Research Article

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Partial regularity of stable solutions to the fractional Gel'fand-Liouville equation

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Abstract: We analyze stable weak solutions to the fractional Gel'fand problem

$$(-\Delta)^s u = e^u \quad \text{in } \Omega \subset \mathbb{R}^n.$$

We prove that the dimension of the singular set is at most $n - 10s$.**Keywords:** Gel'fand equation, stable solution, super critical equation, partial regularity**MSC:** 35B65, 35J60, 35R11

1 Introduction

Let Ω be an open subset of $\mathbb{R}^n$. We consider the following fractional Gel'fand equation

$$(-\Delta)^s u = e^u \quad \text{in } \Omega \subset \mathbb{R}^n, \quad n \geq 1, \quad (1.1)$$

where $s \in (0, 1)$, and u satisfies

$$e^u \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad u \in L_s(\mathbb{R}^n).$$

Here the space $L_s(\mathbb{R}^n)$ is defined by

$$L_s(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}} : \|u\|_{L_s(\mathbb{R}^n)} < \infty \right\}, \quad \|u\|_{L_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < \infty.$$

The non-local operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s \varphi(x) = c_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+2s}} dy,$$

with $c_{n,s} := \frac{2^{2s-1} \Gamma(\frac{n+2s}{2})}{\pi^{n/2} |\Gamma(-s)|}$ being a normalizing constant.

We are interested in the classical question of regularity of solutions to (1.1). More precisely, we aim at studying the partial regularity of stable weak solutions of (1.1). By a weak solution we mean that u satisfies (1.1) in the sense of distribution, that is

$$\int_{\mathbb{R}^n} u(-\Delta)^s \phi dx = \int_{\Omega} e^u \phi dx, \quad \forall \phi \in C_c^\infty(\Omega).$$

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We recall that a solution u to (1.1) is said to be a stable weak solution if $u \in \dot{H}^s(\Omega)$ and

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{n+2s}} dx dy \geq \int_{\mathbb{R}^n} e^u \phi^2 dx, \quad \forall \phi \in C_c^\infty(\Omega), \quad (1.2)$$

where the function space $\dot{H}^s(\Omega)$ is defined by

$$\dot{H}^s(\Omega) := \left\{ u \in L^2_{\text{loc}}(\Omega) \mid \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy < +\infty \right\}.$$

For equation (1.1) in the whole space $\mathbb{R}^n$, the study on the classification of stable solutions and finite Morse index solutions has attracted a lot of attentions in recent decades. In the classical case, that is $s = 1$, Farina [11] and Dancer - Farina [3] established non-existence of stable solutions to (1.1) for $2 \leq n \leq 9$ and non-existence of finite Morse index solutions to (1.1) for $3 \leq n \leq 9$. A counterpart issue for equation (1.1) in a bounded domain is to analyze the extremal solutions. Let us first recall the definition of extremal solutions. Given a bounded smooth domain $\Omega \subset \mathbb{R}^n$, consider the following problem

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is C^1 , non-negative, nondecreasing and superlinear at infinity. Then there exists a constant $\lambda^* > 0$ such that for each $\lambda \in [0, \lambda^*)$ there exists a minimal classical solution $u_\lambda(x)$, and for $\lambda > \lambda^*$ there exists no solution, even in the weak sense, see [9, Proposition 3.3.1]. The family $(u_\lambda)_{0 < \lambda < \lambda^*}$ is monotone increasing with respect to the parameter λ , and the pointwise limit $u_{\lambda^*} := \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a weak stable solution to (1.3). The solution u_{λ^*} is called the extremal solution. When $f(u)$ is the exponential nonlinearity, it is proved that if dimension $n \leq 9$, the extremal solution is always smooth, see [9, Theorem 4.2.1]. For more general nonlinearity $f(u)$, very recently, Cabré et al. [2] showed that the extremal solutions are regular when $n \leq 9$. The restriction on the dimension $n < 10$ is sharp in the sense that there are singular weak stable solutions for $n \geq 10$, for example $u(x) = \log 2(n-2) - 2 \log |x|$ provides a weak stable solution which is not regular.

As there are singular stable weak solutions in dimension $n \geq 10$, a natural question is to understand the partial regularity of stable solutions for $n \geq 10$. In the classical case with exponential nonlinearity, the partial regularity result has been studied by Wang [21, 22], and he proved that the Hausdorff dimension of the singular set for weak stable solutions is at most $n - 10$. In dimension $n = 3$, Da Lio [5] showed that the dimension of singular set for stationary solution can be at most 1. We refer the readers to [7, 10] for the results on the extremal solutions of Liouville system and [4] on the stable solutions of biharmonic Liouville equation.

Let us also mention that the partial regularity of solutions for Lane-Emden equation, that is, the nonlinearity is given by $f(u) = u^p$, has been studied in various settings, e.g., estimating the singular set of stable solutions and stationary solutions, see e.g. [6, 9, 16, 17, 20].

