Research Article

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Global well-posedness of the full compressible Hall-MHD equations

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Abstract: This paper deals with a Cauchy problem of the full compressible Hall-magnetohydrodynamic flows. We establish the existence and uniqueness of global solution, provided that the initial energy is suitably small but the initial temperature allows large oscillations. In addition, the large time behavior of the global solution is obtained.

Keywords: full compressible Hall-MHD equations; global existence; large time behavior

MSC: 35A01, 35Q35, 76W05

1 Introduction

In this paper, we study the three-dimensional full compressible Hall-magnetohydrodynamic (for short, Hall-MHD) system, which is governed by the following equations (see, e.g. [4, 11]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + \nabla P = (\operatorname{curl} \mathbf{B}) \times \mathbf{B}, \\ C_V((\rho \theta)_t + \operatorname{div}(\rho \mathbf{u} \theta)) - \kappa \Delta \theta + \theta \partial_\theta P \operatorname{div} \mathbf{u} = 2\mu |D(\mathbf{u})|^2 + \lambda |\operatorname{div} \mathbf{u}|^2 + \nu |\operatorname{curl} \mathbf{B}|^2, \\ \mathbf{B}_t - \nu \Delta \mathbf{B} + \epsilon \operatorname{curl} \left(\frac{\operatorname{curl} \mathbf{B} \times \mathbf{B}}{\rho} \right) = \operatorname{curl}(\mathbf{u} \times \mathbf{B}), \operatorname{div} \mathbf{B} = 0, \end{cases} \quad (1.1)$$

with $t \geq 0$ and $x \in \mathbb{R}^3$. Here ρ , $\mathbf{u}$, P , θ and $\mathbf{B}$ represent the fluid density, velocity, pressure, absolute temperature and magnetic field, respectively. Deformation tensor $D(\mathbf{u}) := \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$. The constant viscosity coefficients μ and λ satisfy the physical restrictions:

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$

Here, we investigate the ideal polytropic fluids so that the pressure P and the physical constant C_V satisfy

$$P(\rho, \theta) := R\rho\theta, \quad C_V = \frac{R}{\gamma - 1},$$

where $\gamma > 1$ is the adiabatic constant, and for simplicity, we assume $C_V = R = 1$. κ and ν are positive constants. $\epsilon > 0$ is the Hall coefficient.

The Hall-MHD system can be derived from fluid mechanics with appropriate modifications to account for electrical forces and Hall effects. This compressible system(1.1) describes the dynamics of plasma flows with strong shear of magnetic fields such as in the solar flares, neutron stars and geo-dynamo, we refer to

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[1, 3, 10, 12, 23, 26, 30] for the physical background of this system. The Hall term $\text{curl}\left(\frac{(\text{curl}\mathbf{B})\times\mathbf{B}}{\rho}\right)$ in (1.1)₄ has been put forward by Ghosh et al. [10] to restore the influence of the electric current in the Lorentz force occurring in Ohm's law. And the Hall coefficient ϵ is defined by the quotient of Alfvén frequency of the lowest wave number ω_A and the ion cyclotron frequency Ω_i , it means $\epsilon := \frac{\omega_A}{\Omega_i}$. When the Hall effect term is neglected ($\epsilon = 0$), the equations (1.1) is reduced to the well-known compressible full MHD system, whose applications cover a broad range of physical fields from liquid metals to cosmic plasmas. The mathematical results on this compressible heat conducting MHD system can refer for example to [2, 6–8, 15, 16, 20, 25]. Fan and Yu [7] gave the local strong solution and Huang and Li established the blowup criterion. The global weak solutions was proved in [2, 6, 15, 16]. The long time behavior was discussed in [8, 25]. While, Jiang and his collaborators solved the low Mach number limit problem in [19, 20].

