Research Article

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On the set of positive solutions for resonant Robin (p, q) -equations

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Abstract: We consider a nonlinear Robin problem driven by the (p, q) -Laplacian and a parametric reaction exhibiting the competition of a concave term and of a resonant perturbation. We prove a bifurcation-type theorem describing the changes in the set of positive solutions as the parameter λ moves on $\mathbb{R}_+ = (0, +\infty)$. Also, we determine the continuity properties of the solution multifunction.

Keywords: nonlinear regularity, nonlinear maximum principle, resonance, bifurcation-type theorem, continuity of solution multifunction

MSC: Primary: 35J20. Secondary: 35J60

1 Introduction

In this paper we study the following nonlinear Robin problem driven by the (p, q) -Laplacian:

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{\tau-1} + f(z, u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, \quad u > 0, \quad \lambda > 0, \quad 1 < \tau < q < p. \end{array} \right\} \quad (P_\lambda)$$

In this problem, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2 -boundary $\partial\Omega$. For every $r \in (1, +\infty)$, we denote by Δ_r the r -Laplace differential operator defined by

$$\Delta_r u = \operatorname{div} \left(|Du|^{r-2} Du \right) \text{ for all } u \in W^{1,r}(\Omega).$$

In problem (P_λ) we have the sum of two such operators and so the differential operator (left-hand side) in problem (P_λ) is not homogeneous. There is also a potential term $\xi(z)u^{p-1}$ with $\xi \in L^\infty(\Omega)$, $\xi \geq 0$. In the reaction (right-hand side) of problem (P_λ) , we have the combined effects of two distinct terms with different asymptotic behavior as $x \rightarrow +\infty$. The first term, is the parametric $(p-1)$ -sublinear (that is, “concave”) nonlinearity $u \mapsto \lambda u^{\tau-1}$ ($1 < \tau < q < p$) and the second term is a Carathéodory perturbation $f(z, x)$ (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$, the function $x \mapsto f(z, x)$ is continuous), which exhibits $(p-1)$ -linear growth as $x \rightarrow +\infty$ and is resonant with respect to a nonprincipal variational eigenvalue of the operator $u \mapsto -\Delta_p u + \xi(z)|u|^{p-2}u$ with Robin boundary condition. In the Robin boundary condition, $\frac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative corresponding to the (p, q) -Laplacian. This derivative is understood via the nonlinear Green’s identity (see Papageorgiou, Rădulescu & Repovš [20, pp. 33-35]). If $u \in C^1(\overline{\Omega})$, then

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$\frac{\partial u}{\partial n_{pq}} = \left(|Du|^{p-2} + |Du|^{q-2} \right) \frac{\partial u}{\partial n}$, with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Therefore, problem (P_λ) is a nonlinear extension to Robin boundary value problems, of the classical “concave-convex” problem, but in our case the “convex” $((p-1)$ -superlinear) perturbation, is replaced by a $(p-1)$ -linear and resonant one.

Our aim in this paper is to describe the changes in the set of positive solutions of problem (P_λ) as the parameter λ moves on the open semiaxis $\mathbb{R}_+ = (0, +\infty)$ and also determine the continuity properties of the solution multifunction $\lambda \mapsto S_\lambda$, with S_λ being the set of positive solutions of problem (P_λ) . Our work here complements the recent one by Marano, Marino & Papageorgiou [14], which deals with an equation driven by the Dirichlet (p, q) -Laplacian and with a reaction of the form $u \mapsto u^{\tau-1} + \lambda f(z, u)$ with $f(z, \cdot)$ being $(p-1)$ -superlinear. So, the equation in [14] is a “concave-convex” problem with the parameter $\lambda > 0$ multiplying the “convex” perturbation. They prove a bifurcation-type theorem (see [14, Theorem 3.10]), which describes the changes in the set of positive solutions as λ moves in $\mathbb{R}_+$. Having as their starting point the work in [14], Zeng, Gasiński, Nguyen & Bai [30], proved some continuity properties for the solution multifunction. We also mention the related works of Garcia Azorero, Manfredi & Peral Alonso [5], Gasiński, Papageorgiou & Winowski [9], Guo & Zhang [10], Papageorgiou & Scapellato [21], Vetro [29] (p -Laplacian equations with competition phenomena), Papageorgiou, Rădulescu & Repovš [19] (anisotropic p -Laplacian equations with competition phenomena), Papageorgiou, Vetro & Vetro [22], Papageorgiou & Zhang [25] (Dirichlet $(p, 2)$ -equations), Papageorgiou, Vetro & Vetro [24] (weighted (p, q) -equations) and Papageorgiou, Vetro & Vetro [23] (non-homogeneous Robin equations). We also mention the works of Bai, Motreanu & Zeng [2] on the continuity properties of the solution multifunction for a parametric singular problem driven by the Dirichlet p -Laplacian and of Papageorgiou & Zhang [26], where the “concave” contribution is in the boundary condition.

We mention that equations driven by the sum of two differential operators of different nature (such as (p, q) -equations), arise in many mathematical models of physical processes. We refer to the survey papers of Marano & Mosconi [15], Rădulescu [28] and the references therein.

2 Mathematical background-hypotheses

In the analysis of problem (P_λ) , the main spaces are the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$. We denote by $\|\cdot\|$ the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p \right)^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \right\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega} \right\}.$$

We will also use another open cone in $C^1(\overline{\Omega})$

$$D_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega, \left. \frac{\partial u}{\partial n} \right|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

On $\partial\Omega$ we consider the $(N-1)$ -dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the “boundary Lebesgue spaces” $L^r(\partial\Omega)$, $1 \leq r \leq +\infty$. There exists a unique continuous linear map $\hat{\gamma}_0 : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ known as the “trace map” such that $\hat{\gamma}_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. So, the trace map extends the notion of “boundary values” to all Sobolev functions.

We know that $\text{im } \hat{\gamma}_0 = W^{-\frac{1}{p},p}(\partial\Omega)$, $\ker \hat{\gamma}_0 = W_0^{1,p}(\Omega)$ $\left(\frac{1}{p} + \frac{1}{p'} = 1 \right)$. Also $\hat{\gamma}_0(\cdot)$ is compact into $L^r(\partial\Omega)$ for $1 \leq r < \frac{(N-1)p}{N-p}$ if $p < N$ and into $L^r(\Omega)$ for $1 \leq r < +\infty$ if $p \geq N$. In the sequel, for the sake of notational

economy, we drop the use of the map $\hat{\gamma}_0(\cdot)$. All restrictions on $\partial\Omega$ of the Sobolev functions, are understood in the sense of traces.

Given $r \in (1, +\infty)$, we denote by $A_r : W^{1,r}(\Omega) \rightarrow W^{1,r}(\Omega)^*$ the nonlinear operator defined by

$$\langle A_r(u), h \rangle = \int_{\Omega} |Du|^{r-2} (Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W^{1,r}(\Omega).$$

The next proposition summarizes the well-known properties of this operator (see, for example, Gasiński & Papageorgiou [7, p. 279]).

Proposition 1. *The operator $A_r : W^{1,r}(\Omega) \rightarrow W^{1,r}(\Omega)^*$ is bounded (that is, it maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and has the $(S)_+$ -property, that is,*

$$\text{“if } u_n \xrightarrow{w} u \text{ in } W^{1,r}(\Omega) \text{ and } \limsup_{n \rightarrow \infty} \langle A_r(u_n), u_n - u \rangle \leq 0, \text{ then } u_n \rightarrow u \text{ in } W^{1,r}(\Omega).”$$

Given $u, v : \Omega \rightarrow \mathbb{R}$ two measurable functions with $u(z) \leq v(z)$ for a.a. $z \in \Omega$, we introduce the following order intervals in $W^{1,p}(\Omega)$:

$$\begin{aligned} [u, v] &= \left\{ h \in W^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega \right\}, \\ [u] &= \left\{ h \in W^{1,p}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega \right\}, \\ \text{int}_{C^1(\overline{\Omega})}[u, v] &= \text{the interior in } C^1(\overline{\Omega}) \text{ of } [u, v] \cap C^1(\overline{\Omega}). \end{aligned}$$

Also, if $u, v \in W^{1,p}(\Omega)$, we define $u^\pm = \max\{\pm u, 0\}$. We know that $u^\pm \in W^{1,p}(\Omega)$ and $u = u^+ - u^-$, $|u| = u^+ + u^-$.

If X is a Banach space and $\varphi \in C^1(X)$, then we define

$$K_\varphi = \left\{ u \in X : \varphi'(u) = 0 \right\} \text{ (critical set of } \varphi \text{)}.$$

Also we say that $\varphi(\cdot)$ satisfies the “C-condition”, if it has the following property:

$$\begin{aligned} &\text{“Every sequence } \{u_n\}_{n \in \mathbb{N}} \subseteq X \text{ such that} \\ &\quad \{\varphi(u_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R} \text{ is bounded,} \\ &\quad (1 + \|u_n\|_X) \varphi'(u_n) \rightarrow 0 \text{ in } X^* \text{ as } n \rightarrow \infty, \\ &\text{admits a strongly convergent subsequence.”} \end{aligned}$$

This is a compactness-type condition on the functional φ which compensates for the fact that the ambient space X is not in general locally compact (being infinite dimensional in most cases). Using this condition, one can prove a deformation theorem from which follow the minimax theorems of critical theory (see [20, Section 5.4]).

We introduce our hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$.

$H_0 : \xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0, 1)$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

The next two lemmata are useful in estimating the growth of the functionals that arise in the study of problem (P_λ) . The first lemma can be found in Papageorgiou, Rădulescu & Repovš [18, Lemma 2.8].

Lemma 2. *If $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$ and $\xi \not\equiv 0$, then*

$$c_0 \|u\|^p \leq \|u\|_p^p + \int_{\Omega} \xi(z) |u|^p dz$$

for some $c_0 > 0$, all $u \in W^{1,p}(\Omega)$.