In the nonlocal case, Duong - Nguyen [8] studied stable solutions and proved that Eq. (1.1) has no regular stable solution for $n < 10s$. Later, the authors of this paper gave the following optimal condition on n, s

$$n \leq 2s, \quad \text{or } n > 2s \text{ and } \frac{\Gamma(\frac{n}{2})\Gamma(1+s)}{\Gamma(\frac{n-2s}{2})} > \frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})}, \quad (1.4)$$

for which Eq. (1.1) has no stable solutions in the whole space $\mathbb{R}^n$, see [14]. In addition, the authors also showed that weak stable solutions are smooth, whenever $n < 10s$. While for the case of extremal solutions, Ros-Oton - Serra [18] proved that the extremal solutions are regular for $n < 10s$. In a subsequent paper, under certain symmetry assumptions on the domain, Ros-Oton [19] proved regularity of extremal solutions (for the case of exponential nonlinearity) up to the optimal dimension, that is, n and s satisfy (1.4). His arguments are based on the fact that, due to the symmetry assumptions on the domain, the extremal solution can have at most one singular point. For the recent developments on the regularity of extremal solutions for nonlocal systems we refer to [12] and the references therein.

Definition 1.1. A point $x_0 \in \Omega$ is said to be a singular point for a solution u to (1.1) if u is not bounded in any small neighborhood of x_0 . The singular set $\mathcal{S}$ is the collection of all singular points.

It follows from the above definition that the singular set $\mathcal{S}$ is closed in Ω . Moreover, by standard regularity theory one gets that u is regular in $\Omega \setminus \mathcal{S}$.

Our main result is the following:

Theorem 1.1. *Let u be a stable weak solution of (1.1). Then the Hausdorff dimension of the singular set $\mathcal{S}$ is at most $n - 10s$.*

The dimension estimate $n - 10s$ is optimal in the limit $s \uparrow 1$ as there are singular weak stable solutions for $s = 1$. However, due to the regularity results of Ros-Oton [19] for extremal solutions, one would expect that the Hausdorff dimension of the singular set would be at most $n - n^*(s)$, where $n^*(s) > 10s$ is implicitly defined by the second condition in (1.4) with equality. This remains as an open question.

To prove Theorem 1.1 we follow the approach of Wang [21, 22] for the classical case. More precisely, Wang considered the energy

$$\mathcal{F}(u, x_0, r) := r^{2-n} \int_{B_r(x_0)} e^u dx. \quad (1.5)$$

Then he showed that there exists $\varepsilon_0 > 0$ (independent of u, x_0, r) such that if $\mathcal{F}(u, x_0, r) \leq \varepsilon_0$ then u is regular in a small neighborhood of x_0 . This, and the fact that e^u is in L^p_{loc} for every $p < 5$, thanks to Farina's estimate [11], imply that the Hausdorff dimension of the singular set is at most $n - 10$.

Contrary to the above energy (1.5), in our case we shall consider the following energy involving interior and boundary quantities

$$\mathcal{E}(u, x_0, r) := r^{2s-n} \int_{B_r(x_0)} e^u dx + r^{4s-n-2} \int_{B_r^{n+1}(x_0) \cap \mathbb{R}_+^{n+1}} t^{1-2s} e^{\bar{u}} dx dt,$$

where $\bar{u}$ is the Caffarelli-Silvestre extension of u in $\mathbb{R}_+^{n+1}$, i.e.,

$$\bar{u}(X) = \int_{\mathbb{R}^n} P(X, y) u(y) dy, \quad X = (x, t) \in \mathbb{R}^n \times (0, +\infty),$$

where

$$P(X, y) = d_{n,s} \frac{t^{2s}}{|(x - y, t)|^{n+2s}},$$

and $d_{n,s} > 0$ is a normalizing constant such that $\int_{\mathbb{R}^n} P(X, y) dy = 1$, see [1]. The function $\bar{u}$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla \bar{u}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ -\lim_{t \rightarrow 0} t^{1-2s} \partial_t \bar{u} = \kappa_s (-\Delta)^s u = \kappa_s e^u, & \text{in } \Omega, \end{cases}$$

where $\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}$. As in the classical case, the Farina type estimate for the nonlocal case also imply that if u is a stable weak solution then $e^u \in L^p_{\text{loc}}$ for every $p < 5$, see Lemma 2.1. Due to this integrability restrictions we have the dimension estimate $n - 10s$ in Theorem 1.1. Let us also emphasize here that the stability condition will be used only to obtain the Farina type estimate in Lemma 2.1 and to get uniform bounds in the space $L_s(\mathbb{R}^n)$ for a family of rescaled solutions in Lemma 2.3.

