Research Article

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Positive radial symmetric solutions for a class of elliptic problems with critical exponent and -1 growth

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Abstract: In this paper, we consider a class of semilinear elliptic equation with critical exponent and -1 growth. By using the critical point theory for nonsmooth functionals, two positive solutions are obtained. Moreover, the symmetry and monotonicity properties of the solutions are proved by the moving plane method. Our results improve the corresponding results in the literature.

Keywords: Singular elliptic equation; critical growth; radial symmetry; critical point theory for nonsmooth functionals; moving plane method

MSC: 35B33; 35J61; 35J75

1 Introduction and main result

Consider the multiplicity of positive solutions for the following singular elliptic equation with critical growth

$$\begin{cases} \Delta u + u^{2^*-1} + \frac{\mu}{u} = 0, & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \quad (1.1)$$

where $B \subset \mathbb{R}^N$ ($N \geq 3$) is the unit ball, μ is a positive constant. Problem (1.1) has a variational structure given by the functional

$$I(u) = \frac{1}{2} \int_B |\nabla u|^2 dx - \frac{1}{2^*} \int_B |u|^{2^*} dx - \mu \int_B \ln |u| dx$$

for $u \in H_0^1(B)$. It is well known that the singular term leads to the non-differentiability of I on $H_0^1(B)$. In fact, since $I(tu) \rightarrow +\infty$ as $t \rightarrow 0^+$, I is not continuous at the point 0. Therefore, it is difficult to find out the local minimizer and the mountain pass type solutions of problem (1.1). In order to find firstly a local minimizer solution, we consider the following problem

$$\begin{cases} \Delta u + \frac{\mu}{u} = 0, & \text{in } B, \\ u = 0, & \text{on } \partial B. \end{cases} \quad (1.2)$$

According to Theorem 1 of [1], we know that problem (1.2) has a unique positive solution $w_\mu \in C^{2+\alpha}(B) \cap C(\bar{B})$ ($0 < \alpha < 1$) with $w_\mu \geq c\phi_1$ (where ϕ_1 is an eigenfunction corresponding to the smallest eigenvalue λ_1 of

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the problem $\Delta\phi + \mu\phi = 0$, $\phi|_{\partial B} = 0$). Moreover, [1] proved that the following inequality holds

$$\int_B \frac{1}{\phi_1^t} dx < +\infty, \quad (1.3)$$

if and only if $t \in (0, 1)$.

The following singular elliptic problem has been extensively considered

$$\begin{cases} \Delta u + u^p + \frac{\mu}{u^\gamma} = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $0 < p \leq 2^* - 1$ and $\gamma > 0$. For examples, [2-11] studied the case of $0 < \gamma < 1$ for problem (1.4). Particularly, by the variational method and Nehari method, Sun, Wu and Long investigated the multiplicity of positive solutions for the singular elliptic problem for the first time in [8]. And, Yang [11] discussed problem (1.4) with $p = 2^* - 1$ for the first time, and obtained two positive solutions by using the variational method and sub-supersolution method. For the case $\gamma > 1$, problem (1.4) is considered by [1,12,13].

To our best knowledge, problem (1.4) with $\gamma = 1$ is only investigated by [4], and two positive solutions are obtained when $1 < p < 2^* - 1$. A nature question is whether there exist positive solutions for problem (1.4) with $\gamma = 1$ and $p = 2^* - 1$. In the present note, we give a positive answer by the critical point theory for nonsmooth functionals, and obtain two positive solutions for problem (1.1). Moreover, based on the moving plane technique, we study the symmetry and monotonicity properties of positive solutions to problem (1.1).

In order to study problem (1.1), we define $f : B \times \mathbb{R} \rightarrow [0, +\infty)$ by

$$f(x, t) = \begin{cases} 1/t, & \text{if } x \in B \text{ and } t \geq w_\mu(x), \\ 1/w_\mu, & \text{if } x \in B \text{ and } t \leq w_\mu(x). \end{cases}$$

Consider the following auxiliary problem

$$\begin{cases} \Delta u + u^{2^*-1} + \mu f(x, u) = 0, & \text{in } B, \\ u = 0, & \text{on } \partial B. \end{cases} \quad (1.5)$$

Problem (1.5) has a variational structure given by the functional

$$J(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2^*} \int_B (u^+)^{2^*} dx - \mu \int_B F(x, u^+) dx,$$

where $F(x, u^+) = \int_0^{u^+} f(x, t) dt$. Then the functional J is only continuous in $H_0^1(B)$. In the following, we need find out critical points of J , and prove that they are weak solutions of problem (1.1).

Now our main result is as follows.

Theorem 1.1. *There exists $\mu_* > 0$ such that for $0 < \mu < \mu_*$, problem (1.1) has two positive radially symmetric solutions. Moreover, the solutions are monotone decreasing about the origin.*

Remark 1. To the best of our knowledge, problem (1.1) has not been studied up to now. Compared with [4], we generalize the corresponding result to the case of critical exponent.

2 Proof of Theorem 1.1

We divide two parts to prove Theorem 1.1. First, by using the critical point theory for nonsmooth functionals, we prove that problem (1.1) has at least two positive solutions. Then, we prove that the solution of problem (1.1) is radially symmetric and monotone decreasing about the origin by the moving plane method.

2.1 Existence of Two Positive Solutions

By the definition of F , the following statement is valid.