The second lemma can be found in Gasiński & Papageorgiou [8, Proposition 2.4].

Lemma 3. *If $\beta \in L^\infty(\partial\Omega)$, $\beta(z) \geq 0$ σ -a.e. and $\beta \not\equiv 0$, then*

$$u \mapsto |u| = \left(\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \right)^{1/p}$$

is an equivalent norm on $W^{1,p}(\Omega)$.

Let $\gamma_p : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the C^1 -functional defined by

$$\gamma_p(u) = \|Du\|_p^p + \int_{\Omega} \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma$$

for all $u \in W^{1,p}(\Omega)$.

On account of Lemmata 2 and 3, we see that if hypotheses H_0 hold, then we have

$$\hat{c}\|u\|^p \leq \gamma_p(u) \text{ for some } \hat{c} > 0, \text{ all } u \in W^{1,p}(\Omega). \quad (2.1)$$

We consider the following nonlinear eigenvalue problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega. \end{array} \right\}$$

This problem was studied by Fragnelli, Mugnai & Papageorgiou [4], who proved that there exists a smallest eigenvalue $\hat{\lambda}_1$ which is simple, isolated and has the following variational characterization

$$\hat{\lambda}_1 = \left\{ \frac{\gamma_p(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0 \right\}. \quad (2.2)$$

Therefore if hypotheses H_0 hold, then from (2.1) above we see that we have $\hat{\lambda}_1 > 0$. The infimum in (2.2) is realized on the corresponding one-dimensional eigenspace, the elements of which have constant sign. The eigenfunctions corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1$ are all nodal (that is, sign-changing). Moreover, the nonlinear regularity theory of Lieberman [12], implies that all eigenfunctions belong in $C^1(\overline{\Omega})$. Finally, the Ljusternik-Schnirelmann minimax scheme generates a whole sequence $\{\hat{\lambda}_m\}_{m \in \mathbb{N}}$ of distinct eigenvalues, known as “variational eigenvalues”, such that $\hat{\lambda}_m \rightarrow +\infty$ as $m \rightarrow \infty$.

Now let us recall some basic continuity notations for multifunctions. So, let (Y, d) be a metric space. We introduce the following hyperspaces

$$\begin{aligned} P_f(Y) &= \{A \subseteq Y : A \neq \emptyset \text{ and closed}\}, \\ P_k(Y) &= \{A \subseteq Y : A \neq \emptyset \text{ and compact}\}. \end{aligned}$$

Given $B, C \subseteq Y$, we set

$$h^*(B, C) = \sup \{d(b, C) : b \in B\} = \inf \{\varepsilon > 0 : B \subseteq C_\varepsilon\},$$

where $C_\varepsilon = \{y \in Y : d(y, C) < \varepsilon\}$ (the “ ε -enlargement of C ”). Then the “Hausdorff distance” between B, C is defined by

$$h(B, C) = \max \{h^*(B, C), h^*(C, B)\}.$$

It is easy to see that $h(B, C) = 0$ if and only if $\overline{B} = \overline{C}$. So, $h(\cdot, \cdot)$ is a metric on $P_f(Y)$ and if (Y, d) is complete, then so is $(P_f(Y), h)$. Moreover, if Y is separable, then $P_f(Y)$ is an h -closed subset of $P_f(Y)$.

Let D be a metric space and $S : D \rightarrow P_f(Y)$ a multifunction. We introduce the following continuity notions for $S(\cdot)$.

(a) We say that $S(\cdot)$ is “upper semicontinuous” (usc for short), if for all $U \subseteq Y$ open, the set $S^+(U) = \{v \in D : S(v) \subseteq U\}$ is open.

We say that $S(\cdot)$ is “lower semicontinuous” (lsc for short), if for all $C \subseteq Y$ closed, the set $S^+(C) = \{v \in D : S(v) \subseteq C\}$ is closed.

If $S(\cdot)$ is both usc and lsc, then we say that $S(\cdot)$ is “continuous” (or “Vietoris continuous”).

(b) We say that $S(\cdot)$ is “ h -usc”, if for all $y \in Y$, $z \mapsto h^*(S(z), S(y))$ is continuous.

We say that $S(\cdot)$ is “ h -lsc”, if for all $y \in Y$, $z \mapsto h^*(S(y), S(z))$ is continuous.

If $S(\cdot)$ is both h -usc and h -lsc, then we say that $S(\cdot)$ is “ h -continuous” (or “Hausdorff continuous”).

In general, the above continuity notations are distinct and we have:

$$\text{usc} \implies h\text{-usc} \quad \text{and} \quad h\text{-lsc} \implies \text{lsc}.$$

However, if $S(\cdot)$ is $P_k(Y)$ -valued, then

$$\begin{aligned} \text{usc} &\iff h\text{-usc} \quad \text{and} \quad h\text{-lsc} \iff \text{lsc}, \\ &\implies \text{continuous} \iff h\text{-continuous}. \end{aligned}$$

For details, we refer to Hu & Papageorgiou [11].

Next, let us introduce the hypotheses on the perturbation $f(z, x)$.

$H_1 : f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(z, x)| \leq a_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho;$$

(ii) for $m \geq 2$ such that $\hat{\lambda}_m > \|\xi\|_\infty$, we have

$$\lim_{x \rightarrow +\infty} \frac{f(z, x)}{x^{p-1}} = \hat{\lambda}_m \text{ uniformly for a.a. } z \in \Omega;$$

(iii) if $F(z, x) = \int_0^x f(z, s) ds$, then there exists $\eta \in (q, p)$ such that

$$0 < c_1 \leq \liminf_{x \rightarrow +\infty} \frac{pF(z, x) - f(z, x)x}{x^\eta} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) $\lim_{x \rightarrow 0^+} \frac{f(z, x)}{x^{\tau-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(v) for every $\rho > 0$, we can find $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z, x) + \hat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Remark 1. Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. Hypotheses H_1 (ii) implies that our equation is resonant with respect to a nonprincipal variational eigenvalue. Finally, note that the perturbation $f(z, x)$ may be sign-changing. Usually in such problems with concave terms and competition phenomena, the perturbation is nonnegative.

The following function satisfies the above hypotheses. For the sake of simplicity, we drop the z -dependence:

$$f(x) = \begin{cases} c(x^+)^{q-1} + (x^+)^{\theta-1} \ln(x^+) & \text{if } x \leq 1, \\ \hat{\lambda}_m x^{p-1} + (c - \hat{\lambda}_m) x^{s-1} & \text{if } 1 < x, \end{cases}$$

with $c > \hat{\lambda}_m$, $\tau < \theta$, $1 < s < p$. Note that $f(\cdot)$ is sign-changing.

3 Bifurcation-type theorem

In this section we prove a bifurcation-type theorem describing the changes in the set of positive solutions as the parameter λ moves on $\mathbb{R}_+ = (0, +\infty)$.

So, we introduce the following two sets:

$$\begin{aligned}\mathfrak{L} &= \{\lambda > 0 : \text{problem } (P_\lambda) \text{ admits a positive solution}\}, \\ S_\lambda &= \text{set of positive solutions of problem } (P_\lambda).\end{aligned}$$

Proposition 4. *If hypotheses H_0, H_1 hold, then $\mathfrak{L} \neq \emptyset$ and for every $\lambda \in \mathfrak{L}$, $S_\lambda \subseteq \text{int } C_+$.*

Proof. For every $\lambda > 0$, we introduce the C^1 -functional $\varphi_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|_q^q - \frac{\lambda}{\tau} \|u^+\|_\tau^\tau - \int_\Omega F(z, u^+) dz$$

for all $u \in W^{1,p}(\Omega)$.

On account of hypotheses H_1 (i), (ii), (iv), given $r > p$, we can find $c_2 = c_2(\lambda, r) > 0$ such that

$$F(z, x) \leq \frac{\lambda}{\tau} x^\tau + \frac{c_2}{r} x^r \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \quad (3.1)$$

Then for all $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}\varphi_\lambda(u) &\geq \frac{\hat{c}}{p} \|u\|^p - c_3 (2\lambda \|u\|^\tau + \|u\|^r) \text{ for some } c_3 > 0 \text{ (see (2.1) and (3.1))} \\ &= \left(\frac{\hat{c}}{p} - c_3 (2\lambda \|u\|^{\tau-p} + \|u\|^{r-p}) \right) \|u\|^p.\end{aligned} \quad (3.2)$$

We consider the function $\theta_\lambda(t) = 2\lambda t^{\tau-p} + t^{r-p}$ for all $t > 0$. Since $\tau < q < p < r$, we see that

$$\theta_\lambda(t) \rightarrow +\infty \text{ as } t \rightarrow 0^+ \text{ and as } t \rightarrow +\infty.$$

Therefore we can find $t_0 > 0$ such that

$$\begin{aligned}\theta_\lambda(t_0) &= \min_{t>0} \theta_\lambda(t), \\ \Rightarrow \theta'_\lambda(t_0) &= 0, \\ \Rightarrow t_0 &= \left(\frac{2\lambda(p-\tau)}{r-p} \right)^{\frac{1}{r-\tau}}.\end{aligned}$$

We have

$$\theta_\lambda(t_0) = 2\lambda \left(\frac{r-p}{2\lambda(p-\tau)} \right)^{\frac{p-\tau}{r-\tau}} + \left(\frac{2\lambda(p-\tau)}{r-p} \right)^{\frac{r-p}{r-\tau}}.$$

Since $\frac{p-\tau}{r-\tau} < 1$ (recall that $\tau < p < r$), we infer that

$$\theta_\lambda(t_0) \rightarrow 0^+ \text{ as } \lambda \rightarrow 0^+.$$

So, we can find $\lambda_0 > 0$ such that

$$\theta_\lambda(t_0) < \frac{\hat{c}}{c_3 p} \text{ for all } \lambda \in (0, \lambda_0). \quad (3.3)$$