The compressible Hall-MHD equations are also mathematically significant. The solvability and stability of the equations has attracted considerable attention recently. For the isentropic case, Fan et al. [5] studied the global existence of strong solution and established the optimal time decay rates under the small initial perturbation condition. Gao and Yao [9] improved their work and established optimal decay rates for higher-order spatial derivatives of classical solutions. In [29], Tao, Yang and Yao established the global existence, uniqueness and exponential stability of strong solutions with large initial data for the one-dimensional case. Xiang [31] established the uniform estimates and optimal decay rates to global solution with respect to the Hall coefficient ϵ under the condition that H^2 -norm of initial data is small enough. If the temperature is taken into account, Fan et al. [4] first proved the local well-posedness for the full compressible Hall-MHD equations, and obtained a blow-up criterion of strong solution. The boundedness and time decay of the higher-order spatial derivatives of the smooth solution under the condition that H^k -norm ($k \geq 3$) of initial data is small and bounded in $\dot{H}^s$ ($0 < s < \frac{3}{2}$) are established by He, Samet and Zhou in [11]. Recently, Lai, Xu and Zhang [21] generalized Xiang's results [31] into the non-isentropic case. For other works on the compressible Hall-MHD system, we refer to [27, 28] and references therein. However, to our knowledge, all the results on the global smooth solutions for the three dimensional compressible Hall-MHD equation need the initial data has at least H^2 small norm.

In this paper, we consider an initial value problem of the Hall-MHD compressible flows (1.1) supplied with initial data

$$(\rho, \mathbf{u}, \theta, \mathbf{B})(x, 0) = (\rho_0, \mathbf{u}_0, \theta_0, \mathbf{B}_0)(x), \quad \text{for } x \in \mathbb{R}^3 \quad (1.2)$$

and the far field behavior

$$(\rho_0 - 1, \mathbf{u}_0, \theta_0 - 1, \mathbf{B}_0)(x) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty, \quad t \geq 0. \quad (1.3)$$

Motivated by the works for compressible Navier-Stokes equation [14, 18] and the compressible MHD equation [13, 22], we will first establish global existence and uniqueness of solution with smooth initial data which is of small energy. It is worth mentioning that H^2 -norm of the initial data are not necessarily small. Then, the large time behavior of the solution will be given as well. Compared with the works in [13, 14, 18, 22], we also have to deal with the essential difficulties caused by Hall term in the present paper. This term includes the strong coupling between the density and the magnetic field, which together with the second-order derivative structure make the derivations of estimates more difficult. Therefore, the methods used in the MHD equation to show the bounds for the magnetic field are no longer applicable here. In order to overcome the difficulties from the Hall term, we introduce two kinds of estimates for derivatives of the density, which play an important role to establish the time-independent lower-order estimates. In addition, the temperature is considered in this paper, which brings us more nonlinear term, for instance, $|\text{curl}\mathbf{B}|^2$ in temperature equation (1.1)₃ and makes the system more complex.

Throughout this paper, we use $H^s(\mathbb{R}^3)$ ($s \in \mathbb{N}$) to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$ ($1 \leq p \leq \infty$) to denote the L^p spaces with norm $\|\cdot\|_{L^p}$. For given initial data $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{B}_0)$, we define the initial energy E_0 ,

$$E_0 := \int \left(\frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{1}{2} |\mathbf{B}_0|^2 + (1 + \rho_0 \log \rho_0 - \rho_0) + \rho_0 (\theta_0 - \log \theta_0 - 1) \right) dx. \quad (1.4)$$

Now, the main result in this paper is stated as follows.

Theorem 1.1. Assume that for all given $M_1 > 0$ (not necessarily small), the initial data $(\rho_0, \mathbf{u}_0, \theta_0, \mathbf{B}_0)$ satisfies

$$\inf \rho_0 > 0, \quad \inf \theta_0 > 0, \quad (\rho_0 - 1, \mathbf{u}_0, \theta_0 - 1, \mathbf{B}_0) \in H^3, \quad (1.5)$$

$$\|\nabla \rho_0\|_{H^1}^2 + \|\nabla \mathbf{u}_0\|_{H^1}^2 + \|\nabla \mathbf{B}_0\|_{H^1}^2 + \|\nabla \theta_0\|_{L^2}^2 \leq M_1. \quad (1.6)$$

Then, there is a positive constant δ depending only on $\mu, \lambda, \kappa, \nu, \epsilon$ and M_1 , such that if

$$E_0 \leq \delta, \quad (1.7)$$

the Cauchy problem (1.1)–(1.3) has a unique global solution $(\rho, \mathbf{u}, \theta, \mathbf{B})$ in $\mathbb{R}^3 \times [0, \infty)$ satisfying

$$\begin{cases} \rho - 1, \mathbf{u}, \theta - 1, \mathbf{B} \in L^\infty([0, T]; H^3), \\ \nabla \rho \in L^2([0, T]; H^2), \quad \nabla \mathbf{u}, \nabla \theta, \nabla \mathbf{B} \in L^2([0, T]; H^3), \end{cases} \quad (1.8)$$

and the large time behavior:

$$\lim_{t \rightarrow \infty} (\|\rho - 1\|_{L^q}^2 + \|\mathbf{u}\|_{L^q}^2 + \|\theta - 1\|_{L^q}^2 + \|\mathbf{B}\|_{L^q}^2) = 0, \quad \text{for any } q \in (2, \infty]. \quad (1.9)$$

