Research article

Qihan He*, Juntao Lv, and Zongyan Lv

Nontrivial solutions to the p -harmonic equation with nonlinearity asymptotic to $|t|^{p-2}t$ at infinity

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Abstract: We consider the following p -harmonic problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) + m|u|^{p-2}u = f(x, u), & x \in \mathbb{R}^N, \\ u \in W^{2,p}(\mathbb{R}^N), \end{cases}$$

where $m > 0$ is a constant, $N > 2p \geq 4$ and $\lim_{t \rightarrow \infty} \frac{f(x,t)}{|t|^{p-2}t} = l$ uniformly in x , which implies that $f(x, t)$ does not satisfy the Ambrosetti-Rabinowitz type condition. By showing the Pohozaev identity for weak solutions to the limited problem of the above p -harmonic equation and using a variant version of Mountain Pass Theorem, we prove the existence and nonexistence of nontrivial solutions to the above equation. Moreover, if $f(x, u) \equiv f(u)$, the existence of a ground state solution and the nonexistence of nontrivial solutions to the above problem is also proved by using artificial constraint method and the Pohozaev identity.

Keywords: p -harmonic equation, nontrivial solutions, Ambrosetti-Rabinowitz condition

MSC: 35J35, 35J60, 47J30.

1 Introduction

In this paper, we deal with the existence and nonexistence of nontrivial solutions to the following p -harmonic problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) + m|u|^{p-2}u = f(x, u), & x \in \mathbb{R}^N, \\ u \in W^{2,p}(\mathbb{R}^N), \end{cases} \quad (1.1)$$

where $m > 0$ denotes a constant and $N > 2p \geq 4$.

Throughout this paper, we assume that $f(x, t)$ satisfies the following conditions:

(C₁) $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Caratheodory conditions; $f(x, t)t \geq 0$, $\forall (x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $f(x, 0) \equiv 0$, $\forall x \in \mathbb{R}^N$.

(C₂) $\lim_{t \rightarrow 0} \frac{f(x, t)}{|t|^{p-2}t} = 0$ uniformly in $x \in \mathbb{R}^N$.

***Corresponding Author: Qihan He**, College of Mathematics and Information Sciences, Guangxi Center for Mathematical Research, Guangxi University, Guangxi 530003, P. R. China, E-mail: heqihan277@gxu.edu.cn

Juntao Lv, College of Mathematics and Information Sciences, Guangxi Center for Mathematical Research, Guangxi University, Guangxi 530003, P. R. China, E-mail: m18131375275@163.com

Zongyan Lv, College of Mathematics and Information Sciences, Guangxi Center for Mathematical Research, Guangxi University, Guangxi 530003, P. R. China, E-mail: zongyanlv0535@163.com

- (C₃) $\lim_{t \rightarrow \infty} \frac{f(x, t)}{|t|^{p-2}t} = l$ uniformly in $x \in \mathbb{R}^N$ for some $l \in (0, +\infty)$.
- (C₄) For a.e. $x \in \mathbb{R}^N$, $\frac{f(x, t)}{|t|^{p-2}t}$ is nondecreasing with respect to $t > 0$, and nonincreasing with respect to $t < 0$.
- (C₅) $\exists \tilde{f}(t) \in C^1(\mathbb{R})$ such that $|f(x, t)| \geq |\tilde{f}(t)|$, $\forall (x, t) \in \mathbb{R}^N \times \mathbb{R}$, $\text{meas}\{x \in \mathbb{R}^N : |f(x, t)| > |\tilde{f}(t)|\} > 0$, $\forall t \in \mathbb{R} \setminus \{0\}$ and $\lim_{|x| \rightarrow +\infty} f(x, t) = \tilde{f}(t)$ uniformly in $t \in \mathbb{R}$.
- (C₆) $\tilde{f}(t) \in C^1(\mathbb{R})$ satisfies that $(p-1)\tilde{f}(t) > \tilde{f}'(t)t$ for all $t < 0$ and $(p-1)\tilde{f}(t) < \tilde{f}'(t)t$ for all $t > 0$.

Definition 1.1. We call $u \in W^{2,p}(\mathbb{R}^N)$ a (weak) solution of (1.1) if for all $\phi \in W^{2,p}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (|\Delta u|^{p-2} \Delta u \Delta \phi + m|u|^{p-2} u \phi) dx = \int_{\mathbb{R}^N} f(x, u) \phi dx.$$

It is easy to see that any solution of (1.1) corresponds to a critical point of the following energy functional defined on $W^{2,p}(\mathbb{R}^N)$:

$$I(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\Delta u|^p + m|u|^p) dx - \int_{\mathbb{R}^N} F(x, u) dx, \quad (1.2)$$

where $F(x, u) = \int_0^u f(x, s) ds$.

The famous Mountain Pass Theorem proposed in [1] and the constraint minimization are the useful tools to get critical points of $I(u)$. Based on the Mountain Pass Theorem, many results about the existence of solutions to second order nonlinear elliptic problems included p -laplacian or bi-harmonic operators have been obtained (see [2–5] and the references therein).

Evidently, compared with the case of bounded domain, when treating a problem in $\mathbb{R}^N$, the compactness of Sobolev imbedding is absent. Researchers have attempted various methods to study this kind of problem. For example, to the following semilinear elliptic problem

$$\begin{cases} -\Delta u + mu = f(x, u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (1.3)$$

where $f(x, u)$ is spherical symmetrical or autonomous, the existence of nontrivial solutions to (1.3) was considered in the radially symmetrical Sobolev space (see [6–8]). However, this method can not be used to the case of general $f(x, t)$. To dealt with this kind of problem, the concentration-compactness principle was proved by P. L. Lions. Many researchers have studied the variational elliptic problems in $\mathbb{R}^N$ by this principle (see [9, 10] and the references therein).

Yang and Zhu [11] considered the following quasilinear elliptic equation

$$\begin{cases} -\text{div}(|\nabla u|^{p-2} \nabla u) + a(x)|u|^{p-2} u = f(x, u), & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \quad (1.4)$$

where $N > p \geq 2$ and $a(x)$ satisfies the following assumption:

(a) $a(x) \in C(\mathbb{R}^N)$, $a(x) \geq a_0 > 0$ and $\lim_{|x| \rightarrow +\infty} a(x) = \bar{a} > 0$.

Deng, Gao and Jin [12] studied the p -harmonic problem (1.1). The authors in [12] required that $f(x, t)$ is subcritical and satisfies some assumptions similar to (C₁) – (C₂), (C₄) – (C₆) and the (AR) condition:

$$\exists \theta > 0 \text{ s.t. } 0 \leq F(x, t) = \int_0^t f(x, s) ds \leq \frac{1}{p+\theta} f(x, t)t, \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}. \quad (1.5)$$

This condition implies that for some $C > 0$,

$$F(x, t) \geq C|t|^{p+\theta} \quad \text{for } |t| > 0 \text{ large.}$$

However, (C_3) implies that $f(x, t)$ is asymptotic to $|t|^{p-2}t$ at infinity, and (1.5) does not satisfy. During the last twenty years, Researchers have shown a lot of results about (1.3) and (1.4) without the (AR) condition (see [13–17]). Using the concentration-compactness principle together with a variant version of Mountain Pass Theorem, Li and Zhou in [18] proved that (1.3) has a positive solution under similar conditions on $f(x, t)$ as $(C_1) - (C_6)$. After that, He and Li in [19] proved the existence of a nontrivial solution to the p -Laplacian equation under similar assumptions on $f(x, t)$ as $(C_1) - (C_6)$. With the help of Ekeland variational principle and Mountain Pass Theorem, the existence of multiple solutions for boundary value problem of nonhomogeneous p -harmonic equation has been proved (see [20, 21] and the references therein).

This paper is motivated by [12], [18] and [19]. We want to consider the existence and nonexistence of nontrivial solutions to the equation problem (1.1). To our best knowledge, there are few results on problem (1.1) when $f(x, t)$ is asymptotic to $|t|^{p-2}t$ at infinity.

We first define the limited problem of (1.1) as follows:

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) + m|u|^{p-2}u = \bar{f}(u), & x \in \mathbb{R}^N, \\ u \in W^{2,p}(\mathbb{R}^N). \end{cases} \quad (1.6)$$

For any $u \in W^{2,p}(\mathbb{R}^N)$, we define

$$I^\infty(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\Delta u|^p + m|u|^p) dx - \int_{\mathbb{R}^N} \bar{F}(u) dx, \quad (1.7)$$

where $\bar{F}(u) \triangleq \int_0^u \bar{f}(s) ds$. Clearly, $I^\infty \in C^1(W^{2,p}(\mathbb{R}^N), \mathbb{R})$. Denote

$$\Lambda = \{u \in W^{2,p}(\mathbb{R}^N) : \langle I^{\infty'}(u), u \rangle = 0, u \not\equiv 0\}, \quad (1.8)$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $W^{2,p}(\mathbb{R}^N)$ and $(W^{2,p}(\mathbb{R}^N))^{-1}$. $\Lambda \neq \emptyset$ will be shown in Lemma 2.4 below. So

$$J^\infty = \inf\{I^\infty(u) : u \in \Lambda\} \quad (1.9)$$

is well-defined.

Our main results can be stated as follows:

Theorem 1.2. Assume that conditions $(C_1) - (C_6)$ hold. If $l \in (m, +\infty)$, then $J^\infty > 0$ and it is achieved by some $\tilde{u} \in W^{2,p}(\mathbb{R}^N) \setminus \{0\}$, which is a ground state solution for problem (1.6). Moreover, if $l \leq m$, then problem (1.6) has no nontrivial solutions.

Theorem 1.3. Assume that $(C_1) - (C_6)$ hold. Then there exists at least a nontrivial weak solution to (1.1) if $l \in (m, +\infty)$ and there is no nontrivial weak solutions to (1.1) if $l \leq m$.