Lemma 2.1. Assume that $\{u_n\}$ is bounded in $H_0^1(B)$ and $u_n \rightharpoonup u$ in $H_0^1(B)$, then

$$\lim_{n \rightarrow \infty} \int_B F(x, u_n) dx = \int_B F(x, u) dx. \quad (2.1)$$

Proof. When $u < w_\mu$, one has $f(x, u) = \frac{1}{w_\mu}$, so $F(x, u) = \int_0^u f(x, t) dt \leq 1$. When $u > w_\mu$, one has $f(x, u) = \frac{1}{u} = \frac{1}{u^+}$, note that $\ln |x| \leq |x|$. Then

$$\begin{aligned} F(x, u) &= \int_0^{w_\mu} \frac{1}{w_\mu} dt + \int_{w_\mu}^u \frac{1}{t} dt \\ &= 1 + \ln u^+ + 2 \ln \frac{1}{\sqrt{w_\mu}} \\ &\leq 1 + u^+ + \frac{2}{\sqrt{w_\mu}}. \end{aligned}$$

From the above information, one has

$$F(x, u) \leq 1 + u^+ + \frac{2}{\sqrt{w_\mu}}, \quad \text{for } u \in H_0^1(B). \quad (2.2)$$

From [1], we have $\sqrt{w_\mu} \geq c\sqrt{\phi_1}$. By (1.3), it holds that

$$\int_B \frac{1}{\sqrt{w_\mu}} dx \leq c \int_B \frac{1}{\sqrt{\phi_1}} dx < +\infty. \quad (2.3)$$

By the dominated convergence theorem, (2.1) holds. The proof is complete. $\square$

We now recall some concepts adapted from critical point theory for nonsmooth functionals. Let (X, d) be a complete metric space, $f : X \rightarrow \mathbb{R}$ be a continuous functional in X . Denote by $|Df|(u)$ the supremum of δ in $[0, \infty)$ such that there exist $r > 0$, a neighborhood U of $u \in X$, and a continuous map $\sigma : U \times [0, r]$ satisfying

$$\begin{cases} f(\sigma(v, t)) \leq f(v) - \delta t, & (v, t) \in U \times [0, r], \\ d(\sigma(v, t), v) \leq t, & (v, t) \in U \times [0, r]. \end{cases} \quad (2.4)$$

The extended real number $|Df|(u)$ is called the weak slope of f at u , see [14, 15].

A sequence $\{u_n\}$ of X is called Palais-Smale sequence of the functional f , if $|Df|(u_n) \rightarrow 0$ as $n \rightarrow \infty$ and $f(u_n)$ is bounded. We say that $u \in X$ is a critical point of f if $|Df|(u) = 0$. Since $u \rightarrow |Df|(u)$ is lower semicontinuous, any accumulation point of a (PS) sequence is clearly a critical point of f .

Since looking for positive solutions of problem (1.1), we consider the functional J as defined on the closed positive cone P of $H_0^1(B)$

$$P = \{u | u \in H_0^1(B), u(x) \geq 0, \text{ a.e. } x \in B\}.$$

P is a complete metric space and J is a continuous functional on P . Then we have the following conclusion.

Lemma 2.2. Assume that $u \in P$ and $|DJ|(u) < +\infty$, then for any $v \in P$ there holds

$$\begin{aligned} \mu \int_B f(x, u)(v - u) dx &\leq \int_B \nabla u \nabla (v - u) dx - \int_B u^{2^*-1} (v - u) dx \\ &\quad + |DJ|(u) \|v - u\|. \end{aligned} \quad (2.5)$$

Proof. Similar to the proof of Lemma 3.1 in [15]. Let $|DJ|(u) < c$, $\delta < \frac{1}{2} \|v - u\|$, $v \in P$ and $v \neq u$. Define the mapping $\sigma : U \times [0, \delta] \rightarrow P$ by

$$\sigma(z, t) = z + t \frac{v - z}{\|v - z\|},$$

where U is a neighborhood of u . Then $\|\sigma(z, t) - z\| = t$, combining with (2.4), there exists a pair $(z, t) \in U \times [0, \delta]$ such that

$$J(\sigma(z, t)) > J(z) - ct.$$

Consequently, we assume that there exist sequences $\{u_n\} \subset P$ and $\{t_n\} \subset [0, +\infty)$, such that $u_n \rightarrow u$, $t_n \rightarrow 0^+$, and

$$J\left(u_n + t_n \frac{v - u_n}{\|v - u_n\|}\right) \geq J(u_n) - ct_n,$$

that is,

$$J(u_n + s_n(v - u_n)) \geq J(u_n) - cs_n\|v - u_n\|, \quad (2.6)$$

where $s_n = \frac{t_n}{\|v - u_n\|} \rightarrow 0^+$ as $n \rightarrow \infty$. Divided by s_n in (2.6), one has

$$\begin{aligned} & \mu \int_B \frac{F(x, u_n + s_n(v - u_n)) - F(x, u_n)}{s_n} dx \\ & \leq \frac{1}{2} \int_B \frac{|\nabla(u_n + s_n(v - u_n))|^2 - |\nabla u_n|^2}{s_n} dx \\ & - \frac{1}{2^*} \int_B \frac{(u_n + s_n(v - u_n))^{2^*} - u_n^{2^*}}{s_n} dx + c\|v - u_n\|. \end{aligned}$$