Remark 1.1. From (1.6) and the small initial energy, we can find that the initial data in Theorem 1.1 have small H^1 -norm for $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$, which is weaker than that in [21, 31]. Indeed, by Gagliardo-Nirenberg inequality, the smallness of H^2 -norm of the initial data is required in [21, 31]. Moreover, the absolute temperature is considered in this paper and the initial temperature allows large oscillations.

Remark 1.2. Since the Hall term involves second-order derivative of magnetic field and first-order derivative of density, in order to establish the global existence of the solution, we need to get the bound of $\|\nabla \rho\|_{H^1}$, which leads (1.6) including H^2 -norm of initial data.

The rest of this paper is organized as follow. In section 2, we derive the time-independent lower-order estimates and the higher-order estimates depending on time of the solutions. In section 3, the proof of Theorem 1.1 will be showed.

2 Global existence

In this section, a known inequality and some facts are first collected, and then we will establish some suitable *a priori* estimates by the energy method.

2.1 Preliminaries

For the convenience of the proof below, let us rewrite the system (1.1) as follows,

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho \mathbf{u}_t - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + \theta \nabla \rho + \rho \nabla \theta = -\rho \mathbf{u} \cdot \nabla \mathbf{u} + (\operatorname{curl} \mathbf{B}) \times \mathbf{B}, \\ \rho \theta_t - \kappa \Delta \theta = -\rho \mathbf{u} \cdot \nabla \theta - \rho \theta \operatorname{div} \mathbf{u} + 2\mu |D(\mathbf{u})|^2 + \lambda |\operatorname{div} \mathbf{u}|^2 + \nu |\operatorname{curl} \mathbf{B}|^2, \\ \mathbf{B}_t - \nu \Delta \mathbf{B} = -\epsilon \operatorname{curl} \left(\frac{\operatorname{curl} \mathbf{B} \times \mathbf{B}}{\rho} \right) + \operatorname{curl}(\mathbf{u} \times \mathbf{B}), \quad \operatorname{div} \mathbf{B} = 0, \end{cases} \quad (2.1)$$

here we use the equalities

$$(\operatorname{curl} \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla (|\mathbf{B}|^2)$$

and

$$\operatorname{curl}(\mathbf{u} \times \mathbf{B}) = \mathbf{u}(\operatorname{div} \mathbf{B}) - (\mathbf{u} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{u} - \mathbf{B}(\operatorname{div} \mathbf{u}).$$

In addition, the initial data satisfies

$$(\rho, \mathbf{u}, \theta, \mathbf{B})(x, t)|_{t=0} = (\rho_0(x), \mathbf{u}_0(x), \theta_0(x), \mathbf{B}_0(x)) \quad (2.2)$$

and

$$(\rho - 1, \mathbf{u}, \theta - 1, \mathbf{B})(x, t) = (\rho_0 - 1, \mathbf{u}_0, \theta_0 - 1, \mathbf{B}_0)(x) \rightarrow (0, 0, 0, 0), \text{ as } |x| \rightarrow \infty. \quad (2.3)$$

The following Gagliardo-Nirenberg inequality are well-known (see for example [24]).

Lemma 2.1. *Let $0 \leq m, \alpha \leq l$ and the function $f \in C_0^\infty(\mathbb{R}^3)$, then we have*

$$\|\nabla^\alpha f\|_{L^p} \leq C \|\nabla^m f\|_{L^2}^{1-\theta} \|\nabla^l f\|_{L^2}^\theta,$$

where $0 \leq \theta \leq 1$ and α satisfy

$$\frac{1}{p} - \frac{\alpha}{3} = \left(\frac{1}{2} - \frac{m}{3}\right)(1 - \theta) + \left(\frac{1}{2} - \frac{l}{3}\right)\theta.$$