From (3.2) and (3.3), we see that

$$\varphi_\lambda(u) \geq m_0 > 0 \text{ for all } \|u\| = t_0, \text{ all } \lambda \in (0, \lambda_0). \quad (3.4)$$

We fix $\lambda \in (0, \lambda_0)$ and introduce the closed ball

$$\overline{B}_{t_0} = \left\{ u \in W^{1,p}(\Omega) : \|u\| \leq t_0 \right\}.$$

The reflexivity of $W^{1,p}(\Omega)$ and the Eberlein-Smulian theorem imply that $\overline{B}_{t_0}$ is sequentially weakly compact. Moreover, using the Sobolev embedding theorem and the compactness of the trace map, we show that $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\lambda \in \overline{B}_{t_0}$ such that

$$\varphi_\lambda(u_\lambda) = \min \{ \varphi_\lambda(u) : u \in \overline{B}_{t_0} \}. \quad (3.5)$$

On account of hypothesis H_1 (iv), given $\varepsilon \in (0, \lambda)$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z, x) \geq -\frac{\varepsilon}{\tau} x^\tau \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta. \quad (3.6)$$

Let $u \in \text{int } C_+$ and $t \in (0, 1)$ small such that

$$0 \leq tu(z) \leq \delta \text{ for all } z \in \overline{\Omega}, \quad t\|u\| \leq t_0. \quad (3.7)$$

Then we have

$$\varphi_\lambda(tu) \leq \frac{t^p}{p} \gamma_p(u) + \frac{t^q}{q} \|Du\|_q^q - \frac{t^\tau}{\tau} (\lambda - \varepsilon) \|u\|_\tau^\tau \text{ (see (3.6), (3.7)).}$$

Since $t \in (0, 1)$, $q < p$ and $0 < \varepsilon < \lambda$, we can write

$$\varphi_\lambda(tu) \leq c_4 t^q - c_5 t^\tau \text{ for some } c_4, c_5 > 0.$$

Recall that $\tau < q$. So, choosing $t \in (0, 1)$ even smaller, we have

$$\varphi_\lambda(tu) < 0. \quad (3.8)$$

From (3.5) and (3.8), we see that

$$\begin{aligned} \varphi_\lambda(u_\lambda) &< 0 = \varphi_\lambda(0), \\ \Rightarrow u_\lambda &\neq 0, \\ \Rightarrow 0 &< \|u_\lambda\| < t_0 \text{ (see (3.4)).} \end{aligned}$$

Therefore we have

$$\begin{aligned} \varphi'_\lambda(u_\lambda) &= 0, \\ \Rightarrow \langle \gamma'_p(u_\lambda), h \rangle + \langle A_q(u_\lambda), h \rangle &= \int_\Omega \lambda(u_\lambda^+)^{\tau-1} h dz + \int_\Omega f(z, u_\lambda^+) h dz \\ &\text{for all } h \in W^{1,p}(\Omega). \end{aligned} \quad (3.9)$$

In (3.9) we choose the test function $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} \gamma_p(u_\lambda^-) + \|Du_\lambda^-\|_q^q &= 0, \\ \Rightarrow \hat{c} \|u_\lambda^-\|^p &\leq 0 \text{ (see (2.1)),} \\ \Rightarrow u_\lambda &\geq 0, \quad u_\lambda \neq 0. \end{aligned} \quad (3.10)$$

From (3.10) and (3.9), we have that u_λ is a positive solution of problem (P_λ) ($0 < \lambda < \lambda_0$). Therefore $(0, \lambda_0) \subseteq \mathfrak{L} \neq \emptyset$.

If $u \in S_\lambda$, then Proposition 2.10 of Papageorgiou, Rădulescu [16], implies $u_\lambda \in L^\infty(\Omega)$. Then the regularity theory of Lieberman [12] implies that $u_\lambda \in C_+ \setminus \{0\}$. Let $\rho = \|u\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_1 (v). We have

$$\begin{aligned} \Delta_p u_\lambda + \Delta_q u_\lambda &\leq \left(\hat{\xi}_\rho + \|\xi\|_\infty \right) u_\lambda^{p-1} \text{ in } \Omega, \\ \Rightarrow u_\lambda &\in \text{int } C_+ \text{ (see Pucci \& Serrin [27, pp. 111, 120])}, \\ \Rightarrow S_\lambda &\subseteq \text{int } C_+ \text{ for all } \lambda \in \mathfrak{L}. \end{aligned}$$

This proof is now complete. $\square$

Next we show that $\mathfrak{L}$ is an interval.

Proposition 5. *If hypotheses H_0, H_1 hold, $\lambda \in \mathfrak{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathfrak{L}$.*

Proof. Let $u \in S_\lambda \subseteq \text{int } C_+$. We introduce the Carathéodory function $k_\mu(z, x)$ defined by

$$k_\mu(z, x) = \begin{cases} \mu(x^+)^{\tau-1} + f(z, x^+) & \text{if } x \leq u_\lambda(z), \\ \mu u_\lambda(z)^{\tau-1} + f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \quad (3.11)$$

We set $K_\mu(z, x) = \int_0^x k_\mu(z, s) ds$ and consider the C^1 -functional $\psi_\mu : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\psi_\mu(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|_q^q - \int_\Omega K_\mu(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (2.1) and (3.11) it is clear that $\psi_\mu(\cdot)$ is coercive. Also it is sequentially weakly semicontinuous. So, we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\psi_\mu(u_\mu) = \min \left\{ \psi_\mu(u) : u \in W^{1,p}(\Omega) \right\}. \quad (3.12)$$

As before (see the proof of Proposition 4), using hypothesis H_1 (iv) and the fact that $\tau < q < p$, we show that

$$\begin{aligned} \psi_\mu(u_\mu) &< 0 = \psi_\mu(0), \\ \Rightarrow u_\mu &\neq 0. \end{aligned}$$

From (3.12) we have

$$\begin{aligned} \psi'_\mu(u_\mu) &= 0, \\ \Rightarrow \langle \gamma'_p(u_\mu), h \rangle + \langle A_q(u_\mu), h \rangle &= \int_\Omega k_\mu(z, u_\mu) h dz \\ &\text{for all } h \in W^{1,p}(\Omega). \end{aligned} \quad (3.13)$$

In (3.13) first we choose $h = -u_\mu^- \in W^{1,p}(\Omega)$ and obtain $u_\mu \geq 0$, $u_\mu \neq 0$. Next in (3.13) we choose $h = (u_\mu - u_\lambda)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned} &\langle \gamma'_p(u_\mu), (u_\mu - u_\lambda)^+ \rangle + \langle A_q(u_\mu), (u_\mu - u_\lambda)^+ \rangle \\ &= \int_\Omega \left(\mu u_\lambda^{\tau-1} + f(z, u_\lambda) \right) (u_\mu - u_\lambda)^+ dz \text{ (see (3.11))} \\ &\leq \int_\Omega \left(\lambda u_\lambda^{\tau-1} + f(z, u_\lambda) \right) (u_\mu - u_\lambda)^+ dz \text{ (since } \mu < \lambda) \\ &= \langle \gamma'_p(u_\lambda), (u_\mu - u_\lambda)^+ \rangle + \langle A_q(u_\lambda), (u_\mu - u_\lambda)^+ \rangle \text{ (since } u_\lambda \in S_\lambda), \\ &\Rightarrow u_\mu \leq u_\lambda. \end{aligned}$$

So, we have proved that

$$u_\mu \in [0, u_\lambda], \quad u_\mu \neq 0. \quad (3.14)$$

From (3.14), (3.11) and (3.13) it follows that $u_\mu \subseteq S_\mu \subseteq \text{int } C_+$ and so $\mu \in \mathfrak{L}$. $\square$

We set $\lambda^* = \sup \mathfrak{L}$.

Proposition 6. *If hypotheses H_0, H_1 hold, then $\lambda^* < +\infty$.*

Proof. Hypotheses H_1 (i), (ii), (iv) (recall that $\hat{\lambda}_m > \|\xi\|_\infty$), imply that we can find $\tilde{\lambda} > 0$ such that

$$\tilde{\lambda}x^{\tau-1} + f(z, x) \geq \|\xi\|_\infty x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \quad (3.15)$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathfrak{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \text{int } C_+$ and we define

$$m_\lambda = \min_{\bar{\Omega}} u_\lambda > 0.$$

For $\delta \in (0, 1)$, let $m_\lambda^\delta = m_\lambda + \delta$ and for $\rho = \max\{m_\lambda + 1, \|u\|_\infty\}$, let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_1 (v). Then we have

$$\begin{aligned} & -\Delta_p m_\lambda^\delta - \Delta_q m_\lambda^\delta + \left(\xi(z) + \hat{\xi}_\rho\right) (m_\lambda^\delta)^{p-1} \\ & \leq \left(\|\xi\|_\infty + \hat{\xi}_\rho\right) m_\lambda^{p-1} + \chi(\delta) \text{ with } \chi(\delta) \rightarrow 0^+ \text{ as } \delta \rightarrow 0^+ \\ & \leq \tilde{\lambda} m_\lambda^{\tau-1} + f(z, m_\lambda) + \hat{\xi}_\rho m_\lambda^{p-1} + \chi(\delta) \text{ (see (3.15))} \\ & = \lambda m_\lambda^{\tau-1} + f(z, m_\lambda) + \hat{\xi}_\rho m_\lambda^{p-1} - (\lambda - \tilde{\lambda}) m_\lambda^{\tau-1} + \chi(\delta) \\ & < \lambda u_\lambda^{\tau-1} + f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} \text{ for } \delta \in (0, 1) \text{ small} \\ & \quad \text{(recall that } \lambda > \tilde{\lambda} \text{ and see hypothesis } H_1 \text{(v))} \\ & = -\Delta_p u_\lambda - \Delta_q u_\lambda + \left(\xi(z) + \hat{\xi}_\rho\right) u_\lambda^{p-1} \text{ (since } u_\lambda \in S_\lambda). \end{aligned} \quad (3.16)$$

For $\delta \in (0, 1)$ small, we will have $\chi(\delta) < (\lambda - \tilde{\lambda}) m_\lambda$. So, from (3.16) and Proposition 2.10 of Papageorgiou, Rădulescu & Repovš [18], we have

$$u_\lambda(z) > m_\lambda^\delta \text{ for all } z \in \Omega \text{ and all } \delta \in (0, 1) \text{ small,}$$

contradicting the definition of m_λ .