It is important to note that each term in the energy $\mathcal{E}(u, x_0, r)$ controls the other one in a suitable way. More precisely, due to the stability hypothesis, the second term controls the L^2 norm of e^u , see (2.2), whereas, by Jensen's inequality, the second term is controlled by the first one, see Lemma 3.1. This interplay turned out to be very crucial in proving energy decay estimate (see Proposition 3.4 and Lemma 3.5 for precise statements), that is, if $\mathcal{E}(u, x_0, r) \leq \varepsilon_0$ for some sufficiently small $\varepsilon_0 > 0$ then there exists $\theta \in (0, 1)$ such that $\mathcal{E}(u, x_0, \theta r) \leq \frac{1}{2} \mathcal{E}(u, x_0, r)$.

We remark that the energy estimate does not seem to work if we only consider one of the two terms in $\mathcal{E}(u, x_0, r)$. For instance, on one hand, if we only consider the first term (as in the local case), then we lack a Harnack type inequality for non-negative fractional sub-harmonic functions. However, in the fractional case, we do have a Harnack type inequality involving the extension function, see Lemma 3.3. This suggests to consider the second term in the definition of the energy. On the other hand, if we only consider the second term in the energy, then the L^1 norm of $\bar{v}$ (as defined in 3.2) is of the order $\sqrt{\varepsilon}$ due to the fact that there is a sacrifice in the power of ε when we use Hölder inequality, which is not good enough (compare (3.5) and (3.7)) to prove the energy decay estimate in Proposition 3.4.

The article is organized as follows: In section 2 we list some preliminary results, including the conclusions presented in our previous work [14] and some known results that would be used in the current article. In section 3, we give the proof of Theorem 1.1.

Notations:

$X = (x, t)$ represent points in $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times [0, \infty)$ and $\mathbb{R}^n = \partial\mathbb{R}_+^{n+1}$.

$B_r(x)$ the ball centered at x with radius r in $\mathbb{R}^n$, $B_r := B_r(0)$.

$B_r^{n+1}(x)$ the ball centered at x with radius r in $\mathbb{R}^{n+1}$, $B_r^{n+1} := B_r^{n+1}(0)$.

$D_r(x)$ the intersection of $B_r^{n+1}(x)$ and $\mathbb{R}_+^{n+1}$, i.e., $B_r^{n+1}(x) \cap \mathbb{R}_+^{n+1}$, $D_r := D_r(0)$.

u_+ represents the non-negative part of u , i.e., $u_+ = \max(u, 0)$.

C a generic positive constant which may change from line to line.

2 Preliminary results

In this section we present several results that will be used in next section. First, we extend the stability condition in the fractional setting. Notice that $\bar{u}$ is well-defined as $u \in L_s(\mathbb{R}^n)$. Moreover, $t^{\frac{1-2s}{2}} \nabla \bar{u} \in L_{\text{loc}}^2(\Omega \times [0, \infty))$ whenever $u \in \dot{H}^s(\Omega)$. We recall from [14] that the stability condition (1.2) can be generalized to the extended function $\bar{u}$. Precisely, if u is stable in Ω then

$$\int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla \Phi|^2 dx dt \geq \kappa_s \int_{\mathbb{R}^n} e^u \phi^2 dx, \quad (2.1)$$

for every $\Phi \in C_c^\infty(\overline{\mathbb{R}_+^{n+1}})$ satisfying $\phi(\cdot) := \Phi(\cdot, 0) \in C_c^\infty(\Omega)$. The following Farina type estimate has been proven in [14, Lemma 3.4]:

Lemma 2.1 ([14]). *Let $u \in \dot{H}^s(\Omega)$ be a weak stable solution to (1.1). Given a function Φ be of the form $\Phi(x, t) = \phi(x)\eta(t)$ for some $\phi \in C_c^\infty(\Omega)$ and $\eta = 1$ in a small neighborhood of the origin, for every $\alpha \in (0, 2)$ we have*

$$(2 - \alpha)\kappa_s \int_{\mathbb{R}^n} e^{(1+2\alpha)u} \phi^2 dx \leq 2 \int_{\mathbb{R}_+^{n+1}} t^{1-2s} e^{2\alpha\bar{u}} |\nabla \Phi|^2 dx dt - \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} e^{2\alpha\bar{u}} \nabla \cdot [t^{1-2s} \nabla \Phi^2] dx dt. \quad (2.2)$$

Though the following regularity result on Morrey's space is well-known, we give a proof for convenience. We recall that a function f is in the Morrey's space $M^p(\Omega)$ if $f \in L^1(\Omega)$, and it satisfies

$$\int_{\Omega \cap B_r(x_0)} |f| dx \leq Cr^{n(1-\frac{1}{p})} \quad \text{for every } B_r(x_0) \subset \mathbb{R}^n.$$

The norm $\|f\|_{M^p(\Omega)}$ is defined to be the infimum of constants $C > 0$ for which the above inequality holds.

