \leq \int_\Omega \, dy \int_{B_2} \frac{dz}{|z|^{N - sp_1}} = |\Omega| \frac{N|B_1|^{2sp_1}}{sp_1}. \] (3.12)

From (3.7), (3.10) and (3.11), we obtain
\[ \int_\Omega |u|^{p_i(x)} \, dx \leq C \left[ 1 + \int_\Omega |u|^{p(x)} \, dx + \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy \right] \frac{n_i}{p_i} \]
\[ \leq C \left[ 1 + \int_\Omega |u|^{p(x)} \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy \right] \frac{n_i}{p_i}. \]

Summing up for \( i = 1, \ldots, m \), we arrive at
\[ \int_\Omega |u|^{p(x)} \, dx \leq C \left[ 1 + \int_\Omega |u|^{p(x)} \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy \right] \sum_i \frac{n_i}{p_i} \]
\[ < \infty, \]
and so (3.5) is claimed.

**Case 2: \( \Omega = \mathbb{R}^N \)**

Decompose \( \mathbb{R}^N \) by cubes \( \{Q_i\}_{i \in \mathbb{N}} \) with sides of length \( \varepsilon \in (0, 1) \) and parallel to coordinates axes. By the locally critical growth condition (3.3) and the uniform continuity of \( q \) we can choose \( \varepsilon > 0 \) sufficiently small such that
\[ p_i \leq p^*_i \leq q_i \leq q_i \leq \frac{Np_i}{N - sp_i} =: p_{s,i}, \quad \forall i \in \mathbb{N}, \] (3.13)

where
\[ p_i := \inf_{(y,z) \in Q_i \times Q_i} p(y,z), \quad p^*_i := \sup_{(y,z) \in Q_i \times Q_i} p(y,z), \quad q_i := \inf_{x \in Q_i} q(x), \quad \text{and} \quad q_i := \sup_{x \in Q_i} q(x). \]

Set \( v = \frac{u}{\|u\|_{s,p}} \). Thus, \( \|v\|_{s,p} = 1 \) and hence, \( M_{R^N}(v) = 1 \) in view of Proposition 3.1. This yields \( M_{Q_i}(v) \leq 1 \) for all \( i \in \mathbb{N} \) and hence,
\[ \|v\|_{s,p,Q_i} \leq 1, \quad \forall i \in \mathbb{N}. \] (3.14)

We claim that
\[ \|v\|_{L^{p_i}(Q_i)} \leq C \|v\|_{s,p,Q_i}, \quad \forall i \in \mathbb{N}. \] (3.15)
Indeed, let \( i \in \mathbb{N} \) and consider the measure \( \mu \) on \( \mathbb{R}^N \times \mathbb{R}^N \) such that

\[
d\mu(x,y) = \frac{dx \ dy}{|x-y|^{N-2s}}.
\]

As in (3.11) we have

\[
\mu(Q_i \times Q_i) \leq |Q_i| \left| \frac{N|B_1|^{2sp_1}}{sp_1} \right| \left| \frac{N|B_1|^{2sp'}}{sp'} \right|, \quad \forall i \in \mathbb{N}.
\]

(3.16)

Set \( \lambda := [v]_{s,p,Q_i} \) and \( F(x,y) := \frac{|v(x) - v(y)|}{|x-y|^{2s}} \). Invoking Proposition 3.1 and (3.9) we estimate

\[
1 = \int_{Q_i} \int_{Q_i} \frac{|v(x) - v(y)|^{p(x,y)} P(x,y)}{|x-y|^{2s + p(x,y)}} \ dx \ dy
\]

\[
= \int_{Q_i} \int_{Q_i} \frac{|v(x) - v(y)|^{p(x,y)}}{|x-y|^{2s + sp(x,y)}} \ dx \ dy
\]

\[
\geq (C + 1)^{-1} \int_{Q_i} |F(x,y)|^{p(x,y)} \ dx \ dy
\]

\[
= (C + 1)^{-1} \int_{Q_i} |F(x,y)|^{p(x,y)} \ dx \ dy
\]

Thus,

\[
|F|_{L_p^\lambda(Q_i \times Q_i)} \leq (C + 1)^{\frac{1}{p}} \lambda = (C + 1)^{\frac{1}{p}} [v]_{s,p,Q_i}.
\]

Meanwhile, invoking Proposition 2.1 we have

\[
|F|_{L_p^\lambda(Q_i \times Q_i)} \leq 2(1 + \mu(Q_i \times Q_i)|F|_{L_p^\lambda(Q_i \times Q_i)}.
\]

Combining the last two inequalities and (3.16) we obtain

\[
|F|_{L_p^\lambda(Q_i \times Q_i)} \leq C[v]_{s,p,Q_i}.
\]

(3.17)

Noting

\[
|F|_{L_p^\lambda(Q_i \times Q_i)} = \left( \int_{Q_i} \int_{Q_i} \frac{|v(x) - v(y)|^{p(x,y)}}{|x-y|^{2s + sp(x,y)}} \ dx \ dy \right)^{\frac{1}{p}}
\]

\[
= \left( \int_{Q_i} \int_{Q_i} \frac{|v(x) - v(y)|^{p(x)}}{|x-y|^{2s + sp(x)}} \ dx \ dy \right)^{\frac{1}{p}}
\]

\[
= [v]_{s,p,Q_i},
\]

we deduce from (3.17) that

\[
[v]_{s,p,Q_i} \leq C[v]_{s,p,Q_i}.
\]

(3.18)

Combining (3.18) with the following estimate:

\[
|v|_{L_p^\lambda(Q_i)} \leq 2(1 + |Q_i||v|_{L_p^\lambda(Q_i)} \leq 4|v|_{L_p^\lambda(Q_i)}
\]

(see Proposition 2.1) and the relation (3.1), we obtain (3.15).

As in [10, Proof of Theorem 3.5], we can obtain an extension \( \tilde{v} \in W^{s,p}(\mathbb{R}^N) \) with compact support in \( \mathbb{R}^N \) such that \( \tilde{v} = v \) on \( Q_i \), and

\[
|\tilde{v}|_{L_p^\lambda(\mathbb{R}^N)} \leq C|v|_{s,p,Q_i}.
\]

This and (3.12) yield

\[
|v|_{L_p^\lambda(Q_i)} \leq C|v|_{s,p,Q_i}.
\]

(3.19)

Note that by Proposition 2.1 again,

\[
|v|_{L_p^\lambda(Q_i)} \leq 2(1 + |Q_i||v|_{L_p^\lambda(Q_i)} \leq 4|v|_{L_p^\lambda(Q_i)}.
\]

(3.20)
From (3.15), (3.19), and (3.20) we obtain (3.14). Now, for each $i \in \mathbb{N}$, if $|v|_{L^p(Q_i)} \geq 1$, then by (3.13), (3.14) and Proposition 3.1 we have

$$\int_{Q_i} |v|^q(x) \, dx \leq |v|_{L^{q_i}(Q_i)}^{q_i} \leq C |v|_{L^{q_i}_{s,p,Q_i}}^{q_i} \leq C \left( \int_{Q_i} |v|^{p(x)} \, dx + \int_{Q_i} \int_{Q_i} |v(x) - v(y)|^{p(x,y)} \, \frac{dx \, dy}{|x-y|^{N+p(x,y)}} \right)^{\frac{q_i}{p_i}},$$

Similarly, if $|v|_{L^{q_i}(Q_i)} \leq 1$, then

$$\int_{Q_i} |v|^q(x) \, dx \leq |v|_{L^{q_i}(Q_i)}^{q_i} \leq C |v|_{L^{q_i}_{s,p,Q_i}}^{q_i} \leq C \left( \int_{Q_i} |v|^{p(x)} \, dx + \int_{Q_i} \int_{Q_i} |v(x) - v(y)|^{p(x,y)} \, \frac{dx \, dy}{|x-y|^{N+p(x,y)}} \right)^{\frac{q_i}{p_i}}.$$

So in any case,

$$\int_{Q_i} |v|^q(x) \, dx \leq C \left( \int_{Q_i} |v|^{p(x)} \, dx + \int_{Q_i} \int_{Q_i} |v(x) - v(y)|^{p(x,y)} \, \frac{dx \, dy}{|x-y|^{N+p(x,y)}} \right)^{\frac{q_i}{p_i}}.$$

Summing up for $i \in \mathbb{N}$, we obtain

$$\int_{\mathbb{R}^N} |v|^q(x) \, dx \leq C \left( \int_{\mathbb{R}^N} |v|^{p(x)} \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(x) - v(y)|^{p(x,y)} \, \frac{dx \, dy}{|x-y|^{N+p(x,y)}} \right)^{\frac{q_i}{p_i}},$$

which implies (3.5) with $\Omega = \mathbb{R}^N$. The proof is complete. $\square$

We conclude this section with a compact imbedding from $W^{s,p(\cdot)}(\mathbb{R}^N)$ into the weighted Lebesgue spaces with variable exponents.

**Theorem 3.4.** Assume that $(P_1)$, $(Q_1)$, and the log-Hölder continuity condition (3.2) hold. Let $w \in L^\infty$ for some $r \in C_+(\mathbb{R}^N)$ such that $\inf_{x \in \mathbb{R}^N} q(x) - r(x) > 0$. Then, it holds that

$$W^{s,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N).$$

A proof of Theorem 3.4 can be obtained in a similar fashion to that of [27, Lemma 4.1] and we omit it.

## 4 The concentration-compactness principles for $W^{s,p(\cdot)}(\mathbb{R}^N)$

In this section we establish two Lions type concentration-compactness principles for the spaces $W^{s,p(\cdot)}(\mathbb{R}^N)$.

### 4.1 Statements of the concentration-compactness principles

Let $\mathcal{M}(\mathbb{R}^N)$ be the space of all signed finite Radon measures on $\mathbb{R}^N$ endowed with the total variation norm. Note that we may identify $\mathcal{M}(\mathbb{R}^N)$ with the dual of $C_0(\mathbb{R}^N)$, the completion of all continuous functions $u : \mathbb{R}^N \to \mathbb{R}$ whose support is compact relative to the supremum norm $\| \cdot \|_{\infty}$ (see, e.g., [32, Section 1.3.3]).
In the rest of this paper, we always assume that the variable exponents $p$ and $q$ satisfy the following assumptions.

$$(p_2) \quad p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \text{ is uniformly continuous and symmetric such that}$$

\[
P^{-} := \inf_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x,y) \leq \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x,y) =: \overline{p} < \frac{N}{s};
\]

there exists $\varepsilon_0 \in (0, \frac{1}{2})$ such that $p(x,y) = \overline{p}$ for all $x, y \in \mathbb{R}^N$ satisfying $|x-y| < \varepsilon_0$ and $\sup_{y \in \mathbb{R}^N} p(x,y) = \overline{p}$ for all $x \in \mathbb{R}^N$; and $\{|x \in \mathbb{R}^N : p(x) \neq \overline{p}\} < \infty$, where $p(x) := \inf_{y \in \mathbb{R}^N} p(x,y)$ for $x \in \mathbb{R}^N$.

$$(\Omega_2) \quad q : \mathbb{R}^N \to \mathbb{R} \text{ is uniformly continuous such that } p^{*}(x) \leq q(x) \leq \overline{p}_{\mathcal{S}} := \frac{N\overline{p}}{N-3p} \text{ for all } x \in \mathbb{R}^N \text{ and } \mathcal{C} := \{x \in \mathbb{R}^N : q(x) = \overline{p}_{\mathcal{S}}\} \neq \emptyset.$$

It is clear that if $p$ satisfies $(p_2)$, then $p(x,x) = \overline{p}$ for all $x \in \mathbb{R}^N$ and $p$ satisfies $(p_1)$ and (3.2). Hence, by Theorem 3.3, we have

$$W^{s,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{p^{*}}(\mathbb{R}^N), \quad (4.1)$$

On the other hand, by $(p_2)$ we have that for any $u \in L^{\overline{p}}(\mathbb{R}^N)$,

\[
\int_{\mathbb{R}^N} |u(x)|^{p^{*}(x)} \, dx = \int_{\{p(x) = \overline{p}\}} |u(x)|^{p^{*}(x)} \, dx + \int_{\{p(x) \neq \overline{p}\}} |u(x)|^{p^{*}(x)} \, dx \\
\leq \int_{\{p(x) = \overline{p}\}} |u(x)|^{\overline{p}} \, dx + \int_{\{p(x) \neq \overline{p}\}} \left[1 + |u(x)|^{\overline{p}} \right] \, dx \\
= \left|\{x \in \mathbb{R}^N : p^{*}(x) \neq \overline{p}\}\right| + \int_{\mathbb{R}^N} |u(x)|^{\overline{p}} \, dx < \infty.
\]

Hence, $L^{\overline{p}}(\mathbb{R}^N) \subset L^{p^{*}(\cdot)}(\mathbb{R}^N)$. From this and (4.1) we obtain

$$W^{s,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{t(\cdot)}(\mathbb{R}^N), \quad (4.2)$$

for any $t \in C(\mathbb{R}^N)$ satisfying $p^{*}(x) \leq t(x) \leq \overline{p}_{\mathcal{S}}$ for all $x \in \mathbb{R}^N$. In particular, $(\Omega_2)$ yields

$$S_{\overline{p}} := \inf_{u \in W^{s,p(\cdot)}(\mathbb{R}^N) \setminus \{0\}} \frac{|u|^s}{\|u\|_{W^{s,p(\cdot)}(\mathbb{R}^N)}} > 0, \quad (4.3)$$

Our main results in this sections are the following CCPs for $W^{s,p(\cdot)}(\mathbb{R}^N)$.