Set

$$I_{1,n} = \int_B \frac{F(x, u_n + s_n(v - u_n)) - F(x, (1 - s_n)u_n)}{s_n} dx$$

and

$$I_{2,n} = \int_B \frac{F(x, (1 - s_n)u_n) - F(x, u_n)}{s_n} dx.$$

Notice that

$$I_{1,n} = \int_B \frac{f(x, \xi_n)s_nv}{s_n} dx = \int_B f(x, \xi_n)v dx,$$

where $\xi_n \in (u_n - s_n u_n, u_n + s_n(v - u_n))$, which implies that $\xi_n \rightarrow u$ ($u_n \rightarrow u$) as $s_n \rightarrow 0^+$. Note that $F(x, t)$ is increasing in t , then $I_{1,n} \geq 0$ for all n . Applying Fatou's Lemma to $I_{1,n}$, we obtain

$$\liminf_{n \rightarrow \infty} I_{1,n} \geq \int_B f(x, u)v dx$$

for $v \in P$. For $I_{2,n}$, by the differential mean value theorem, we have

$$\lim_{n \rightarrow \infty} I_{2,n} = - \int_B f(x, u)u dx.$$

From the above information, one has

$$\begin{aligned} \mu \int_B f(x, u)(v - u) dx & \leq \liminf_{n \rightarrow \infty} (I_{1,n} + I_{2,n}) \\ & \leq \int_B \nabla u \nabla (v - u) dx - \int_B u^{2^*-1}(v - u) dx + c\|v - u\| \end{aligned}$$

for every $v \in P$. Since $|DJ|(u) < c$ is arbitrary, this leads us to the proof of Lemma 2.2. $\square$

Lemma 2.3. (i) Assume that u is a critical point of J , then u is a weak solution of problem (1.5), that is, for all $\varphi \in H_0^1(B)$ it holds that

$$\int_B \nabla u \nabla \varphi dx = \int_B u^{2^*-1} \varphi dx + \mu \int_B f(x, u) \varphi dx. \quad (2.7)$$

Moreover, $u \geq w_\mu$ a.e. in B .

(ii) If u is a critical point of J , then u is a positive solution of problem (1.1).

Proof. (i) Let u be a critical point of J . By Lemma 2.2, for $\varphi \in H_0^1(B)$, $s > 0$, taking $v = (u + s\varphi)^+ \in P$ as test function in (2.5), one has

$$\begin{aligned} 0 &\leq \int_B \nabla u \nabla ((u + s\varphi)^+ - u) dx \\ &\quad - \int_B u^{2^*-1} ((u + s\varphi)^+ - u) dx - \mu \int_B f(x, u) ((u + s\varphi)^+ - u) dx \\ &= s \left[\int_B \nabla u \nabla \varphi dx - \int_B u^{2^*-1} \varphi dx - \mu \int_B f(x, u) \varphi dx \right] - \int_{\{u+s\varphi < 0\}} \nabla u \nabla (u + s\varphi) dx \\ &\quad + \int_{\{u+s\varphi < 0\}} u^{2^*-1} (u + s\varphi) dx + \mu \int_{\{u+s\varphi < 0\}} f(x, u) (u + s\varphi) dx \\ &\leq s \left[\int_B \nabla u \nabla \varphi dx - \int_B u^{2^*-1} \varphi dx - \mu \int_B f(x, u) \varphi dx \right] \\ &\quad - s \int_{\{u+s\varphi < 0\}} \nabla u \nabla \varphi dx. \end{aligned}$$

Since $\nabla u(x) = 0$ for a.e. $x \in B$ with $u(x) = 0$ and $\text{meas}\{x \in B | u(x) + s\varphi(x) < 0, u(x) > 0\} \rightarrow 0$ as $s \rightarrow 0$, we have

$$\int_{\{u+s\varphi < 0\}} \nabla u \nabla \varphi dx = \int_{\{u+s\varphi < 0, u > 0\}} \nabla u \nabla \varphi dx \rightarrow 0 \text{ as } s \rightarrow 0.$$

Therefore

$$0 \leq s \left(\int_B \nabla u \nabla \varphi dx - \int_B u^{2^*-1} \varphi dx - \mu \int_B f(x, u) \varphi dx \right) + o(s)$$

as $s \rightarrow 0$. We obtain

$$\int_B \nabla u \nabla \varphi dx - \int_B u^{2^*-1} \varphi dx - \mu \int_B f(x, u) \varphi dx \geq 0.$$

By the arbitrariness of the sign of φ , we can deduce that (2.7) holds. Thus, u is a weak solution of problem (1.5).

Next, we prove that $u \geq w_\mu$ a.e. in B . Choosing $\varphi = (u - w_\mu)^-$ in (2.7), one has

$$\int_{\{u < w_\mu\}} (\nabla u, \nabla(u - w_\mu)) dx = \int_{\{u < w_\mu\}} (u^{2^*-1} + \frac{\mu}{w_\mu})(u - w_\mu) dx. \quad (2.8)$$

Notice that

$$\int_{\{u < w_\mu\}} (\nabla w_\mu, \nabla(u - w_\mu)) dx = \mu \int_{\{u < w_\mu\}} \frac{u - w_\mu}{w_\mu} dx. \quad (2.9)$$

Hence, it follows from (2.8) and (2.9) that

$$\int_{\{u < w_\mu\}} |\nabla(u - w_\mu)|^2 dx = \int_{\{u < w_\mu\}} u^{2^*-1} (u - w_\mu) dx \leq 0,$$

which implies that $\|(u - w_\mu)^-\| = 0$, that is, $u(x) \geq w_\mu(x)$ a.e. in B . Thus, $u \in P$.