To show the Theorems, we need to prove the following Proposition:

Proposition 1.4. (Pohozaev identity for weak solution) Suppose that $\bar{f}$ is a continuous function satisfying the following growth condition :

$$|\bar{f}(t)| \leq c|t|^{p-1} + C|t|^{d_0-1} \text{ for all } t \in \mathbb{R},$$

where $c, C > 0$ and $d_0 = \frac{Np-p}{N-2p}$. If u is a weak solution of the p -harmonic problem (1.6), that is

$$\int_{\mathbb{R}^N} (|\Delta u|^{p-2}\Delta u \Delta \phi + m|u|^{p-2}u \phi) dx = \int_{\mathbb{R}^N} \bar{f}(u) \phi dx, \quad \forall \phi \in W^{2,p}(\mathbb{R}^N), \quad (1.10)$$

then u satisfies the Pohozaev type identity:

$$\frac{N-2p}{Np} \int_{\mathbb{R}^N} |\Delta u|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} m|u|^p dx = \int_{\mathbb{R}^N} \bar{F}(u) dx. \quad (1.11)$$

Remark 1.5. The role of exponent $d_0 = \frac{Np-p}{N-2p}$ is to guarantee that (2.7) is true. It is easy to verify that $p^* := \frac{Np}{N-p} < d_0 < \frac{Np}{N-2p} =: p^{**}$ and (2.7) holds naturally for $d_0 = p^{**}$ if the weak solution $u \in L_{Loc}^\infty(\mathbb{R}^N)$. So the Pohozaev type identity (1.11) is also true for $d_0 = p^{**}$ if $u \in L_{Loc}^\infty(\mathbb{R}^N)$.

Following from the Proposition 1.4, we can get the following Corollary directly.

Corollary 1.6. Assume that $u \in W^{2,p}(\mathbb{R}^N)$ is a weak solution of

$$\Delta(|\Delta u|^{p-2}\Delta u) = \lambda|u|^{q-2}u, \quad x \in \mathbb{R}^N$$

where $\lambda \in \mathbb{R}$ and $q \in [p, d_0]$. Then $u \equiv 0$.

In the proofs of our main results, we are faced with several difficulties:

Firstly, the method in [19] can not be applied directly to the p -harmonic problem. For example, for the quasilinear elliptic problem (1.4), $u \in W^{1,p}(\mathbb{R}^N)$ implies that $|u|, u^+, u^- \in W^{1,p}(\mathbb{R}^N)$, where $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$. From this fact, we can prove that a solution can be taken to be positive. While for the p -harmonic problem (1.1), this way fails completely since $u \in W^{2,p}(\mathbb{R}^N)$ does not imply that $|u|, u^+, u^- \in W^{2,p}(\mathbb{R}^N)$. To this end, we need to take prolongation on $f(x, t)$ for $t < 0$.

Secondly, as we can see in [18], the Pohozaev identity played an important role in the proofs of the main results, which is slightly different from what [19] did. However, we can not get the Pohozaev identity of p -harmonic problem (1.6) as usual, since there is less information on the regularity of solutions to (1.6). Due to the lack of regularity of the solutions to problem (1.6), the usual method to derive the corresponding Pohozaev identity can not work. Inspired by [22], we take $\phi := \psi \sum_{j=1}^N x_j D_j^h u$ as a test function to derive the corresponding Pohozaev identity, which relaxes the restriction on the regularity of the solution and where $D_j^h u$ denotes the difference quotient and ψ is a given cut-off function. But our problem (1.6) is a quasilinear elliptic equation of fourth order or more, we need to estimate the higher order difference. Thus more delicate analysis is needed.

Thirdly, $W^{2,p}(\mathbb{R}^N)$ is not a Hilbert space in general. It is not clear that, up to a subsequence,

$$|\Delta u_n|^{p-2}\Delta u_n \rightharpoonup |\Delta u|^{p-2}\Delta u \quad \text{in } L^{\frac{p}{p-1}}(\mathbb{R}^N),$$

even if the (PS) sequence $\{u_n\}$ of the functional I is bounded in $W^{2,p}(\mathbb{R}^N)$ and

$$u_n \rightharpoonup u \quad \text{in } W^{2,p}(\mathbb{R}^N).$$

Hence, more delicate analysis is needed to prove that

$$\Delta u_n \rightarrow \Delta u \quad \text{a.e. in } \mathbb{R}^N.$$

Finally, since $f(x, t)$ and $\tilde{f}(t)$ are asymptotic to $|t|^{p-2}t$ at infinity, the (AR) condition (1.5) does not satisfy so that showing the boundedness of any $(PS)_c$ sequence for $I(u)$ or $I^\infty(u)$ in $W^{2,p}(\mathbb{R}^N)$ has become one of the main difficulties for studying the existence of nontrivial weak solutions to (1.1) or (1.6) in $W^{2,p}(\mathbb{R}^N)$. To show Theorem 1.2, inspired by [18], we apply Ekeland's variational principle to get a minimizing sequence $\{u_n\}$ for J^∞ with $I^{\infty'}(u_n) \rightarrow 0$ in $(W^{2,p}(\mathbb{R}^N))^{-1}$, which guarantees that we can show $J^\infty > 0$. Then we can show that J^∞ is achieved by some u_0 . As to Theorem 1.3, motivated by [19], we would prove it by a mountain pass theorem without Cerami condition together with the concentration-compactness principle. With the help of the ground state solution $\tilde{u}$ to (1.6) obtained in Theorem 1.2, we construct the mountain pass level c as

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], W^{2,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = t_0 \tilde{u}\}$ for some $t_0 > 0$ large enough and which is slightly different from what [18] did. Liu and Zhou [18] use $\tilde{u}(\frac{x}{t_0})$ instead of $t_0 \tilde{u}$ in the definition of Γ and prove that $I(\tilde{u}(\frac{x}{t_0})) < 0$ for t_0 large enough. But for the solutions of (1.6), we can not obtain the regularity result to

ensure that $\gamma_0(t) = \begin{cases} \tilde{u}(\frac{x}{t\tilde{t}_0}), & t \in (0, 1], \\ 0, & t = 0, \end{cases}$ belongs to Γ , which is a key point to show $c < J^\infty$. So the method in [18] does not work here. Due to $\lim_{t \rightarrow \infty} \frac{\tilde{f}(t)}{|t|^{p-2}t} = l$, (C_3) and (C_5) , we can show that $I(t\tilde{u}) < I^\infty(t\tilde{u}) < 0$ for $t > 0$ large enough, which implies that the mountain pass level c defined above is well-defined and $c < J^\infty$. By the mountain pass theorem without the Cerami condition, we can see that there exists a Cerami sequence $\{u_n\} \subset W^{2,p}(\mathbb{R}^N)$ of $I(u)$ at the level c , that is

$$I(u_n) \rightarrow c \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{(W^{2,p}(\mathbb{R}^N))^{-1}} \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \quad (1.12)$$

Then we can apply the fact that $c < J^\infty$ and the concentration-compactness principle to prove that $\{u_n\}$ is bounded in $W^{2,p}(\mathbb{R}^N)$ and the weak limit of such subsequence of $\{u_n\}$ is a nontrivial weak solution of (1.1).

This paper is organized as follows. In Section 2, we first derive the Pohozaev identity for the weak solutions of problem (1.6), and some preliminary lemmas are presented. The proof of Theorem 1.2 is put into the Section 3. In Section 4, by using a variant version of Mountain Pass Theorem, we devote to prove Theorem 1.3.

2 Some notations and preliminary Lemmas

In this section, we devote to some notations and preliminary Lemmas, which are crucial in our proofs of main results.

In the sequel, C represents positive constant. We denote the norm of $u \in L^p(\mathbb{R}^N)$ by

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty,$$

and the norm of $u \in W^{2,p}(\mathbb{R}^N)$ by

$$\|u\| = \left[\int_{\mathbb{R}^N} (|\Delta u|^p + m|u|^p) dx \right]^{\frac{1}{p}}, \quad 2 \leq p < +\infty.$$

Let $N > 2p$ and denote $p^{**} = \frac{Np}{N-2p}$. It follows from $(C_1) - (C_5)$ that for any $\varepsilon > 0$, $\tau \in (p, p^{**})$, there exists $C_\varepsilon > 0$ such that for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}$,

$$|f(x, t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{\tau-1}, \quad |\tilde{f}(t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{\tau-1}, \quad (2.1)$$

$$F(x, t) \leq \varepsilon |t|^p + C_\varepsilon |t|^\tau, \quad \bar{F}(t) \leq \varepsilon |t|^p + C_\varepsilon |t|^\tau, \quad (2.2)$$

$$|f(x, t)| \leq l |t|^{p-1}, \quad |\tilde{f}(t)| \leq l |t|^{p-1}, \quad (2.3)$$

where $\bar{F}(t) = \int_0^t \tilde{f}(s) ds$. Combining (C_2) , (C_3) and (C_5) , we see that

$$\lim_{t \rightarrow 0} \frac{\tilde{f}(t)}{|t|^{p-2}t} = 0, \quad \lim_{t \rightarrow \infty} \frac{\tilde{f}(t)}{|t|^{p-2}t} = l. \quad (2.4)$$

According to (C_6) , ones can see that

$$\begin{cases} \frac{\tilde{f}(t)}{|t|^{p-2}t} \text{ is strictly increasing in } t > 0, \text{ and strictly decreasing in } t < 0, \\ \bar{F}(t) < \frac{1}{p} \tilde{f}(t)t, \quad \forall t \neq 0. \end{cases} \quad (2.5)$$

Based on the above observations, we are going to prove Proposition 1.4.