Lemma 2.2. Let $f \in M^{\frac{n}{2s-\delta}}(B_3)$ for some $\delta > 0$. We set

$$\mathcal{R}_s f(x) := \int_{B_1} \frac{1}{|x-y|^{n-2s}} |f(y)| dy.$$

Then we have

$$\|\mathcal{R}_s f\|_{L^\infty(B_1)} \leq C(n, s, \delta) \|f\|_{M^{\frac{n}{2s-\delta}}(B_3)}.$$

Proof. We set

$$F(r) = \int_{B_r(x)} |f(y)| dy.$$

Then

$$\mathcal{R}_s f(x) \leq \int_{B_2} \frac{1}{|y|^{n-2s}} |f(x-y)| dy = \int_0^2 \rho^{2s-n} F'(\rho) d\rho, \quad x \in B_1, \quad (2.3)$$

where $\rho = |y|$. We can derive from (2.3) and integration by parts that

$$\begin{aligned} |\mathcal{R}_s f| &\leq \int_0^2 \rho^{2s-n} F'(\rho) d\rho = 2^{2s-n} F(2) + (n-2s) \int_0^2 \rho^{2s-n-1} F(\rho) d\rho \\ &\leq \|f\|_{M^{\frac{n}{2s-\delta}}(B_3)} 2^{2s-n} 2^{n+\delta-2s} + (n-2s) \|f\|_{M^{\frac{n}{2s-\delta}}(B_2)} \int_0^2 \rho^{\delta-1} d\rho \\ &\leq C \|f\|_{M^{\frac{n}{2s-\delta}}(B_3)}. \end{aligned}$$

Thus we finish the proof. $\square$

For any $x_0 \in B_1$, we set

$$u^\lambda(x) := u(x_0 + \lambda x) + 2s \log \lambda. \quad (2.4)$$

The following lemma is crucial for the proof of Lemma 3.5.

Lemma 2.3. Let u be a stable solution to (1.1) with $\Omega = B_1$. Let u^λ be defined as in (2.4) for some $|x_0| < 1$ and $0 < \lambda < (1 - |x_0|)^{1+\frac{n}{2s}}$. Then

$$\|u_+^\lambda\|_{L_s(\mathbb{R}^n)} \leq C (1 + \|u_+\|_{L_s(\mathbb{R}^n)}),$$

for some $C > 0$ independent of u .

Proof. It is easy to see that u^λ is a stable solution to (1.1) on B_R with $R := \frac{1}{\lambda}(1 - |x_0|)$. Hence, by (2.1)

$$\int_{B_\rho} e^{u^\lambda} dx \leq C \rho^{n-2s} \quad \text{for } 0 < \rho \leq \frac{R}{2}.$$

This would imply that

$$\int_{B_{R/2}} \frac{u_+^\lambda(x)}{1 + |x|^{n+2s}} dx \leq \int_{B_{R/2}} \frac{e^{u^\lambda(x)}}{1 + |x|^{n+2s}} dx \leq C.$$

As $\lambda < 1$ we get

$$u_+^\lambda(x) < u_+(x_0 + \lambda x).$$

Therefore, changing the variable $x_0 + \lambda x \mapsto y$ we obtain

$$\int_{B_{R/2}^c} \frac{u_+^\lambda(x)}{1 + |x|^{n+2s}} dx \leq \lambda^{-n} \left(\int_{\{|y| \leq 2\} \cap \{|x_0 - y| \geq \frac{1}{2} R \lambda\}} + \int_{\{|y| > 2\}} \right) \frac{u_+(y)}{1 + \left(\frac{|x_0 - y|}{\lambda}\right)^{n+2s}} dy$$

$$\begin{aligned} &\leq \frac{C}{\lambda^n R^{n+2s}} \int_{|y| \leq 2} u_+(y) dy + C\lambda^{2s} \int_{|y| > 2} \frac{u_+(y)}{|y|^{n+2s}} dy \\ &\leq C \|u_+\|_{L_s(\mathbb{R}^n)}. \end{aligned}$$

We conclude the lemma. $\square$

3 Proof of Theorem 1.1

We start with the following refinement version of [14, Lemma 3.3]:

Lemma 3.1. *Let $e^{au} \in L^1(B_1)$. Then $t^{1-2s} e^{a\bar{u}} \in L^1_{\text{loc}}(B_1 \times [0, \infty))$. Moreover,*

i) *there exists $\delta = \delta(n, s) > 0$ and $C = C(n, s, \|u_+\|_{L_s(\mathbb{R}^n)}) > 0$ such that*

$$\|t^{1-2s} e^{a\bar{u}}\|_{L^1(D_{1/2})} \leq C \left(\|e^{au}\|_{L^1(B_1)} + \|e^{au}\|_{L^1(B_1)}^\delta \right),$$

ii) *for every $\delta_0 > 0$ small there exists $r_0 = r_0(n, s, \delta_0) > 0$ small such that*

$$\int_0^{r_0} \int_{B_{1/2}} t^{1-2s} e^{a\bar{u}} dx dt \leq C \left(\|e^{au}\|_{L^1(B_1)} + \|e^{au}\|_{L^1(B_1)}^{1-\delta_0} \right).$$