(ii) This follows from (i). The proof of Lemma 2.3 is completed. $\square$

Let S be the best Sobolev constant, namely

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} |u|^{2^*} dx\right)^{\frac{2}{2^*}}}. \quad (2.10)$$

Lemma 2.4. Assume that $0 < \mu < 1$ and $J(u_n) \rightarrow c < \frac{1}{N} S^{\frac{N}{2}} - \Theta \mu$, $|DJ|(u_n) \rightarrow 0$ as $n \rightarrow \infty$, then $u_n \rightarrow u$ in $H_0^1(B)$, where $\Theta = \Theta(N, S, |B|, \int_B \frac{1}{\sqrt{\phi_1}} dx)$.

Proof. Let $\{u_n\} \subset P \subset H_0^1(B)$ be such that $|DJ|(u_n) \rightarrow 0$, $J(u_n) \rightarrow c$ as $n \rightarrow \infty$. By Lemma 2.2, we have

$$\begin{aligned} \mu \int_B f(x, u_n)(v - u_n) dx &\leq \int_B [\nabla u_n \nabla(v - u_n) - u_n^{2^*-1}(v - u_n)] dx \\ &\quad + |DJ|(u_n) \|v - u_n\|. \end{aligned} \quad (2.11)$$

Taking $v = 2u_n \in P$ in (2.11), we have

$$\|u_n\|^2 - \int_B u_n^{2^*} dx - \mu \int_B f(x, u_n) u_n dx + |DJ|(u_n) \|u_n\| \geq 0. \quad (2.12)$$

By (2.2) and (2.3)

$$\begin{aligned} J(u_n) + \frac{1}{2^*} |DJ|(u_n) \|u_n\| &\geq \frac{1}{N} \|u_n\|^2 - \mu \int_B f(x, u_n) dx \\ &\geq \frac{1}{N} \|u_n\|^2 - c \|u_n\|, \end{aligned}$$

for some positive constant c . Therefore, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Thus, there exist a subsequence, still denoted by $\{u_n\}$, and u such that $u_n \rightharpoonup u$ in $H_0^1(B)$. Taking $v = 0 \in P$ in (2.11) again, one has

$$\|u_n\|^2 - \int_B u_n^{2^*} dx - \mu \int_B f(x, u_n) u_n dx + |DJ|(u_n) \|u_n\| \leq 0. \quad (2.13)$$

It follows from (2.12) and (2.13) that

$$\|u_n\|^2 - \int_B u_n^{2^*} dx - \mu \int_B f(x, u_n) u_n dx + |DJ|(u_n) \|u_n\| = 0. \quad (2.14)$$

Since $f(x, u_n) u_n \leq 1$ for all n , by the dominated convergence theorem, one gets

$$\int_B f(x, u_n) u_n dx \rightarrow \int_B f(x, u) u dx.$$

Set $w_n = u_n - u$ and $\lim_{n \rightarrow \infty} \|w_n\| = l > 0$, by Brézis-Lieb's lemma, Lemma 2.1 and (2.14), we have

$$\|w_n\|^2 + \|u\|^2 - \int_B w_n^{2^*} dx - \int_B u^{2^*} dx - \mu \int_B f(x, u) u dx = o(1). \quad (2.15)$$

On the other hand, for $\varphi \in P$, taking $v = u_n + \varphi$ in (2.11) and letting $n \rightarrow \infty$, by Fatou's lemma, we obtain

$$\mu \int_B f(x, u) \varphi dx \leq \int_B \nabla u \nabla \varphi dx - \int_B u^{2^*-1} \varphi dx, \text{ for } \varphi \in P. \quad (2.16)$$

Denote $u_n^T = \min\{u_n, T\}$, $T > 0$. Taking $v = u_n - u_n^T \in P$ in (2.11), one has

$$-\mu \int_B f(x, u_n) u_n^T dx \leq - \int_B \nabla u_n \nabla u_n^T dx + \int_B u_n^{2^*-1} u_n^T dx + |DJ|(u_n) \|u_n^T\|.$$

Taking the limit $n \rightarrow \infty$ first, then $T \rightarrow \infty$, one gets

$$-\mu \int_B f(x, u) u dx \leq - \int_B |\nabla u|^2 dx + \int_B u^{2^*} dx. \quad (2.17)$$

It follows from (2.16) and (2.17) that

$$\mu \int_B f(x, u)(\varphi - u) dx \leq \int_B \nabla u \nabla(\varphi - u) dx - \int_B u^{2^*-1}(\varphi - u) dx, \text{ for } \varphi \in P.$$

Therefore, similar to the proof of (i) in Lemma 2.3, for all $\varphi \in H_0^1(B)$, we have

$$\int_B \nabla u \nabla \varphi dx - \int_B u^{2^*-1} \varphi dx - \mu \int_B f(x, u) \varphi dx = 0.$$