Hence we infer that $\lambda \notin \mathfrak{L}$ and so $\lambda^* \leq \tilde{\lambda} < +\infty$. □

According to the above proposition, we have

$$(0, \lambda^*) \subseteq \mathfrak{L} \subseteq (0, \lambda^*].$$

Next, we will produce a lower bound for the elements of S_λ . This will be useful in producing a minimal positive solution for problem (P_λ) and in the study of the continuity properties of the solution multifunction.

Hypotheses H_1 (i), (ii), (iv), imply that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$\begin{aligned} f(z, x) & \geq -\varepsilon x^{\tau-1} - c_\varepsilon x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \\ \Rightarrow \lambda x^{\tau-1} + f(z, x) & \geq (\lambda - \varepsilon) x^{\tau-1} - c_\varepsilon x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we choose $\varepsilon \in (0, \lambda)$ and we have

$$\begin{aligned} \lambda x^{\tau-1} + f(z, x) & \geq c_6(\lambda) x^{\tau-1} - c_7 x^{p-1} \\ & \text{for a.a. } z \in \Omega, \text{ all } x \geq 0 \text{ and some } c_6(\lambda) > 0, c_7 > 0. \end{aligned} \quad (3.17)$$

Motivated by this unilateral growth restriction, we introduce the following auxiliary Robin (p, q) -equation:

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) + \xi(z) u(z)^{p-1} = c_6(\lambda) u(z)^{\tau-1} - c_7 u(z)^{p-1} \text{ in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} + \beta(z) u^{p-1} = 0 \text{ on } \partial\Omega, u > 0. \end{array} \right\} \quad (Q_\lambda)$$

Proposition 7. *If hypotheses H_0 hold and $\lambda > 0$, then problem (Q_λ) admits a unique positive solution $\bar{u}_\lambda \in \text{int } C_+$ and the map $\mathring{\mathbb{R}}_+ \ni \lambda \rightarrow \bar{u}_\lambda \in \text{int } C_+$ is nondecreasing (that is, $0 < \lambda < \lambda' \Rightarrow \bar{u}_\lambda \leq \bar{u}_{\lambda'}$).*

Proof. First we show the existence of a positive solution for problem (Q_λ) . To this end, we introduce the C^1 -functional $\hat{\eta}_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\hat{\eta}_\lambda(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|_q^q + \frac{c_7}{p} \|u^+\|_p^p - \frac{c_6(\lambda)}{\tau} \|u^+\|_\tau^\tau.$$

Since $\tau < q < p$, we see that $\hat{\eta}_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\bar{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \hat{\eta}_\lambda(\bar{u}_\lambda) &= \min \left\{ \hat{\eta}_\lambda(u) : u \in W^{1,p}(\Omega) \right\} < 0 = \hat{\eta}_\lambda(0), \\ \Rightarrow \bar{u}_\lambda &\neq 0. \end{aligned}$$

We have

$$\begin{aligned} \hat{\eta}'_\lambda(\bar{u}_\lambda) &= 0, \\ \Rightarrow \langle \gamma'_p(\bar{u}_\lambda), h \rangle + \langle A_q(\bar{u}_\lambda), h \rangle &= c_6(\lambda) \int_\Omega (u_\lambda^+)^{\tau-1} h dz - c_7 \int_\Omega (u_\lambda^+)^{p-1} h dz \\ &\text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

Choosing $h = -\bar{u}_\lambda \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned} \gamma_p(\bar{u}_\lambda) + \|D\bar{u}_\lambda\|_q^q &= 0, \\ \Rightarrow \hat{c} \|\bar{u}_\lambda\|^p &\leq 0 \text{ (see (2.1))}, \\ \Rightarrow \bar{u}_\lambda &\geq 0, \bar{u}_\lambda \neq 0. \end{aligned}$$

Therefore $\bar{u}_\lambda$ is a positive solution of problem (Q_λ) and the nonlinear regularity theory implies that $\bar{u}_\lambda \in C_+ \setminus \{0\}$. We have

$$\begin{aligned} \Delta_p \bar{u}_\lambda + \Delta_q \bar{u}_\lambda &\leq (\|\xi\|_\infty + c_7) \bar{u}_\lambda^{p-1}, \\ \Rightarrow \bar{u}_\lambda &\in \text{int } C_+ \text{ (see Pucci & Serrin [27, pp. 111, 120])}. \end{aligned}$$

Next we show the uniqueness of this positive solution. For this purpose, we introduce the integral functional $j : L^1(\Omega) \rightarrow \mathbb{R} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p} \gamma_p(u^{1/q}) + \frac{1}{q} \|Du^{1/q}\|_q^q & \text{if } u \geq 0, u^{1/q} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Díaz & Saá [3, Lemma 1], we know that $j(\cdot)$ is convex.

Suppose that $\bar{v}_\lambda \in W^{1,p}(\Omega)$ is another positive solution of problem (Q_λ) . The nonlinear regularity theory (see [12]) and the nonlinear maximum principle (see [27]), imply that $\bar{v}_\lambda \in \text{int } C_+$. Then using Proposition 4.1.22, of Papageorgiou, Rădulescu & Repovš [20, p. 274], we have

$$\frac{\bar{u}_\lambda}{\bar{v}_\lambda} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\bar{v}_\lambda}{\bar{u}_\lambda} \in L^\infty(\Omega).$$

Therefore, if we set $h = \bar{u}_\lambda^q - \bar{v}_\lambda^q \in W^{1,p}(\Omega)$ and define $\text{dom } j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$), then for $|t| < 1$ small we have

$$\bar{u}_\lambda^q + th \in \text{dom } j \quad \text{and} \quad \bar{v}_\lambda^q + th \in \text{dom } j.$$

From this and the convexity of $j(\cdot)$, it follows that $j(\cdot)$ is Gâteaux differentiable at $\bar{u}_\lambda^q$ and at $\bar{v}_\lambda^q$ in the direction h . Then using the chain rule and the nonlinear Green's identity (see [20, p. 35]), we have

$$j'(\bar{u}_\lambda^q)(h) = \frac{1}{q} \int_\Omega \frac{-\Delta_p \bar{u}_\lambda - \Delta_q \bar{u}_\lambda + \xi(z) \bar{u}_\lambda^{p-1}}{\bar{u}_\lambda^{q-1}} h dz,$$

$$\begin{aligned}
&= \frac{1}{q} \int_{\Omega} \left(\frac{c_6(\lambda)}{\bar{u}_{\lambda}^{q-\tau}} - c_7 \bar{u}_{\lambda}^{p-q} \right) h dz, \\
j'(\bar{v}_{\lambda}^q)(h) &= \frac{1}{q} \int_{\Omega} \frac{-\Delta_p \bar{v}_{\lambda} - \Delta_q \bar{v}_{\lambda} + \xi(z) \bar{v}_{\lambda}^{p-1}}{\bar{v}_{\lambda}^{q-1}} h dz, \\
&= \frac{1}{q} \int_{\Omega} \left(\frac{c_6(\lambda)}{\bar{v}_{\lambda}^{q-\tau}} - c_7 \bar{v}_{\lambda}^{p-q} \right) h dz.
\end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence

$$\begin{aligned}
0 &\leq \int_{\Omega} \left(c_6(\lambda) \left(\frac{1}{\bar{u}_{\lambda}^{q-\tau}} - \frac{1}{\bar{v}_{\lambda}^{q-\tau}} \right) - c_7 (\bar{u}_{\lambda}^{p-q} - \bar{v}_{\lambda}^{p-q}) \right) (\bar{u}_{\lambda}^q - \bar{v}_{\lambda}^q) dz \leq 0 \\
&\quad (\text{since } \tau < q < p), \\
&\Rightarrow \bar{u}_{\lambda} = \bar{v}_{\lambda}.
\end{aligned}$$

This proves the uniqueness of the positive solution $\bar{u}_{\lambda} \in \text{int } C_+$ of problem (Q_{λ}) .

If $\lambda < \lambda'$, then $c_6(\lambda) < c_6(\lambda')$ and choosing $\varepsilon < \lambda < \lambda'$, we have the same $c_7 > 0$. Then

$$-\Delta_p \bar{u}_{\lambda'} - \Delta_q \bar{u}_{\lambda'} \geq c_6(\lambda) \bar{u}_{\lambda'}^{\tau-1} - c_7 \bar{u}_{\lambda'}^{p-1} \text{ in } \Omega. \quad (3.18)$$

Then truncating the reaction of problem (Q_{λ}) at $\bar{u}_{\lambda'}(z)$ and using the direct method of the calculus of variations (as in the proof of Proposition 5), we obtain that $\tilde{u}_{\lambda} \in W^{1,p}(\Omega)$ is a solution of problem (Q_{λ}) such that

$$\begin{aligned}
&\tilde{u}_{\lambda} \in [0, \bar{u}_{\lambda'}] \cap \text{int } C_+, \\
&\Rightarrow \tilde{u}_{\lambda} = \bar{u}_{\lambda} \text{ and so } \bar{u}_{\lambda} \leq \bar{u}_{\lambda'}.
\end{aligned}$$

This proof is now complete. □

This solution $\bar{u}_{\lambda}$ is a lower bound for the solution set S_{λ} .