The proof of Proposition 1.4. Set $g(u) = \tilde{f}(u) - m|u|^{p-2}u$, and $G(u) = \int_0^u g(t)dt$. Denote $D_i^h u(x) = \frac{u(x+he_i) - u(x)}{h}$, where $h \in \mathbb{R} \setminus \{0\}$ and e_i denotes the unit vector along coordinate x_i . We take a cut-off function $\psi \in C_0^\infty(\mathbb{R}^N)$ satisfying that $\psi(x) = 1$ for $|x| \leq R$, $\psi(x) = 0$ for $|x| \geq 2R$, $|\nabla \psi| \leq \frac{2}{R}$ and $|\Delta \psi| \leq \frac{2}{R^2}$. Following the idea in [22], we set $\phi = \psi \sum_{j=1}^N x_j D_j^h u$, then we have $\phi \in W^{2,p}(\mathbb{R}^N)$. Taking such ϕ as a test function in (1.10), we have

$$\int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u \Delta \left(\psi \sum_{j=1}^N x_j D_j^h u \right) dx = \int_{\mathbb{R}^N} g(u) \psi \sum_{j=1}^N x_j D_j^h u dx. \quad (2.6)$$

A straightforward computation gives us

$$\begin{aligned} & \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u \Delta \left(\psi \sum_{j=1}^N x_j D_j^h u \right) dx \\ &= \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u \Delta (\psi x_j D_j^h u) dx \\ &= \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u D_j^h u \Delta(\psi x_j) dx + \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u \Delta(D_j^h u) \psi x_j dx \\ & \quad + 2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u D_i(\psi x_j) D_i(D_j^h u) dx \\ & \triangleq I_1 + I_2 + I_3, \end{aligned}$$

where $D_i u$ denotes $\nabla u \cdot e_i$. Now we estimate the three terms I_1 , I_2 and I_3 .

$$\begin{aligned} I_1 &= \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u D_j^h u \Delta(\psi x_j) dx \\ &= \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u D_j^h u (\Delta \psi x_j + 2 D_j \psi) dx. \end{aligned}$$

Since $\|D_j^h u\|_p \leq C \|Du\|_p$, we have $D_j^h u \rightharpoonup D_j u$ weakly in $L^p(\mathbb{R}^N)$, which means that, as $h \rightarrow 0$,

$$I_1 \rightarrow \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla u, x) \Delta \psi dx + 2 \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla u, \nabla \psi) dx.$$

$$\begin{aligned} I_2 &= \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u \Delta(D_j^h u) \psi x_j dx \\ &= \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} D_j^h (|\Delta u|^p) \psi x_j dx \\ & \quad - \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} \left(D_j^h (|\Delta u|^p) - p |\Delta u|^{p-2} \Delta u \Delta(D_j^h u) \right) \psi x_j dx \\ &= I_{21} - I_{22} \end{aligned}$$

and

$$\begin{aligned} I_{21} &= \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} D_j^h (|\Delta u|^p) \psi x_j dx \\ &= \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} D_j^h (|\Delta u|^p \psi x_j) dx - \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u(x + he_j)|^p D_j^h (\psi x_j) dx \\ &= -\frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u(x + he_j)|^p D_j^h (\psi x_j) dx. \end{aligned}$$

It follows from $\|D_j^h(\psi x_j)\|_p \leq C\|D_j(\psi x_j)\|_p$ that $D_j^h(\psi x_j) \rightharpoonup D_j(\psi x_j)$ weakly in $L^p(\mathbb{R}^N)$, which implies that when $h \rightarrow 0$,

$$\begin{aligned} I_{21} &\rightarrow -\frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^p D_j(\psi x_j) dx \\ &= -\frac{N}{p} \int_{\mathbb{R}^N} |\Delta u|^p \psi dx - \frac{1}{p} \int_{\mathbb{R}^N} |\Delta u|^p (\nabla \psi, x) dx. \end{aligned}$$

Since u satisfies (1.10), we have

$$\begin{aligned} I_{22} &= \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} (D_j^h(|\Delta u|^p) - p|\Delta u|^{p-2} \Delta u \Delta(D_j^h u)) \psi x_j dx \\ &= \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} \frac{|\Delta u(x + he_j)|^p - |\Delta u(x)|^p}{h} \psi x_j dx \\ &\quad - \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} p|\Delta u|^{p-2} \Delta u \frac{\Delta u(x + he_j) - \Delta u(x)}{h} \psi x_j dx \\ &= \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} \frac{p|\tilde{\Delta u}|^{p-2} \tilde{\Delta u} (\Delta u(x + he_j) - \Delta u(x))}{h} \psi x_j dx \\ &\quad - \frac{1}{p} \sum_{j=1}^N \int_{\mathbb{R}^N} p|\Delta u|^{p-2} \Delta u \frac{\Delta u(x + he_j) - \Delta u(x)}{h} \psi x_j dx \\ &= \sum_{j=1}^N \int_{\mathbb{R}^N} |\tilde{\Delta u}|^{p-2} \tilde{\Delta u} \left[\Delta \left(\frac{u(x + he_j) \psi x_j}{h} \right) - \frac{u(x + he_j) \Delta(\psi x_j)}{h} \right] dx \\ &\quad - \sum_{j=1}^N \int_{\mathbb{R}^N} |\tilde{\Delta u}|^{p-2} \tilde{\Delta u} \left[\Delta \left(\frac{u(x) \psi x_j}{h} \right) - \frac{u(x) \Delta(\psi x_j)}{h} \right] dx \\ &\quad - 2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} |\tilde{\Delta u}|^{p-2} \tilde{\Delta u} D_i(D_j^h u) D_i(\psi x_j) dx \\ &\quad - \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u \left[\Delta \left(\frac{u(x + he_j) \psi x_j}{h} \right) - \frac{u(x + he_j) \Delta(\psi x_j)}{h} \right] dx \\ &\quad + \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u \left[\Delta \left(\frac{u(x) \psi x_j}{h} \right) - \frac{u(x) \Delta(\psi x_j)}{h} \right] dx \\ &\quad + 2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u D_i(D_j^h u) D_i(\psi x_j) dx \\ &= \sum_{j=1}^N \int_{\mathbb{R}^N} (g(\tilde{u}) - g(u)) D_j^h u \psi x_j dx \\ &\quad + \sum_{j=1}^N \int_{\mathbb{R}^N} (|\Delta u|^{p-2} \Delta u - |\tilde{\Delta u}|^{p-2} \tilde{\Delta u}) D_j^h u \Delta(\psi x_j) dx \\ &\quad + 2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} (|\Delta u|^{p-2} \Delta u - |\tilde{\Delta u}|^{p-2} \tilde{\Delta u}) D_i(D_j^h u) D_i(\psi x_j) dx, \end{aligned}$$

where $\tilde{u} = \lambda u + (1 - \lambda)u(\cdot + he_j)$, $\lambda \in [0, 1]$. By the Hölder inequality, we have

$$|I_{22}| \leq \sum_{j=1}^N \left(\int_{\mathbb{R}^N} |D_j^h u|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |g(\tilde{u}) - g(u)|^q |\psi x_j|^q dx \right)^{\frac{1}{q}}$$

$$\begin{aligned}
& + \sum_{j=1}^N \left(\int_{\mathbb{R}^N} |D_j^h u|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |\Delta(\psi x_j)|^q |\Delta u|^{p-2} \Delta u - |\Delta \tilde{u}|^{p-2} \Delta \tilde{u}|^q dx \right)^{\frac{1}{q}} \\
& + 2 \sum_{i,j=1}^N \left(\int_{\mathbb{R}^N} |D_i(D_j^h u)|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |D_i(\psi x_j)|^q |\Delta u|^{p-2} \Delta u - |\Delta \tilde{u}|^{p-2} \Delta \tilde{u}|^q dx \right)^{\frac{1}{q}}
\end{aligned}$$

Note that $\|D_j^h u\|_p \leq C \|\nabla u\|_p < +\infty$, $\|D_i(D_j^h u)\|_p \leq C \|\Delta u\|_p < +\infty$, then we have

$$\begin{aligned}
|I_{22}| & \leq C \left(\int_{\mathbb{R}^N} |g(\tilde{u}) - g(u)|^q |\psi x_j|^q dx \right)^{\frac{1}{q}} \\
& + C \left(\int_{\mathbb{R}^N} |\Delta(\psi x_j)|^q |\Delta u|^{p-2} \Delta u - |\Delta \tilde{u}|^{p-2} \Delta \tilde{u}|^q dx \right)^{\frac{1}{q}} \\
& + C \left(\int_{\mathbb{R}^N} |D_i(\psi x_j)|^q |\Delta u|^{p-2} \Delta u - |\Delta \tilde{u}|^{p-2} \Delta \tilde{u}|^q dx \right)^{\frac{1}{q}} \rightarrow 0, \text{ as } h \rightarrow 0,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Additionally,

$$\begin{aligned}
I_3 & = 2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u D_i(\psi x_j) D_i(D_j^h u) dx \\
& = 2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u D_i(D_j^h u) (D_i \psi x_j + \psi \delta_{ij}) dx,
\end{aligned}$$

where δ_{ij} is the Kronecker symbol, that is $\delta_{ij} = 1$ when $i = j$ and 0 otherwise. By $\|D_i(D_j^h u)\|_p \leq C \|D_i(D_j u)\|_p$, we obtain $D_i(D_j^h u) \rightharpoonup D_i(D_j u)$ weakly in $L^p(\mathbb{R}^N)$, which means that

$$I_3 \rightarrow 2 \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla \psi, \nabla(D_j u)) x_j dx + 2 \int_{\mathbb{R}^N} |\Delta u|^p \psi dx.$$

Next we turn to estimate the right hand side of (2.6).

Since $u \in W^{2,p}(\mathbb{R}^N)$, $D_i u \in L^d(\mathbb{R}^N)$ and $u \in L^q(\mathbb{R}^N)$, where $d \in [p, \frac{Np}{N-p}]$, $q \in [p, \frac{Np}{N-2p}]$. It follows from the definition of $D_i^h u(x)$, we have $\|D_i^h u(x)\|_{L^d(\mathbb{R}^N)} \leq C \|D_i u\|_{L^d(\mathbb{R}^N)}$. Therefore, $D_i^h u(x) \rightharpoonup D_i u$ weakly in $L^d(\mathbb{R}^N)$, which implies that, as $h \rightarrow 0$,

$$\begin{aligned}
& \int_{\mathbb{R}^N} |g(u) \psi x_i (D_i^h u(x) - D_i u)| dx \\
& \leq \int_{\mathbb{R}^N} 2R(c|u|^{p-1} + C|u|^{d_0-1}) |D_i^h u(x) - D_i u| dx \\
& \leq C \int_{\mathbb{R}^N} |u|^{p-1} |D_i^h u(x) - D_i u| dx + C \int_{\mathbb{R}^N} |u|^{d_0-1} |D_i^h u(x) - D_i u| dx \\
& \rightarrow 0,
\end{aligned} \tag{2.7}$$

where we have used the facts that $|u|^{p-1} \in L^{\frac{p}{p-1}}(\mathbb{R}^N) = (L^p(\mathbb{R}^N))^{-1}$, $d_0 = \frac{Np-p}{N-2p}$ and $|u|^{d_0-1} \in L^{\frac{Np}{Np-N+p}}(\mathbb{R}^N) = (L^{\frac{Np}{N-p}}(\mathbb{R}^N))^{-1}$. So we have

$$\begin{aligned} \int_{\mathbb{R}^N} g(u) \psi \sum_{j=1}^N x_j D_j^h u dx &\rightarrow \int_{\mathbb{R}^N} g(u) \psi (\nabla u, x) dx \\ &= \int_{\mathbb{R}^N} \psi (\nabla G(u), x) dx \\ &= \int_{\mathbb{R}^N} \operatorname{div}(G(u) \psi x) dx - \int_{\mathbb{R}^N} G(u) (\nabla \psi, x) dx \\ &\quad - N \int_{\mathbb{R}^N} G(u) \psi dx \\ &= - \int_{\mathbb{R}^N} G(u) (\nabla \psi, x) dx - N \int_{\mathbb{R}^N} G(u) \psi dx. \end{aligned}$$