In particular, one has

$$\|u\|^2 - \int_B u^{2^*} dx - \mu \int_B f(x, u) u dx = 0. \quad (2.18)$$

It follows from (2.15) and (2.18) that

$$\|w_n\|^2 - \int_B w_n^{2^*} dx = o(1). \quad (2.19)$$

Consequently, it follows from (2.10) that $l^2 \geq S^{\frac{N}{2}}$. Then, by (2.2), (2.3), (2.18) and the Young inequality, one has

$$\begin{aligned} J(u) &\geq \frac{1}{N} \|u\|^2 - \mu \int_B F(x, u^+) dx \\ &\geq \frac{1}{N} \|u\|^2 - C\mu \\ &\geq -\Theta\mu, \end{aligned}$$

where $\Theta = \Theta(N, S, |B|, \int_B \frac{1}{\sqrt{\phi_1}} dx)$. Besides, by Lemma 2.1, (2.19) and the condition $c < \frac{1}{N} S^{\frac{N}{2}} - \Theta\mu$, one has

$$\begin{aligned} J(u) &= J(u_n) - \frac{1}{2} \|w_n\|^2 + \frac{1}{2^*} \int_B w_n^{2^*} dx + o(1) \\ &\leq c - \frac{1}{N} l^2 \\ &< \frac{1}{N} S^{\frac{N}{2}} - \Theta\mu - \frac{1}{N} S^{\frac{N}{2}} \\ &= -\Theta\mu, \end{aligned}$$

which contradicts the above inequality. Therefore, $l = 0$, which implies that $u_n \rightarrow u$ in $H_0^1(B)$. This completes the proof of Lemma 2.4. $\square$

Now, we show that the functional J satisfies the Mountain-pass lemma.

Lemma 2.5. *There exist constants $r, \rho, \mu_0 > 0$ such that the functional J satisfies the following conclusions:*

- (i) $J|_{u \in S_\rho} \geq r > 0$ and $\inf_{u \in B_\rho} J(u) < 0$ for $\mu \in (0, \mu_0)$.
- (ii) *There exists $e \in H_0^1(B)$ such that $\|e\| > \rho$ and $J(e) < 0$.*

Proof. (i) By (2.2) and (2.3), one has

$$J(u) \geq \frac{1}{2} \|u\|^2 - \frac{S^{-\frac{2^*}{2}}}{2^*} \|u\|^{2^*} - C\mu.$$

We see that there exist constants $\rho, r, \Lambda_0 > 0$, such that $J(u)|_{S_\rho} \geq r$ for every $\mu \in (0, \mu_0)$. Moreover, for $u \in H_0^1(B) \setminus \{0\}$, it holds

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{J(tu)}{t} &= -\mu \lim_{t \rightarrow 0^+} \int_B \frac{\int_0^{tu^+} f(x, s) ds}{t} dx \\ &= -\mu \int_B f(x, 0) u^+ dx \\ &= -\mu \int_B \frac{u^+}{w_\mu} dx \\ &< 0. \end{aligned}$$

So we obtain $J(tu) < 0$ for all $u^+ \neq 0$ and t small enough. Therefore, for $\|u\|$ small enough, one has

$$d \triangleq \inf_{u \in B_\rho} J(u) < 0. \quad (2.20)$$

(ii) For every $u \in H_0^1(B)$, $u^+ \neq 0$, we have $J(tu) \rightarrow -\infty$ as $t \rightarrow +\infty$. Thus, there is $e \in H_0^1(B)$ such that $\|e\| > \rho$ and $J(e) < 0$. The proof is completed. $\square$

According to Lemma 2.1, similar to [16], we can easily obtain that d is attained at some $u^* \in B_\rho$. According to Lemma 2.3 and Lemma 2.5 (i), we obtain the following result.

Theorem 2.6. For $0 < \mu < \mu_0$, problem (1.1) has a positive solution u^* with $J(u^*) = d < 0$.

Denote

$$U_\varepsilon(x) = \frac{[N(N-2)]^{\frac{N-2}{4}} \varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2}{2}}}, \quad x \in \mathbb{R}^N, \quad \varepsilon > 0.$$

Then, the infimum (2.10) is attained at $U(x)$. Choosing $\eta \in C_0^\infty(B_\delta(x_0), [0, 1])$ such that $\eta(x) = 1$ near $x = x_0$, where $B_\delta(x_0) \subset B$. Let $\varphi_\varepsilon = U_\varepsilon \eta$, we have the following conclusion.

Lemma 2.7. There holds $\sup_{t \geq 0} J(u^* + t\varphi_\varepsilon) \leq \frac{1}{N} S^{\frac{N}{2}} - \Theta\mu$ for sufficient small $\mu > 0$.

Proof. From [17], we have

$$\int_B |\nabla \varphi_\varepsilon|^2 dx = S^{\frac{N}{2}} + O(\varepsilon^{N-2}), \quad \int_B \varphi_\varepsilon^{2^*} dx = S^{\frac{N}{2}} + O(\varepsilon^N), \quad \int_B \varphi_\varepsilon^{2^*-1} dx = c\varepsilon^{\frac{N-2}{2}}, \quad (2.21)$$

where $c > 0$ is a constant. Since $J(u^* + t\varphi_\varepsilon) \rightarrow -\infty$ as $t \rightarrow \infty$, $J(u^*) < 0$, according to Lemma 2.5 (i), we can assume that there exist $t_1, t_2 > 0$ such that $\sup_{t \geq 0} J(u^* + t\varphi_\varepsilon) = \sup_{t \in [t_1, t_2]} J(u^* + t\varphi_\varepsilon)$.