Proposition 8. *If hypotheses H_0, H_1 hold and $\lambda \in \mathfrak{L}$, then $\bar{u}_{\lambda} \leq u$ for all $u \in S_{\lambda}$.*

Proof. Let $u \in S_{\lambda}$ and introduce the Carathéodory function $b_{\lambda}(z, x)$ defined by

$$b_{\lambda}(z, x) = \begin{cases} c_6(\lambda)(x^+)^{\tau-1} - c_7(x^+)^{p-1} & \text{if } x \leq u(z), \\ c_6(\lambda)u(z)^{\tau-1} - c_7u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \quad (3.19)$$

We set $B_{\lambda}(z, x) = \int_0^x b_{\lambda}(z, s) ds$ and consider the C^1 -functional $\tilde{\eta}_{\lambda} : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tilde{\eta}_{\lambda}(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} B_{\lambda}(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Evidently $\tilde{\eta}_{\lambda}(\cdot)$ is coercive (see (2.1) and (3.19)) and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_{\lambda} \in W^{1,p}(\Omega)$ such that

$$\tilde{\eta}_{\lambda}(\tilde{u}_{\lambda}) = \min \left\{ \tilde{\eta}_{\lambda}(u) : u \in W^{1,p}(\Omega) \right\}.$$

Since $\tau < q < p$, we have that $\tilde{\eta}_{\lambda}(\tilde{u}_{\lambda}) < 0 = \tilde{\eta}_{\lambda}(0)$ and so $\tilde{u}_{\lambda} \neq 0$. Also we have

$$\tilde{\eta}'_{\lambda}(\tilde{u}_{\lambda}) = 0,$$

$$\Rightarrow \langle \gamma'_p(\tilde{u}_\lambda), h \rangle + \langle A_q(\tilde{u}_\lambda), h \rangle = \int_{\Omega} b_\lambda(z, \tilde{u}_\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega).$$

Choosing $h = -\tilde{u}_\lambda^- \in W^{1,p}(\Omega)$, we obtain $\tilde{u}_\lambda \geq 0$, $\tilde{u}_\lambda \neq 0$.

Also, choosing $h = (\tilde{u}_\lambda - u)^+ \in W^{1,p}(\Omega)$, we have

$$\begin{aligned} & \langle \gamma'_p(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+ \rangle + \langle A_q(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+ \rangle \\ &= \int_{\Omega} \left(c_6(\lambda) u^{\tau-1} - c_7 u^{p-1} \right) (\tilde{u}_\lambda - u)^+ dz \text{ (see (3.19))} \\ &\leq \int_{\Omega} \left(\lambda u^{\tau-1} + f(z, u) \right) (\tilde{u}_\lambda - u)^+ dz \text{ (see (3.17))} \\ &= \langle \gamma'_p(u), (\tilde{u}_\lambda - u)^+ \rangle + \langle A_q(u), (\tilde{u}_\lambda - u)^+ \rangle \text{ (since } u \in S_\lambda), \\ &\Rightarrow \tilde{u}_\lambda \leq u. \end{aligned}$$

So, we have proved that

$$\begin{aligned} & \tilde{u}_\lambda \in [0, u], \tilde{u}_\lambda \neq 0, \\ & \Rightarrow \tilde{u}_\lambda = \bar{u}_\lambda \in \text{int } C_+ \text{ (see (3.19) and Proposition 7),} \\ & \Rightarrow \bar{u}_\lambda \leq u \text{ for all } u \in S_\lambda. \end{aligned}$$

This proof is now complete. $\square$

Using this lower bound, we can show that for $\lambda \in \mathfrak{L}$, S_λ has a smallest element (minimal positive solution).

Proposition 9. *If hypotheses H_0, H_1 hold and $\lambda \in \mathfrak{L}$, then problem (P_λ) has a smallest positive solution $u_\lambda^* \in S_\lambda \subseteq \text{int } C_+$, that is, $u_\lambda^* \leq u$ for all $u \in S_\lambda$.*

Proof. From Papageorgiou, Rădulescu & Repovš [17] (see the proof Proposition 7) we know that the solution set S_λ is downward directed (that is, if $u_1, u_2 \in S_\lambda$, then we can find $u \in S_\lambda$ such that $u \leq u_1, u \leq u_2$). Invoking Lemma 3.10, of Hu & Papageorgiou [11, p. 178], we can find $\{u_n\}_{n \in \mathbb{N}} \subseteq S_\lambda$ decreasing such that

$$\begin{aligned} & \inf_{n \in \mathbb{N}} u_n = \inf S_\lambda \\ & \bar{u}_\lambda \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N} \text{ (see Proposition 8).} \end{aligned} \quad (3.20)$$

We have

$$\langle \gamma'_p(u_n), h \rangle + \langle A_q(u_n), h \rangle = \int_{\Omega} \left(\lambda u_n^{\tau-1} + f(z, u_n) \right) h dz \quad (3.21)$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

Choosing $h = u_n \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned} & \gamma_p(u_n) + \|Du_n\|_q^q \leq \lambda \|u_n\|_\tau^\tau + c_8 \text{ for some } c_8 > 0, \text{ all } n \in \mathbb{N}, \\ & \text{(see (3.20) and hypothesis } H_1(i)), \\ & \Rightarrow \hat{c} \|u_n\|^p \leq c_9 (\lambda \|u_n\|^\tau + 1) \text{ for some } c_9 > 0, \text{ all } n \in \mathbb{N}, \\ & \Rightarrow \{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded (since } \tau < p). \end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow u_\lambda^* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \quad (3.22)$$

In (3.21) we choose $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.22). Then

$$\lim_{n \rightarrow \infty} \left(\langle A_p(u_n), u_n - u_\lambda^* \rangle + \langle A_q(u_n), u_n - u_\lambda^* \rangle \right) = 0,$$

$$\begin{aligned}
&\Rightarrow \limsup_{n \rightarrow \infty} \left(\langle A_p(u_n), u_n - u_\lambda^* \rangle + \langle A_q(u_\lambda^*), u_n - u_\lambda^* \rangle \right) \leq 0 \\
&\quad (\text{using the monotonicity of } A_q(\cdot)), \\
&\Rightarrow \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u_\lambda^* \rangle \leq 0 \text{ (see (3.22))}, \\
&\Rightarrow u_n \rightarrow u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}.
\end{aligned} \tag{3.23}$$

If in (3.21) we pass to the limit as $n \rightarrow \infty$ and use (3.23), we obtain

$$\langle \gamma_p'(u_\lambda^*), h \rangle + \langle A_q(u_\lambda^*), h \rangle = \int_{\Omega} \left(\lambda (u_\lambda^*)^{\tau-1} + f(z, u_\lambda^*) \right) h dz$$

for all $h \in W^{1,p}(\Omega)$.

Also from (3.20) and (3.23), we have $\bar{u}_\lambda \leq u_\lambda^*$. Therefore $u_\lambda^* \in S_\lambda \subseteq \text{int } C_+$ and $u_\lambda^* = \inf S_\lambda$. $\square$

When $\lambda \in (0, \lambda^*)$, we can prove a multiplicity result.

Proposition 10. *If hypotheses H_0, H_1 hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has at least two positive solutions*

$$u_0, \hat{u} \in \text{int } C_+, \quad u_0 \neq \hat{u}.$$

Proof. Let $\mu < \lambda < \theta < \lambda^*$. From Proposition 5 and its proof, we know that we can find $u_\theta \in S_\theta \subseteq \text{int } C_+$, $u_0 \in S_\lambda \subseteq \text{int } C_+$, $u_\mu \in S_\mu \subseteq \text{int } C_+$ such that

$$u_\mu \leq u_0 \leq u_\theta. \tag{3.24}$$

Let $\rho = \|u_\theta\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_1(v)$. We have

$$\begin{aligned}
& -\Delta_p u_0 - \Delta_q u_0 + \left(\xi(z) + \hat{\xi}_\rho \right) u_0^{p-1} \\
&= \lambda u_0^{\tau-1} + f(z, u_0) + \hat{\xi}_\rho u_0^{p-1} \text{ (since } u_0 \in S_\lambda) \\
&= \theta u_0^{\tau-1} + f(z, u_0) + \hat{\xi}_\rho u_0^{p-1} - (\theta - \lambda) u_0^{\tau-1} \\
&\leq \theta u_\theta^{\tau-1} + f(z, u_\theta) + \hat{\xi}_\rho u_\theta^{p-1} \\
&\quad (\text{see (3.24), hypothesis } H_1(v) \text{ and recall } \lambda < \theta) \\
&= -\Delta_p u_\theta - \Delta_q u_\theta + \left(\xi(z) + \hat{\xi}_\rho \right) u_\theta^{p-1} \text{ in } \Omega \text{ (since } u_\theta \in S_\theta).
\end{aligned} \tag{3.25}$$

Let $m_0 = \min_{\bar{\Omega}} u_0 > 0$ (since $u_0 \in \text{int } C_+$). So, since $\lambda < \theta$, we have

$$0 < (\theta - \lambda) m_0^{\tau-1} \leq (\theta - \lambda) u_0(z)^{\tau-1} \text{ for all } z \in \bar{\Omega}.$$

Then from (3.25) and Proposition 2.10 of Papageorgiou, Rădulescu & Repovš [18], we have

$$u_\theta - u_0 \in D_+. \tag{3.26}$$

In a similar fashion, we show that

$$u_0 - u_\mu \in D_+. \tag{3.27}$$

From (3.26) and (3.27) we infer that

$$u_0 \in \text{int}_{C^1(\bar{\Omega})} [u_\mu, u_\theta]. \tag{3.28}$$

We introduce the Carathéodory functions $i_\lambda(z, x)$ and $\hat{i}_\lambda(z, x)$ defined by

$$i_\lambda(z, x) = \begin{cases} \lambda u_\mu(z)^{\tau-1} + f(z, u_\mu(z)) & \text{if } x \leq u_\mu(z), \\ \lambda x^{\tau-1} + f(z, x) & \text{if } u_\mu(z) < x \end{cases} \tag{3.29}$$

and

$$\hat{i}_\lambda(z, x) = \begin{cases} i_\lambda(z, x) & \text{if } x \leq u_\theta(z), \\ i_\lambda(z, u_\theta(z)) & \text{if } u_\theta(z) < x, \end{cases} \quad (\text{see (3.24)}). \quad (3.30)$$

We set $I_\lambda(z, x) = \int_0^x i_\lambda(z, s) ds$ and $\hat{I}_\lambda(z, x) = \int_0^x \hat{i}_\lambda(z, s) ds$ and consider the C^1 -functionals $l_\lambda, \hat{l}_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} l_\lambda(u) &= \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|_q^q - \int_\Omega I_\lambda(z, u) dz, \\ \hat{l}_\lambda(u) &= \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|_q^q - \int_\Omega \hat{I}_\lambda(z, u) dz \end{aligned}$$

for all $u \in W^{1,p}(\Omega)$.