Altogether, as $h \rightarrow 0$, by (2.6), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla u, x) \Delta \psi dx + 2 \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla u, \nabla \psi) dx \\ &\quad - \frac{N}{p} \int_{\mathbb{R}^N} |\Delta u|^p \psi dx - \frac{1}{p} \int_{\mathbb{R}^N} |\Delta u|^p (\nabla \psi, x) dx \\ &\quad + 2 \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla \psi, \nabla (D_j u)) x_j dx + 2 \int_{\mathbb{R}^N} |\Delta u|^p \psi dx \\ &= - \int_{\mathbb{R}^N} G(u) (\nabla \psi, x) dx - N \int_{\mathbb{R}^N} G(u) \psi dx. \end{aligned}$$

Thus

$$\begin{aligned} &\frac{N-2p}{p} \int_{\mathbb{R}^N} |\Delta u|^p \psi dx + \frac{1}{p} \int_{\mathbb{R}^N} |\Delta u|^p (\nabla \psi, x) dx \\ &= \int_{\mathbb{R}^N} G(u) (\nabla \psi, x) dx + N \int_{\mathbb{R}^N} G(u) \psi dx + \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla u, x) \Delta \psi dx \\ &\quad + 2 \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla u, \nabla \psi) dx + 2 \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla \psi, \nabla (D_j u)) x_j dx. \end{aligned}$$

Notice that $|\nabla \psi| \leq \frac{2}{R}$, $|\Delta \psi| \leq \frac{2}{R^2}$ and $\operatorname{supp} \nabla \psi, \operatorname{supp} \Delta \psi \subset \subset \{x \in \mathbb{R}^N : R \leq |x| \leq 2R\}$, then

$$\int_{\mathbb{R}^N} |\Delta u|^p (\nabla \psi, x) dx \leq 4 \int_{R \leq |x| \leq 2R} |\Delta u|^p dx \rightarrow 0 \text{ as } R \rightarrow +\infty.$$

Similarly we have

$$\begin{aligned} &\int_{\mathbb{R}^N} G(u) (\nabla \psi, x) dx \rightarrow 0 \text{ as } R \rightarrow +\infty, \\ &\int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla u, x) \Delta \psi dx \rightarrow 0 \text{ as } R \rightarrow +\infty, \\ &\int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla u, \nabla \psi) dx \rightarrow 0 \text{ as } R \rightarrow +\infty, \end{aligned}$$

and

$$\begin{aligned} & \sum_{j=1}^N \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u (\nabla \psi, \nabla (D_j u)) x_j dx \\ & \leq 4 \left(\int_{R \leq |x| \leq 2R} |\Delta u|^p dx \right)^{\frac{p-1}{p}} \left(\int_{R \leq |x| \leq 2R} |\nabla(\nabla u)|^p dx \right)^{\frac{1}{p}} \rightarrow 0 \text{ as } R \rightarrow +\infty. \end{aligned}$$

On the other hand, as $R \rightarrow +\infty$,

$$\int_{\mathbb{R}^N} |\Delta u|^p \psi dx \rightarrow \int_{\mathbb{R}^N} |\Delta u|^p dx, \quad \int_{\mathbb{R}^N} G(u) \psi dx \rightarrow \int_{\mathbb{R}^N} G(u) dx.$$

Therefore we obtain the Pohozaev identity

$$\frac{N-2p}{p} \int_{\mathbb{R}^N} |\Delta u|^p dx = N \int_{\mathbb{R}^N} G(u) dx,$$

which gives (1.11) since $G(u) = \bar{F}(u) - \frac{1}{p} m|u|^p$. $\square$

We list some useful Lemmas.

Lemma 2.1. (Lemma 1.1 of [9]) Let $\{\rho_n\} \subset L^1(\mathbb{R}^N)$ be a bounded sequence and $\rho_n \geq 0$, then there exists a subsequence, still denoted by $\{\rho_n\}$, such that one of the following two possibilities occurs:

(i) (Vanishing): $\lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} \rho_n(x) dx = 0$ for all $0 < R < +\infty$.

(ii) (Nonvanishing): There exist $\alpha > 0$, $0 < R < +\infty$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow +\infty} \int_{B_R(y_n)} \rho_n(x) dx \geq \alpha > 0.$$

Lemma 2.2. (Lemma 2.1 of [12]) Let $p \geq 2$, $p \leq \tau < p^{**} = \frac{Np}{N-2p}$. Assuming that $\{u_n\}$ is bounded in $W^{2,p}(\mathbb{R}^N)$ and as $n \rightarrow +\infty$

$$\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^\tau dx \rightarrow 0 \text{ for some } R > 0,$$

then $u_n \rightarrow 0$ in $L^\alpha(\mathbb{R}^N)$ as $n \rightarrow +\infty$, for any $\alpha \in (p, p^{**})$.

Lemma 2.3. For the functional I^∞ defined by (1.7), if $u_n \in W^{2,p}(\mathbb{R}^N)$ satisfies $\langle I^{\infty'}(u_n), u_n \rangle = 0$ for all $n \geq 1$, then $I^\infty(tu_n) \leq I^\infty(u_n)$ for all $t > 0$.

Proof. A similar proof can be found in [19]. We omit it here. $\square$

In the following, we give several results for the set Λ and the minimization problem (1.9).

Lemma 2.4. Assume that $(C_1) - (C_3)$ hold and $l > m$. Then $\Lambda \neq \emptyset$.

Proof. Let

$$[d(N)]^p = \int_{\mathbb{R}^N} e^{-p|x|^p} dx.$$

For $\alpha > 0$, we set

$$\omega_\alpha(x) = [d(N)]^{-1} \alpha^{\frac{N}{p^2}} e^{-\alpha|x|^p}$$

and

$$D(N) = p^p [d(N)]^{-p} \int_{\mathbb{R}^N} [p|x|^{2p-2} e^{-|x|^p} - (p-2+N)|x|^{p-2} e^{-|x|^p}]^p dx.$$

It is easy to see that $\omega_\alpha \in W^{2,p}(\mathbb{R}^N)$. Straightforward calculations show that

$$\|\omega_\alpha\|_p = 1, \quad \|\Delta\omega_\alpha\|_p^p = \alpha^2 D(N).$$

By using (2.1), for some $\tau \in (p, p^{**})$, we have

$$\begin{aligned} \langle I^{\infty'}(t\omega_\alpha), t\omega_\alpha \rangle &= t^p \|\omega_\alpha\|_p^p - \int_{\mathbb{R}^N} \bar{f}(t\omega_\alpha) t\omega_\alpha dx \\ &\geq t^p \|\omega_\alpha\|_p^p - \int_{\mathbb{R}^N} \varepsilon t^p |\omega_\alpha|^p dx - \int_{\mathbb{R}^N} C_\varepsilon t^\tau |\omega_\alpha|^\tau dx \\ &= t^p (\alpha^2 D(N) + m - \varepsilon) - C_\varepsilon t^\tau \int_{\mathbb{R}^N} |\omega_\alpha|^\tau dx > 0, \end{aligned} \quad (2.8)$$

if we take $t > 0$ small enough. On the other hand, if we choose $\alpha \in (0, [\frac{l-m}{D(N)}]^{\frac{1}{2}})$, where l is given by (C_3) , then

$$\|\Delta\omega_\alpha\|_p^p < l - m. \quad (2.9)$$

whence, by Fatou's lemma together with (2.4), one has

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{\langle I^{\infty'}(t\omega_\alpha), t\omega_\alpha \rangle}{t^p} &= \|\omega_\alpha\|_p^p - \lim_{t \rightarrow +\infty} \int_{\mathbb{R}^N} \frac{\bar{f}(t\omega_\alpha) t\omega_\alpha}{t^p} dx \\ &\leq \|\omega_\alpha\|_p^p - \int_{\mathbb{R}^N} \lim_{t \rightarrow +\infty} \frac{\bar{f}(t\omega_\alpha) \omega_\alpha^p}{(t\omega_\alpha)^{p-1}} dx \\ &= \|\Delta\omega_\alpha\|_p^p + m - l < 0. \end{aligned}$$

So $\langle I^{\infty'}(t\omega_\alpha), t\omega_\alpha \rangle \rightarrow -\infty$, as $t \rightarrow +\infty$, which, together with (2.8), implies that there exists a $t^* > 0$ such that $\langle I^{\infty'}(t^*\omega_\alpha), t^*\omega_\alpha \rangle = 0$, that is $t^*\omega_\alpha \in \Lambda$. $\square$

Now we continue to study Λ . By $\bar{f}(t) \in C^1(\mathbb{R})$, it is easy to see that the functional $g(u) \triangleq \langle I^{\infty'}(u), u \rangle = \int_{\mathbb{R}^N} (|\Delta u|^p + m|u|^p) dx - \int_{\mathbb{R}^N} \bar{f}(u) u dx \in C^1(W^{2,p}(\mathbb{R}^N), \mathbb{R})$. Then by (C_6) , for any $u \in \Lambda$

$$\begin{aligned} \langle g'(u), u \rangle &= p \int_{\mathbb{R}^N} (|\Delta u|^p + m|u|^p) dx - \int_{\mathbb{R}^N} [\bar{f}(u)u + \bar{f}'(u)u^2] dx \\ &= \int_{\mathbb{R}^N} [(p-1)\bar{f}(u)u - \bar{f}'(u)u^2] dx \\ &= \int_{\mathbb{R}^N} [(p-1)\bar{f}(u) - \bar{f}'(u)u] u dx < 0. \end{aligned}$$

So Λ is a closed complete submanifold of $W^{2,p}(\mathbb{R}^N)$ with the natural Finsler structure (see [23]). Therefore, using the standard Ekeland's variational principle on Finsler manifold (see Lemma 2.4 in [18] and Corollary 1.3 in Chapter 2 of [23]) and Lemma 2.4, we can deduce the following Lemma.