Since u^* is a positive solution of problem (1.1), we have

$$\begin{cases} J(u^*) < 0, \\ \int_B \nabla u^* \nabla \varphi_\varepsilon dx = \int_B (u^*)^{2^*-1} \varphi_\varepsilon dx + \mu \int_B \frac{\varphi_\varepsilon}{u^*} dx, \\ \frac{\varphi_\varepsilon}{u^*} \in L^1(B), \text{ that is } \int_B \frac{\varphi_\varepsilon}{u^*} dx \leq C \text{ for some } C > 0. \end{cases} \quad (2.22)$$

Moreover, according to $u^* > w_\mu$, we deduce that

$$\begin{aligned} F(x, u^*) &= \int_0^{w_\mu} \frac{1}{w_\mu} dt + \int_{w_\mu}^{u^*} \frac{1}{t} dt \\ &= 1 + \ln u^* - \ln w_\mu. \end{aligned}$$

Consequently, one has

$$\begin{aligned} \int_B [F(x, u^* + t\varphi_\varepsilon) - F(x, u^*)] dx &= \int_B [1 + \ln(u^* + t\varphi_\varepsilon) - \ln w_\mu] dx \\ &= \int_B (1 + \ln u^* - \ln w_\mu) dx \\ &= \int_B \ln \left(1 + \frac{t\varphi_\varepsilon}{u^*} \right) dx \\ &\geq 0 \end{aligned} \quad (2.23)$$

for all $t \geq 0$. Therefore, from (2.22), (2.23) and the following inequality

$$(a + b)^{2^*} \geq a^{2^*} + 2^* a^{2^*-1} b + 2^* a b^{2^*-1} + b^{2^*}, \text{ for } a, b \geq 0,$$

we have

$$\begin{aligned} J(u^* + t\varphi_\varepsilon) &= \frac{1}{2} \|u^* + t\varphi_\varepsilon\|^2 - \frac{1}{2^*} \int_B (u^* + t\varphi_\varepsilon)^{2^*} dx - \mu \int_B F(x, u^* + t\varphi_\varepsilon) dx \\ &= \frac{1}{2} \|u^*\|^2 + \frac{t^2}{2} \|\varphi_\varepsilon\|^2 + t \int_B \nabla u^* \nabla \varphi_\varepsilon dx \\ &\quad - \frac{1}{2^*} \int_B (u^* + t\varphi_\varepsilon)^{2^*} dx - \mu \int_B F(x, u^* + t\varphi_\varepsilon) dx \\ &= J(u^*) + \frac{t^2}{2} \|\varphi_\varepsilon\|^2 \\ &\quad - \frac{1}{2^*} \int_B [(u^* + t\varphi_\varepsilon)^{2^*} - (u^*)^{2^*} - 2^* (u^*)^{2^*-1} t\varphi_\varepsilon] dx \\ &\quad - \mu \int_B [F(x, u^* + t\varphi_\varepsilon) - F(x, u^*)] dx + t\mu \int_B \frac{\varphi_\varepsilon}{u^*} dx \\ &\leq \frac{t^2}{2} \|\varphi_\varepsilon\|^2 - \frac{t^{2^*}}{2^*} \int_B \varphi_\varepsilon^{2^*} dx - t^{2^*-1} \int_B u^* \varphi_\varepsilon^{2^*-1} dx + t\mu \int_B \frac{\varphi_\varepsilon}{u^*} dx. \end{aligned} \quad (2.24)$$

For $t \geq 0$, let

$$g(t) = \frac{t^2}{2} \|\varphi_\varepsilon\|^2 - \frac{t^{2^*}}{2^*} \int_B \varphi_\varepsilon^{2^*} dx.$$

Then, we have $g'(t) = t \left[\|\varphi_\varepsilon\|^2 - t^{2^*-2} \int_B \varphi_\varepsilon^{2^*} dx \right]$. Let $g'(t) = 0$, one has

$$t_{\max} = \left(\frac{\|\varphi_\varepsilon\|^2}{\int_B \varphi_\varepsilon^{2^*} dx} \right)^{\frac{1}{2^*-2}}$$

such that $g'(t) > 0$ for $0 < t < t_{\max}$ and $g'(t) < 0$ for $t > t_{\max}$. Moreover, $g(t_{\max}) = \max_{t \geq 0} g(t)$. By a standard regularity argument, one has $u^* \in C^1(B, \mathbb{R}^+)$ and there exists a positive constant C (independent of ε) such that $u^* < C$. Consequently, it follows from (2.21)-(2.24) that

$$\begin{aligned} \sup_{t \in [t_1, t_2]} J(u^* + t\varphi_\varepsilon) &\leq \sup_{t \geq 0} \left(\frac{t^2}{2} \|\varphi_\varepsilon\|^2 - \frac{t^{2^*}}{2^*} \int_B \varphi_\varepsilon^{2^*} dx \right) - Ct_1^{2^*-1} \int_B \varphi_\varepsilon^{2^*-1} dx \\ &\quad + t_2 \mu \int_B \frac{\varphi_\varepsilon}{u^*} dx \\ &\leq g(t_{\max}) + C_1 \mu - C_2 \varepsilon^{\frac{N-2}{2}} \\ &= \frac{1}{N} S^{\frac{N}{2}} + C_3 \varepsilon^{N-2} + C_1 \mu - C_2 \varepsilon^{\frac{N-2}{2}}. \end{aligned}$$

Let $\varepsilon = \mu^{\frac{1}{N-2}}$, $\mu_1 = \min\{1, (\frac{C_2}{C_3+C_1+\Theta})^2\}$, we have

$$\begin{aligned} C_3\varepsilon^{N-2} + C_1\mu - C_2\varepsilon^{\frac{N-2}{2}} &= (C_3 + C_1)\mu - C_2\mu^{\frac{1}{2}} \\ &= \mu(C_3 + C_1 - C_2\mu^{-\frac{1}{2}}) \\ &\leq -\Theta\mu \end{aligned}$$

for all $\mu \in (0, \mu_1)$. This leads us to the proof of Lemma 2.7. $\square$

Theorem 2.8. *There exists $\mu_* > 0$ such that for $0 < \mu < \mu_*$, problem (1.1) has at least two positive solutions.*