Proof. For $X = (x, t) \in D_{1/2}$ we have

$$\bar{u}(x, t) \leq C \|u_+\|_{L_s(\mathbb{R}^n)} + \int_{B_1} u(y) P(X, y) dy = C + \int_{B_1} g(x, t) u(y) \frac{P(X, y)}{g(x, t)} dy,$$

where

$$1 > g(x, t) := \int_{B_1} P(X, y) dy \geq \delta, \quad (3.1)$$

for some positive constant δ depending on n and s only. The last inequality easily follows from

$$\lim_{t \downarrow 0} \int_{B_1} P(X, y) dy = 1.$$

Therefore, by Jensen's inequality

$$\begin{aligned} \int_{B_{1/2}} e^{a\bar{u}(x, t)} dx &\leq C \int_{B_{1/2}} \int_{B_1} e^{ag(x, t)u(y)} P(X, y) dy dx \\ &\leq C \int_{B_1} \max \left\{ e^{au(y)}, e^{a\delta u(y)} \right\} \int_{B_{1/2}} P(X, y) dx dy \\ &\leq C \left(\int_{B_1} e^{\delta au(y)} + \int_{B_1} e^{au(y)} dy \right) \\ &\leq C \left(\|e^{au}\|_{L^1(B_1)} + \|e^{au}\|_{L^1(B_1)}^\delta \right), \end{aligned}$$

where the last inequality follows from Hölder inequality. Integrating the above inequality with respect to t on the interval $[0, \frac{1}{2}]$ we obtain i).

To to prove ii), we notice that for a given $\delta_0 > 0$ small we can choose $r_0 > 0$ sufficiently small such that (3.1) holds with $\delta = 1 - \delta_0$ for every $x \in B_{1/2}$ and $0 < t \leq r_0$. Then ii) follows in a similar way. $\square$

Up to a translation and dilation of the domain Ω , we can simply assume that $B_1 \subset \Omega$. For fixed $0 < r < 1$ we decompose

$$\bar{u} = \bar{v} + \bar{w},$$

where

$$\bar{v}(x, t) := C(n, s) \int_{B_r} \frac{1}{(|x - y|^2 + t^2)^{\frac{n-2s}{2}}} e^{u(y)} dy, \quad x \in \mathbb{R}^n, \quad (3.2)$$

where $C(n, s) > 0$ is a dimensional constant such that

$$-\lim_{t \rightarrow 0} t^{1-2s} \partial_t \bar{v} = \kappa_s e^u \quad \text{on } B_r.$$

Then $\bar{w}$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla \bar{w}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \lim_{t \rightarrow 0} t^{1-2s} \partial_t \bar{w} = 0 & \text{in } B_r. \end{cases} \quad (3.3)$$

Notice that $\bar{w}$ is continuous up to the boundary B_r .

Now we prove some elementary properties of the functions $\bar{v}$ and $\bar{w}$.

Lemma 3.2. *Setting $v := \bar{v}(x, 0)$ we have*

$$\|t^{\frac{1-2s}{2}} \bar{v}\|_{L^2(D_r)} + \|v\|_{L^2(B_r)} \leq C \|e^u\|_{L^1(B_r)}^\gamma \|e^u\|_{L^2(B_r)}^{1-\gamma}, \quad 0 < r \leq 1,$$

for some $\gamma \in (0, 1)$ and $C > 0$ independent of u .

Proof. The function v can be written as a convolution, and in fact,

$$v \chi_{B_r} \leq (\Gamma \chi_{B_{2r}}) * (e^u \chi_{B_r}), \quad \Gamma(x) := \frac{C(n, s)}{|x|^{n-2s}},$$

where χ_A denotes the characteristic function of the set A . On the one hand, by Young's inequality, we obtain

$$\|v\|_{L^p(B_r)} \leq \|\Gamma\|_{L^q(B_{2r})} \|e^u\|_{L^2(B_r)}, \quad (3.4)$$

where p, q verify the following conditions

$$1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{2}, \quad 1 < q < \frac{n}{n-2s}.$$

On the other hand, it is easy to see that

$$\|v\|_{L^1(B_r)} \leq \|\Gamma\|_{L^1(B_{2r})} \|e^u\|_{L^1(B_r)}. \quad (3.5)$$