**Theorem 4.1.** Assume that $(p_2)$ and $(\Omega_2)$ hold. Let $\{u_n\}$ be a bounded sequence in $W^{s,p(\cdot)}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \text{ in } W^{s,p(\cdot)}(\mathbb{R}^N), \quad (4.4)$$

\[
|u_n|^{\overline{p}} + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N \cdot sp(x,y)}} \, dy \rightharpoonup \mu \text{ in } \mathcal{M}(\mathbb{R}^N), \quad (4.5)
\]

\[
|u_n|^{q(x)} \rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^N), \quad (4.6)
\]

Then, there exist sets $\{\mu_i\}_{i \in I} \subset (0, \infty)$, $\{v_i\}_{i \in I} \subset (0, \infty)$ and $\{x_i\}_{i \in I} \subset \mathcal{C}$, where $I$ is an at most countable index set, such that

\[
\mu \geq |u|^\overline{p} + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N \cdot sp(x,y)}} \, dy + \sum_{i \in I} \mu_i \delta_{x_i},
\]

\[
\nu = |u|^{q(x)} + \sum_{i \in I} v_i \delta_{x_i}, \quad (4.8)
\]

\[
S_{\overline{p}}^\frac{1}{s} \leq \mu_i^\frac{1}{s}, \quad \forall i \in I. \quad (4.9)
\]
For possible loss of mass at infinity, we have the following.

**Theorem 4.2.** Assume that \((P_2)\) and \((\Omega_2)\) hold. Let \(\{u_n\}\) be a sequence in \(W^{s,p(\cdot)}(\mathbb{R}^N)\) as in Theorem 4.1. Set

\[
\nu_\infty := \lim_{R \to \infty} \limsup_{n \to \infty} \int_{B_R} |u_n|^{p(x)} \, dx,
\]

\[
\mu_\infty := \lim_{R \to \infty} \limsup_{n \to \infty} \int_{B_R} \left[ |u_n|^{\overline{p}} + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \right] \, dx.
\]

Then

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{\overline{p}} \, dx = \nu(\mathbb{R}^N) + \nu_\infty,
\]

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^{p(x,y)} \, dy \, dx = \mu(\mathbb{R}^N) + \mu_\infty.
\]

Assume in addition that

\((E_\infty)\) There exist \(p(x,y) = \overline{p}\) and \(q(x) = q_\infty\) for \(\overline{p}\) given by \((P_2)\) and some \(q_\infty \in (1, \infty)\).

Then

\[
S_q \nu_\infty \leq \nu_\infty.
\]

The following example provides a nonconstant exponent \(p\) that fulfills the conditions in Theorems 4.1 and 4.2.

**Example 4.3.** Let \(p(x,y) = \overline{p} - \xi(|x-y|)\phi(x,y)\), where \(\xi \in C^\infty(\mathbb{R})\) such that \(0 \leq \xi(t) \leq 1\) for all \(t \in \mathbb{R}\), \(\xi(t) = 0\) for \(t \leq \varepsilon_0\) and \(\xi(t) = 1\) for \(t \geq 1\); \(\phi \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)\), \(\phi(x,y) = \psi(y,x)\) and \(0 \leq \psi(y,x) < \overline{p} - 1\) for all \((x,y) \in \mathbb{R}^N \times \mathbb{R}^N\). Here \(\varepsilon_0\) and \(\overline{p}\) are as in \((P_2)\).

### 4.2 Auxiliary lemmas and proofs of the concentration-compactness principles

The following auxiliary lemmas are useful to prove Theorems 4.1 and 4.2.

**Lemma 4.4.** Let \(x_0 \in \mathbb{R}^N\) be fixed and let \(\psi \in C^\infty(\mathbb{R}^N)\) be such that \(0 \leq \psi \leq 1\), \(\psi \equiv 1\) on \(B_1\), \(\text{supp}(\psi) \subset B_2\) and \(|\nabla \psi|_\infty \leq 2\). For \(p > 0\), define \(\psi_p(x) := \psi\left(\frac{x-x_0}{p}\right)\) for \(x \in \mathbb{R}^N\). Let \((P_2)\) hold and let \(\{u_n\}\) be as in Theorem 4.1. Then, we have

\[
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(x)|^{p(x,y)} \frac{\psi_p(x) - \psi_p(y)}{|x-y|^{N+sp(x,y)}} \, dy \, dx = 0.
\]

**Lemma 4.5.** Let \(\phi \in C^\infty(\mathbb{R}^N)\) be such that \(0 \leq \phi \leq 1\), \(\phi \equiv 0\) on \(B_1\), \(\phi \equiv 1\) on \(\mathbb{R}^N \setminus B_2\), and \(|\nabla \phi|_\infty \leq 2\). For \(R > 0\), define \(\phi_R(x) := \phi\left(\frac{x}{R}\right)\) for \(x \in \mathbb{R}^N\). Let \((P_2)\) hold and let \(\{u_n\}\) be as in Theorem 4.1. Then, we have

\[
\lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(x)|^{p(x,y)} \frac{\phi_R(x) - \phi_R(y)}{|x-y|^{N+sp(x,y)}} \, dy \, dx = 0.
\]

**Proof of Lemma 4.4.** Set

\[
J(n, \rho) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(x)|^{p(x,y)} \frac{\psi_p(x) - \psi_p(y)}{|x-y|^{N+sp(x,y)}} \, dy \, dx.
\]

Let \(K > 4\) be arbitrary and fixed and let \(\rho \in (0, \frac{\varepsilon_0}{2K})\). Clearly,

\[
\mathbb{R}^N \times \mathbb{R}^N = \left[ (\mathbb{R}^N \setminus B_{2p}(x_0)) \times (\mathbb{R}^N \setminus B_{2p}(x_0)) \right] \cup \left[ B_{Kp}(x_0) \times \mathbb{R}^N \right] \cup \left[ (\mathbb{R}^N \setminus B_{Kp}(x_0)) \times B_{2p}(x_0) \right]
\]
From this and the fact that \(|\psi_p(x) - \psi_p(y)| = 0\) on \((\mathbb{R}^N \setminus B_{2\rho}(x_0)) \times (\mathbb{R}^N \setminus B_{2\rho}(x_0))\), we have

\[
J(n, \rho) = \int_{B_{\rho}(x_0)} \int_{\mathbb{R}^N} \left| u_n(x) \right|^p |\psi_p(x) - \psi_p(y)|^{p(x,y)} |x - y|^{-N + sp(x,y)} \, dy \, dx \\
+ \int_{B_{\rho}(x_0)} \int_{B_{\rho}(x_0)} |u_n(x)|^p |\psi_p(x) - \psi_p(y)|^{p(x,y)} |x - y|^{-N + sp(x,y)} \, dy \, dx \\
= J_1(n, \rho) + J_2(n, \rho).
\]  (4.17)

We first estimate \(J_1(n, \rho)\). Decompose

\[
J_1(n, \rho) = \int_{B_{\rho}(x_0)} \int_{\{x - y| < \rho\}} |u_n(x)|^p |\psi_p(x) - \psi_p(y)|^{p(x,y)} |x - y|^{-N + sp(x,y)} \, dy \, dx \\
+ \int_{B_{\rho}(x_0)} \int_{\{\rho < |x - y| < \epsilon_0\}} |u_n(x)|^p |\psi_p(x) - \psi_p(y)|^{p(x,y)} |x - y|^{-N + sp(x,y)} \, dy \, dx \\
+ \int_{B_{\rho}(x_0)} \int_{\{|x - y| < \epsilon_0\}} |u_n(x)|^p |\psi_p(x) - \psi_p(y)|^{p(x,y)} |x - y|^{-N + sp(x,y)} \, dy \, dx \\
= \sum_{i=1}^{3} f_1^{(i)}(n, \rho).
\]  (4.18)

By (P2) and the choice of \(\psi_p\), we have \(p(x, y) = \overline{p}\) and \(|\psi_p(x) - \psi_p(y)| \leq \frac{2}{\overline{p}} |x - y|\) on \(\{|x - y| < \rho\}\). Thus, we obtain

\[
f_1^{(1)}(n, \rho) \leq \left( \frac{2}{\overline{p}} \right)^p \int_{B_{\rho}(x_0)} |u_n(x)|^p \int_{\{x - y| < \rho\}} |x - y|^{-N + (1 - s)\overline{p}} \, dy \, dx.
\]

Hence,

\[
f_1^{(1)}(n, \rho) \leq \frac{2^p N |B_1|}{(1 - s)\overline{p}} \rho^{1-\overline{p}} \int_{B_{\rho}(x_0)} |u_n(x)|^p \, dx.
\]  (4.19)

By (4.4), we have that \(u_n \to u\) in \(L^\overline{p}(B_{\rho}(x_0))\) in view of and Theorem 3.2 (iii). From this fact and (4.19) we obtain

\[
\lim_{n \to \infty} f_1^{(1)}(n, \rho) \leq \frac{2^p N |B_1|}{(1 - s)\overline{p}} \rho^{1-\overline{p}} \int_{B_{\rho}(x_0)} |u(x)|^p \, dx.
\]  (4.20)

Using the Hölder inequality we have

\[
\int_{B_{\rho}(x_0)} |u(x)|^p \, dx \leq |B_1|^{\frac{2}{\overline{p}}} K^p \rho^\overline{p} \left( \int_{B_{\rho}(x_0)} |u(x)|^{\overline{p}} \, dx \right)^{\frac{p}{\overline{p}}}
\]  (4.21)

From (4.20), (4.21) and the fact that \(u \in L^\overline{p}(\mathbb{R}^N)\) (see (4.1)) we arrive at

\[
\limsup_{\rho \to 0} \limsup_{n \to \infty} f_1^{(1)}(n, \rho) = 0.
\]  (4.22)

On the other hand, from \(p(x, y) = \overline{p}\) and \(|\psi_p(x) - \psi_p(y)|^{p(x,y)} \leq 1\) on \(\{|x - y| < \epsilon_0\}\) we have

\[
f_1^{(2)}(n, \rho) \leq \int_{B_{\rho}(x_0)} |u_n(x)|^p \int_{\{\rho < |x - y| < \epsilon_0\}} |x - y|^{-N + \epsilon_0 \over\overline{p}} \, dy \, dx.
\]

That is,

\[
f_1^{(2)}(n, \rho) \leq \frac{N |B_1|}{s \over\overline{p}} \int_{B_{\rho}(x_0)} |u_n(x)|^p \left( \rho^\overline{p} - \epsilon_0^\overline{p} \right) \, dx.
\]  (4.23)

Arguing as that obtained (4.20) we deduce from (4.23) that

\[
\limsup_{n \to \infty} f_1^{(2)}(n, \rho) \leq \frac{N |B_1|}{s \over\overline{p}} \int_{B_{\rho}(x_0)} |u(x)|^p \left( \rho^\overline{p} - \epsilon_0^\overline{p} \right) \, dx.
\]  (4.24)
Then, using \( (4.21) \) and the fact that \( u \in L^p(\mathbb{R}^N) \) again we obtain from \( (4.29) \) that
\[
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} f^{(2)}_1(n, \rho) = 0. \tag{4.25}
\]
In order to estimate \( f^{(3)}_1(n, \rho) \), using the following estimations on \( \{|x - y| \geq \varepsilon_0\} \):
\[
|u_n(x)|^{p(x,y)} \leq 1 + |u_n(x)|^p
\]
and
\[
\frac{|\psi_\rho(x) - \psi_\rho(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \leq \frac{1}{|x - y|^{N + sp}} + \frac{1}{|x - y|^{N + \bar{p}}},
\]
we first estimate
\[
f^{(3)}_1(n, \rho) \leq \int_{B_{\varepsilon_0}(x_0)} \left( 1 + |u_n(x)|^p \right) \int_{\{|x - y| < \varepsilon_0\}} \left( \frac{1}{|x - y|^{N + sp}} + \frac{1}{|x - y|^{N + \bar{p}}} \right) dy \, dx
\]
\[
\leq 2N|B_1|^{1 - sp} |\varepsilon_0|^{1 - sp} \int_{B_{\varepsilon_0}(x_0)} \left( 1 + |u_n(x)|^p \right) \, dx.
\]
Then, arguing as before we obtain
\[
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} f^{(3)}_1(n, \rho) = 0. \tag{4.26}
\]
Utilizing \( (4.22) \), \( (4.25) \) and \( (4.26) \), we infer from \( (4.18) \) that
\[
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} J_1(n, \rho) = 0. \tag{4.27}
\]
Next, we estimate \( J_2(n, \rho) \). Note that
\[
J_2(n, \rho) \leq \sum_{t \in \{p, \bar{p}\}} \int_{\mathbb{R}^N \setminus B_{\varepsilon_0}(x_0)} \int_{B_{\varepsilon_0}(x_0)} |u_n(x)|^t \left( \frac{|\psi_\rho(x) - \psi_\rho(y)|^t}{|x - y|^{N + st(x)}} \right) dy \, dx
\]
\[
=: \sum_{t \in \{p, \bar{p}\}} J_2(n, \rho, t). \tag{4.28}
\]
Let \( t \in \{p, \bar{p}\} \). Using the fact that
\[
|x - y| \geq |x - x_0| - |y - x_0| \geq |x - x_0| - 2p \geq \frac{1}{2} |x - x_0|, \quad \forall (x, y) \in (\mathbb{R}^N \setminus B_{\varepsilon_0}(x_0)) \times B_{2\rho}(x_0),
\]
we have
\[
J_2(n, \rho, t) \leq 2^{N + \bar{p}} \int_{\mathbb{R}^N \setminus B_{\varepsilon_0}(x_0)} \frac{|u_n(x)|^t}{|x - x_0|^{N + st(x)}} \, dy \, dx.
\]
That is,
\[
J_2(n, \rho, t) \leq 2^{N + \bar{p}} |B_1| \int_{\mathbb{R}^N \setminus B_{\varepsilon_0}(x_0)} \frac{|u_n(x)|^t}{|x - x_0|^{N + st(x)}} \, dx.
\]
Invoking Proposition 2.3 again and using the boundedness of \( \{u_n\} \) in \( L^p(\mathbb{R}^N) \), we deduce from the last inequality that
\[
J_2(n, \rho, t) \leq C_1 \left\| |u_n|^t \right\|_{L^p(\mathbb{R}^N \setminus B_{\varepsilon_0}(x_0))} \left\| |x - x_0|^{-N - st(x)} \right\|_{L^{p/\gamma}(\mathbb{R}^N \setminus B_{\varepsilon_0}(x_0))} \frac{p^\gamma}{p - \gamma} \left( \frac{p^\gamma}{p - \gamma} \right)^+. \tag{4.29}
\]