Now, we are ready to define some functions which will be frequently used later. First of all, let $\sigma = \sigma(t) = \min\{1, t\}$ and $\sigma' = \frac{d}{dt}\sigma(t)$. Then, we set:

$$\begin{aligned} A_1(T) &= \sup_{0 \leq t \leq T} \int ((\rho - 1)^2 + (\theta - 1)^2) dx + \int_0^T \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{B}|^2 + |\nabla \theta|^2) dx dt, \\ A_2(T) &= \sup_{0 \leq t \leq T} \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{B}|^2 + |\nabla \rho|^2) dx + \int_0^T \int (|\nabla^2 \mathbf{u}|^2 + |\nabla^2 \mathbf{B}|^2) dx dt, \\ A_3(T) &= \sup_{0 \leq t \leq T} \int (|\nabla^2 \mathbf{u}|^2 + |\nabla^2 \mathbf{B}|^2 + |\nabla^2 \rho|^2) dx, \\ A_4(T) &= \sup_{0 \leq t \leq T} (\sigma^4 \|\nabla^2 \theta\|_{L^2}^2). \end{aligned}$$

In what follows, we denote the generic constant and suitably small constant by $C > 0$ and $\delta_1 > 0$ depending only on some known constants μ, λ, k, ν and ϵ but independent of time t , respectively. Particularly, we will use $C(M)$ to emphasize that C may depend on $M = \max\{(1 + C_1^{-1})M_1, (1 + C_3)M_1\}$, where the given constants C_1 and C_3 are defined in Lemma 2.8 and 2.10.

2.2 Time-independent lower-order estimates.

The aim of this subsection is to derive the lower-order estimates on the solutions which are independent of time. Now, let $(\rho, \mathbf{u}, \theta, \mathbf{B})$ be a solution to system (2.1)–(2.3) on $\mathbb{R}^3 \times (0, T)$ for some positive time $T > 0$ and without loss of generality, let $E_0 \leq 1$.

Proposition 2.1. *Assume that the solution $(\rho, \mathbf{u}, \theta, \mathbf{B})$ satisfies*

$$A_1(T) \leq 4E_0^{\frac{1}{6}}, \quad A_2(T) \leq 2M, \quad A_3(T) \leq 2M, \quad A_4(T) \leq 2E_0^{\frac{1}{32}}, \quad (2.4)$$

for all $(x, t) \in \mathbb{R}^3 \times (0, T)$, and the initial data satisfies (1.5)–(1.6), then it holds that

$$A_1(T) \leq 2E_0^{\frac{1}{6}}, \quad A_2(T) \leq \frac{3}{2}M, \quad A_3(T) \leq \frac{3}{2}M, \quad A_4(T) \leq E_0^{\frac{1}{32}}, \quad (2.5)$$

provided $E_0 \leq \delta$, where δ is a positive constant depending on $\mu, \lambda, \kappa, \nu, \epsilon$ and M_1 but independent of T .

The proof of Proposition 2.1 consists of Lemma 2.2–2.12 and is to be completed by the end of this subsection.

Lemma 2.2. *Under all the assumption of Proposition 2.1, it holds that*

$$\frac{1}{2} < \rho < \frac{3}{2} \quad (2.6)$$

in $\mathbb{R}^3 \times (0, T)$, provided $E_0 \leq (4(C(M)))^{-24}$.

Proof: By (2.4) and Sobolev inequality, we obtain

$$\|\rho - 1\|_{L^\infty} \leq C \|\rho - 1\|_{L^2}^{\frac{1}{4}} \|\nabla^2 \rho\|_{L^2}^{\frac{3}{4}} \leq C(M) E_0^{\frac{1}{24}},$$

provided $E_0 \leq (4C(M))^{-24}$. Thus, (2.6) holds. $\square$

Lemma 2.3. *Under the conditions of Proposition 2.1, it holds that*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{2} |\mathbf{B}|^2 + (1 + \rho \log \rho - \rho) + \rho(\theta - \log \theta - 1) \right) dx \\ & + \int_0^T \int \left(\theta^{-1} (\mu |\nabla \mathbf{u}|^2 + (\lambda + \mu) |\operatorname{div} \mathbf{u}|^2 + \frac{\nu}{2} |\nabla \mathbf{B}|^2) + \kappa \theta^{-2} |\nabla \theta|^2 \right) dx dt \leq E_0. \end{aligned} \quad (2.7)$$