Therefore, together with the interpolation inequality

$$\|v\|_{L^2(B_r)} \leq \|v\|_{L^1(B_r)}^\gamma \|v\|_{L^p(B_r)}^{1-\gamma},$$

with γ, p satisfying

$$\frac{1}{2} = \gamma + \frac{1-\gamma}{p}, \quad 0 < \gamma < 1,$$

we obtain

$$\|v\|_{L^2(B_r)} \leq C \|e^u\|_{L^1(B_r)}^\gamma \|e^u\|_{L^2(B_r)}^{1-\gamma}, \quad \forall r \in (0, 1].$$

Here we choose p slightly bigger than 2 in (3.4), and use the fact that the L^q and L^1 norms of Γ are uniformly bounded in B_r if r stays bounded. The lemma follows as $\bar{v}(x, t) \leq v(x)$. $\square$

Lemma 3.3. Setting $w = \bar{w}(x_0, 0)$ we have for every $0 < \rho < R := (r - |x_0|)$

$$c_s e^{w(x_0)} \leq \rho^{2s-n-2} \int_{D_\rho(x_0)} t^{1-2s} e^{\bar{w}} dx dt \leq R^{2s-n-2} \int_{D_R(x_0)} t^{1-2s} e^{\bar{u}} dx dt, \quad x_0 \in B_r,$$

where

$$c_s = \int_{D_1} t^{1-2s} dx dt.$$

Proof. We prove the lemma only for $x_0 = 0$. From (3.3) we have that

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla e^{\bar{w}}) = t^{1-2s} e^{\bar{w}} |\nabla \bar{w}|^2 \geq 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \lim_{t \rightarrow 0} t^{1-2s} \partial_t e^{\bar{w}} = 0 & \text{in } B_r. \end{cases}$$

Therefore, for $0 < \rho < r$

$$\begin{aligned} 0 &\leq \int_{D_\rho} \operatorname{div}(t^{1-2s} \nabla e^{\bar{w}(x,t)}) dx dt = \int_{\partial D_\rho \setminus B_\rho} t^{1-2s} \partial_\nu e^{\bar{w}(x,t)} d\sigma(X) \\ &= \rho^{n+1-2s} \partial_\rho \int_{\partial D_1 \setminus B_1} t^{1-2s} e^{\bar{w}(\rho x, \rho t)} d\sigma. \end{aligned}$$

This implies that $\rho^{2s-n-1} \int_{\partial D_\rho \setminus B_\rho} t^{1-2s} e^{\bar{w}} d\sigma$ is monotone increasing with respect to ρ . As a consequence, we have that $\rho^{2s-n-2} \int_{D_\rho} t^{1-2s} e^{\bar{w}} dx dt$ is increasing in ρ . Hence, using that $\bar{w} < \bar{u}$ and $\bar{w}$ is continuous up to the boundary B_r , where the former conclusion follows from the fact that $\bar{v} > 0$, we get

$$c_s e^{\bar{w}(0)} \leq \rho^{2s-n-2} \int_{D_\rho} t^{1-2s} e^{\bar{w}} dx dt \leq \rho^{2s-n-2} \int_{D_\rho} t^{1-2s} e^{\bar{u}} dx dt.$$

It completes the proof. $\square$

We now prove the following energy decay estimate for $\mathcal{E}(u, x_0, r)$. For simplicity, we write $\mathcal{E}(u, x_0, r)$ by $\mathcal{E}(x_0, r)$, and consider $x_0 = 0$ and $r = 1$ in the following proposition.

Proposition 3.4. Let u be a stable solution to (1.1) with $\Omega = B_1$ for some $u \in L_s(\mathbb{R}^n)$. Then there exists $\varepsilon_0 \in (0, 1)$ and $\theta \in (0, 1)$ depending only on n, s and $\|u_+\|_{L_s(\mathbb{R}^n)}$ such that if

$$\varepsilon := \mathcal{E}(0, 1) = \int_{B_1} e^u dx + \int_{D_1} t^{1-2s} e^{\bar{u}} dx dt \leq \varepsilon_0$$

then

$$\mathcal{E}(0, \theta) \leq \frac{1}{2} \mathcal{E}(0, 1).$$

Proof. It follows from (2.2) that

$$\int_{B_{1/2}} e^{2u} dx \leq C \int_{D_1} t^{1-2s} e^{\bar{u}} dx dt \leq C\varepsilon. \quad (3.6)$$

Then writing $\bar{u} = \bar{v} + \bar{w}$, where $\bar{v}$ is given by (3.2) with $r = \frac{1}{2}$, we get from Lemma 3.2 that

$$\|v\|_{L^2(B_{1/2})} + \|t^{\frac{1-2s}{2}} \bar{v}\|_{L^2(D_{1/2})} \leq C\varepsilon^{\frac{1+\gamma}{2}}. \quad (3.7)$$