Lemma 2.5. *If $(C_1) - (C_3)$ and $(C_5) - (C_6)$ hold, then there exists a sequence $\{u_n\} \subset \Lambda$ such that as $n \rightarrow +\infty$*

$$I^\infty(u_n) \rightarrow J^\infty = \inf_{u \in \Lambda} I^\infty(u) \quad \text{and} \quad I^{\infty'}(u_n) \rightarrow 0 \quad \text{in } (W^{2,p}(\mathbb{R}^N))^{-1}.$$

Lemma 2.6. *Assume that $(C_1) - (C_3)$ and $(C_5) - (C_6)$ hold. Then we have that $J^\infty > 0$.*

Proof. If $J^\infty = 0$, then it follows from Lemma 2.5 that there exists a sequence $\{u_n\} \subset \Lambda$ such that as $n \rightarrow +\infty$

$$I^\infty(u_n) = \frac{1}{p} \|u_n\|_p^p - \int_{\mathbb{R}^N} \bar{F}(u_n) dx \rightarrow J^\infty = 0, \quad (2.10)$$

$$\langle I^{\infty'}(u_n), u_n \rangle = \|u_n\|^p - \int_{\mathbb{R}^N} \tilde{f}(u_n) u_n dx = 0, \quad (2.11)$$

and

$$I^{\infty'}(u_n) \rightarrow 0 \text{ in } (W^{2,p}(\mathbb{R}^N))^{-1}. \quad (2.12)$$

First, we show the boundedness of $\{u_n\}$ in $W^{2,p}(\mathbb{R}^N)$. Seeking a contradiction, we assume that

$$\|u_n\| \rightarrow +\infty, \text{ as } n \rightarrow +\infty, \quad (2.13)$$

and for any fixed $\alpha > 0$, let

$$t_n = \frac{\alpha}{\|u_n\|}, \quad \omega_n(x) = t_n u_n(x) = \frac{\alpha u_n(x)}{\|u_n(x)\|}. \quad (2.14)$$

Clearly $\{\omega_n\}$ is bounded in $W^{2,p}(\mathbb{R}^N)$. For $\rho_n = |\omega_n|^p$, then $\{\rho_n\} \subset L^1(\mathbb{R}^N)$ is a bounded sequence and $\rho_n \geq 0$, which implies that either (i) or (ii) of Lemma 2.1 holds. We will get contradictions by showing none of these alternatives can occur.

Case 1 Vanishing: In this case, it follows from Lemma 2.2 and (2.2) that

$$\int_{\mathbb{R}^N} \tilde{F}(\omega_n) dx \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Then by (2.14), we have

$$I^\infty(\omega_n) = \frac{1}{p} \|\omega_n\|^p - \int_{\mathbb{R}^N} \tilde{F}(\omega_n) dx = \frac{1}{p} \|\omega_n\|^p + o_n(1) = \frac{1}{p} \alpha^p + o_n(1), \quad (2.15)$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow +\infty$. But by Lemma 2.3 and (2.10), one has

$$I^\infty(\omega_n) = I^\infty(t_n u_n) \leq I^\infty(u_n) \rightarrow J^\infty = 0, \text{ as } n \rightarrow +\infty, \quad (2.16)$$

which contradicts to (2.15).

Case 2 Nonvanishing: In this case, there exist $\eta > 0$, $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow +\infty} \int_{B_R(y_n)} |\omega_n|^p dx \geq \eta > 0. \quad (2.17)$$

Set $\tilde{\omega}_n(x) = \omega_n(x + y_n)$, then $\|\tilde{\omega}_n\| = \|\omega_n\| = \alpha$ and by Sobolev imbedding, we may assume that for some $\tilde{\omega} \in W^{2,p}(\mathbb{R}^N)$, such that as $n \rightarrow +\infty$

$$\begin{cases} \tilde{\omega}_n \rightharpoonup \tilde{\omega} & \text{in } W^{2,p}(\mathbb{R}^N), \\ \tilde{\omega}_n \rightarrow \tilde{\omega} & \text{in } L^p_{loc}(\mathbb{R}^N), \\ \tilde{\omega}_n \rightarrow \tilde{\omega} & \text{a.e. in } \mathbb{R}^N. \end{cases}$$

It follows from (2.17) that

$$\tilde{\omega} \not\equiv 0. \quad (2.18)$$

By (2.5), for n large enough, we have

$$t_n = \frac{\alpha}{\|u_n\|} \in (0, 1) \text{ and } \frac{\tilde{f}(t_n u_n)}{|t_n u_n|^{p-2} t_n u_n} \leq \frac{\tilde{f}(u_n)}{|u_n|^{p-2} u_n}.$$

Hence from (2.11)

$$\begin{aligned} \| \omega_n \|^p - \int_{\mathbb{R}^N} \tilde{f}(\omega_n) \omega_n dx &= t_n^p \| u_n \|^p - \int_{\mathbb{R}^N} \tilde{f}(t_n u_n) t_n u_n dx \\ &= t_n^p \left[\| u_n \|^p - \int_{\mathbb{R}^N} \frac{\tilde{f}(t_n u_n)}{|t_n u_n|^{p-2} t_n u_n} |u_n|^p dx \right] \\ &\geq t_n^p \left[\| u_n \|^p - \int_{\mathbb{R}^N} \frac{\tilde{f}(u_n)}{|u_n|^{p-2} u_n} |u_n|^p dx \right] = 0. \end{aligned}$$

Then Fatou's Lemma and (2.5) imply that

$$\begin{aligned} I^\infty(\omega_n) &= \frac{1}{p} \|\omega_n\|^p - \int_{\mathbb{R}^N} \bar{F}(\omega_n) dx \\ &\geq \frac{1}{p} \int_{\mathbb{R}^N} \bar{f}(\omega_n) \omega_n dx - \int_{\mathbb{R}^N} \bar{F}(\omega_n) dx \\ &\geq \int_{\mathbb{R}^N} \left[\frac{1}{p} \bar{f}(\tilde{\omega}) \tilde{\omega} - \bar{F}(\tilde{\omega}) \right] dx + o_n(1). \end{aligned}$$

So by (2.5) and (2.16), we have

$$0 \leq \int_{\mathbb{R}^N} \left[\frac{1}{p} \bar{f}(\tilde{\omega}) \tilde{\omega} - \bar{F}(\tilde{\omega}) \right] dx \leq 0,$$

which means $\tilde{\omega} \equiv 0$, contradicting to (2.18).

Thus, $\{u_n\}$ is bounded in $W^{2,p}(\mathbb{R}^N)$.

Next, we prove that $J^\infty > 0$. To this end, we take $\rho_n = |u_n|^p$. Following from Sobolev imbedding, we have that $\{\rho_n\}$ is bounded in $L^1(\mathbb{R}^N)$. Therefore, Lemma 2.1 implies that for some subsequence of $\{\rho_n\}$ either *Vanishing* or *Nonvanishing* occurs. We will show that both *Vanishing* and *Nonvanishing* are impossible if $J^\infty = 0$.

Case I *Vanishing* is impossible:

In fact, if *Vanishing* occurs, by Lemma 2.2 and (2.2), we have

$$I^\infty(u_n) = \frac{1}{p} \|u_n\|^p + o_n(1). \quad (2.19)$$

Taking $\varepsilon = \frac{m}{2}$ in (2.1), it follows from (2.11) that

$$\|u_n\|^p = \int_{\mathbb{R}^N} \bar{f}(u_n) u_n dx \leq \frac{1}{2} \|u_n\|^p + C \|u_n\|^\tau, \quad \text{for } \tau \in (p, p^{**}),$$

which means that there exists a $\delta > 0$ such that

$$\|u_n\| \geq \delta. \quad (2.20)$$

So, if $I^\infty(u_n) \rightarrow J^\infty = 0$ as $n \rightarrow +\infty$, (2.19) and (2.20) can deduce a contradiction.

Case II *Nonvanishing* is also impossible:

In fact, if *Nonvanishing* occurs, there exist $\eta > 0$, $R > 0$, $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow +\infty} \int_{B_R(y_n)} |u_n|^p dx \geq \eta > 0. \quad (2.21)$$

Let $\tilde{u}_n(x) = u_n(x + y_n)$. Since $\{u_n\}$ is bounded in $W^{2,p}(\mathbb{R}^N)$, $\{\tilde{u}_n\}$ is also bounded in $W^{2,p}(\mathbb{R}^N)$, then by Sobolev imbedding, we may assume that there exists $0 \neq \tilde{u} \in W^{2,p}(\mathbb{R}^N)$, such that as $n \rightarrow +\infty$

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u} & \text{in } W^{2,p}(\mathbb{R}^N), \\ \tilde{u}_n \rightarrow \tilde{u} & \text{in } L^p_{loc}(\mathbb{R}^N), \\ \tilde{u}_n \rightarrow \tilde{u} & \text{a.e. in } \mathbb{R}^N. \end{cases} \quad (2.22)$$

It is easy to see that

$$I^\infty(u_n) = I^\infty(\tilde{u}_n) \quad \text{and} \quad \langle I^{\infty'}(u_n), u_n \rangle = \langle I^{\infty'}(\tilde{u}_n), \tilde{u}_n \rangle.$$

So by (2.5), (2.10)-(2.11) and Fatou's Lemma, we have

$$\begin{aligned} 0 < \int_{\mathbb{R}^N} \left[\frac{1}{p} \tilde{f}(\tilde{u}) \tilde{u} - \tilde{F}(\tilde{u}) \right] dx &\leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \left[\frac{1}{p} \tilde{f}(\tilde{u}_n) \tilde{u}_n - \tilde{F}(\tilde{u}_n) \right] dx \\ &= \liminf_{n \rightarrow +\infty} \left[\frac{1}{p} \|\tilde{u}_n\|^p - \int_{\mathbb{R}^N} \tilde{F}(\tilde{u}_n) dx \right] \\ &= J^\infty = 0, \end{aligned}$$

which means that *Nonvanishing* is also impossible.

Above arguments show that both Vanishing and Nonvanishing are impossible if $J^\infty = 0$. This contradiction gives that $J^\infty > 0$. $\square$

3 Proof of Theorem 1.2

This section will devote to the proof of Theorem 1.2.

The Proof of Theorem 1.2.: By Lemma 2.6, we have $J^\infty > 0$. By Lemma 2.5, there exists $\{u_n\} \subset W^{2,p}(\mathbb{R}^N)$ such that as $n \rightarrow +\infty$

$$I^\infty(u_n) \rightarrow J^\infty > 0, \quad (3.1)$$

$$I^{\infty'}(u_n) \rightarrow 0 \text{ in } (W^{2,p}(\mathbb{R}^N))^{-1}, \quad (3.2)$$

i.e., $\{u_n\}$ is a $(PS)_{J^\infty}$ sequence. Now we divide the proof into two steps.