Here and in the remainder of the proof $C_i$ ($i \in \mathbb{N}$) is a positive constant independent of $n$, $\rho$ and $K$. By changing variable $x = x_0 + \rho z$ we have

$$
\int_{\mathbb{R}^n \setminus B_R(x_0)} |x - x_0|^{-\alpha} \cdot \frac{N |x|^{p(x,y)} |\nabla u_n|^p}{\rho^p R^{p(x,y)}} \, dx
= \int_{\{|z| > K\}} |z|^{-\alpha} \cdot \frac{N |x|^{p(x,y)} |\nabla u_n|^p}{\rho^p R^{p(x,y)}} \, dz.
$$

(4.30)

Note that for any $x \in \mathbb{R}^N$, it holds that $N - \frac{N |x|^{p(x,y)} |\nabla u_n|^p}{\rho^p R^{p(x,y)}} \geq 0$ due to $t(x) \leq \overline{p}$ and $\frac{(N+\varepsilon)(x)}{\overline{p} - t(x)} = N + \frac{t(x)(N+\varepsilon)}{\overline{p} - t(x)} > N + \alpha$, where $\alpha := N - \frac{N |x|^{p(x,y)} |\nabla u_n|^p}{\rho^p R^{p(x,y)}} > 0$. Plugging this into (4.30) we obtain

$$
\int_{\mathbb{R}^n \setminus B_R(x_0)} |x - x_0|^{-\alpha} \cdot \frac{N |x|^{p(x,y)} |\nabla u_n|^p}{\rho^p R^{p(x,y)}} \, dx \leq \int_{\{|z| > K\}} |z|^{-N-\alpha} \, dz = \frac{N |B_1|}{\alpha} K^{-\alpha}.
$$

Combining this with (4.28) and (4.29) we derive

$$
J_2(n, \rho) \leq C_3 K^{-\alpha} \left( \frac{x_0^\alpha}{R} \right)
$$

for all $n \in \mathbb{N}$ and all $\rho \in (0, \frac{\rho_0}{2R})$. Thus,

$$
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} J_2(n, \rho) \leq C_3 K^{-\alpha} \left( \frac{x_0^\alpha}{R} \right).
$$

Since $K > 4$ was chosen arbitrarily, the last inequality yields

$$
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} J_2(n, \rho) = 0.
$$

(4.31)

From (4.17), (4.27), and (4.31), we obtain (4.16) and the proof is complete.

Proof of Lemma 4.5. Let $R > 2$ and decompose

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u_n(x)|^{p(x,y)} |\phi_R(x) - \phi_R(y)|^{p(x,y)} \frac{|x-y|^{N+sp(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \, dx
= \int_{B_R^c} \int_{\mathbb{R}^n} |u_n(x)|^{p(x,y)} |\phi_R(x) - \phi_R(y)|^{p(x,y)} \frac{|x-y|^{N+sp(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \, dx
+ \int_{B_R} \int_{\mathbb{R}^n} |u_n(x)|^{p(x,y)} |\phi_R(x) - \phi_R(y)|^{p(x,y)} \frac{|x-y|^{N+sp(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \, dx
=: I_1(n, R) + I_2(n, R).
$$

(4.32)

First, we estimate $I_1(n, R).$ By rearranging

$$
I_1(n, R) = \int_{B_R^c} \int_{\mathbb{R}^n} \left( \frac{|u_n(x)| |\phi_R(x) - \phi_R(y)|}{|x-y|^{\frac{1}{p}}} \right)^{p(x,y)} \frac{1}{|x-y|^N} \, dy \, dx,
$$

we easily get

$$
I_1(n, R) \leq \int_{B_R^c} \int_{\mathbb{R}^n} \left[ \frac{|u_n(x)|^p |\phi_R(x) - \phi_R(y)|^p}{|x-y|^{sp}} + \frac{|u_n(x)|^{p(x,y)} |\phi_R(x) - \phi_R(y)|^{p(x,y)} |x-y|^{N+sp(x,y)}}{|x-y|^{N+sp(x,y)}} \right] \, dy \, dx
= \int_{B_R^c} \int_{\mathbb{R}^n} |u_n(x)|^p \frac{|\phi_R(x) - \phi_R(y)|^p}{|x-y|^{N+sp(x,y)}} \, dy \, dx
+ \int_{B_R^c} \int_{\mathbb{R}^n} |u_n(x)|^{p(x,y)} \frac{|\phi_R(x) - \phi_R(y)|^{p(x,y)} |x-y|^{N+sp(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \, dx
$$
\begin{equation}
\tag{4.33}
=: I_1(n, R, \bar{p}) + I_1(n, R, p^*).
\end{equation}

By \eqref{4.2} and the boundedness of \( \{u_n\} \) in \( W_{p}\), we can find \( M > 0 \) such that
\begin{equation}
\tag{4.34}
\max_{t \in \{p, p^*\}} \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} |u_n(x)|^t dx \leq M.
\end{equation}

Let \( t \in \{p, p^*\} \). We have
\begin{equation}
\tag{4.35}
I_1(n, R, t) = \int_{B_R^c} \int_{\mathbb{R}^N} |u_n(x)|^t \frac{\phi_R(x) - \phi_R(y)}{|x - y|^{N+st(x)}} dy dx
= \int_{B_R^c} |u_n(x)|^t \int_{\{x-y| \leq R\} |x-y|^{N+st(x)}} \frac{\phi_R(x) - \phi_R(y)}{|x-y|^{N+st(x)}} dy dx
+ \int_{B_R^c} |u_n(x)|^t \int_{\{x-y| > R\} |x-y|^{N+st(x)}} \frac{\phi_R(x) - \phi_R(y)}{|x-y|^{N+st(x)}} dy dx
=: I_1^{(1)}(n, R, t) + I_1^{(2)}(n, R, t).
\end{equation}

We have
\begin{equation}
\tag{4.36}
\sup_{n \in \mathbb{N}} I_1^{(1)}(n, R, t) \leq \int_{B_R^c} |u_n(x)|^t \int_{B_R^c} \frac{dz}{|z|^{N+st(x)}} dx = N|B_1| \int_{B_R^c} \frac{|u_n(x)|^t}{st(x)R^{st(x)}} dx \leq \frac{N|B_1|}{sp^*} R^{-sp^*}.
\end{equation}

On the other hand, using estimation:
\begin{equation}
\tag{4.37}
\frac{|\phi_R(x) - \phi_R(y)|}{|x-y|^{N+st(x)}} \leq |x-y|^{-N+1-s)}/N| \nabla \phi_R||^{t(x)}_{\infty}
\leq |x-y|^{-N+1-s)} \left( \frac{2}{R} \right)^{t(x)} \nabla \phi_R(x), \forall (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \ x \neq y,
\end{equation}
we have
\begin{equation}
\tag{4.38}
I_1^{(2)}(n, R, t) \leq \int_{B_R^c} |u_n(x)|^t \int_{\{x-y| > R\} |x-y|^{N+st(x)}} \left( \frac{2}{R} \right)^{t(x)} dx
\leq \frac{2^\tau N|B_1|}{1-s} R^{-sp^*} \int_{B_R^c} |u_n(x)|^t dx.
\end{equation}

Combining this with \eqref{4.34} yields
\begin{equation}
\tag{4.39}
\sup_{n \in \mathbb{N}} I_1^{(2)}(n, R, t) \leq \frac{2^\tau N|B_1|}{1-s} R^{-sp^*}.
\end{equation}

From \eqref{4.33}, \eqref{4.35}, \eqref{4.36} and \eqref{4.38} we obtain
\begin{equation}
\sup_{n \in \mathbb{N}} I_1(n, R) \leq \frac{2^\tau N|B_1|}{1-s} R^{-sp^*}
\end{equation}
and hence,
\begin{equation}
\lim_{R \to \infty} \limsup_{n \to \infty} I_1(n, R) = 0.
\end{equation}
Next, we estimate $I_2(n, R)$. Fix $\sigma \in (0, 1/2)$ and decompose

$$I_2(n, R) = \int_{B_R \setminus B_{\sigma R}} \left| \frac{\phi_R(y) - \phi_R(y)}{|y|^{N+s}} \right| \frac{|\phi_R(x) - \phi_R(y)| |\phi_R(x)|}{|x - y|^{N+s}} \, dy \, dx$$

$$+ \int_{B_R \setminus B_{\sigma R}} \left| \frac{\phi_R(y) - \phi_R(y)}{|y|^{N+s}} \right| \frac{|\phi_R(x) - \phi_R(y)|}{|x - y|^{N+s}} \, dy \, dx$$

$$+ \int_{B_{\sigma R}} \left| \frac{\phi_R(y) - \phi_R(y)}{|y|^{N+s}} \right| \frac{|\phi_R(x) - \phi_R(y)|}{|x - y|^{N+s}} \, dy \, dx$$

$$=: I_2(1)(n, R, \sigma) + I_2(2)(n, R, \sigma) + I_2(3)(n, R, \sigma).$$

(4.40)

Note that as in (4.37), we have

$$\frac{|\phi_R(x) - \phi_R(y)|}{|x - y|^{N+s}} \leq |x - y|^{-N} \left| \frac{2}{R} \right|^{p(x)}$$

$$\leq 2^p \sum_{t \in \mathbb{N}} R^{-t} |x - y|^{-N+(1-s)t}, \forall (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, x \neq y.$$

Thus, for $x \in B_R \setminus B_{\sigma R}$ we have

$$\int_{\{|x-y|<\frac{R}{2}\}} \frac{|\phi_R(x) - \phi_R(y)|}{|x - y|^{N+s}} \, dy \leq 2^p \sum_{t \in \mathbb{N}} R^{-t} |x - y|^{-N+(1-s)t} \, dy$$

$$\leq 2^{2^p} N |B_1| R^{sp} \sum_{t \in \mathbb{N}} \frac{R^{-t}}{(1-s)t}.$$

From this and (4.34) we obtain

$$I_2^{(1)}(n, R, \sigma) \leq \int_{B_R \setminus B_{\sigma R}} \left[ |u_n(x)|^p + |u_n(x)|^{p(x)} \right] \int_{\{|x-y|<\frac{R}{2}\}} \frac{|\phi_R(x) - \phi_R(y)|}{|x - y|^{N+s}} \, dy \, dx$$

$$\leq 2^{1+2^p} N |B_1| R^{sp} \int_{B_R \setminus B_{\sigma R}} \left[ |u_n(x)|^p + |u_n(x)|^{p(x)} \right] \, dx$$

$$\leq 2^{2+2^p} N |B_1| M R^{sp} \sum_{n \in \mathbb{N}} \frac{R^{-sp}}{(1-s)p^s}, \forall n \in \mathbb{N}.$$ (4.41)

Using (4.34) again, we have

$$I_2^{(2)}(n, R, \sigma) \leq \int_{B_R \setminus B_{\sigma R}} \left[ |u_n(x)|^p + |u_n(x)|^{p(x)} \right] \int_{\{|x-y|>\frac{R}{2}\}} \frac{dy}{|x - y|^{N+s}} \, dx$$

$$\leq \int_{B_R \setminus B_{\sigma R}} \left[ |u_n(x)|^p + |u_n(x)|^{p(x)} \right] \int_{|z|>\frac{R}{2}} \frac{dz}{|z|^{N+s+p}} \, dx$$

$$\leq 2^{1+2^p} N |B_1| M R^{sp} \sum_{n \in \mathbb{N}} \frac{R^{-sp}}{(1-s)p^s}, \forall n \in \mathbb{N}.$$ (4.42)

Finally, to estimate $I_2^{(3)}(n, R, \sigma)$ we first note that $\phi_R(x) - \phi_R(y) = 0$ for all $(x, y) \in B_{\sigma R} \times B_R$ and $|x - y| \geq (1-\sigma)R$ for all $(x, y) \in B_{\sigma R} \times B_R^c$. Using these facts and invoking (4.34) again, we have

$$I_2^{(3)}(n, R, \sigma) = \int_{B_R} \int_{B_R^c} \left| \frac{\phi_R(x) - \phi_R(y)|}{|x - y|^{N+s}} \right| \frac{|\phi_R(x) - \phi_R(y)|}{|x - y|^{N+s}} \, dy \, dx$$

$$\leq \int_{B_R} \int_{B_R^c} \left| \frac{\phi_R(x) - \phi_R(y)|}{|x - y|^{N+s}} \right| \frac{|\phi_R(x)|}{|x - y|^{N+s}} \, dy \, dx$$

$$\leq \int_{B_R} \int_{B_R^c} \left| \frac{\phi_R(x) - \phi_R(y)|}{|x - y|^{N+s}} \right| \frac{|\phi_R(x)|}{|x - y|^{N+s}} \, dy \, dx$$

$$\leq 2N |B_1|(1-\sigma)^{-sp} M R^{sp}.$$ (4.43)
Making use of (4.41)-(4.43), we deduce from (4.40) that
\[
\lim_{R \to \infty} \limsup_{n \to \infty} I_2(n, R) = 0. \tag{4.44}
\]
Finally, (4.15) follows from (4.32), (4.39) and (4.44). The proof is complete.

We now prove the first concentration-compactness principle.