We shall give estimation on $\mathcal{E}(0, r)$ for suitable r . Using Lemma 3.1 one can find $r_1 = r_1(n, s, \gamma) \in (0, \frac{1}{2})$ such that for every $0 < r \leq r_1$

$$\int_{D_r} t^{1-2s} e^{2\bar{u}(x,t)} dx dt \leq C\varepsilon^{1-\frac{\gamma}{2}},$$

where $C > 0$ depends on n, s and $\|u_+\|_{L_s(\mathbb{R}^n)}$. Together with (3.6), (3.7) and Hölder inequality

$$\int_{B_r} v e^u dx + \int_{D_r} t^{1-2s} \bar{v} e^{\bar{u}} dx dt \leq C \varepsilon^{1+\frac{\gamma}{4}}.$$

This in turn implies that

$$r^{4s-n-2} \int_{D_r \cap \{\bar{v} \geq 1\}} t^{1-2s} e^{\bar{u}} dx dt \leq C r^{4s-n-2} \varepsilon^{1+\frac{\gamma}{4}}.$$

Moreover, by Lemma 3.3

$$r^{4s-n-2} \int_{D_r \cap \{\bar{v} \leq 1\}} t^{1-2s} e^{\bar{u}} dx dt \leq C r^{4s-n-2} \int_{D_r} t^{1-2s} e^{\bar{w}} dx dt \leq C r^{2s} \varepsilon.$$

Combining the above two estimates

$$r^{4s-n-2} \int_{D_r} t^{1-2s} e^{\bar{u}} dx dt \leq C(r^{2s} \varepsilon + r^{4s-n-2} \varepsilon^{1+\frac{\gamma}{4}}).$$

In a similar way, by Lemma 3.3 we can also obtain,

$$\begin{aligned} r^{2s-n} \int_{B_r} e^u dx &\leq C r^{2s-n} \int_{B_r \cap \{v \leq 1\}} e^w dx + r^{2s-n} \int_{B_r \cap \{v \geq 1\}} v e^u dx \\ &\leq C r^{2s-n} r^n \int_{D_1} t^{1-2s} e^{\bar{u}} dx dt + C r^{2s-n} \varepsilon^{1+\frac{\gamma}{4}} \\ &\leq C(r^{2s} \varepsilon + r^{2s-n} \varepsilon^{1+\frac{\gamma}{4}}). \end{aligned}$$

Thus, for $0 < r \leq r_1$,

$$\mathcal{E}(0, r) \leq C(r^{2s} \varepsilon + r^{4s-n-2} \varepsilon^{1+\frac{\gamma}{4}}),$$

for some $C = C(n, s, \|u_+\|_{L_s(\mathbb{R}^n)}) > 0$, where we used $s < 1$ and $r_1 \leq \frac{1}{2}$.

Finally, we first choose $\theta > 0$ small enough such that $C\theta^{2s} = \frac{1}{4}$, and then choose $\varepsilon_0 > 0$ small such that $C\theta^{4s-n-2} \varepsilon_0^{\frac{\gamma}{4}} = \frac{1}{4}$. Then for $\varepsilon \leq \varepsilon_0$ we obtain

$$\mathcal{E}(0, \theta) \leq \frac{1}{2} \varepsilon = \frac{1}{2} \mathcal{E}(0, 1).$$

Hence, we finish the proof. $\square$

Though the proof of the following lemma is quite standard, thanks to Proposition 3.4 and Lemma 2.3, for completeness we give a sketch of the proof.

Lemma 3.5. *Let $u \in L_s(\mathbb{R}^n)$ be a stable solution to (1.1) with $\Omega = B_1$. Then there exists $\hat{\varepsilon}_0 > 0$ depending only on n, s and $\|u_+\|_{L_s(\mathbb{R}^n)}$ such that if*

$$0 < \mathcal{E}(u, 0, 1) \leq \hat{\varepsilon}_0,$$

then u is continuous in $B_{1/6}$.

Proof. For $x_0 \in B_{\frac{1}{2}}$ and $0 < \lambda \leq \lambda_0 := 2^{-2-\frac{n}{2s}}$ we set

$$u^\lambda(x) = u(x_0 + \lambda x) + 2s \log \lambda.$$

Then by Lemma 2.3 we see that

$$\|u_+^\lambda\|_{L_s(\mathbb{R}^n)} \leq C_1 \quad \text{for every } x_0 \in B_{\frac{1}{2}}, 0 < \lambda \leq \lambda_0,$$

for some $C_1 > 0$ independent of $x_0 \in B_{\frac{1}{2}}$ and $0 < \lambda \leq \lambda_0$. Therefore, by Proposition 3.4, we can find $\varepsilon_0 > 0$ and $\theta \in (0, 1)$ (depending only on n, s and C_1) such that

$$\mathcal{E}(u^\lambda, 0, 1) \leq \varepsilon_0 \quad \Rightarrow \quad \mathcal{E}(u^\lambda, 0, \theta) \leq \frac{1}{2} \mathcal{E}(u^\lambda, 0, 1). \quad (3.8)$$