Step 1: $\{u_n\}$ is bounded in $W^{2,p}(\mathbb{R}^N)$.

If $\|u_n\| \rightarrow +\infty$, as $n \rightarrow +\infty$, we let

$$t_n = \frac{\alpha}{\|u_n\|}, \quad \omega_n(x) = t_n u_n(x) = \frac{\alpha u_n(x)}{\|u_n\|}, \quad (3.3)$$

where α is chosen such that $\alpha^p \geq 2pJ^\infty > 0$. Set $\rho_n = |\omega_n|^p$, if there is a subsequence, still denoted by $\{\rho_n\}$, such that Lemma 2.1 holds, then by the similar arguments of (2.15)-(2.16), we know that *Vanishing* doesn't occur.

If *Nonvanishing* occurs, i.e. there exist $\eta > 0$, $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow +\infty} \int_{B_R(y_n)} |\omega_n|^p dx \geq \eta > 0,$$

$$\tilde{\omega}_n = \omega_n(x + y_n), \quad \|\tilde{\omega}_n\| = \|\omega_n\| = \alpha.$$

Hence, there exists some $0 \neq \tilde{\omega} \in W^{2,p}(\mathbb{R}^N)$ satisfying as $n \rightarrow +\infty$

$$\begin{cases} \tilde{\omega}_n \rightharpoonup \tilde{\omega} & \text{in } W^{2,p}(\mathbb{R}^N), \\ \tilde{\omega}_n \rightarrow \tilde{\omega} & \text{in } L^1_{loc}(\mathbb{R}^N), \quad \tau \in (p, p^{**}), \\ \tilde{\omega}_n \rightarrow \tilde{\omega} & \text{a.e. in } \mathbb{R}^N. \end{cases} \quad (3.4)$$

Set $\tilde{u}_n(x) = u_n(x + y_n)$. Then $\tilde{\omega}_n = t_n \tilde{u}_n$, and it is not difficult to verify that $\forall \phi \in C_0^\infty(\mathbb{R}^N)$

$$\langle I^{\infty'}(\tilde{u}_n), \phi \rangle = o_n(1), \quad (3.5)$$

that is

$$\int_{\mathbb{R}^N} (|\Delta \tilde{u}_n|^{p-2} \Delta \tilde{u}_n \Delta \phi + m |\tilde{u}_n|^{p-2} \tilde{u}_n \phi) dx - \int_{\mathbb{R}^N} \tilde{f}(\tilde{u}_n) \phi dx = o_n(1).$$

Then, $\forall \phi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (|\Delta \tilde{\omega}_n|^{p-2} \Delta \tilde{\omega}_n \Delta \phi + m |\tilde{\omega}_n|^{p-2} \tilde{\omega}_n \phi) dx - \int_{\mathbb{R}^N} \frac{\tilde{f}(\tilde{u}_n)}{|\tilde{u}_n|^{p-2} \tilde{u}_n} |\tilde{\omega}_n|^{p-2} \tilde{\omega}_n \phi dx = o_n(1). \quad (3.6)$$

Now we claim that as $n \rightarrow +\infty$

$$\frac{\tilde{f}(\tilde{u}_n)}{|\tilde{u}_n|^{p-2} \tilde{u}_n} |\tilde{\omega}_n|^{p-2} \tilde{\omega}_n \rightarrow l |\tilde{\omega}|^{p-2} \tilde{\omega} \quad \text{a.e. } x \in \mathbb{R}^N. \quad (3.7)$$

Indeed, let

$$\Omega_1 = \{x \in \mathbb{R}^N : \tilde{\omega}(x) \neq 0\} \quad \text{and} \quad \Omega_2 = \{x \in \mathbb{R}^N : \tilde{\omega}(x) = 0\}.$$

If $x \in \Omega_1$, then we have $|\tilde{u}_n(x)| = \alpha^{-1} \|u_n(x)\| \cdot |\tilde{\omega}_n(x)| \rightarrow +\infty$, as $n \rightarrow +\infty$. Hence by (2.4), we have as $n \rightarrow +\infty$

$$\frac{\tilde{f}(\tilde{u}_n)}{|\tilde{u}_n|^{p-2} \tilde{u}_n} |\tilde{\omega}_n|^{p-2} \tilde{\omega}_n \rightarrow l |\tilde{\omega}|^{p-2} \tilde{\omega}, \quad \text{a.e. } x \in \Omega_1.$$

Since $\frac{|\tilde{f}(t)|}{|t|^{p-1}} \leq l$ and $\tilde{\omega}_n(x) \rightarrow \tilde{\omega}(x) = 0$ a.e. $x \in \Omega_2$, it follows that as $n \rightarrow +\infty$

$$\frac{\tilde{f}(\tilde{u}_n)}{|\tilde{u}_n|^{p-2} \tilde{u}_n} |\tilde{\omega}_n|^{p-2} \tilde{\omega}_n \rightarrow 0 = l |\tilde{\omega}|^{p-2} \tilde{\omega}, \quad \text{a.e. } x \in \Omega_2.$$

Therefore, (3.7) is proved. Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be arbitrary and fixed, and let $\Omega \subset \mathbb{R}^N$ be a compact set such that $\text{supp } \phi \subset \Omega$. The compactness of the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow L^{p-1}(\Omega)$ implies $\tilde{\omega}_n \rightarrow \tilde{\omega}$ strongly in $L^{p-1}(\Omega)$. Therefore, it follows from [24], by using the Lebesgue Dominated Theorem, then as $n \rightarrow +\infty$

$$\int_{\mathbb{R}^N} \frac{\tilde{f}(\tilde{u}_n)}{|\tilde{u}_n|^{p-2} \tilde{u}_n} |\tilde{\omega}_n|^{p-2} \tilde{\omega}_n \phi dx \rightarrow l \int_{\mathbb{R}^N} |\tilde{\omega}|^{p-2} \tilde{\omega} \phi dx. \quad (3.8)$$

Similarly, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, as $n \rightarrow +\infty$

$$\int_{\mathbb{R}^N} m |\tilde{\omega}_n|^{p-2} \tilde{\omega}_n \phi dx \rightarrow \int_{\mathbb{R}^N} m |\tilde{\omega}|^{p-2} \tilde{\omega} \phi dx. \quad (3.9)$$

On the other hand, by (3.4)-(3.5), we have that

$$\langle I^{\infty'}(\tilde{\omega}_n) - I^{\infty'}(\tilde{\omega}), \eta_\rho(\tilde{\omega}_n - \tilde{\omega}) \rangle = o_n(1) \quad (3.10)$$

for any cut-off function $\eta_\rho \in C_0^\infty(\mathbb{R}^N)$ with $0 \leq \eta_\rho \leq 1$, $\eta_\rho = 1$ on $B_\rho(0) = \{x \in \mathbb{R}^N : |x| \leq \rho\}$. Next we are going to prove that as $n \rightarrow +\infty$

$$\Delta \tilde{\omega}_n \rightarrow \Delta \tilde{\omega} \quad \text{a.e. in } \mathbb{R}^N. \quad (3.11)$$

Following the idea in [17, 25], we let

$$P_n(x) = (|\Delta \tilde{\omega}_n|^{p-2} \Delta \tilde{\omega}_n - |\Delta \tilde{\omega}|^{p-2} \Delta \tilde{\omega})(\Delta \tilde{\omega}_n - \Delta \tilde{\omega}).$$

By the well-known inequality

$$(|\xi|^{\gamma-2} \xi - |\eta|^{\gamma-2} \eta)(\xi - \eta) > 0$$

for any $\gamma > 1$ and $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, we have $P_n(x) \geq 0$. By (3.4) and (3.10), it is easy to see that

$$\lim_{n \rightarrow +\infty} \int_{B_\rho(0)} P_n(x) dx = 0 \quad \text{for any } \rho > 0.$$

Then as $n \rightarrow +\infty$

$$\Delta \tilde{\omega}_n \rightharpoonup \Delta \tilde{\omega} \quad \text{in } L^p(B_\rho(0)).$$

Since $\rho > 0$ is arbitrary, we know that (3.11) holds. Hence as $n \rightarrow +\infty$

$$|\Delta \tilde{w}_n|^{p-2} \Delta \tilde{w}_n \rightharpoonup |\Delta \tilde{w}|^{p-2} \Delta \tilde{w} \quad \text{in } L^{\frac{p}{p-1}}(\mathbb{R}^N).$$

So for any $\phi \in C_0^\infty(\mathbb{R}^N)$, as $n \rightarrow +\infty$

$$\int_{\mathbb{R}^N} |\Delta \tilde{w}_n|^{p-2} \Delta \tilde{w}_n \Delta \phi \, dx \rightarrow \int_{\mathbb{R}^N} |\Delta \tilde{w}|^{p-2} \Delta \tilde{w} \Delta \phi \, dx. \quad (3.12)$$

Combining (3.6), (3.8), (3.9) and (3.12), we have for any $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\Delta \tilde{w}|^{p-2} \Delta \tilde{w} \Delta \phi \, dx = (l - m) \int_{\mathbb{R}^N} |\tilde{w}|^{p-2} \tilde{w} \Delta \phi \, dx,$$

which contradicts to Corollary 1.6, and Step 1 is completed.

Step 2: J^∞ is achieved by some $\tilde{u} \in W^{2,p}(\mathbb{R}^N) \setminus \{0\}$.

Let $\rho_n = |u_n|^p$. Up to a subsequence, we may assume that $\{\rho_n\}$ satisfies Lemma 2.1.

If *Vanishing* occurs, then it follows from Lemma 2.2 and (2.1)-(2.2) that

$$\int_{\mathbb{R}^N} \bar{F}(u_n) \, dx \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \bar{f}(u_n) u_n \, dx \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Since

$$0 = \langle I^{\infty'}(u_n), u_n \rangle = \|u_n\|^p - \int_{\mathbb{R}^N} \bar{f}(u_n) u_n \, dx,$$

then as $n \rightarrow +\infty$

$$I^\infty(u_n) = \frac{1}{p} \|u_n\|^p - \int_{\mathbb{R}^N} \bar{F}(u_n) \, dx = \int_{\mathbb{R}^N} \left[\frac{1}{p} \bar{f}(u_n) u_n - \bar{F}(u_n) \right] \, dx \rightarrow 0,$$

which contradicts to (3.1).