Proof. Let $0 < \mu < \mu_* = \min\{\mu_0, \mu_1\}$. By Lemma 2.4 and Lemma 2.7, J satisfies the Palais-Smale condition at the level c . By Lemma 2.5, there exists a Palais-Smale sequence $\{u_n\}$ such that $|DJ|(u_n) \rightarrow 0$, $J(u_n) \rightarrow c$ as $n \rightarrow \infty$. Up to a subsequence, $u_n \rightarrow v^*$ in $H_0^1(B)$, and $J(v^*) = \lim_{n \rightarrow \infty} J(u_n) = c > 0$, $|DJ|(u_n) \rightarrow 0$. Applying the Mountain pass lemma [14] and Lemma 2.3, v^* is a positive solution of problem (1.1). Combining with Theorem 2.6, the proof of Theorem 2.8 is completed. $\square$

2.2 Radially Symmetric and Monotone Decreasing Solution

In this part, we shall prove that every positive solution u of problem (1.1) is radially symmetric and monotone decreasing about the origin. Let $u = \mu^{\frac{1}{2}}v$, then we deduce that v satisfies the following equation

$$\begin{cases} \Delta v + \mu^{\frac{2^*-2}{2}} v^{2^*-1} + \frac{1}{v} = 0, & \text{in } B, \\ v = 0, & \text{on } \partial B. \end{cases} \quad (2.25)$$

Therefore, it is sufficient to prove that v is radially symmetric and monotone decreasing about the origin. First, we introduce some notations.

Choose any direction to be the x_1 direction. Let

$$T_\lambda = \{x \in \mathbb{R}^n | x_1 = \lambda, \text{ for some } \lambda \in \mathbb{R}\}$$

be the moving plane, and the set

$$\Sigma_\lambda = \{x \in B_1(0) | x_1 < \lambda\}$$

be the region to the left of the plane. We use the standard notation

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$$

for the reflection of x about the plane T_λ .

We denote $v(x^\lambda) = v_\lambda(x)$, then compare the values of $v(x)$ and $v_\lambda(x)$, let

$$w_\lambda(x) = v_\lambda(x) - v(x).$$

Otherwise, after a direct calculation, we derive that $w_\lambda(x^\lambda) = -w_\lambda(x)$, hence it is said to be anti-symmetric.

We carry out the proof in two steps. To begin with, we show that for λ sufficiently close to -1 , we have

$$w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \quad (2.26)$$

This provides a starting point to move the plane T_λ .

Next, we move the plane T_λ along the x_1 direction to the right as long as inequality (2.26) holds. The plane T_λ will eventually stop at some limiting position λ_0 , where

$$\lambda_0 = \sup\{\lambda | w_\rho(x) \geq 0, \rho \leq \lambda\},$$

then we are able to claim that

$$w_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}.$$

The symmetry and monotonicity of solution v about T_λ follow naturally from the proof. Also, because of the arbitrariness of the x_1 direction, we conclude that v must be radially symmetric and monotone decreasing about the origin.

Step 1. We show that for λ sufficiently close to -1 , we have

$$w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \quad (2.27)$$

If not, we denote there exists some point $x_0 \in \Sigma_\lambda$, such that

$$w_\lambda(x_0) = \min_{\Sigma_\lambda} w_\lambda(x) < 0.$$

By problem (2.25) and the Mean Value Theorem, we can easy derive that

$$\begin{aligned} -\Delta w_\lambda(x_0) &= -\Delta v_\lambda(x_0) - (-\Delta v(x_0)) \\ &= \frac{1}{v_\lambda(x_0)} + \mu^{\frac{2^*-2}{2}} v_\lambda^{2^*-1}(x_0) - \left[\frac{1}{v(x_0)} + \mu^{\frac{2^*-2}{2}} v^{2^*-1}(x_0) \right] \\ &= v_\lambda^{-1}(x_0) - v^{-1}(x_0) + \mu^{\frac{2^*-2}{2}} [v_\lambda^{2^*-1}(x_0) - v^{2^*-1}(x_0)] \\ &= -\xi_\lambda^{-2}(x_0) w_\lambda(x_0) + (2^* - 1) \mu^{\frac{2^*-2}{2}} \eta_\lambda^{2^*-2}(x_0) w_\lambda(x_0) \\ &\geq [-v^{-2}(x_0) + (2^* - 1) \mu^{\frac{2^*-2}{2}} v^{2^*-2}(x_0)] w_\lambda(x_0), \end{aligned} \quad (2.28)$$

where $\xi_\lambda(x_0)$, $\eta_\lambda(x_0)$ are in between $v_\lambda(x_0)$ and $v(x_0)$, and the last inequality is due to the negative point x_0 of $w_\lambda(x)$. That is

$$v_\lambda(x_0) < v(x_0),$$

we have

$$0 < v_\lambda(x_0) \leq \xi_\lambda(x_0), \eta_\lambda(x_0) \leq v(x_0).$$

Then we obtain that

$$-\Delta w_\lambda(x_0) + c(x_0) w_\lambda(x_0) \geq 0, \quad (2.29)$$

where

$$c(x_0) = v^{-2}(x_0) - (2^* - 1) \mu^{\frac{2^*-2}{2}} v^{2^*-2}(x_0) = \frac{1 - (2^* - 1) \mu^{\frac{2^*-2}{2}} v^{2^*}(x_0)}{v^2(x_0)}.$$

Since $v \in C^\infty(B) \cap L^\infty(B)$ (see [4]), $v(x_0)$ is bounded. We can see that $c(x_0) > 0$ provided μ enough small.