**Proof of Theorem 4.1.** Let \( v_n = u_n - u \). Then,
\[
v_n \to 0 \quad \text{in} \quad W^{k,p}((\mathbb{R}^N)). \tag{4.45}
\]
Invoking Theorem 3.2, we deduce from (4.45) that
\[
v_n \to 0 \quad \text{in} \quad L_{\text{loc}}^r(\mathbb{R}^N) \tag{4.46}
\]
for any \( r \in C_+(\mathbb{R}^N) \) satisfying \( r(x) < p^* \) for all \( x \in \mathbb{R}^N \). Hence, up to a subsequence we have
\[
v_n(x) \to 0 \quad \text{for a.e.} \quad x \in \mathbb{R}^N. \tag{4.47}
\]
Using (4.6), (4.45), (4.47) and arguing as in [27], we have
\[
|v_n|^{(x)} \ast v - |u|^{(x)} =: \bar{v} \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N). \tag{4.48}
\]
Obviously, \( \left\{ |v_n|^p \ast \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{p(x,y)}}{|x-y|^N} \, dy \right\} \) is bounded in \( L^1(\mathbb{R}^N) \). So up to a subsequence, we have
\[
|v_n|^p \ast \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{p(x,y)}}{|x-y|^N} \, dy \ast \bar{\mu} \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N) \tag{4.49}
\]
for some nonnegative finite Radon measure \( \bar{\mu} \) on \( \mathbb{R}^N \). Let \( \phi \in C_+^\infty(\mathbb{R}^N) \) and let \( R > 2 \) be such that
\[
\text{supp}(\phi) \subset B_R \quad \text{and} \quad d := \text{dist}(B_R, \text{supp}(\phi)) \geq 1 + \frac{R}{2}. \tag{4.50}
\]
By (4.3), we have
\[
S_q |\phi v_n|_{L^p([\infty, |x|])} \leq \|\phi v_n\|_{L^p}. \tag{4.51}
\]
Set \( \bar{v}_n := |v_n|^{(x)} \), \( \bar{\mu}_n := |v_n(x)|^p \ast \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{p(x,y)}}{|x-y|^N} \, dy \), and \( \lambda_n := \|\phi v_n\|_{L^p}. \) Let \( \varepsilon > 0 \) be arbitrary and fixed. Then, there exists \( \delta(\varepsilon) \in (2, \infty) \) such that
\[
|a + b|^{p(x,y)} \leq (1 + \varepsilon)|a|^{p(x,y)} + C(\varepsilon)|b|^{p(x,y)}, \quad \forall a, b \in \mathbb{R}, \quad \forall x, y \in \mathbb{R}^N. \tag{4.52}
\]
Invoking Proposition 3.1 and (4.52) we have
\[
1 = \int_{\mathbb{R}^N} \left( \frac{\phi v_n}{\lambda_n} \right)^p \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(\phi v_n)(x) - (\phi v_n)(y)|^{p(x,y)}}{\lambda_n^{p(x,y)}|x-y|^N} \, dy \, dx
\]
\[
\leq \int_{\mathbb{R}^N} \left( \frac{\phi v_n}{\lambda_n} \right)^p \, dx + (1 + \varepsilon) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\lambda_n^{p(x,y)}|v_n(x) - v_n(y)|^{p(x,y)}}{\lambda_n^{p(x,y)}|x-y|^N} \, dy \, dx
\]
\[
+ C(\varepsilon) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(y)|^{p(x,y)}|\phi(x) - \phi(y)|^{p(x,y)}}{\lambda_n^{p(x,y)}|x-y|^N} \, dy \, dx. \tag{4.53}
\]
Set
\[
I_n := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(y)|^{p(x,y)}|\phi(x) - \phi(y)|^{p(x,y)}}{\lambda_n^{p(x,y)}|x-y|^N} \, dy \, dx.
\]
Then, invoking Proposition 3.1 again we deduce from (4.53) that
\[
1 \leq \frac{(1 + \varepsilon)(|\phi|^p + 1)}{\min\{\lambda_n^p, \lambda_n^p\}} \left( 1 + ||v_n||_{L^p} \right) + C(\varepsilon)I_n. \tag{4.54}
\]
By the symmetry of $p$ we also have
\[
I_n = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{p(x,y)}|\phi(x) - \phi(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \, dx.
\]
Thus, by the facts that $\text{supp}(\phi) \subset B_R$ and $\lambda_n^{p(x,y)} \geq \min\{\lambda_n^p, \lambda_n^\infty\}$ for all $x, y \in \mathbb{R}^N$,
\[
I_n \leq \frac{1}{\min\{\lambda_n^p, \lambda_n^\infty\}} \left[ \int_{B_R^c} \left( |v_n(x)|^{p(x)} + |v_n(x)|^{p(y)} \right) \int_{B_R} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \right. \\
+ \int_{B_R} \left( |v_n(x)|^{p(x)} + |v_n(x)|^{p(y)} \right) \int_{B_R^c} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \\
- \left. \left( |v_n(x)|^{p(x)} + |v_n(x)|^{p(y)} \right) \int_{B_R} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \right]. 
\] (4.55)

We estimate each integral in the right-hand side of (4.55) as follows. Arguing as that obtained (4.34) we have
\[
\max_{t \in \{p, p^*\}} \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} |v_n(x)|^{t(x)} \, dx \leq C_1. 
\] (4.56)

Here and in the rest of the proof, $C_i (i \in \mathbb{N})$ denotes a positive constant independent of $n$ and $R$ while $C_i(R)$ ($i \in \mathbb{N}$) denotes a positive constant independent of $n$. Let $t \in \{p, p^*\}$. Using (4.50) and (4.56), we have
\[
\int_{B_R^c} \left| v_n(x) \right|^{t(x)} \left( \int_{B_R} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \right) \, dx \\
= \int_{B_R^c} \left| v_n(x) \right|^{t(x)} \left( \int_{\text{supp}(\phi)} \frac{|\phi(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \right) \, dx \\
\leq \left( 1 + |\phi|_{p}^{p(x)} \right) \int_{B_R^c} \left| v_n(x) \right|^{t(x)} \left( \int_{\text{supp}(\phi)} \left( \frac{2}{R} \right)^{N + sp(x,y)} \, dy \right) \, dx \\
\leq \left( 1 + |\phi|_{p}^{p(x)} \right) \int_{B_R^c} \left| v_n(x) \right|^{t(x)} \, dx \leq \frac{C_2}{R^p}. 
\] (4.57)

Before estimating the remaining integrals, we note that by (4.50) again,
\[
\int_{B_R} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy = \int_{B_R} \frac{|\phi(x)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \\
\leq \left( 1 + |\phi|_{p}^{p(x)} \right) \int_{\{|z| \leq 1\}} \frac{dz}{|z|^{N + sp}} = \left( 1 + |\phi|_{p}^{p(x)} \right) \frac{|B_1|}{sp}, \ \forall x \in B_R. 
\] (4.58)

Using (4.58), we have
\[
\int_{B_R} \left| v_n(x) \right|^{t(x)} \left( \int_{B_R^c} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \right) \, dx \leq C_3 \int_{B_R} \left| v_n(x) \right|^{t(x)} \, dx. 
\] (4.59)

To estimate the last integral in the right-hand side of (4.55) we notice that for $x \in B_R$,
\[
\int_{B_R} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \leq \left( 1 + |\nabla \phi|_{p}^{p(x)} \right) \int_{B_R} \frac{1}{|x - y|^{N + (s-1)p(x,y)}} \, dy \\
\leq \left( 1 + |\nabla \phi|_{p}^{p(x)} \right) \left[ |B_R| + \int_{B_R} \frac{dz}{|z|^{N + (s-1)p}} \right] \\
= \left( 1 + |\nabla \phi|_{p}^{p(x)} \right) \left[ |B_R| + \frac{N|B_1|(2R)^{(1-s)p}}{(1-s)p} \right].
\]
Using this, (4.57) and (4.59), we obtain from (4.55) that

This yields

Combining this with (4.56) and the boundedness of \( \{v_n\} \) in \( W^{s,p(\cdot)}(\mathbb{R}^N) \), we deduce from (4.54) that

and hence

Thus \( \{\lambda_n\} \) is a bounded sequence in \( \mathbb{R} \) and hence, up to a subsequence, we may assume that there exists \( \lambda^* \in [0, \infty) \) such that

Suppose that \( \lambda^* > 0 \). From (4.53) and (4.60) we obtain

Letting \( n \to \infty \) in the last inequality, noticing (4.49), (4.61) and \( \lim_{n \to \infty} \int_{B_R} |v_n(x)|^{t(x)} \, dx = 0 \) for \( t \in \{p, p^*\} \) (see (4.46)), we obtain

Letting \( R \to \infty \) and then letting \( \varepsilon \to 0^+ \), we deduce from the last inequality that

Invoking Proposition 2.2, we easily obtain from the last estimate that

From (4.48), (4.51), (4.61) and the last inequality, we arrive at

If \( \lambda^* = 0 \) then by (4.48) and (4.51), we get \( |\phi|_{L_{p^*}^{s(\cdot)}(\mathbb{R}^N)} = 0 \); hence, (4.62) also holds. That is, (4.62) holds for any \( \phi \in C_0^\infty(\mathbb{R}^N) \) and hence, (4.8) follows by invoking Proposition A.1 and the definition of \( \mathcal{V} \) (see (4.48)).

The fact that \( \{x_i\}_{i \in I} \subset \mathcal{C} \) can be obtained by an argument similar to that of [27, Theorem 3.3] and we omit the proof. Next, we obtain the relation (4.9). Let \( i \in I \) and for \( \rho > 0 \), define \( \psi_\rho \) as in Lemma 4.4 with \( x_0 \) replaced by \( x_i \). Thus \( \psi_\rho \in C_0^\infty(\mathbb{R}^N) \), \( 0 \leq \psi_\rho \leq 1 \), \( \psi_\rho \equiv 1 \) on \( B_\rho(x_i) \), \( \text{supp}(\psi_\rho) \subset B_{2\rho}(x_i) \). Using (4.3) again, we have

\[ S_q \|\psi_\rho u_n\|_{L^{t(x)}(\mathbb{R}^N)} \leq \|\psi_\rho u_n\|_{s,p}. \]
Taking the limit inferior as $n \to \infty$ in the above inequality and using (4.6) we obtain
\[
S_q |\psi_0|_{L^q(B_0(x_i))} \leq \liminf_{n \to \infty} \|\psi_0 u_n\|_{s,p}.
\] (4.63)

Hence,
\[
S_q \limsup_{\rho \to 0} |\psi_0|_{L^q(B_0(x_i))} \leq \limsup_{\rho \to 0} \liminf_{n \to \infty} \|\psi_0 u_n\|_{s,p}.
\] (4.64)

Invoking Proposition 2.2, we have
\[
|\psi_0|_{L^q(B_0(x_i))} \geq \min\left\{ \left( \int_{B_0(x_i)} |\psi_0|^q(\xi)\,d\nu \right)^{\frac{1}{q}}, \left( \int_{B_0(x_i)} |\psi_0|^q(\xi)\,d\nu \right)^{\frac{1}{q+1}} \right\}
\geq \min\left\{ \nu(B_0(x_i))^{\frac{1}{q}}, \nu(B_0(x_i))^{\frac{1}{q+1}} \right\},
\] (4.65)

where $q_{i,\rho} := \max_{x \in B_0(x_i)} q(x)$, $q_{i,\rho}^+ := \min_{x \in B_0(x_i)} q(x)$. Thus, we obtain a lower bound of the left-hand side of (4.64) as follows:
\[
\limsup_{\rho \to 0} |\psi_0|_{L^q(B_0(x_i))} \geq \nu_i^{\frac{1}{q+1}} = \nu_i^{\frac{1}{p+1}}.
\] (4.66)

due to the continuity of $q$ and the fact that $x_i \in \mathcal{C}$. To obtain an upper bound of the right-hand side of (4.64), we first prove that there exist $\rho_0 \in (0, 1)$ and $\lambda_0 \in (0, \infty)$ such that
\[
0 < \frac{S_q}{2} \nu_i^{\frac{1}{p+1}} \leq \liminf_{n \to \infty} \lambda_{n,\rho} =: \lambda_{\ast,\rho} \leq \lambda_0 \text{ for any } \rho \in (0, \rho_0),
\] (4.67)

where $\lambda_{n,\rho} := \|\psi_0 u_n\|_{s,p}$.

Indeed, by the continuity of $q$ and the positiveness of $\nu_i$, we can choose $\rho_0 \in (0, 1)$ such that
\[
S_q \min\left\{ \nu_i^{\frac{1}{q}}, \nu_i^{\frac{1}{q+1}} \right\} > \frac{S_q}{2} \nu_i^{\frac{1}{p+1}}, \quad \forall \rho \in (0, \rho_0)
\] (4.68)

From (4.63), (4.65) and (4.68), we infer $\frac{S_q}{2} \nu_i^{\frac{1}{p+1}} \leq \lambda_{\ast,\rho}$ for all $\rho \in (0, \rho_0)$. On the other hand, by choosing $\rho_0$ smaller if necessary we have
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(y)|^{p(x,y)} \frac{|\psi_0(x) - \psi_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \,dy \,dx < 1, \quad \forall \rho \in (0, \rho_0)
\] (4.69)
in view of Lemma 4.4. Note that
\[
\int_{\mathbb{R}^N} |\psi_0 u_n|^p \,dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(\psi_0 u_n)(x) - (\psi_0 u_n)(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \,dy \,dx
\leq \int_{\mathbb{R}^N} |\psi_0 u_n|^p \,dx + 2^{p-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\psi_0(x)|^{p(x,y)} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \,dy \,dx
+ 2^{p-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(y)|^{p(x,y)} \frac{|\psi_0(x) - \psi_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \,dy \,dx.
\] (4.70)

Using (4.69), (4.70), the boundedness of $\{u_n\}$ in $W^{s,p(-)}(\mathbb{R}^N)$ and invoking Proposition 3.1, we can easily show that there exists $\lambda_0 \in (0, \infty)$ such that $\lambda_{n,\rho} < \lambda_0$ for all $n \in \mathbb{N}$ and $\rho \in (0, \rho_0)$. Thus, (4.67) has been proved.