So only *Nonvanishing* occurs. Similar to the arguments in Step 1, there exists $0 \neq \tilde{u} \in W^{2,p}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (|\Delta \tilde{u}|^{p-2} \Delta \tilde{u} \Delta \phi + m |\tilde{u}|^{p-2} \tilde{u} \phi) \, dx = \int_{\mathbb{R}^N} \bar{f}(\tilde{u}) \phi \, dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N). \quad (3.13)$$

Hence $\tilde{u} \in \Lambda$ and $I^\infty(\tilde{u}) \geq J^\infty > 0$. On the other hand, since

$$0 = \langle I^{\infty'}(u_n), u_n \rangle = \langle I^{\infty'}(\tilde{u}_n), \tilde{u}_n \rangle = \|\tilde{u}_n\|^p - \int_{\mathbb{R}^N} \bar{f}(\tilde{u}_n) \tilde{u}_n \, dx,$$

then by (2.5) and Fatous's Lemma, one has

$$\begin{aligned} J^\infty &= I^\infty(u_n) + o_n(1) = I^\infty(\tilde{u}_n) + o_n(1) \\ &= \frac{1}{p} \|\tilde{u}_n\|^p - \int_{\mathbb{R}^N} \bar{F}(\tilde{u}_n) \, dx + o_n(1) \\ &= \int_{\mathbb{R}^N} \left[\frac{1}{p} \bar{f}(\tilde{u}_n) \tilde{u}_n - \bar{F}(\tilde{u}_n) \right] \, dx + o_n(1) \\ &\geq \int_{\mathbb{R}^N} \left[\frac{1}{p} \bar{f}(\tilde{u}) \tilde{u} - \bar{F}(\tilde{u}) \right] \, dx + o_n(1) \\ &= I^\infty(\tilde{u}) + o_n(1), \end{aligned}$$

which implies that $J^\infty \geq I^\infty(\tilde{u})$.

So $I^\infty(\tilde{u}) = J^\infty$ with $\tilde{u} \neq 0$.

For the case $l \leq m$, we assume that Problem (1.6) has a nontrivial weak solution $u \in W^{2,p}(\mathbb{R}^N)$. Then, following from Proposition 1.4, (2.3) and (2.5), we have

$$\begin{aligned} 0 &\leq \frac{N-2p}{Np} \int_{\mathbb{R}^N} |\Delta u|^p dx \\ &= \int_{\mathbb{R}^N} (\bar{F}(u) - \frac{1}{p} m |u|^p) dx \\ &< \frac{1}{p} \int_{\mathbb{R}^N} (\bar{f}(u)u - m |u|^p) dx \\ &\leq \frac{1}{p} \int_{\mathbb{R}^N} (l - m) |u|^p dx \\ &\leq 0, \end{aligned}$$

which is impossible. So when $l \leq m$, there is no nontrivial weak solutions to (1.6).

We complete the proof. $\square$

4 Proof of Theorem 1.3

We will put the proof of Theorem 1.3 into this section. First, we can verify that the functional I defined in (1.2) exhibits the Mountain Pass geometry.

Lemma 4.1. Assume that $(C_1) - (C_3)$ and (C_5) hold. Then the functional I satisfies

- (i) there exist α_0 and $\rho_0 > 0$ such that $I(u) \geq \alpha_0$ for all $\|u\| = \rho_0$;
- (ii) there exists $w \in W^{2,p}(\mathbb{R}^N)$ such that $\|w\| > \rho_0$ and $I(w) \leq 0$.

Proof. By (2.2), for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for some $\tau \in (p, p^{**})$

$$\begin{aligned} I(u) &= \frac{1}{p} \int_{\mathbb{R}^N} (|\Delta u|^p + m |u|^p) dx - \int_{\mathbb{R}^N} F(x, u) dx \\ &\geq \frac{1}{p} \int_{\mathbb{R}^N} (|\Delta u|^p + m |u|^p) dx - \int_{\mathbb{R}^N} (\varepsilon |u|^p + C_\varepsilon |u|^\tau) dx \\ &\geq C \|u\|^p - C \|u\|^\tau. \end{aligned}$$

If we choose $\|u\| = \rho_0$ small, then $I(u) \geq \alpha_0 > 0$. We can get the result (i).

On the other hand, Let $\tilde{u}$ be the ground state solution obtained in Theorem 1.2. We have that if $x \in \Omega_0 := \{x \in \mathbb{R}^N : \tilde{u}(x) = 0\}$, then

$$\frac{\bar{F}(t\tilde{u})}{t^p} = 0 = \frac{1}{p} \bar{f}(\tilde{u})\tilde{u},$$

and if $x \in \Omega := \{x \in \mathbb{R}^N : \tilde{u}(x) \neq 0\}$, then

$$\liminf_{t \rightarrow +\infty} \frac{\bar{F}(t\tilde{u})}{t^p} = \lim_{t \rightarrow +\infty} \frac{\bar{f}(t\tilde{u})\tilde{u}}{p t^{p-1}} = \frac{1}{p} |\tilde{u}|^p \lim_{t \rightarrow +\infty} \frac{\bar{f}(t\tilde{u})}{|t\tilde{u}|^{p-2} t \tilde{u}} \geq \frac{1}{p} |\tilde{u}|^p \frac{\bar{f}(\tilde{u})}{|\tilde{u}|^{p-2} \tilde{u}} = \frac{1}{p} \bar{f}(\tilde{u})\tilde{u}.$$

Thus,

$$\limsup_{t \rightarrow +\infty} \int_{\mathbb{R}^N} \left(\frac{1}{p} \bar{f}(\tilde{u})\tilde{u} - \frac{\bar{F}(t\tilde{u})}{t^p} \right) dx \leq \int_{\mathbb{R}^N} \left(\frac{1}{p} \bar{f}(\tilde{u})\tilde{u} - \liminf_{t \rightarrow +\infty} \frac{\bar{F}(t\tilde{u})}{t^p} \right) dx \leq 0,$$

which implies that, for t large enough,

$$\begin{aligned} I^\infty(t\tilde{u}) &= \frac{1}{p} t^p \int_{\mathbb{R}^N} (|\Delta \tilde{u}|^p + m|\tilde{u}|^p) dx - \int_{\mathbb{R}^N} \bar{F}(t\tilde{u}) dx \\ &= \frac{1}{p} t^p \int_{\mathbb{R}^N} \bar{f}(\tilde{u}) \tilde{u} dx - \int_{\mathbb{R}^N} \bar{F}(t\tilde{u}) dx \\ &= t^p \int_{\mathbb{R}^N} \left(\frac{1}{p} \bar{f}(\tilde{u}) \tilde{u} - \frac{\bar{F}(t\tilde{u})}{t^p} \right) dx \\ &\leq 0. \end{aligned} \quad (4.1)$$

So there exists $w = t_0 \tilde{u} \in W^{2,p}(\mathbb{R}^N)$ (t_0 is large enough) such that

$$I^\infty(w) \leq 0 \text{ and } \|w\| > \rho_0.$$

By (C_5) , we have

$$I(w) \leq I^\infty(w) \leq 0.$$

We complete the proof. $\square$

As a consequence of Lemma 4.1 and the Mountain Pass Theorem without Cerami condition, founded in [26], for the constant

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad (4.2)$$

where

$$\Gamma = \{\gamma \in C([0, 1], W^{2,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = w\},$$

there exists a Cerami sequence $\{u_n\} \subset W^{2,p}(\mathbb{R}^N)$ at the level c , that is

$$I(u_n) \rightarrow c \quad \text{and} \quad (1 + \|u_n\|) \|I'(u_n)\|_{(W^{2,p}(\mathbb{R}^N))^{-1}} \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \quad (4.3)$$

Lemma 4.2. *The sequence $\{u_n\}$ obtained in (4.3) is bounded.*

Proof. Just replacing J^∞ by c , and following exactly the same procedures as in Step 1 in the proof of Theorem 1.2, we know that if $\|u_n\| \rightarrow +\infty$ as $n \rightarrow +\infty$, by Lemma 2.1, *Vanishing* can not happen. If *Nonvanishing* occurs, then there exist $\eta > 0$, $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow +\infty} \int_{B_R(y_n)} |\omega_n|^p dx \geq \eta > 0.$$

Let $\tilde{\omega}_n(x) = \omega_n(x + y_n)$, then $\|\tilde{\omega}_n\| = \|\omega_n\|$ and there exists $0 \neq \tilde{\omega} \in W^{2,p}(\mathbb{R}^N)$ such that as $n \rightarrow +\infty$

$$\begin{cases} \tilde{\omega}_n \rightharpoonup \tilde{\omega} & \text{in } W^{2,p}(\mathbb{R}^N), \\ \tilde{\omega}_n \rightarrow \tilde{\omega} & \text{in } L^{\tau}_{loc}(\mathbb{R}^N), \quad \tau \in (p, p^{**}), \\ \tilde{\omega}_n \rightarrow \tilde{\omega} & \text{a.e. in } \mathbb{R}^N. \end{cases} \quad (4.4)$$

Let $\tilde{u}_n(x) = u_n(x + y_n)$, and for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\langle I'(u_n), \phi(x - y_n) \rangle = o_n(1).$$

Hence

$$\int_{\mathbb{R}^N} (|\Delta \tilde{u}_n|^{p-2} \Delta \tilde{u}_n \Delta \phi + m|\tilde{u}_n|^{p-2} \tilde{u}_n \phi) dx - \int_{\mathbb{R}^N} f(x + y_n, \tilde{u}_n) \phi dx = o_n(1),$$

that is, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (|\Delta \tilde{w}_n|^{p-2} \Delta \tilde{w}_n \Delta \phi + m |\tilde{w}_n|^{p-2} \tilde{w}_n \phi) dx - \int_{\mathbb{R}^N} \frac{f(x + y_n, \tilde{u}_n)}{|\tilde{u}_n|^{p-2} \tilde{u}_n} |\tilde{w}_n|^{p-2} \tilde{w}_n \phi dx = o_n(1). \quad (4.5)$$

Similar to the *Nonvanishing* case in Step 1 of Theorem 1.2, we can also prove that for any $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\Delta \tilde{w}|^{p-2} \Delta \tilde{w} \Delta \phi dx = (l - m) \int_{\mathbb{R}^N} |\tilde{w}|^{p-2} \tilde{w} \phi dx,$$

which contradicts to Corollary 1.6, and the Lemma is proved. $\square$

Lemma 4.3. Assume that the assumptions of Theorem 1.3 hold. Then $c < J_\infty$.