Proof: It follows from [18] and maximum principle, we have $\theta > 0$ for all $(x, t) \in \mathbb{R}^3 \times (0, T)$. Multiplying (2.1)₂–(2.1)₄ by $\mathbf{u}$, $1 - \theta^{-1}$ and $\mathbf{B}$, respectively, then adding them up and integrating by parts over $\mathbb{R}^3$, using (2.1)₁ and the equality

$$\int \operatorname{curl} \left(\frac{\operatorname{curl} \mathbf{B} \cdot \mathbf{B}}{\rho} \right) \cdot \mathbf{B} dx = 0,$$

we have

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{2} |\mathbf{B}|^2 + (1 + \rho \log \rho - \rho) + \rho(\theta - \log \theta - 1) \right) dx \\ & = - \int \left(\theta^{-1} (\mu |\nabla \mathbf{u}|^2 + (\lambda + \mu) |\operatorname{div} \mathbf{u}|^2 + \frac{\nu}{2} |\nabla \mathbf{B}|^2) + \kappa \theta^{-2} |\nabla \theta|^2 \right) dx. \end{aligned}$$

We thus derive (2.7) directly by integrating the above equality over $(0, T)$ and finish the proof of Lemma 2.3. $\square$

Lemma 2.4. *Under the assumptions of Proposition 2.1, it holds that*

$$\sup_{0 \leq t \leq T} (\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2) \leq C E_0 \quad (2.8)$$

and

$$\|\theta - 1\|_{L^2} \leq C E_0^{\frac{1}{2}} + C E_0^{\frac{1}{3}} \|\nabla \theta\|_{L^2}. \quad (2.9)$$

The proof of Lemma 2.4 is the same as Lemma 3.1 in [18], so we omit it for brevity.

The following lemma is given to estimate $A_1(T)$.

Lemma 2.5. *Under the assumptions of Proposition 2.1, it holds that*

$$\sup_{0 \leq t \leq T} \int (\theta - 1)^2 dx + \int_0^T \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{B}|^2 + |\nabla \theta|^2) dx dt \leq E_0^{\frac{1}{6}}, \quad (2.10)$$

provided $E_0 \leq \min\{(2C)^{-\frac{1}{24}}, (\frac{\kappa}{C(M)})^{12}\}$.

Proof: The lemma can be established by a similar way in [14]. For completeness, we prove it here. Multiplying (2.1)₂ by $\mathbf{u}$, integrating the result over $\mathbb{R}^3$ and using integration by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int (\rho |\mathbf{u}|^2 + |\mathbf{B}|^2) dx + \int (\mu |\nabla \mathbf{u}|^2 + (\lambda + \mu) |\operatorname{div} \mathbf{u}|^2 + \nu |\nabla \mathbf{B}|^2) dx \leq - \int \nabla(\rho \theta) \cdot \mathbf{u} dx. \quad (2.11)$$

By Hölder inequality and using (2.8) and (2.9), the right-hand side of (2.11) is estimated as follows,

$$\begin{aligned} - \int \nabla(\rho \theta) \cdot \mathbf{u} dx &= \int (\rho(\theta - 1) + \rho - 1) \operatorname{div} \mathbf{u} dx \\ &\leq C(\|\theta - 1\|_{L^2} + \|\rho - 1\|_{L^2}) \|\nabla \mathbf{u}\|_{L^2} \\ &\leq \frac{\mu}{2} \|\nabla \mathbf{u}\|_{L^2}^2 + CE_0 + CE_0^{\frac{2}{3}} \|\nabla \theta\|_{L^2}^2. \end{aligned} \quad (2.12)$$

Taking (2.12) into (2.11), we have

$$\frac{1}{2} \frac{d}{dt} \int (\rho |\mathbf{u}|^2 + |\mathbf{B}|^2) dx + \int \left(\frac{\mu}{2} |\nabla \mathbf{u}|^2 + \nu |\nabla \mathbf{B}|^2 \right) dx \leq CE_0 + CE_0^{\frac{2}{3}} \|\nabla \theta\|_{L^2}^2. \quad (2.13)$$

On the other hand, multiplying (2.1)₃ by $\theta - 1$, integrating the resulting inequality over $\mathbb{R}^3$, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \rho(\theta - 1)^2 dx + \kappa \|\nabla \theta\|_{L^2}^2 \\ &\leq C \int \rho \theta |\theta - 1| |\operatorname{div} \mathbf{u}| dx + C \int |\theta - 1| (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{B}|^2) dx \\ &:= E_1 + E_2. \end{aligned} \quad (2.14)$$