Using (3.29), (3.30) and the nonlinear regularity theory, we obtain easily that

$$K_{l_\lambda} \subseteq [u_\mu] \cap \text{int } C_+, \quad K_{\hat{l}_\lambda} \subseteq [u_\mu, u_\theta] \cap \text{int } C_+. \quad (3.31)$$

From (3.28), (3.29) and (3.31), we see that we may assume that

$$K_{l_\lambda} \cap [u_\mu, u_\theta] = \{u_0\}. \quad (3.32)$$

Otherwise on account of (3.29) and (3.31), we see that we already have a second positive smooth solution and so we are done.

In a similar fashion we see that we may assume that

$$K_{l_\lambda} \text{ is finite.} \quad (3.33)$$

Otherwise on account of (3.31) and (3.29), we see that we already have a whole sequence of distinct positive smooth solutions and so we are done.

From (3.29) and (3.30) we see that we have

$$l_\lambda|_{[0, u_\theta]} = \hat{l}_\lambda|_{[0, u_\theta]} \quad \text{and} \quad l'_\lambda|_{[0, u_\theta]} = \hat{l}'_\lambda|_{[0, u_\theta]}. \quad (3.34)$$

Evidently $\hat{l}_\lambda(\cdot)$ is coercive (see (2.1) and (3.30)). Also it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \hat{l}_\lambda(\hat{u}_0) &= \min \left\{ \hat{l}_\lambda(u) : u \in W^{1,p}(\Omega) \right\}, \\ \Rightarrow \hat{u}_0 &\in K_{\hat{l}_\lambda} \subseteq [u_\mu, u_\theta] \cap \text{int } C_+ \quad (\text{see (3.31)}). \end{aligned} \quad (3.35)$$

From (3.34) and (3.32), we infer that

$$\hat{u}_0 = u_0 \in \text{int } C_+. \quad (3.36)$$

Then (3.36), (3.35), (3.34) and (3.28) imply that

$$\begin{aligned} u_0 &\text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } l_\lambda, \\ \Rightarrow u_0 &\text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } l_\lambda. \end{aligned} \quad (3.37)$$

(see Papageorgiou & Rădulescu [16, Proposition 2.12]).

Then from (3.37), (3.33) and Theorem 5.7.6, of Papageorgiou, Rădulescu & Repovš [20, p. 449], we can find $\rho \in (0, 1)$ small such that

$$l_\lambda(u_0) < \inf \{ l_\lambda(u) : \|u - u_0\| = \rho \} = \tilde{m}_\lambda. \quad (3.38)$$

Let $\hat{u}_1$ be the positive L^p -normalized (that is, $\|\hat{u}_1\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1 > 0$. We know that $\hat{u}_1 \in \text{int } C_+$ (see [12]). Then hypotheses $H_1(\text{ii})$, (iii) and Proposition 3.2 of Liu & Papageorgiou [13], imply that

$$l_\lambda(t\hat{u}_1) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \quad (3.39)$$

Claim: The functional $l_\lambda(\cdot)$ satisfies the C-condition.

We consider a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega)$ such that $\{l_\lambda(u_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)l'_\lambda(u_n) \rightarrow 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \rightarrow \infty. \quad (3.40)$$

From (3.40), we have

$$\left| \langle \gamma'_p(u_n), h \rangle + \langle A_q(u_n), h \rangle - \int_\Omega i_\lambda(z, u_n) h dz \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad (3.41)$$

for all $h \in W^{1,p}(\Omega)$, with $\varepsilon_n \rightarrow 0^+$.

In (3.41) we choose $h = -u_n^- \in W^{1,p}(\Omega)$ and obtain

$$\begin{aligned} \gamma_p(u_n^-) + \|Du_n^-\|_q^q &\leq c_{10} \|u_n^-\| \text{ for some } c_{10} > 0, \text{ all } n \in \mathbb{N} \text{ (see (3.29))}, \\ \Rightarrow \hat{c} \|u_n^-\|^p &\leq c_{10} \|u_n^-\| \text{ for all } n \in \mathbb{N}, \\ \Rightarrow \{u_n^-\}_{n \in \mathbb{N}} &\subseteq W^{1,p}(\Omega) \text{ is bounded.} \end{aligned} \quad (3.42)$$

Suppose that

$$\|u_n^+\| \rightarrow +\infty \text{ as } n \rightarrow \infty. \quad (3.43)$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. We have $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \rightarrow y \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \quad (3.44)$$

From (3.40) and (3.42) we have

$$\begin{aligned} \left| \langle \gamma'_p(u_n^+), h \rangle + \langle A_q(u_n^+), h \rangle - \int_\Omega i_\lambda(z, u_n^+) h dz \right| &\leq c_{11} \|h\| \\ \text{for some } c_{11} > 0, \text{ all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \\ \Rightarrow \left| \langle \gamma'_p(y_n), h \rangle + \frac{1}{\|u_n^+\|^{p-q}} \langle A_q(y_n), h \rangle - \int_\Omega \frac{i_\lambda(z, u_n^+)}{\|u_n^+\|^{p-1}} h dz \right| &\leq \frac{c_{11} \|h\|}{\|u_n^+\|^{p-1}} \\ \text{for all } n \in \mathbb{N}. \end{aligned} \quad (3.45)$$

Hypotheses $H_1(\text{ii})$ implies that

$$\frac{i_\lambda(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \hat{\lambda}_m y^{p-1} \text{ in } L^{p'}(\Omega) \quad (3.46)$$

(see Aizicovici, Papageorgiou & Staicu [1], proof of Proposition 16).

In (3.45) we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.44). Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle A_p(y_n), y_n - y \rangle &\leq 0 \text{ (see the proof of Proposition 9),} \\ \Rightarrow y_n &\rightarrow y \text{ in } W^{1,p}(\Omega) \text{ and so } \|y\| = 1, y \geq 0. \end{aligned} \quad (3.47)$$

If in (3.45) we pass to the limit as $n \rightarrow \infty$ and use (3.47), (3.46), (3.43) (recall $q < p$), we obtain

$$\begin{aligned} \langle \gamma'_p(y), h \rangle &= \hat{\lambda}_m \int_{\Omega} y^{p-1} h dz \text{ for all } h \in W^{1,p}(\Omega), \\ \Rightarrow -\Delta_p y + \xi(z)y^{p-1} &= \hat{\lambda}_m y^{p-1} \text{ in } \Omega, \\ \frac{\partial y}{\partial n_p} + \beta(z)y^{p-1} &= 0 \text{ on } \partial\Omega, \\ \Rightarrow y \text{ must be nodal (since } m \geq 2) \text{ or } y &= 0, \end{aligned}$$

both conclusions contradicting (3.47). Therefore

$$\begin{aligned} \{u_n^+\}_{n \in \mathbb{N}} &\subseteq W^{1,p}(\Omega) \text{ is bounded,} \\ \Rightarrow \{u_n\}_{n \in \mathbb{N}} &\subseteq W^{1,p}(\Omega) \text{ is bounded (see (3.42)).} \end{aligned}$$

Then we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \quad (3.48)$$

In (3.41) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.48). As in the proof of Proposition 9, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle &\leq 0, \\ \Rightarrow u_n &\rightarrow u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1),} \\ \Rightarrow l_{\lambda}(\cdot) &\text{ satisfies the C-condition.} \end{aligned}$$

This proves the Claim.

Then (3.38), (3.39) and the Claim permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \hat{u} &\in K_{l_{\lambda}} \subseteq [u_{\mu}] \cap \text{int } C_+ \text{ (see (3.31)),} \\ l_{\lambda}(u_0) &< \tilde{m}_{\lambda} \leq l_{\lambda}(\hat{u}) \text{ (see (3.38)).} \end{aligned}$$

From (3.29) it follows that $\hat{u} \in S_{\lambda} \subseteq \text{int } C_+$, $\hat{u} \neq u_0$. □

Next we show the admissibility of the critical parameter λ^* .

Proposition 11. *If hypotheses H_0, H_1 hold, then $\lambda^* \in \mathfrak{L}$.*

Proof. Let $\lambda_n \in (0, \lambda^*)$, $n \in \mathbb{N}$ such that $\lambda_n \uparrow \lambda^*$. We can find $u_n \in S_{\lambda_n} \subseteq \text{int } C_+$, $n \in \mathbb{N}$. Suppose that $\|u_n\| \rightarrow +\infty$ and set $y_n = \frac{u_n}{\|u_n\|}$, $y_n \geq 0$, $n \in \mathbb{N}$. Reasoning as in the proof of Proposition 10, we obtain

$$y_n \rightarrow y \text{ in } W^{1,p}(\Omega) \text{ and } \|y_n\| = 1, y \geq 0, \quad (3.49)$$

$$\left\{ \begin{array}{l} -\Delta_p y + \xi(z)y^{p-1} = \hat{\lambda}_m y^{p-1} \text{ in } \Omega, \\ \frac{\partial y}{\partial n_p} + \beta(z)y^{p-1} = 0 \text{ on } \partial\Omega. \end{array} \right\} \quad (3.50)$$

From (3.50) and since $m \geq 2$, we have that y must be nodal or $y = 0$, both contradicting (3.49). Therefore $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. Then as before using Proposition 1, at least for a subsequence we have

$$\begin{aligned} u_n &\rightarrow u^* \text{ in } W^{1,p}(\Omega), \\ \Rightarrow \langle \gamma'_p(u^*), h \rangle + \langle A_q(u^*), h \rangle &= \int_{\Omega} \left(\lambda^*(u^*)^{r-1} + f(z, u^*) \right) h dz \end{aligned}$$

for all $h \in W^{1,p}(\Omega)$.