Next, let $\varepsilon > 0$ be arbitrary and fixed. We have
\[
1 = \int_{\mathbb{R}^N} \frac{|\psi_0 u_n|^p}{\lambda_{n,\rho}} \,dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(\psi_0 u_n)(x) - (\psi_0 u_n)(y)|^{p(x,y)}}{\lambda_{n,\rho}^{p(x,y)} |x-y|^{N+sp(x,y)}} \,dy \,dx
\]
\[
= \int_{\mathbb{R}^N} \frac{|\psi_0 u_n|^p}{\lambda_{n,\rho}} \,dx + 2 \int_{B_{\varepsilon,0}(x_i)} \int_{B_{\varepsilon,0}(x_i)} \frac{|(\psi_0 u_n)(x) - (\psi_0 u_n)(y)|^{p(x,y)}}{\lambda_{n,\rho}^{p(x,y)} |x-y|^{N+sp(x,y)}} \,dy \,dx
+ \int_{B_{\varepsilon,0}(x_i)} \int_{B_{\varepsilon,0}(x_i)} \frac{|(\psi_0 u_n)(x) - (\psi_0 u_n)(y)|^{p(x,y)}}{\lambda_{n,\rho}^{p(x,y)} |x-y|^{N+sp(x,y)}} \,dy \,dx.
\]
Hence, by utilizing (4.52) again we have
\[
1 \leq \int_{\mathbb{R}^N} \frac{\psi_n u_n}{\lambda_{n,\rho}} \| \psi_n u_n \|_{\mathbb{R}^N} \, dx + 2 \int_{B_{\rho}(x)} \int_{\mathbb{R}^N} \left| u_n(x) \right|^{p(x,y)} \left| \psi_n(x) - \psi_n(y) \right|^{p(x,y)} \frac{dx}{|x-y|^{N+sp(x,y)}} \, dy \\
+ (1 + \varepsilon) \int_{B_{\rho}(x)} \int_{B_{\rho}(x)} \left| \psi_n(x) \right|^{p(x,y)} \left| u_n(x) - u_n(y) \right|^{p(x,y)} \frac{dx}{|x-y|^{N+sp(x,y)}} \, dy \\
+ C(\varepsilon) \int_{B_{\rho}(x)} \int_{B_{\rho}(x)} \left| u_n(y) \right|^{p(x,y)} \left| \psi_n(x) - \psi_n(y) \right|^{p(x,y)} \frac{dx}{|x-y|^{N+sp(x,y)}} \, dy.
\]
Combining this with the fact that \(0 \leq \psi_n \leq 1\) yields
\[
1 \leq \frac{C(\varepsilon)}{\min \{ \lambda_{n,\rho}^p, \lambda_{n,\rho}^{p_i} \}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| u_n(x) \right|^{p(x,y)} \left| \psi_n(x) - \psi_n(y) \right|^{p(x,y)} \frac{dx}{|x-y|^{N+sp(x,y)}} \, dy \\
+ \frac{1 + \varepsilon}{\min \{ \lambda_{n,\rho}^p, \lambda_{n,\rho}^{p_i} \}} \int_{\mathbb{R}^N} \psi_n(x) u_n(x) \, dx,
\]
where \(p_i = \inf_{x \in B_{\rho}(x)} p(x,y)\). Here and in what follows, for brevity we denote
\[
U_n(x) := \left| u_n(x) \right|^{p_i} + \int_{\mathbb{R}^N} \left| u_n(x) - u_n(y) \right|^{p(x,y)} \frac{dx}{|x-y|^{N+sp(x,y)}} \, dy, \quad \forall x \in \mathbb{R}^N, \forall n \in \mathbb{N}.
\]
Using (4.67), we deduce from (4.71) that
\[
1 \leq \frac{C(\varepsilon)}{\min \{ \lambda_{n,\rho}^p, \lambda_{n,\rho}^{p_i} \}} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| u_n(x) \right|^{p(x,y)} \left| \psi_n(x) - \psi_n(y) \right|^{p(x,y)} \frac{dx}{|x-y|^{N+sp(x,y)}} \, dy \\
+ \frac{1 + \varepsilon}{\min \{ \lambda_{n,\rho}^p, \lambda_{n,\rho}^{p_i} \}} \int_{\mathbb{R}^N} \psi_n(x) u_n(x) \, dx, \quad \forall \rho \in (0, \rho_0).
\]
Hence,
\[
\min \{ \lambda_{n,\rho}^p, \lambda_{n,\rho}^{p_i} \} \leq C(\varepsilon) \limsup_{n \to \infty} \int_{\mathbb{R}^N} \left| u_n(x) \right|^{p(x,y)} \left| \psi_n(x) - \psi_n(y) \right|^{p(x,y)} \frac{dx}{|x-y|^{N+sp(x,y)}} \, dy \\
+ (1 + \varepsilon) \int_{\mathbb{R}^N} \psi_n(x) u_n(x) \, dx \Rightarrow \text{Taking limit superior as } \rho \to 0^+ \text{ in the last inequality and invoking Lemma 4.4, we obtain}
\]
\[
\lambda_{n,\rho}^p \leq (1 + \varepsilon) \mu_1 \text{ i.e., } \lambda_{n,\rho} \leq (1 + \varepsilon) \mu_1^p,
\]
where \(\lambda_\ast := \limsup_{\rho \to 0^+} \lambda_{n,\rho}\) and \(\mu_1 := \lim_{\rho \to 0^+} \mu(B_{2\rho}(x_i))\). Combining this with (4.64) and (4.66) together with the fact that \(\varepsilon\) was chosen arbitrarily we obtain (4.9). Hence, \(\{x_i\}_{i \in I}\) are also atoms of \(\mu\).

Finally, to obtain (4.7) we note that for each \(\phi \in C_0(\mathbb{R}^N)\), \(\phi \geq 0\), the functional
\[
u \mapsto \int_{\mathbb{R}^N} \phi(x) \left[ \left| u(x) \right|^{p_i} + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \right] \, dx
\]
is convex and differentiable on \(W^{s,p}((\mathbb{R}^N)^N)\). From this and (4.5) we infer
\[
\int_{\mathbb{R}^N} \phi(x) \left[ \left| u(x) \right|^{p_i} + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \right] \, dx \\
\leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} \phi(x) \left[ \left| u_n(x) \right|^{p_i} + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \right] \, dx \\
= \int_{\mathbb{R}^N} \phi \, d\mu.
\]
Thus,
\[
\mu \geq |u|^{p_i} + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy.
\]
Extracting \(\mu\) to its atoms, we get (4.7) and the proof is complete.
We conclude this section by proving Theorem 4.2.

**Proof of Theorem 4.2.** For each \( R > 0 \), define \( \phi_R \) as in Lemma 4.5. Thus \( \phi_R \in C_0^\infty(\mathbb{R}^N) \), \( 0 \leq \phi_R \leq 1 \), \( \phi_R \equiv 0 \) on \( B_R \) and \( \phi_R \equiv 1 \) on \( B_R^c \), and \( |\nabla \phi_R|_\infty \leq \frac{2}{R} \). In order to obtain (4.13), we decompose

\[
\int_{\mathbb{R}^N} U_n(x) \, dx = \int_{\mathbb{R}^N} \phi_R(x) U_n(x) \, dx + \int_{\mathbb{R}^N} (1 - \phi_R(x)) U_n(x) \, dx, 
\]

where \( U_n \) is given by (4.72). By (4.11) and the fact that

\[
\int_{B_R^c} U_n(x) \, dx \leq \int_{\mathbb{R}^N} \phi_R(x) U_n(x) \, dx \leq \int_{B_R} U_n(x) \, dx
\]

for all \( n \in \mathbb{N} \) and \( R > 0 \), we obtain

\[
\mu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \phi_R(x) U_n(x) \, dx.
\]

On the other hand, the fact that \( 1 - \phi_R \in C_0^\infty(\mathbb{R}^N) \) gives

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} (1 - \phi_R(x)) U_n(x) \, dx = \int_{\mathbb{R}^N} (1 - \phi_R(x)) \, d\mu.
\]

Meanwhile,

\[
\lim_{R \to \infty} \int_{\mathbb{R}^N} \phi_R(x) \, d\mu = 0
\]

in view of the Lebesgue dominated convergence theorem. From the last two equalities, we obtain

\[
\lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} (1 - \phi_R(x)) U_n(x) \, dx = \mu(\mathbb{R}^N).
\]

From this and (4.73)-(4.75) we obtain (4.13).

In order to prove (4.12), we decompose

\[
\int_{\mathbb{R}^N} |u_n(x)|^{q(x)} \, dx = \int_{\mathbb{R}^N} \phi_R^{q(x)}(x) |u_n(x)|^{q(x)} \, dx + \int_{\mathbb{R}^N} \left(1 - \phi_R^{q(x)}(x)\right) |u_n(x)|^{q(x)} \, dx.
\]

From the definition (4.10) of \( \nu_\infty \) and the estimate

\[
\int_{B_{R^c}} |u_n(x)|^{q(x)} \, dx \leq \int_{\mathbb{R}^N} \phi_R^{q(x)}(x) |u_n(x)|^{q(x)} \, dx \leq \int_{B_R} |u_n(x)|^{q(x)} \, dx
\]

for all \( n \in \mathbb{N} \) and \( R > 0 \), we deduce

\[
\nu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \phi_R^{q(x)}(x) |u_n(x)|^{q(x)} \, dx.
\]

Arguing as that obtained (4.13) above for which \( \phi_R \) is replaced with \( \phi_R^{q(x)} \), we obtain (4.12).

We conclude the proof by proving (4.14). Without loss of generality we assume \( \nu_\infty > 0 \). Let \( \varepsilon \in (0, 1) \) be arbitrary and fixed. By (\( E_\infty \)), we can choose \( R_\varepsilon > 1 \) such that

\[
|p(x, y) - \overline{p}| < \varepsilon \quad \text{and} \quad |q(x) - q_\infty| < \varepsilon \quad \text{for all} \quad |x|, |y| > R_\varepsilon.
\]

From (4.3), we have

\[
S_{q/2} \| \phi_R u_n \|_{L^{q/(q-1)}(\mathbb{R}^N)} \leq \| \phi_R u_n \|_{s,p}.
\]

For \( R > R_\varepsilon \), using (4.78) and Proposition 2.2 we have

\[
\| \phi_R u_n \|_{L^{q/(q-1)}(\mathbb{R}^N)} = \| \phi_R u_n \|_{L^{q/(q-1)}(B_R)} \geq \min \left\{ \left( \int_{B_R} \phi_R^{q(x)}(x) |u_n|^{q(x)} \, dx \right)^{\frac{1}{q-1}}, \left( \int_{B_R^c} \phi_R^{q(x)}(x) |u_n|^{q(x)} \, dx \right)^{\frac{1}{q-1}} \right\}
\]

\[
\geq \min \left\{ \left( \int_{B_R^c} |u_n|^{q(x)} \, dx \right)^{\frac{1}{q-1}}, \left( \int_{B_R} |u_n|^{q(x)} \, dx \right)^{\frac{1}{q-1}} \right\}.
\]
Thus, \[
\liminf_{R \to \infty} \limsup_{n \to \infty} \|\phi_R u_n\|_{L^q(\mathbb{R}^N)} \geq \min \left\{ \frac{V_{s,q}}{\nu_{s,q}}, \frac{V_{s,q}}{\nu_{s,q}} \right\}.
\] (4.80)

Next, we estimate the right-hand side of (4.79). To this end, denote \(\sigma_{n,R} := \|\phi_R u_n\|_{s,p} \) for brevity. We will show that there exist \(R_2 \in (R_1, \infty)\) and \(\sigma \in (0, \infty)\) such that
\[
0 < S_q \left( \frac{1}{q} \nu_{s,q} \right)^{\frac{1}{q}} \leq \sigma_{s,R} := \limsup_{n \to \infty} \sigma_{n,R} < \sigma, \quad \forall R \in (R_2, \infty).
\] (4.81)

Indeed, we first choose \(\varepsilon > 0\) sufficiently small such that
\[
\min \left\{ \left( \frac{V_{s,q}}{2} \right)^{\frac{1}{\nu_{s,q}}}, \left( \frac{V_{s,q}}{2} \right)^{\frac{1}{\nu_{s,q}}} \right\} > \left( \frac{V_{s,q}}{4} \right)^{\frac{1}{\nu_{s,q}}}.
\] (4.82)

Then we can find \(R_2 > R_1\) such that
\[
\|\phi_R u_n\|_{L^q(\mathbb{R}^N)} \geq \min \left\{ \left( \int_{B_{R_2}^c} \phi_R^{q(x)} |u_n|^{q(x)} \, dx \right)^{\frac{1}{q(x)}}, \left( \int_{B_{R_2}^c} \phi_R^{q(x)} |u_n|^{q(x)} \, dx \right)^{\frac{1}{q(x)}} \right\}
\] (4.83)

for all \(R > R_2\). Finally, by (4.77), we can find \(R_2 > R_2\) such that
\[
\limsup_{n \to \infty} \int_{B_{R_2}} \phi_R^{q(x)} |u_n|^{q(x)} \, dx = \limsup_{n \to \infty} \int_{B_{R_2}} \phi_R^{q(x)} |u_n|^{q(x)} \, dx > \frac{V_{s,q}}{2}
\] (4.84)

for all \(R > R_2\). From (4.83) and (4.84) we get
\[
\limsup_{n \to \infty} \|\phi_R u_n\|_{L^q(\mathbb{R}^N)} \geq \min \left\{ \left( \frac{V_{s,q}}{2} \right)^{\frac{1}{\nu_{s,q}}}, \left( \frac{V_{s,q}}{2} \right)^{\frac{1}{\nu_{s,q}}} \right\}
\]

and hence, by (4.82),
\[
\limsup_{n \to \infty} \|\phi_R u_n\|_{L^q(\mathbb{R}^N)} \geq \left( \frac{V_{s,q}}{4} \right)^{\frac{1}{\nu_{s,q}}}
\]

for all \(R > R_2\). This and (4.79) yield \(S_q \left( \frac{1}{q} \nu_{s,q} \right)^{\frac{1}{q}} \leq \sigma_{s,R} \) for all \(R \in (R_2, \infty)\). By a similar argument to that obtained (4.67), invoking Lemma 4.5 and choosing \(R_2\) larger if necessary, we can show that there exists \(\sigma \in (0, \infty)\) such that \(\sigma_{s,R} < \sigma\) for all \(R \in (R_2, \infty)\). Thus, (4.81) has been proved.