Proof. Assume that $w \in W^{2,p}(\mathbb{R}^N)$ is the function given in Lemma 4.1, we set $\tilde{\gamma}(t) = tw$. Then $\tilde{\gamma} \in \Gamma$. Hence, by the definition of c and Lemma 2.3,

$$c \leq \max_{t \in [0,1]} I(\tilde{\gamma}(t)) \leq \max_{t > 0} I(t\tilde{u}(x)) < \max_{t > 0} I^\infty(t\tilde{u}(x)) = I^\infty(\tilde{u}(x)) = J^\infty.$$

We complete the proof. $\square$

Finally, we give the proof of Theorem 1.3.

The Proof of Theorem 1.3.: Since $\{u_n\}$ is bounded in $W^{2,p}(\mathbb{R}^N)$, then there exist some $u_0 \in W^{2,p}(\mathbb{R}^N)$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that as $n \rightarrow +\infty$

$$\begin{cases} u_n \rightharpoonup u_0 & \text{in } W^{2,p}(\mathbb{R}^N), \\ u_n \rightarrow u_0 & \text{in } L_{loc}^\tau(\mathbb{R}^N), \quad \tau \in (p, p^{**}), \\ u_n \rightarrow u_0 & \text{a.e. in } \mathbb{R}^N. \end{cases} \quad (4.6)$$

By (4.6) and similar to the proof of Step 1 in Theorem 1.2, we can prove that

$$\Delta u_n \rightarrow \Delta u_0 \text{ a.e. in } \mathbb{R}^N, \text{ as } n \rightarrow +\infty.$$

By (2.1)-(2.2), (4.3) and Lebesgue Dominated Theorem, we have that for any $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned} & \langle I'(u_n) - I'(u_0), \phi \rangle \\ &= \int_{\mathbb{R}^N} (|\Delta u_n|^{p-2} \Delta u_n - |\Delta u_0|^{p-2} \Delta u_0) \Delta \phi dx + \int_{\mathbb{R}^N} m(|u_n|^{p-2} u_n - |u_0|^{p-2} u_0) \phi dx \\ & \quad - \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_0)] \phi dx \rightarrow 0, \text{ as } n \rightarrow +\infty. \end{aligned}$$

Thus $I'(u_0) = 0$. In order to complete the proof of Theorem 1.3, we must show u_0 is nontrivial. To this end, we suppose by contradiction that $u_0 \equiv 0$.

Claim I: $u_n \rightarrow 0$ in $W^{2,p}(\mathbb{R}^N)$ as $n \rightarrow +\infty$.

Assume that $u_n \rightarrow 0$ in $W^{2,p}(\mathbb{R}^N)$ as $n \rightarrow +\infty$. Then $\lim_{n \rightarrow +\infty} I(u_n) = 0$, which contradicts to $\lim_{n \rightarrow +\infty} I(u_n) = c > 0$.

Claim II: If $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ as $n \rightarrow +\infty$, then

$$\lim_{n \rightarrow +\infty} [I(u_n) - I^\infty(u_n)] = 0, \quad \lim_{n \rightarrow +\infty} [\langle I'(u_n), u_n \rangle - \langle I^{\infty'}(u_n), u_n \rangle] = 0. \quad (4.7)$$

In fact, for any given $R > 0$, we have that

$$\begin{aligned} |I(u_n) - I^\infty(u_n)| &\leq \int_{\mathbb{R}^N} |F(x, u_n) - \bar{F}(u_n)| dx \\ &\leq \int_{|x| \leq R} |F(x, u_n) - \bar{F}(u_n)| dx + \int_{|x| \geq R} |F(x, u_n) - \bar{F}(u_n)| dx \\ &\triangleq I_n^1 + I_n^2. \end{aligned}$$

Since $u_n \rightharpoonup 0$ in $W^{2,p}(\mathbb{R}^N)$ as $n \rightarrow +\infty$, by Sobolev imbedding and up to a subsequence, we have that $u_n \rightarrow 0$ in $L_{loc}^\tau(\mathbb{R}^N)$ as $n \rightarrow +\infty$, where $\tau \in [p, p^{**})$. Hence (2.2) implies that $\lim_{n \rightarrow +\infty} I_n^1 = 0$.

On the other hand, by $(C_2) - (C_3)$ and (C_5) for any $\delta > 0$, we have

$$\begin{aligned} I_n^2 &= \left[\int_{\{|x| \geq R: |u_n| < \delta\}} + \int_{\{|x| \geq R: \delta \leq |u_n| \leq \frac{1}{\delta}\}} + \int_{\{|x| \geq R: |u_n| > \frac{1}{\delta}\}} \right] |F(x, u_n) - \bar{F}(u_n)| dx \\ &\leq \varepsilon_1(\delta) \int_{\mathbb{R}^N} |u_n|^p dx + \varepsilon_2(R) \delta^{-p} \int_{\mathbb{R}^N} |u_n|^p dx + \varepsilon_3(\delta) \int_{\mathbb{R}^N} |u_n|^{p^{**}} dx, \end{aligned}$$

where

$$\begin{aligned} \varepsilon_1(\delta) &= \sup_{\{|x| \geq R: |t| < \delta\}} \frac{|F(x, t) - \bar{F}(t)|}{|t|^p} \rightarrow 0 \text{ as } \delta \rightarrow 0^+, \\ \varepsilon_3(\delta) &= \sup_{\{|x| \geq R: |t| > \frac{1}{\delta}\}} \frac{|F(x, t) - \bar{F}(t)|}{|t|^{p^{**}}} \rightarrow 0 \text{ as } \delta \rightarrow 0^+ \end{aligned}$$

and

$$\varepsilon_2(R) = \sup_{\{|x| \geq R: \delta \leq |t| \leq \frac{1}{\delta}\}} \frac{|F(x, t) - \bar{F}(t)|}{|t|^p} \rightarrow 0 \text{ as } R \rightarrow +\infty, \text{ for fixed } \delta.$$

Therefore, $\lim_{n \rightarrow +\infty} I_n^2 = 0$ and hence $\lim_{n \rightarrow +\infty} [I(u_n) - I^\infty(u_n)] = 0$. Similarly, we can deduce that $\lim_{n \rightarrow +\infty} [\langle I'(u_n), u_n \rangle - \langle I^\infty(u_n), u_n \rangle] = 0$. So (4.7) is proved.

Claim III: There is a $A > 0$ such that

$$\int_{\mathbb{R}^N} |\Delta u_n|^p dx \geq A > 0, \quad \forall n \geq 1. \quad (4.8)$$

In fact, if (4.8) is false, then $\int_{\mathbb{R}^N} |\Delta u_n|^p dx \rightarrow 0$ as $n \rightarrow +\infty$. Using Claim I, we may assume that for some $\eta > 0$

$$u_n \in W^{2,p}(\mathbb{R}^N) \setminus \{0\} \text{ for all } n \geq 1 \text{ and } \lim_{n \rightarrow +\infty} \|u_n\| = \eta,$$

which, together with $\int_{\mathbb{R}^N} |\Delta u_n|^p dx \rightarrow 0$ as $n \rightarrow +\infty$, implies that

$$\left(\int_{\mathbb{R}^N} m |u_n|^p dx \right)^{\frac{1}{p}} \rightarrow \eta > 0 \text{ as } n \rightarrow +\infty.$$

By $(C_1) - (C_3)$, we have that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$f(x, u_n) u_n \leq \varepsilon |u_n|^p + C_\varepsilon |u_n|^{p^{**}}.$$

Then

$$\begin{aligned} o_n(1) &= \langle I'(u_n), u_n \rangle \\ &\geq \int_{\mathbb{R}^N} (|\Delta u_n|^p + m |u_n|^p) dx - \int_{\mathbb{R}^N} (\varepsilon |u_n|^p + C_\varepsilon |u_n|^{p^{**}}) dx \\ &\geq \int_{\mathbb{R}^N} (|\Delta u_n|^p + m |u_n|^p) dx - \varepsilon \int_{\mathbb{R}^N} |u_n|^p dx - C \left(\int_{\mathbb{R}^N} |\Delta u_n|^p dx \right)^{\frac{p^{**}}{p}} \\ &\geq C \eta^p + o_n(1), \end{aligned}$$

which is a contradiction. So (4.8) is true.

Based on the above three Claims, we would show that there is a contradiction, which implies that $u_0 \neq 0$. By (4.3), (4.6) and (4.7), we have

$$\begin{cases} c = I(u_n) + o_n(1) = I^\infty(u_n) + o_n(1), \\ \langle I^{\infty'}(u_n), u_n \rangle = o_n(1). \end{cases} \quad (4.9)$$

Let $\bar{u}_n(x) = u_n(t_n x)$, where $t_n > 0$ will be determined later. Then

$$I^\infty(\bar{u}_n) = \frac{1}{p} t_n^{-N} (t_n^{2p} - 1) \int_{\mathbb{R}^N} |\Delta u_n|^p dx + t_n^{-N} I^\infty(u_n) \quad (4.10)$$

and

$$\langle I^{\infty'}(\bar{u}_n), \bar{u}_n \rangle = t_n^{-N} \left[(t_n^{2p} - 1) \int_{\mathbb{R}^N} |\Delta u_n|^p dx + \langle I^{\infty'}(u_n), u_n \rangle \right]. \quad (4.11)$$

Taking $t_n = \left| \left(1 - \frac{\langle I^{\infty'}(u_n), u_n \rangle}{\int_{\mathbb{R}^N} |\Delta u_n|^p dx} \right) \right|^{\frac{1}{2p}}$, then by (4.8), (4.9) and (4.11) we know that

$$t_n \rightarrow 1 \text{ as } n \rightarrow +\infty, \text{ and } \bar{u}_n \in \Lambda \text{ for } n \text{ large enough.} \quad (4.12)$$

Combining (4.9)-(4.10) and (4.12), one has

$$c = I^\infty(u_n) + o_n(1) = I^\infty(\bar{u}_n) + o_n(1) \geq J^\infty + o_n(1), \quad (4.13)$$

which contradicts to Lemma 4.3.

Therefore $u_0 \in W^{2,p}(\mathbb{R}^N) \setminus \{0\}$.

Going on as the proof of Theorem 1.2, we can get the nonexistence of nontrivial weak solutions for the case $l \leq m$. So the proof of Theorem 1.3 is complete. $\square$

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