Note that if $0 < \mu < \lambda_1$, then

$$\begin{aligned} \bar{u}_\mu &\leq u_n \text{ for all } n \in \mathbb{N} \text{ (see Proposition 7),} \\ \Rightarrow \bar{u}_\mu &\leq u^*, \\ \Rightarrow u^* &\in S_{\lambda^*} \text{ and so } \lambda^* \in \mathcal{L}. \end{aligned}$$

This proof is now complete. $\square$

We have proved that $\mathcal{L} = (0, \lambda^*]$ and we can state the following bifurcation-type result for problem (P_λ) .

Theorem 12. *If hypotheses H_0, H_1 hold, then there exists $\lambda^* > 0$ such that*

- (a) *for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in \text{int } C_+$, $u_0 \neq \hat{u}$;*
- (b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution $u^* \in \text{int } C_+$;*
- (c) *for $\lambda > \lambda^*$ problem (P_λ) has no positive solution.*

4 The solution multifunction

In this section we establish the continuity properties of the solution multifunction $\mathcal{L} \ni \lambda \mapsto S_\lambda \subseteq C^1(\bar{\Omega})$.

We start by proving an important topological property of the solution set S_λ .

Proposition 13. *If hypotheses H_0, H_1 hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then $S_\lambda \in P_k \left(C^1(\bar{\Omega}) \right)$.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subseteq S_\lambda \subseteq \text{int } C_+$. As before (see the proof of Proposition 10) using a contradiction argument and the fact that $m \geq 2$, we show that

$$\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

Then Proposition 2.10 of Papageorgiou & Rădulescu [16], implies that we can find $c_{12} > 0$ such that

$$u_n \in L^\infty(\Omega), \quad \|u_n\|_\infty \leq c_{12} \text{ for all } n \in \mathbb{N}. \quad (4.1)$$

From (4.1) and the nonlinear regularity theory of Lieberman [12] we know that we can find $\alpha \in (0, 1)$ and $c_{13} > 0$ such that

$$u_n \in C^{1,\alpha}(\bar{\Omega}), \quad \|u_n\|_{C^{1,\alpha}(\bar{\Omega})} \leq c_{13} \text{ for all } n \in \mathbb{N}.$$

Since $C^{1,\alpha}(\bar{\Omega}) \hookrightarrow C^1(\bar{\Omega})$ compactly, by passing to a subsequence if necessary, we can say that

$$u_n \rightarrow \tilde{u} \text{ in } C^1(\bar{\Omega}).$$

We have that

$$\langle \gamma'_p(\tilde{u}), h \rangle + \langle A_q(\tilde{u}), h \rangle = \int_{\Omega} \left(\lambda \tilde{u}^{r-1} + f(z, \tilde{u}) \right) h dz$$

for all $h \in W^{1,p}(\Omega)$,

$\bar{u}_\lambda \leq \tilde{u}$ (see Proposition 8).

Therefore $\tilde{u} \in S_\lambda$ and we conclude that $S_\lambda \in P_k \left(C^1(\bar{\Omega}) \right)$. $\square$

So, the solution multifunction $\lambda \mapsto S_\lambda$ is defined on $\mathcal{L} = (0, \lambda^*]$ and has values in $P_k \left(C^1(\bar{\Omega}) \right)$.

Proposition 14. *If hypotheses H_0, H_1 hold, then the solution multifunction $\lambda \mapsto S_\lambda$ is usc and h-usc.*

Proof. A careful reading of the proof of Proposition 13 reveals that the solution multifunction is locally compact, that is for all $\lambda \in \mathcal{L}$ we can find U an open neighborhood of λ such that $\overline{\cup_{\lambda' \in \mathcal{L} \cap U} S_{\lambda'}} \in P_k(C^1(\Omega))$. Therefore according to Proposition 2.23, of Hu & Papageorgiou [11, p. 43], in order to show the upper semicontinuity of the solution multifunction, it suffices to show that the set $G = \{(\lambda, u) \in \mathcal{L} \times C^1(\overline{\Omega}) : u \in S_{\lambda}\}$ (the graph of the solution multifunction) is closed in $\mathcal{L} \times C^1(\overline{\Omega})$.

So, let $\{(\lambda_n, u_n)\}_{n \in \mathbb{N}} \subseteq G$ and assume that $\lambda_n \rightarrow \lambda \in \mathcal{L}$ in $\mathbb{R}_+$ and $u_n \rightarrow u$ in $C^1(\overline{\Omega})$. We have

$$\begin{aligned} \langle \gamma'_p(u_n), h \rangle + \langle A_q(u_n), h \rangle &= \int_{\Omega} \left(\lambda_n u_n^{\tau-1} + f(z, u_n) \right) h dz \\ &\text{for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \\ \Rightarrow \langle \gamma'_p(u), h \rangle + \langle A_q(u), h \rangle &= \int_{\Omega} \left(\lambda u^{\tau-1} + f(z, u) \right) h dz \\ &\text{for all } h \in W^{1,p}(\Omega). \end{aligned} \quad (4.2)$$

Moreover, if $\mu \leq \inf \{\lambda_n\}_{n \in \mathbb{N}}$, then $\bar{u}_{\mu} \leq u_n$ for all $n \in \mathbb{N}$ (see Proposition 8) and so $\bar{u}_{\mu} \leq u$. This means that $u \in S_{\lambda} \subseteq \text{int } C_+$ (see (4.2)). So $(\lambda, u) \in G$ and we have proved the upper semicontinuity of the solution multifunction. Recall that upper semicontinuity implies h -upper semicontinuity (see Section 2). Hence the solution multifunction $\lambda \mapsto S_{\lambda}$ is also h -usc. $\square$

Next we will show the lower semicontinuity of the solution multifunction. We will do this via an iterative process which requires that we have $f(z, x) \geq 0$ for a.a. $z \in \Omega$, all $x \geq 0$.

So, the new hypotheses on the perturbation $f(z, x)$ are the following

$H'_1 : f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_{\rho} \in L^{\infty}(\Omega)$ such that

$$0 \leq f(z, x) \leq a_{\rho}(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho;$$

hypotheses $H'_1(\text{ii}) \rightarrow (\text{v})$ are the same as the corresponding hypotheses $H_1(\text{ii}) \rightarrow (\text{v})$.

Remark 2. In this case the lower bound for the elements of S_{λ} is given by the unique solution $\bar{u}_{\lambda} \in \text{int } C_+$ of the following auxiliary Robin problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{\tau-1} \text{ in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, u > 0, \lambda > 0. \end{array} \right\}$$

Proposition 15. If hypotheses H_0, H'_1 and $m \geq 2$ is big enough, then the solution multifunction $\lambda \mapsto S_{\lambda}$ is lsc and h -lsc.

Proof. Let $\{\lambda_n, \lambda\}_{n \in \mathbb{N}} \subseteq \mathcal{L}$ and suppose that $\lambda_n \rightarrow \lambda$. According to Proposition 2.6, of Hu & Papageorgiou [11, p. 37] in order to show the lower semicontinuity of the solution multifunction, it suffices to show that given $u \in S_{\lambda}$, we can find $u_n \in S_{\lambda_n}$, $n \in \mathbb{N}$ such that

$$u_n \rightarrow u \text{ in } C^1(\overline{\Omega}) \text{ as } n \rightarrow \infty.$$

To this end, let $u \in S_{\lambda}$ and consider the following Robin problem

$$\left\{ \begin{array}{l} -\Delta_p v(z) - \Delta_q v(z) + \xi(z)v(z)^{p-1} = \lambda_n u(z)^{\tau-1} + f(z, u(z)) \text{ in } \Omega, \\ \frac{\partial v}{\partial n_{pq}} + \beta(z)v^{p-1} = 0 \text{ on } \partial\Omega, v > 0, n \in \mathbb{N}. \end{array} \right\} \quad (4.3)$$

Consider the operator $K : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ defined by

$$\langle K(v), h \rangle = \langle \gamma'_p(v), h \rangle + \langle A_q(v), h \rangle \text{ for all } v, h \in W^{1,p}(\Omega).$$

Using Proposition 1, we see that $K(\cdot)$ is continuous, strictly monotone (hence maximal monotone too) and coercive. Therefore $K(\cdot)$ is surjective (see [20, Corollary 2.8.8, p. 135]). So, we can find $v_n \in W^{1,p}(\Omega)$ such that

$$K(v_n) = \lambda_n u^{\tau-1} + N_f(u)$$

with $N_f(\cdot)$ being the Nemitsky map corresponding to f , that is, $N_f(y)(\cdot) = f(\cdot, y(\cdot))$ for all $y \in W^{1,p}(\Omega)$. The strict monotonicity of $K(\cdot)$ implies that this solution is unique. Moreover, as before (see the proof of Proposition 4) using the nonlinear regularity theory (see [12]) and the nonlinear maximum principle (see [27]), we obtain that $v_n \in \text{int } C_+$. In fact we have $\{v_n\}_{n \in \mathbb{N}} \subseteq C^1(\overline{\Omega})$ is relatively compact (see the proof Proposition 13). So, by passing to a subsequence if necessary, we have

$$v_n \rightarrow v \text{ in } C^1(\overline{\Omega}).$$

We obtain

$$\left\{ \begin{array}{l} -\Delta_p v(z) - \Delta_q v(z) + \xi(z)v(z)^{p-1} = \lambda u(z)^{\tau-1} + f(z, u(z)) \text{ in } \Omega, \\ \frac{\partial v}{\partial n_{pq}} + \beta(z)v^{p-1} = 0 \text{ on } \partial\Omega. \end{array} \right\}$$