We now turn to estimate the right-hand side of (4.79). For each \(R > R_2\) given, let \(n_k = n_k(R)\) \((k = 1, 2, \cdots)\) be a sequence such that
\[
\lim_{k \to \infty} \sigma_{n_k,R} = \limsup_{n \to \infty} \sigma_{n,R} = \sigma_{s,R}.
\] (4.85)

Utilizing Proposition 3.1 and (4.52) again, we have
\[
1 = \int_{\mathbb{R}^N} \frac{|\phi_R(x)u_n(x)|^p}{\sigma_{n,R}^p} \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(\phi_R u_n)(x) - (\phi_R u_n)(y)|^{p(x,y)}}{\sigma_{n,R}^{p(x,y)} |x - y|^{N + sp(x,y)}} \, dx \, dy
\]
\[
= \int_{\mathbb{R}^N} \frac{|\phi_R(x)|^p u_n(x)^p}{\sigma_{n,R}^p} \, dx + 2 \int_{B_{R_2}} \int_{B_{R_2}} \frac{|\phi_R^{p(x)}u_n(x)|^{p(x,y)}}{\sigma_{n,R}^{p(x,y)} |x - y|^{N + sp(x,y)}} \, dy \, dx
\]
\[
+ \int_{B_{R_2}} \int_{B_{R_2}^c} \frac{|(\phi_R u_n)(x) - (\phi_R u_n)(y)|^{p(x,y)}}{\sigma_{n,R}^{p(x,y)} |x - y|^{N + sp(x,y)}} \, dy \, dx
\]
\[
\leq \int_{B_{R_2}} \frac{|\phi_R(x)|^p u_n(x)^p}{\sigma_{n,R}^p} \, dx + 2 \int_{B_{R_2}} \int_{B_{R_2}} \frac{|u_n(x)|^{p(x,y)} |\phi_R(x) - \phi_R(y)|^{p(x,y)}}{\sigma_{n,R}^{p(x,y)} |x - y|^{N + sp(x,y)}} \, dy \, dx
\]
\[
+ C(\varepsilon) \int_{B_{R_2}} \int_{B_{R_2}^c} \frac{|u_n(x)|^{p(x,y)} |\phi_R(x) - \phi_R(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dy \, dx
\]
\[
+ (1 + \varepsilon) \int_{B_{R_2}} \int_{B_{R_2}} \frac{|\phi_R(y)|^{p(x,y)} |u_n(x) - u_n(y)|^{p(x,y)}}{\sigma_{n,R}^{p(x,y)} |x - y|^{N + sp(x,y)}} \, dx \, dy.
\]
This and the fact that \( 0 \leq \phi_R \leq 1 \) yield

\[
1 \leq \frac{C(\epsilon)}{\min\{\sigma_{n,R}^{p}, \sigma_{n,R}^{p-}\}} \int_{B_n^c} \int_{\mathbb{R}^N} \frac{|\phi_R(x) - \phi_R(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \, dx
+ \frac{1+\epsilon}{\min\{\sigma_{n,R}^{p+e}, \sigma_{n,R}^{p-e}\}} \int_{B_n^c} \phi_R(x)U_n(x) \, dx.
\]

Taking limit superior as \( k \to \infty \) in the last inequality with noticing (4.81) and (4.85) we obtain

\[
1 \leq \frac{C(\epsilon)}{\min\{\sigma_{n,R}^{p+e}, \sigma_{n,R}^{p-e}\}} \limsup_{n \to \infty} \int_{B_n^c} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dy \, dx
+ \frac{1+\epsilon}{\min\{\sigma_{n,R}^{p+e}, \sigma_{n,R}^{p-e}\}} \limsup_{n \to \infty} \int_{B_n^c} \phi_R(x)U_n(x) \, dx.
\] (4.86)

Now, taking the limit as \( R \to \infty \) in (4.86) with taking Lemma 4.5 and (4.74) into account, we deduce

\[
1 \leq \frac{1+\epsilon}{\min\{\sigma_{n,R}^{p+e}, \sigma_{n,R}^{p-e}\}} \mu_{\infty}, \text{ i.e., } \sigma^{*} \leq \left(1 + \epsilon\right)^{\frac{1}{p^*}} \max \left\{ \frac{\mu_{\infty}}{\mu_{\infty}}, \frac{\mu_{\infty}}{\mu_{\infty}} \right\},
\]

where \( \sigma^{*} := \liminf_{R \to \infty} \sigma^{*}_R \) and hence, \( 0 < \sigma^{*} < \sigma \) due to (4.81). From this, (4.79) and (4.80) we obtain

\[
S_q \min \left\{ \frac{\mu_{\infty}}{\mu_{\infty}}, \frac{\mu_{\infty}}{\mu_{\infty}} \right\} \leq \left(1 + \epsilon\right)^{\frac{1}{p^*}} \max \left\{ \frac{\mu_{\infty}}{\mu_{\infty}}, \frac{\mu_{\infty}}{\mu_{\infty}} \right\}.
\]

Since \( \epsilon \) was chosen arbitrarily in the last inequality, (4.14) follows. The proof of Theorem 4.2 is complete.

\[ \square \]

## 5 Application

### 5.1 The existence of solutions

In this section, we investigate the existence and multiplicity of solutions to the following problem

\[
\begin{align*}
\mathcal{L} u + |u|^{p(x)-2} u &= f(x, u) + \lambda |u|^{q(x)-2} u \quad \text{in } \mathbb{R}^N, \\
u &\in W^{p, p(x)}(\mathbb{R}^N),
\end{align*}
\] (5.1)

where \( s, p, q \) satisfy \((P_2), (Q_2)\) and \((\mathcal{E}_\infty)\) with \( p^* < q^* \); the operator \( \mathcal{L} \) is defined as in (1.2); \( \lambda \) is a positive real parameter; and the nonlinear term \( f \) satisfies the following assumptions.

1. \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that \( f \) is odd with respect to the second variable.

2. \( \text{There exist functions } r_j, a_j \text{ with } r_j \in C_c(\mathbb{R}^N), \text{ inf } q(x) - r_j(x) > 0, a_j \in L^{q(x)}(\mathbb{R}^N) (j = 1, \cdots, m), \text{ and } \max_{1 \leq j \leq m} r_j > p^- \) such that

\[
|f(x, u)| \leq \sum_{j=1}^{m} a_j(x)|u|^{r_j(x)-1} \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and all } u \in \mathbb{R}.
\]

3. \( \text{There exist } B_\epsilon(x_0) \text{ and } a \in L^{\frac{q(x)}{r_j(x)}}(B_\epsilon(x_0)) \text{ such that } \sup_{|u| < M} |F(x, u)| \in L^1(B_\epsilon(x_0)) \text{ for each } M > 0, \text{ and } \lim_{|u| \to \infty} \frac{F(x, u)}{|u|^{p^*}} = \infty \text{ uniformly for a.e. } x \in B_\epsilon(x_0), \)

where \( F(x, u) := \int_0^u f(x, r) \, dr. \)
Lemma 5.2 which is a variant of Theorem 2.19 in [33] (see also [34, Theorem 10.20]).

In order to prove Theorem 5.1, we will make use of the following abstract result for symmetric $C^1$ functionals, which is a variant of Theorem 2.19 in [33] (see also [34, Theorem 10.20]).

Lemma 5.2 ([33]). Let $E = V \oplus X$, where $E$ is a real Banach space and $V$ is finite dimensional. Suppose that $J \in C^1(E, \mathbb{R})$ is an even functional satisfying $J(0) = 0$ and

(J1) there exist constants $\rho, \beta > 0$ such that $J(u) \geq \beta$ for all $u \in \partial B_\rho \cap X$;
(J2) there exists a subspace $E$ of $E$ with $\dim V < \dim E < \infty$ and $\{ u \in \hat{E} : J(u) \geq 0 \}$ is bounded in $E$;
(J3) for $\beta$ and $\hat{E}$ respectively given in (J1) and (J2), $J$ satisfies the (PS) condition for any $c \in [0, L]$ with $L := \sup_{u \in \hat{E}} J(u)$.

Then $J$ possesses at least $\dim \hat{E} - \dim V$ pairs of nontrivial critical points.

Proof. The proof is similar to that of [34, Theorem 10.20] for which we take $E^m$ in the proof of [34, Lemma 10.19] as $E^m = \text{span}\{e_1, \cdots, e_m\}$, where $\{e_k\}_{k=1}^{\dim \hat{E}}$ is a basis of $\hat{E}$. \qed

To determine solutions to problem (5.1), we will apply Lemma 5.2 for $E := W^{s,p}(-\cdot)(\mathbb{R}^N)$ endowed with the norm $\| \cdot \| := \| \cdot \|_{s,p}$ and $J = J_\lambda$, where $J_\lambda : E \to \mathbb{R}$ is the energy functional associated with problem (5.1) defined as

$$J_\lambda(u) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}|p(x,y)|}{|x-y|^{N+sp(x,y)}} \ dx \ dy + \int_{\mathbb{R}^N} \frac{1}{p(x)} |u|^p \ dx - \int_{\mathbb{R}^N} F(x, u) \ dx - \lambda \int_{\mathbb{R}^N} \frac{1}{q(x)} |u|^q \ dx, \ u \in E. \quad (5.2)$$

It is clear that under the assumptions (J1) – (J2), $J_\lambda$ is of class $C^1(E, \mathbb{R})$ and its Fréchet derivative $J'_\lambda : E \to E^*$ is given by

$$(J'_\lambda(u), v) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp(x,y)}} \ dx \ dy + \int_{\mathbb{R}^N} |u|^{p-2} uv \ dx.$$
From (5.5) and (5.6) we obtain by the relation between modular and norm (see Proposition 3.1) and
\[ C \]
Here in the remaining proof, for any given \( \lambda > 0 \), \( J_\lambda \) satisfies the (PS)_c condition provided
\[ c \left( \frac{1}{\alpha} - \frac{1}{q} \right) \min \left\{ S_q^{(q)} \right\} \min \left\{ \lambda^{-h'}, \lambda^{-h} \right\} \geq \frac{\| g \|_1}{\alpha}, \]
(5.4)
where \( h(x) := \frac{P}{q(x)} \) for \( x \in \mathbb{R}^N \) and \( S_q \) is defined as in (4.3).

**Proof.** Let \( \{ u_n \} \) be a (PS)_c sequence for \( J_\lambda \) with \( c \) satisfying (5.4). We first claim that \( \{ u_n \} \) is bounded in \( E \). Indeed, by (5.4) and invoking Proposition 3.1 we have that for \( n \) large,
\[
c + 1 + \| u_n \| \geq J_\lambda(u_n) - \frac{1}{\alpha} J'_\lambda(u_n), u_n
\]
\[
\geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( \frac{1}{p(x,y)} - \frac{1}{q} \right) \frac{u_n(x) - u_n(y) p(x,y)}{|x - y|^{N + sp(x,y)}} \ dx \ dy + \int_{\mathbb{R}^N} \left( \frac{1}{\alpha} - \frac{1}{q} \right) |u|^q \ dx
\]
\[
+ \lambda \int_{\mathbb{R}^N} \left( \frac{1}{\alpha} - \frac{1}{q(x)} \right) |u_n|^{q(x)} \ dx + \int_{\mathbb{R}^N} \left[ \lambda f(x, u_n) - F(x, u_n) \right] \ dx
\]
\[
\geq \lambda \left( \frac{1}{\alpha} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u_n|^{q(x)} \ dx - \frac{1}{\alpha} \| g \|_1.
\]
That is,
\[
\lambda \left( \frac{1}{\alpha} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u_n|^{q(x)} \ dx \leq C_1 + \| u_n \|, \text{ for all } n \in \mathbb{N} \text{ large.}
\]
(5.5)
Here and in the remaining proof, \( C_i \) \( (i \in \mathbb{N}) \) denotes a positive constant independent of \( n \). On the other hand, by the relation between modular and norm (see Proposition 3.1) and (5.2) we have that for \( n \) large,
\[
\frac{1}{\alpha} \left( \| u_n \|^{p'} - 1 \right) \leq J_\lambda(u_n) + \int_{\mathbb{R}^N} F(x, u_n) \ dx + \lambda \int_{\mathbb{R}^N} \frac{1}{q(x)} |u_n|^{q(x)} \ dx
\]
\[
\leq c + 1 + \sum_{j=1}^m \int_{\mathbb{R}^N} \frac{a_j(x)}{r_j(x)} |u_n|^{r_j(x)} \ dx + \lambda \int_{\mathbb{R}^N} |u_n|^{q(x)} \ dx.
\]
Then, by the Young inequality we easily get
\[
\| u_n \|^{p'} \leq C_2 \left( 1 + \int_{\mathbb{R}^N} |u_n|^{q(x)} \ dx \right).
\]
(5.6)
From (5.5) and (5.6) we obtain
\[
\| u_n \|^{p'} \leq C_3 \left( 1 + \| u_n \| \right), \text{ for all } n \in \mathbb{N} \text{ large.}
\]
This implies the boundedness of \( \{ u_n \} \) since \( p' > 1 \) and hence,
\[
C_* := \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \ dx \ dy + \int_{\mathbb{R}^N} |u_0|^{p'} \ dx < \infty
\]
(5.7)
in view of Proposition 3.1. Then, invoking Theorems 3.2, 4.1 and 4.2, up to a subsequence, we have
\[
u_n(x) \to u(x) \text{ for a.e. } x \in \mathbb{R}^N,
\]
(5.8)
\[
u_n \to u \text{ in } E
\]
(5.9)
\[
U_n(x) \to^* \mu U(x) + \sum_{i \in I} \mu_i \delta_{x_i} \text{ in } M(\mathbb{R}^N)
\]
(5.10)
\[ |u_n|^{q(x)} \rightharpoonup v = |u|^{q(x)} + \sum_{i \in I} v_i \delta_{x_i} \text{ in } M(\mathbb{R}^N), \]  
\[ S_q v_i^{\frac{1}{q(x)}} \leq \mu_i^{\frac{1}{q(x)}}, \quad \forall i \in I, \]  
where \( U_n(x) := |u_n(x)|^q + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^q(x,y)}{|x-y|^{sp(x,y)}} \, dy \) and \( U(x) := |u(x)|^q + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q(x,y)}{|x-y|^{sp(x,y)}} \, dy \) for \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^N \). Moreover, we have