It follows from the definition of the energy $\mathcal{E}$ that

$$\mathcal{E}(u^\lambda, 0, 1) = \mathcal{E}(u, x_0, \lambda), \quad \mathcal{E}(u, x_0, \lambda_0) \leq C(\lambda_0) \mathcal{E}(u, 0, 1).$$

Consequently, if we choose $0 < \hat{\varepsilon}_0 < \frac{\varepsilon_0}{C(\lambda_0)}$, then from (3.8) we get

$$\mathcal{E}(u, x_0, \lambda_0 \theta) = \mathcal{E}(u^{\lambda_0}, 0, \theta) \leq \frac{1}{2} \mathcal{E}(u^{\lambda_0}, 0, 1) = \frac{1}{2} \mathcal{E}(u, x_0, \lambda_0) \leq \frac{1}{2} \varepsilon_0.$$

By an iteration argument one obtains

$$\mathcal{E}(u, x_0, \lambda_0 \theta^k) \leq \frac{1}{2^k} \mathcal{E}(u, x_0, \lambda_0) \leq \frac{\varepsilon_0}{2^k}, \quad k = 1, 2, 3, \dots$$

In particular, setting $\alpha := -\frac{\log 2}{\log \theta} > 0$ we get

$$\mathcal{E}(u, x_0, r) \leq Cr^\alpha \mathcal{E}(u, 0, 1) \quad \text{for every } x_0 \in B_{\frac{1}{2}} \text{ and } 0 < r \leq \frac{1}{4},$$

which gives

$$\int_{B_r(x)} e^{u(y)} dy \leq Cr^{n-2s+\alpha} \quad \text{for every } x \in B_{\frac{1}{2}} \text{ and } 0 < r \leq \frac{1}{4}. \quad (3.9)$$

Now we decompose $u = u_1 + u_2$, with

$$u_1(x) = c(n, s) \int_{B_{1/3}} \frac{1}{|x-y|^{n-2s}} e^{u(y)} dy, \quad x \in \mathbb{R}^n,$$

where $c(n, s)$ is chosen such that

$$c(n, s)(-\Delta)^s \frac{1}{|x-y|^{n-2s}} = \delta(x-y).$$

By (3.9) and Lemma 2.2 we conclude that u_1 is bounded in $B_{1/3}$. While u_2 satisfies $(-\Delta)^s u_2 = 0$ in $B_{1/3}$, and it implies that u_2 is smooth in $B_{1/6}$. As a consequence, we get that u is continuous in $B_{1/6}$. $\square$

We recall that if u is stable in Ω , then $e^u \in L_{\text{loc}}^p(\Omega)$ for every $p \in [1, 5)$ by Lemma 2.1. Consequently, the small energy regularity results can be stated as follows:

Lemma 3.6. *For every $1 \leq p < 5$ there exists $\varepsilon_p > 0$ depending only on n, s and $\|u\|_{L^s(\mathbb{R}^n)}$ such that if*

$$\int_{B_1} e^{pu} dx \leq \varepsilon_p,$$

then u is continuous on $B_{1/12}$.

Proof. By Hölder inequality we get that

$$\int_{B_1} e^u dx \leq C\varepsilon_p^{\frac{1}{p}}.$$

Hence, by Lemma 3.1

$$\int_{D_{1/2}} t^{1-2s} e^{\bar{u}} dx dt \leq C\varepsilon_p^{\frac{\delta}{p}},$$

for some $\delta > 0$. This shows that

$$\mathcal{E}(0, \frac{1}{2}) \leq C\varepsilon_p^{\frac{\delta}{p}}.$$

Then by Lemma 3.5 we get that u is continuous in $B_{1/12}$ provided $C\varepsilon_p^{\frac{\delta}{p}}$ is sufficiently small. $\square$

Proof of Theorem 1.1. The problem (1.1) is invariant under the rescaling

$$u^\lambda(x) := u(x_0 + \lambda x) + 2s \log \lambda.$$

Therefore, if

$$r^{2ps-n} \int_{B_r(x_0)} e^{pu} dx \leq \varepsilon_p,$$

for some $p \in [1, 5)$, then u is continuous in $B_{r/12}(x_0)$, thanks to Lemma 3.6. Thus, if $x_0 \in \mathcal{S}$ ($\mathcal{S}$ is the singular set) we see that for every $r > 0$ small

$$r^{2ps-n} \int_{B_r(x_0)} e^{pu} dx > \varepsilon_p.$$

Hence, by the well-known Besicovitch covering theorem we have that the Hausdorff dimension of $\mathcal{S}$ is at most $n - 2ps$ with $p \in [1, 5)$, see for instance [15, Lemma 1.3.5, Lemma 2.1.1] or [13, Proposition 9.21]. Hence, we conclude that the Hausdorff dimension of $\mathcal{S}$ is at most $n - 10s$ and it completes the proof. $\square$

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