So, v is the unique positive solution of problem (4.3) with λ_n replaced by λ . Since $u \in S_\lambda$, it follows that

$$v = u.$$

Therefore by the Urysohn criterion for the convergence of sequences (see, for example, Gasiński & Papageorgiou [6, p. 31]), for the initial sequence we have

$$v_n \rightarrow u \text{ in } C^1(\overline{\Omega}) \text{ as } n \rightarrow \infty. \quad (4.4)$$

Next we consider the following auxiliary Robin problem

$$\left\{ \begin{array}{l} -\Delta_p v(z) - \Delta_q v(z) + \xi(z)v(z)^{p-1} = \lambda_n v_n(z)^{\tau-1} + f(z, v_n(z)) \text{ in } \Omega, \\ \frac{\partial v}{\partial n_{pq}} + \beta(z)v^{p-1} = 0 \text{ on } \partial\Omega. \end{array} \right\} \quad (4.5)$$

As we did for problem (4.3), we show that problem (4.5) has a unique solution $v_n^1 \in \text{int } C_+$ and

$$v_n^1 \rightarrow u \text{ in } C^1(\overline{\Omega}) \text{ as } n \rightarrow \infty.$$

We continue to this way and produce a sequence $\{v_n^k\}_{n, k \in \mathbb{N}} \subseteq \text{int } C_+$ such that (with $v_n^0 = v_n$)

$$\left\{ \begin{array}{l} -\Delta_p v_n^k(z) - \Delta_q v_n^k(z) + \xi(z)(v_n^k(z))^{p-1} = \lambda_n (v_n^{k-1}(z))^{\tau-1} + f(z, v_n^{k-1}(z)) \text{ in } \Omega, \\ \frac{\partial v_n^k}{\partial n_{pq}} + \beta(z)(v_n^k)^{p-1} = 0 \text{ on } \partial\Omega. \end{array} \right\} \quad (4.6)$$

$$v_n^k \rightarrow u \text{ in } C^1(\overline{\Omega}) \text{ as } n \rightarrow \infty, \text{ for all } k \in \mathbb{N}. \quad (4.7)$$

Now we fix $n \in \mathbb{N}$ and consider the sequence $\{v_n^k\}_{k \in \mathbb{N}} \subseteq W^{1,p}(\Omega)$. We will show that this sequence is bounded. Arguing by contradiction suppose that the sequence is not bounded in $W^{1,p}(\Omega)$. Then we may assume that

$$\|v_n^k\| \rightarrow +\infty \text{ as } k \rightarrow \infty, \left\{ \|v_n^k\| \right\}_{k \in \mathbb{N}} \text{ is nondecreasing.} \quad (4.8)$$

We set $y_k = \frac{v_n^k}{\|v_n^k\|}$, $k \in \mathbb{N}$. Then $\|y_k\| = 1$, $y_k \geq 0$ for all $k \in \mathbb{N}$. So, we may assume that

$$y_k \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_k \rightarrow y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \quad (4.9)$$

We have

$$\begin{aligned} \langle \gamma'_p(v_n^k), h \rangle + \langle A_q(v_n^k), h \rangle &= \int_{\Omega} \left(\lambda_n (v_n^{k-1})^{\tau-1} + f(z, v_n^{k-1}) \right) h dz \\ \text{for all } h &\in W^{1,p}(\Omega), \\ \Rightarrow \langle \gamma'_p(y_k), h \rangle + \frac{1}{\|v_n^k\|^{p-q}} \langle A_q(y_k), h \rangle &= \int_{\Omega} \left(\frac{\lambda_n}{\|v_n^k\|^{p-1}} (v_n^{k-1})^{\tau-1} + \frac{f(z, v_n^{k-1})}{\|v_n\|^{p-1}} \right) h dz \\ \text{for all } h &\in W^{1,p}(\Omega), \text{ all } k \in \mathbb{N}. \end{aligned} \quad (4.10)$$

Note that

$$\left\{ \frac{f(\cdot, v_n^{k-1}(\cdot))}{\|v_n^k\|^{p-1}} \right\}_{k \in \mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded (see (4.8)).} \quad (4.11)$$

If in (4.10) we choose $h = y_k - y \in W^{1,p}(\Omega)$, pass to the limit as $k \rightarrow \infty$ and use (4.8) (recall that $\tau < q < p$), (4.9) and (4.11), we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \langle A_p(y_k), y_k - y \rangle &= 0, \\ \Rightarrow y_k &\rightarrow y \text{ in } W^{1,p}(\Omega), \|y\| = 1, y \geq 0. \end{aligned} \quad (4.12)$$

From (4.11) and hypothesis H_1' (ii) we have

$$\frac{f(\cdot, v_n^{k-1}(\cdot))}{\|v_n^k\|^{p-1}} \xrightarrow{w} \hat{\lambda}_m (\eta y)^{p-1} \text{ in } L^{p'}(\Omega) \text{ with } \eta \in [0, 1]. \quad (4.13)$$

So, if we pass to the limit as $k \rightarrow \infty$ in (4.10) and use (4.12) and (4.13), we obtain

$$\left\{ \begin{array}{l} -\Delta_p y(z) + \xi(z) y(z)^{p-1} = \hat{\lambda}_m (\eta y(z))^{p-1} \text{ in } \Omega, \\ \frac{\partial y}{\partial n_p} + \beta(z) y^{p-1} = 0 \text{ on } \partial\Omega. \end{array} \right\} \quad (4.14)$$

If $\eta = 0$, then $y = 0$ (see (2.1)), a contradiction to (4.12). So, suppose $\eta \in (0, 1]$. Let $m \geq 2$ be big enough so that $\hat{\lambda}_m \eta^{p-1} > \hat{\lambda}_1$ (recall $\hat{\lambda}_m \rightarrow +\infty$ as $m \rightarrow \infty$) and of course $\hat{\lambda}_m > \|\xi\|_{\infty}$ (see H_1' (ii) = H_1 (ii)). Then from (4.14) we have

y is nodal or $y = 0$, contradicting (4.12).

Therefore $\left\{ v_n^k \right\}_{k \in \mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded for every $n \in \mathbb{N}$. Then as before the nonlinear regularity, implies that at least for a subsequence we have

$$v_n^k \rightarrow u_n \text{ in } C^1(\bar{\Omega}) \text{ as } k \rightarrow \infty. \quad (4.15)$$

In the limit as $k \rightarrow \infty$ in (4.6), we obtain

$$\left\{ \begin{array}{l} -\Delta_p u_n(z) - \Delta_q u_n(z) + \xi(z) u_n(z)^{p-1} = \lambda_n u_n(z)^{\tau-1} + f(z, u_n(z)) \text{ in } \Omega, \\ \frac{\partial u_n}{\partial n_{pq}} + \beta(z) u_n^{p-1} = 0 \text{ on } \partial\Omega. \end{array} \right\} \quad (4.16)$$

Consider $0 < \tilde{\lambda} \leq \inf_{n \geq 1} \lambda_n$ and let $\tilde{u} \in \text{int } C_+$ be the unique positive solution of the Robin problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) + \xi(z) u(z)^{p-1} = \tilde{\lambda} (v_n^{k-1}(z))^{\tau-1} \text{ in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} + \beta(z) u^{p-1} = 0 \text{ on } \partial\Omega, u > 0, k, n \in \mathbb{N}. \end{array} \right\}$$

Since $f \geq 0$, from (4.6), for every $k, n \in \mathbb{N}$, we have

$$-\Delta_p v_n^k(z) - \Delta_q v_n^k(z) + \xi(z) (v_n^k(z))^{p-1} \geq \tilde{\lambda} (v_n^{k-1}(z))^{\tau-1} \text{ in } \Omega.$$

So, considering the Carathéodory function

$$e_n^k(z, x) = \begin{cases} \tilde{\lambda}(x^+)^{\tau-1} & \text{if } x \leq v_n^{k-1}(z), \\ \tilde{\lambda}(v_n^{k-1}(z))^{\tau-1} & \text{if } v_n^{k-1}(z) < x \end{cases}$$

and using the direct method of the calculus of variations as in the proof Proposition 8, we obtain

$$\begin{aligned} \tilde{u} &\leq v_n^{k-1} \text{ for all } k, n \in \mathbb{N}, \\ \Rightarrow \tilde{u} &\leq u_n \text{ for all } n \in \mathbb{N} \text{ (see (4.15)),} \\ \Rightarrow u_n &\in S_{\lambda_n} \subseteq \text{int } C_+ \text{ (see (4.16)).} \end{aligned}$$

As in the first part of the proof, via a contradiction argument we show that $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded and so by the nonlinear regularity theory, we have that $\{u_n\}_{n \in \mathbb{N}} \subseteq C^1(\overline{\Omega})$ is relatively compact. Then by (4.7) and the double limit lemma (see Gasiński & Papageorgiou [6, p. 61]), we have $u_n \rightarrow u$ in $C^1(\overline{\Omega})$ and so we have proved the lower semicontinuity of the solution multifunction. Since the multifunction is $P_k(C^1(\overline{\Omega}))$ -valued it follows that $\lambda \mapsto S_\lambda$ is h -lsc too (see Section 2). $\square$

Remark 3. *It is interesting to know if the result remains true if the perturbation f is sign-changing.*

Summarizing we can state the following theorem about the continuity properties of the solution multifunction $\lambda \mapsto S_\lambda$.

Theorem 16. (a) *If hypotheses H_0, H_1 hold, then the solution multifunction is usc and h -usc from $\mathcal{L} = (0, \lambda^*)$ into $P_k(C^1(\overline{\Omega}))$;*
 (b) *If hypotheses H_0, H'_1 hold, then the solution multifunction is continuous and h -continuous from $\mathcal{L} = (0, \lambda^*)$ into $P_k(C^1(\overline{\Omega}))$.*

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