\[ \limsup_{n \to \infty} \int_{\mathbb{R}^N} U_n(x) \, dx = \mu(\mathbb{R}^N) + \mu_\infty, \]  
\[ \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{q(x)} \, dx = v(\mathbb{R}^N) + v_\infty, \]  
\[ S_q v_\infty^{\frac{1}{q(x)}} \leq \mu_\infty. \]  

We will show that \( I = \emptyset \) and \( v_\infty = 0 \). For this purpose we invoke (\( F4 \)) to estimate

\[ c = \lim_{n \to \infty} I_1(u_n) - \frac{1}{\alpha} I_2(u_n, u_n) \geq \left( \frac{1}{\alpha} - \frac{1}{q} \right) \lambda \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{q(x)} \, dx - \frac{||g||_1}{\alpha}. \]  

Combining this with (5.14) gives

\[ c \geq \left( \frac{1}{\alpha} - \frac{1}{q} \right) \lambda \left[ v(\mathbb{R}^N) + v_\infty \right] - \frac{||g||_1}{\alpha}. \]  

We now suppose on the contrary that \( I \neq \emptyset \). Let \( i \in I \) and for \( \rho > 0 \), define \( \psi_\rho \) as in Lemma 4.4 with \( x_0 \) replaced by \( x_i \). For an arbitrary and fixed \( \rho \), it is not difficult to see that \( \{u_n \psi_\rho\} \) is a bounded sequence in \( E \). Hence,

\[ o_n(1) = \langle f_\rho(u_n), u_n \psi_\rho \rangle = \int_{\mathbb{R}^N} \psi_\rho U_n \, dx - \lambda \int_{\mathbb{R}^N} \psi_\rho |u_n|^{q(x)} \, dx - \int_{\mathbb{R}^N} f(x, u_n) u_n \psi_\rho \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^{q(x,y) - 2}(u_n(x) - u_n(y))u_n(y)(\psi_\rho(x) - \psi_\rho(y)) \frac{1}{|x-y|^{sp(x,y)}} \, dx \, dy. \]  

This yields

\[ \left| \int_{\mathbb{R}^N} \psi_\rho(x) \, d\mu - \lambda \int_{\mathbb{R}^N} \psi_\rho(x) \, dv \right| \leq \limsup_{n \to \infty} |I_1(n, \psi_\rho)| + \limsup_{n \to \infty} |I_2(n, \psi_\rho)|, \]  

where

\[ I_1(n, \psi_\rho) := \int_{\mathbb{R}^N} f(x, u_n) u_n \psi_\rho \, dx \]  

and

\[ I_2(n, \psi_\rho) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{q(x,y) - 2}(u_n(x) - u_n(y))u_n(y)(\psi_\rho(x) - \psi_\rho(y))}{|x-y|^{sp(x,y)}} \, dx \, dy. \]  

Note that the boundedness of \( \{u_n\} \) in \( E \) implies the boundedness of \( \{u_n\} \) in \( L^{q(x)}(\mathbb{R}^N) \) due to Theorem 3.3. Hence, from (\( F2 \)) and invoking Propositions 2.2 and 2.3 we have

\[ |I_1(n, \psi_\rho)| \leq \sum_{j=1}^{m} \int_{\mathbb{R}^N} a_j(x)|u_n|^{q(x)} \psi_\rho \, dx \]  

\[ \leq \sum_{j=1}^{m} 2||a_j||_{L^{q(x)\cap (B_{B^2}(x_i))}} |||u_n||_{L^{q(x)}(B_{B^2}(x_i))} \]  

\[ \leq \sum_{j=1}^{m} 2||a_j||_{L^{q(x)\cap (B_{B^2}(x_i))}} \left[ 1 + ||u_n||_{L^{q(x)}(\mathbb{R}^N)} \right] \]  

\[ \leq C_n \sum_{j=1}^{m} ||a_j||_{L^{q(x)\cap (B_{B^2}(x_i))}}, \quad \forall n \in \mathbb{N}. \]  

\[ (5.18) \]
Here and in the remaining proof, $C_i$ ($i \in \mathbb{N}$) denotes a positive constant independent of $n$ and $\rho$. From (5.18), we obtain
\[
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} |I_1(n, \psi_\rho)| = 0. \tag{5.19}
\]

In order to estimate $I_2(n, \psi_\rho)$, let $\delta > 0$ be arbitrary and fixed. By (5.7) and the Young inequality we have
\[
|I_2(n, \psi_\rho)| \leq \delta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy + C_5 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi_\rho(x) - \psi_\rho(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy \leq C_\delta + C_5 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi_\rho(x) - \psi_\rho(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy. \tag{5.20}
\]

Taking limit superior in (5.20) as $n \to \infty$ then taking limit superior as $\rho \to 0^+$ with taking Lemma 4.4 into account, we arrive at
\[
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} |I_2(n, \psi_\rho)| \leq C \cdot \delta. \tag{5.21}
\]

Since $\delta > 0$ was chosen arbitrarily we obtain
\[
\limsup_{\rho \to 0^+} \limsup_{n \to \infty} |I_2(n, \psi_\rho)| = 0. \tag{5.22}
\]

Now, by taking limit superior in (5.17) as $\rho \to 0^+$ with taking (5.19) and (5.21) into account, we obtain
\[
\mu_i = \lambda \nu_i.
\]

Plugging this into (5.12) we get
\[
\mu_i \geq S^{\frac{\alpha_i}{\alpha(q)}} \lambda^{-\frac{\alpha - 1}{\alpha(q)}}.
\]

This yields
\[
\lambda \nu_i = \mu_i \geq \min\{S^{\alpha(q)}\} \min\{\lambda^{h'}, \lambda^{-h'}\}. \tag{5.23}
\]

From (5.22) and (5.16), we obtain
\[
c \geq \left(\frac{1}{a} - \frac{1}{q'}\right) \lambda \nu_i - \frac{\|g\|_1}{a} \geq \left(\frac{1}{a} - \frac{1}{q}\right) \min\{S^{\alpha(q)}\} \min\{\lambda^{h'}, \lambda^{-h'}\} - \frac{\|g\|_1}{a},
\]

which is a contradiction to (5.4), and hence, $I = \emptyset$. Next, we prove that $v_{\infty} = 0$. Suppose on the contrary that $v_{\infty} > 0$. Let $\phi_R$ be the same in Lemma 4.5. Using a similar argument to that obtained (5.17), we arrive at
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \phi_R u_n \, dx \leq \lambda \limsup_{n \to \infty} \int_{\mathbb{R}^N} \phi_R |u_n|^{q(x)} \, dx + \limsup_{n \to \infty} |I_3(n, \phi_R)| + \limsup_{n \to \infty} |I_6(n, \phi_R)|, \tag{5.24}
\]

and
\[
\lambda \limsup_{n \to \infty} \int_{\mathbb{R}^N} \phi_R |u_n|^{q(x)} \, dx \leq \limsup_{n \to \infty} \int_{\mathbb{R}^N} \phi_R u_n \, dx + \limsup_{n \to \infty} |I_3(n, \phi_R)| + \limsup_{n \to \infty} |I_6(n, \phi_R)|, \tag{5.25}
\]

where
\[
I_3(n, \phi_R) := \int_{\mathbb{R}^N} f(x, u_n) \phi_R \, dx
\]

and
\[
I_6(n, \phi_R) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}(u_n(x) - u_n(y))\phi_R(x) - \phi_R(y)}{|x - y|^{N + sp(x,y)}} \, dx \, dy.
\]

\[
|I_2(n, \psi_\rho)| \leq \frac{C_5}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi_\rho(x) - \psi_\rho(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy + C_5 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy.
\]
We claim that
\[
\lim_{R \to \infty} \limsup_{n \to \infty} |I_4(n, \phi_R)| = 0
\]  
(5.25)
and
\[
\lim_{R \to \infty} \limsup_{n \to \infty} |I_5(n, \phi_R)| = 0.
\]  
(5.26)
Indeed, the equality (5.25) can be obtained in a similar fashion to (5.19). To prove (5.26), we proceed as in (5.20) to get
\[
|I_4(n, \phi_R)| \leq C_\delta + C_5 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u_n(y)|^{p(x,y)} \frac{|\phi_R(x) - \phi_R(y)|}{|x - y|^N} \, dx \, dy
\]
for each \( \delta > 0 \) arbitrary and fixed. Taking limit superior in the last estimate as \( n \to \infty \) and then taking limit as \( R \to \infty \) with taking Lemma 4.5 into account, we obtain
\[
\lim_{R \to \infty} \limsup_{n \to \infty} |I_4(n, \phi_R)| \leq C_\delta.
\]
Since \( \delta > 0 \) in the last inequality can be taken arbitrarily we deduce (5.26). Using (5.25) and (5.26) and letting \( R \to \infty \) in (5.23) and (5.24) we obtain
\[
\mu_\infty = \lambda \nu_\infty.
\]  
(5.27)
Here we have used (4.74) and the fact that
\[
\nu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{q(x)} \phi_R \, dx,
\]
which can be seen by using \( \phi_R \) in place of \( \phi_R^{q(x)} \) in (4.76). Combining (5.27) with (5.15) gives
\[
\lambda \nu_\infty = \mu_\infty \geq S \frac{N \alpha}{N - \alpha} A - \frac{1}{\alpha}.
\]  
(5.28)
The fact that \( q_\infty = \lim_{|x| \to \infty} q(x) \in [q^-, q^+] \) yields
\[
(qh)^- \leq \frac{q_\infty \bar{p}}{q_\infty - \bar{p}} \leq (qh)^+ \text{ and } \bar{h}^- \leq \frac{\bar{p}}{q_\infty - \bar{p}} \leq \bar{h}^+.
\]
From this and (5.28) one has
\[
\lambda \nu_\infty \geq \min \{ S(qh)^-, S(qh)^+ \} \min \{ \lambda^{-h^-}, \lambda^{-h^+} \}.
\]
Utilizing this estimate we deduce from (5.16) that
\[
c \geq \left( \frac{1}{a} - \frac{1}{q} \right) \lambda \nu_\infty - \| g \|_1 a \geq \left( \frac{1}{a} - \frac{1}{q} \right) \min \{ S(qh)^-, S(qh)^+ \} \min \{ \lambda^{-h^-}, \lambda^{-h^+} \} - \| g \|_1 a,
\]
which is a contradiction to (5.4), and hence; \( \nu_\infty = 0 \).
Combining the facts that \( I = 0 \) and \( \nu_\infty = 0 \) with (5.11) and (5.14), we obtain
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{q(x)} \, dx = \int_{\mathbb{R}^N} |u|^{q(x)} \, dx.
\]
Invoking the Fatou lemma we get from (5.8) that
\[
\int_{\mathbb{R}^N} |u|^{q(x)} \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{q(x)} \, dx.
\]
Thus,
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{q(x)} \, dx = \int_{\mathbb{R}^N} |u|^{q(x)} \, dx.
\]
By a Brezis-Lieb type result for the Lebesgue spaces with variable exponents (see e.g., [27, Lemma 3.9]), it follows from the last equality and (5.8) that
\[ \int_{\mathbb{R}^N} |u_n - u|^{q(x)} \, dx \to 0, \quad \text{i.e., } u_n \to u \text{ in } L^{q(\cdot)}(\mathbb{R}^N). \]

Consequently, we have \( \int_{\mathbb{R}^N} |u_n|^{p(x)} - 2 u_n (u_n - u) \, dx \to 0 \) by invoking Proposition 2.3 and the boundedness of \( \{u_n\} \) in \( L^{q(\cdot)}(\mathbb{R}^N) \). Also, we easily obtain \( \int_{\mathbb{R}^N} f(x, u_n) (u_n - u) \, dx \to 0 \) by using (5.2), (5.9), Proposition 2.3 and Theorem 3.4. We therefore have
\[ \begin{align*}
&\int_{\mathbb{R}^N} |u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y)) \left( ((u_n - u)(x) - (u_n - u)(y)) \right) \, dx \, dy \\
&+ \int_{\mathbb{R}^N} |u_n|^{q(x)} - 2 u_n (u_n - u) \, dx = (f_\lambda(u_n), u_n - u) + \int_{\mathbb{R}^N} f(x, u_n) (u_n - u) \, dx \\
&\quad + \lambda \int_{\mathbb{R}^N} |u_n|^{q(x)} - 2 u_n (u_n - u) \, dx \to 0.
\end{align*} \]

Hence, \( u_n \to u \) in \( E \) in view of [9, Lemma 4.2 (i)]. The proof is complete. \( \square \)

We now conclude this section by proving Theorem 5.1.