On the other hand, at the minmum point, we have that

$$-\Delta w_\lambda(x_0) + c(x_0) w_\lambda(x_0) < 0,$$

which contradicts with (2.29). Therefore (2.27) hods. This completes the preparation for the moving of planes.

Step 2. Inequality (2.27) provides a starting point, from which we move the plane T_λ toward the right as long as (2.27) holds to its limiting position to show that v is monotone decreasing about the origin. More precisely, we define

$$\lambda_0 = \sup\{\lambda | w_\rho(x) \geq 0, \rho \leq \lambda\}.$$

We will show that

$$\lambda_0 = 0, \text{ and } w_{\lambda_0}(x) \equiv 0. \quad (2.30)$$

If not, then $\lambda_0 < 0$, we show that the plane can be moved further right to cause a contradiction with the definition of λ_0 . More precisely, there exists a small $\varepsilon > 0$ such that for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ such that

$$w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda, \quad (2.31)$$

which contradicts the definition of λ_0 . Hence, (2.30) must be true.

Now we need to prove (2.31) is true. First, by the definition of λ_0 , we have $w_{\lambda_0}(x) \geq 0$ in Σ_{λ_0} . Moreover, by the strong maximum principle, we obtain

$$w_{\lambda_0}(x) > 0, x \in \Sigma_{\lambda_0}.$$

We claim that there exists a constant C_0 such that

$$w_{\lambda_0}(x) \geq C_0 > 0, x \in \Sigma_{\lambda_0-\delta}.$$

Due to the continuity of $w_{\lambda_0}(x)$ with respect to λ , for sufficiently small ϵ and any $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, one obtains

$$w_{\lambda}(x) \geq 0, x \in \Sigma_{\lambda_0-\delta}. \quad (2.32)$$

Denote $\Omega_{\lambda} = \Sigma_{\lambda} \setminus \Sigma_{\lambda_0-\delta}$ this is a narrow region, we need to prove that w_{λ} satisfies the conditions of narrow region principle. From the (2.28), we know that, for any $x \in \Omega_{\lambda}$,

$$\begin{aligned} -\Delta w_{\lambda}(x) &= -\Delta v_{\lambda}(x) - (-\Delta v(x)) \\ &= \frac{1}{v_{\lambda}(x)} + \mu^{\frac{2^*-2}{2}} v_{\lambda}^{2^*-1}(x) - \left[\frac{1}{v(x)} + \mu^{\frac{2^*-2}{2}} v^{2^*-1}(x) \right] \\ &= v_{\lambda}^{-1}(x) - v^{-1}(x) + \mu^{\frac{2^*-2}{2}} [v_{\lambda}^{2^*-1}(x) - v^{2^*-1}(x)] \\ &= -\xi_{\lambda}^{-2}(x) w_{\lambda}(x) + (2^* - 1) \mu^{\frac{2^*-2}{2}} \eta_{\lambda}^{2^*-2}(x) w_{\lambda}(x) \\ &= [-\xi_{\lambda}^{-2}(x) + (2^* - 1) \mu^{\frac{2^*-2}{2}} \eta_{\lambda}^{2^*-2}(x)] w_{\lambda}(x), \end{aligned}$$

where $\xi_{\lambda}(x)$, $\eta_{\lambda}(x)$ are in between $v_{\lambda}(x)$ and $v(x)$. Consequently, one has

$$-\Delta w_{\lambda}(x) + c(x) w_{\lambda}(x) \geq 0, x \in \Omega_{\lambda},$$

where $c(x) = \xi_{\lambda}^{-2}(x) - (2^* - 1) \mu^{\frac{2^*-2}{2}} \eta_{\lambda}^{2^*-2}(x)$. Noting that $v \in C^{\infty}(B) \cap L^{\infty}(B)$, we have $c(x)$ is bounded in Ω_{λ} . So w_{λ} satisfies the narrow region principle

$$\begin{cases} -\Delta w_{\lambda}(x) + c(x) w_{\lambda}(x) \geq 0, & x \in \Omega_{\lambda}, \\ w_{\lambda}(x) \geq 0, & x \in \partial\Omega_{\lambda}, \end{cases} \quad (2.33)$$

then we derive that

$$w_{\lambda}(x) \geq 0, x \in \Omega_{\lambda}.$$

Combining with (2.32), we have

$$w_{\lambda}(x) \geq 0, x \in \Sigma_{\lambda}.$$

This (2.31) is true. Therefore, we must have

$$\lambda_0 = 0.$$

Further more, we have

$$w_0(x) \geq 0, x \in \Sigma_0.$$

If we move the plane from λ is sufficiently close to 1, then we move the plane T_{λ} along the x_1 direction to the left. By a similar argument, we can derive

$$w_0(x) \leq 0, x \in \Sigma_0,$$

then we have

$$w_0(x) \equiv 0, x \in \Sigma_0.$$

We drive that $v(x)$ is symmetric about the plane T_0 . Moreover, the arbitrariness of the x_1 direction leads to the radial symmetry of $v(x)$ about the origin, so does u . The monotonicity comes directly from the argument. This completes the proof.

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