**Proof of Theorem 5.1.** We will show that conditions (j1)–(j3) of Lemma 5.2 are fulfilled with \( E := W^{s,p(\cdot)}(\mathbb{R}^N) \) and \( I = I_1 \). In order to verify (j1), let \( \{e_n\}_{n=1}^{\infty} \) be a Schauder basis of \( E \) and let \( \{e_n^*\}_{n=1}^{\infty} \subset E' \) be such that for each \( n \in \mathbb{N} \),
\[ e_n^*(u) = a_n \quad \text{for } u = \sum_{k=1}^{\infty} a_k e_k \in E. \]

For each \( k \in \mathbb{N} \), define
\[ V_k := \{ u \in E : e_n^*(u) = 0, \quad \forall n > k \}, \]
\[ X_k := \{ u \in E : e_n^*(u) = 0, \quad \forall n \leq k \}, \]
and
\[ \xi_k := \sup_{u \in X_k} \max_{1 \leq j \leq m} \|u\|_{L^{q_j}(\mathbb{R}^N)} \] (5.29)

Then,
\[ E = V_k \oplus X_k, \quad \forall k \in \mathbb{N}. \]

Since \( X_{k+1} \subset X_k \) (\( k \in \mathbb{N} \)), we have
\[ 0 \leq \xi_{k+1} \leq \xi_k, \quad \forall k \in \mathbb{N}. \]

Thus, the sequence \( \{\xi_k\} \) converges to some \( \xi^* \) as \( k \to \infty \). We claim that \( \xi^* = 0 \). Indeed, for each \( k \in \mathbb{N} \) there exists \( u_k \in X_k \) such that \( \|u_k\| \leq 1 \) and
\[ 0 \leq \xi_k \leq \max_{1 \leq j \leq m} \|u_k\|_{L^{q_j}(\mathbb{R}^N)} + \frac{1}{k}. \] (5.30)

Since \( \{u_k\} \) is bounded in \( E \), up to a subsequence we have
\[ u_k \to u \quad \text{in } E \] (5.31)
and hence, by Theorem 3.4,
\[ u_k \to u \quad \text{in } L^{q_j}(\mathbb{R}^N), \quad \forall j \in \{1, \cdots, m\}. \] (5.32)

From (5.31) and the definition of \( X_k \) we have that for any \( n \in \mathbb{N} \),
\[ \langle e_n^*, u \rangle = \lim_{k \to \infty} \langle e_n^*, u_k \rangle = 0. \] (5.33)
This yields \( u = 0 \). This fact together with (5.32) and (5.30) infer \( \xi = 0 \). That is, we have just proved that

\[
\lim_{k \to \infty} \xi_k = 0. \tag{5.34}
\]

For \( u \in X_k \) with \( \|u\| = \rho_k > 1 \), by (\( J_2 \)) and invoking Proposition 2.2 and Theorem 3.4 we have

\[
J_\lambda(u) \geq \frac{1}{\beta} \left( \|u\|^p - 1 \right) - \sum_{j=1}^{m} \frac{1}{r_j} \int_{\mathbb{R}^N} a_j(x) |u|^{r_j(x)} \, dx - \frac{\lambda}{q^*} \int_{\mathbb{R}^N} |u|^{q^*} \, dx
\]

\[
\geq \frac{1}{\beta} \|u\|^p - \frac{1}{\beta} \frac{1}{r^*} \sum_{j=1}^{m} \left( \|u\|^{r_j}_{L^{r_j}(\mathbb{R}^N)} + 1 \right) - \frac{\lambda}{q^*} \max \left\{ \|u\|^{q^*}_{L^{q^*}(\mathbb{R}^N)}, \|u\|^{q^*}_{L^{q^*}(\mathbb{R}^N)} \right\}
\]

\[
\geq \frac{1}{\beta} \|u\|^p - \frac{m}{\beta} \xi_k^p \|u\|^r - \left( \frac{1}{\beta} + \frac{m}{r^*} \right) - \frac{\max \{s_q^{q^*}, s_q^{q^*} \}}{q^*} \lambda \|u\|^{q^*},
\]

where \( r^* := \min \frac{1}{r_j} \) and \( r^* := \max \frac{1}{r_j} \). Let \( \rho_k > 0 \) be such that

\[
\frac{m}{\beta} \xi_k^p \frac{r^*}{r} = \frac{1}{\beta^2} \rho_k^p \text{ i.e., } \rho_k = \left( \frac{r^*}{2mp^*} \right)^{\frac{1}{p}}.
\]

Note that \( \xi_k \to 0 \) as \( k \to \infty \) by (5.34) and hence, \( \rho_k \to \infty \) as \( k \to \infty \). Thus, we can fix \( k_0 \in \mathbb{N} \) such that

\[
\rho_{k_0} > 1 \text{ and } \frac{1}{\beta^2} \rho_{k_0}^p - \left( \frac{1}{\beta} + \frac{m}{r^*} \right) > \frac{1}{\beta^2} \rho_{k_0}^p.
\]

Then, (5.35) yields

\[
J_\lambda(u) \geq \frac{1}{\beta^2} \rho_{k_0}^p \max \left\{ S_q^{q^*}, S_q^{q^*} \right\} \rho_{k_0}^p \lambda, \quad \forall u \in X_{k_0} \cap B_{\rho_{k_0}}. \tag{5.37}
\]

Therefore, by choosing \( V := V_{k_0}, X := X_{k_0} \) and \( \lambda^* := \frac{q^* \rho_{k_0}^p}{q^* \max \{S_q^{q^*}, S_q^{q^*} \}}, \) we have that for any \( \lambda \in (0, \lambda^*), \)

\[
J_\lambda(u) \geq \beta, \quad \forall u \in X \cap B_{\rho}
\]

with \( \rho = \rho_{k_0} \) and \( \beta = \max \{S_q^{q^*}, S_q^{q^*} \} \rho_{k_0}^p (\lambda^* - \lambda). \) That is, \( J_\lambda \) verifies (\( J_1 \)) in Lemma 5.2.

Next, we show that \( J_\lambda \) verifies (\( J_2 \)) and (\( J_3 \)) in Lemma 5.2. Let \( \varphi_k \) be the \( k \)th eigenpair of the following eigenvalue problem

\[
\begin{cases}
-\Delta u = \gamma u & \text{in } B_{x_0}, \\
u = 0 & \text{on } \partial B_{x_0}.
\end{cases}
\]

Extend \( \varphi_k(x) \) to \( \mathbb{R}^N \) by putting \( \varphi_k(x) = 0 \) for \( x \in \mathbb{R}^N \setminus B_{x_0} \). Clearly, \( \{ \varphi_k \} \subset E \). Define

\[
E_k := \text{span} \{ \varphi_1, \ldots, \varphi_k \} \quad (k \in \mathbb{N}).
\]

Let \( k \in \mathbb{N} \) be arbitrary and fixed. We claim that there exists \( R_k > 0 \) independent of \( \lambda \) such that

\[
J_\lambda(u) \leq 0, \quad \forall u \in E_k \setminus B_{R_k}. \tag{5.38}
\]

Indeed, since all norms on \( E_k \) are equivalent we can find \( \zeta_k > 0 \) such that

\[
\zeta_k \|u\| \leq \|u\|_{L^p(a, B_{x_0})}, \quad \forall u \in E_k. \tag{5.39}
\]

Choose \( \theta_k > 0 \) such that

\[
\frac{1}{\beta^2} - \theta_k \zeta_k^p < 0. \tag{5.40}
\]
By (3), we can choose \( M_k > 0 \) such that
\[
F(x, u) \geq \theta_k a(x) |u|^p \quad \text{for a.e. } x \in B_k(x_0) \text{ and all } |u| \geq M_k.
\]
This infers
\[
F(x, u) \geq \theta_k a(x) |u|^p - \sup_{|u| \leq M_k} |F(x, u)| \quad \text{for a.e. } x \in B_k(x_0) \text{ and all } u \in \mathbb{R}.
\] (5.41)
From (5.40), (5.41) and invoking Proposition 2.2 again, we have that for any \( u \in E_k \) with \( \|u\| \geq R_k > 1 \),
\[
J_\lambda(u) \leq \frac{1}{p} |u|^p - \theta_k \int_{B_k(x_0)} a(x) |u|^p \, dx + \int_{B_k(x_0)} \sup_{|t| \leq M_k} |F(x, t)| \, dx
\leq \left( \frac{1}{p} - \theta_k c_k^p \right) |u|^p + \int_{B_k(x_0)} \sup_{|t| \leq M_k} |F(x, t)| \, dx < 0
\]
provided \( R_k \) large enough. Clearly, \( R_k \) can be chosen independently of \( \lambda \). That is, we have just obtained (5.38).

Noting \( J_\lambda(0) = 0 \), we deduce from (5.38) that
\[
\sup_{u \in E_k} J_\lambda(u) = \max_{\|u\| \leq R_k} J_\lambda(u) \leq \max_{u \in B_k} \left\{ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p(x,y)}{|x-y|^{N+sp(x,y)}} \, dx \, dy + \int_{\mathbb{R}^N} \frac{1}{p} |u|^p \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx \right\} =: L_k.
\]
It is clear that for all \( k \in \mathbb{N} \), \( L_k \) is independent of \( \lambda \) and \( L_k \in [0, \infty) \) due to (3). Finally, let \( \{\lambda_k\}_{k=1}^\infty \subset (0, \lambda_\ast) \) be such that for any \( k \in \mathbb{N} \),
\[
\begin{cases}
L_{k_0+k} < \left( \frac{1}{\alpha} - \frac{1}{q} \right) \min \left\{ S_q^{(h^\ast)}, S_q^{(h^\ast)} \right\} \min \left\{ \lambda_{k_0}^{h^\ast}, \lambda_{k_0}^{h^\ast} \right\} - \|g\|_1 \alpha, \\
\lambda_{k+1} \leq \lambda_k.
\end{cases}
\] (5.42)
Then, for any \( \lambda \in (\lambda_{k+1}, \lambda_k) \) we have
\[
L_{k_0+k} < \left( \frac{1}{\alpha} - \frac{1}{q} \right) \min \left\{ S_q^{(h^\ast)}, S_q^{(h^\ast)} \right\} \min \left\{ \lambda_{k}^{h^\ast}, \lambda_{k}^{h^\ast} \right\} - \|g\|_1 \alpha
\]
and hence, \( J_\lambda \) satisfies the (PS)_\( c \) for any \( c \in [0, L_{k_0+k}] \) in view of Lemma 5.3. Thus, \( J_\lambda \) satisfies (2) and (3) with \( \bar{E} = E_{k_0+k} \) and \( L = L_{k_0+k} \). So, \( J_\lambda \) admits at least \( \dim \bar{E} - \dim V = k \) pairs of nontrivial critical points in view of Lemma 5.2; hence, problem (5.1) has at least \( k \) pairs of nontrivial solutions. The proof is complete.

\[\square\]

A An auxiliary result

In this appendix, we state a result for the Radon measures on \( \mathbb{R}^N \), which is necessary for proving Theorem 4.1.

**Proposition A.1.** Let \( p, q, r \in C_\ast(\mathbb{R}^N) \) such that \( \inf_{x \in \mathbb{R}^N} \{ r(x) - \max\{p(x), q(x)\} \} > 0 \). Let \( \mu, \nu \) be two finite and nonnegative Radon measures on \( \mathbb{R}^N \) such that
\[
|\phi|_{L^r_\mu(\mathbb{R}^N)} \leq C \max \left\{ |\phi|_{L^p_\mu(\mathbb{R}^N)}, |\phi|_{L^q_\mu(\mathbb{R}^N)} \right\}, \quad \forall \phi \in C_\infty^\ast(\mathbb{R}^N),
\]
for some constant \( C > 0 \). Then there exist an at most countable set \( \{x_i\}_{i \in I} \) of distinct point in \( \mathbb{R}^N \) and \( \{v_i\}_{i \in I} \subset (0, \infty) \) such that
\[
\nu = \sum_{i \in I} v_i \delta_{x_i}.
\]
In order to prove Proposition A.1, we will make use of the following result.

**Lemma A.2.** Let $p, q, r \in C_{c}(\mathbb{R}^N)$ such that $\inf_{x \in \mathbb{R}^N} \{ r(x) - \max \{ p(x), q(x) \} \} > 0$. Let $\nu$ be a finite nonnegative Radon measure on $\mathbb{R}^N$ such that

$$|\phi|_{L^r(\mathbb{R}^N)} \leq C \max \{ |\phi|_{L^p(\mathbb{R}^N)}, |\phi|_{L^q(\mathbb{R}^N)} \}, \quad \forall \phi \in C_c^\infty(\mathbb{R}^N).$$

Then $\nu = 0$ or there exist $\{ x_i \}_{i=1}^n$ of distinct points in $\mathbb{R}^N$ and $\{ \nu_i \}_{i=1}^n \subset (0, \infty)$ such that $\nu = \sum_{i=1}^n \nu_i \delta_{x_i}$.

The proof of Lemma A.2 is similar to that of [27, Lemma 3.8] and using this result we can prove Proposition A.1 via the same method as in [21, Lemma 3.2] and we leave the proofs to the reader.

**Acknowledgements:** The first author was supported by University of Economics Ho Chi Minh City, Vietnam. The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2019R1A1057775).

**References**