By (2.4), (2.8) and (2.9), using Hölder, Sobolev and Young inequalities, we can bound E_1 as follows,

$$\begin{aligned} E_1 &\leq C \int |\theta - 1|^2 |\operatorname{div} \mathbf{u}| dx + \int |\theta - 1| |\operatorname{div} \mathbf{u}| dx \\ &\leq C \|\theta - 1\|_{L^3} \|\theta - 1\|_{L^6} \|\nabla \mathbf{u}\|_{L^2} + C \|\theta - 1\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \\ &\leq C \|\theta - 1\|_{L^2}^{\frac{1}{2}} \|\nabla \theta\|_{L^2}^{\frac{3}{2}} \|\nabla \mathbf{u}\|_{L^2} + C \|\theta - 1\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \\ &\leq C(E_0^{\frac{1}{6}} + E_0^{\frac{1}{6}} \|\nabla \theta\|_{L^2}^{\frac{1}{2}}) \|\nabla \theta\|_{L^2}^{\frac{3}{2}} \|\nabla \mathbf{u}\|_{L^2} + C(E_0^{\frac{1}{6}} + E_0^{\frac{1}{6}} \|\nabla \theta\|_{L^2}^{\frac{1}{2}}) \|\nabla \mathbf{u}\|_{L^2} \\ &\leq CE_0^{\frac{1}{6}} \|\nabla \theta\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^2} + CE_0^{\frac{1}{6}} \|\nabla \mathbf{u}\|_{L^2} + CE_0^{\frac{1}{6}} (\|\nabla \theta\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2) \\ &\leq C(M) E_0^{\frac{1}{6}} + C(M) E_0^{\frac{1}{6}} \|\nabla \theta\|_{L^2}^2. \end{aligned}$$

Similarly, we have following estimate for E_2 ,

$$\begin{aligned} E_2 &\leq C \|\theta - 1\|_{L^3} (\|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^6} + \|\nabla \mathbf{B}\|_{L^2} \|\nabla \mathbf{B}\|_{L^6}) \\ &\leq C \|\theta - 1\|_{L^2}^{\frac{1}{2}} \|\nabla \theta\|_{L^2}^{\frac{1}{2}} (\|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^6} + \|\nabla \mathbf{B}\|_{L^2} \|\nabla \mathbf{B}\|_{L^6}) \\ &\leq C(E_0^{\frac{1}{6}} \|\nabla \theta\|_{L^2}^{\frac{1}{2}} + E_0^{\frac{1}{6}} \|\nabla \theta\|_{L^2}^{\frac{1}{2}}) (\|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^6} + \|\nabla \mathbf{B}\|_{L^2} \|\nabla \mathbf{B}\|_{L^6}) \\ &\leq C(M) E_0^{\frac{1}{6}} + CE_0^{\frac{1}{12}} \|\nabla \theta\|_{L^2}^2 + C(M) E_0^{\frac{1}{6}} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2). \end{aligned}$$

Inserting the estimates of E_1 and E_2 into (2.14), we get

$$\frac{d}{dt} \int \rho(\theta - 1)^2 dx + \kappa \|\nabla \theta\|_{L^2}^2 \leq C(M) (E_0^{\frac{1}{6}} + E_0^{\frac{1}{6}} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2)), \quad (2.15)$$

provided that $E_0 \leq (\frac{\kappa}{C(M)})^{12}$. Summing up (2.13) and (2.15), we obtain

$$\begin{aligned} &\frac{d}{dt} \int (\rho |\mathbf{u}|^2 + |\mathbf{B}|^2 + \rho(\theta - 1)^2) dx + \int (\mu |\nabla \mathbf{u}|^2 + \nu |\nabla \mathbf{B}|^2 + \kappa |\nabla \theta|^2) dx \\ &\leq C(M) (E_0^{\frac{1}{6}} + E_0^{\frac{1}{6}} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2)). \end{aligned} \quad (2.16)$$

Integrating (2.16) over $(0, \sigma(T))$, it follows from (2.4) that

$$\begin{aligned} & \sup_{0 \leq t \leq \sigma(T)} \frac{1}{2} \int (\rho |\mathbf{u}|^2 + |\mathbf{B}|^2 + \rho(\theta - 1)^2) dx + \int_0^{\sigma(T)} \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{B}|^2 + |\nabla \theta|^2) dx dt \\ & \leq C(M) E_0^{\frac{1}{4}}. \end{aligned} \quad (2.17)$$

For $t \in (\sigma(T), T)$, by (2.4), we have the following estimate

$$\sup_{\sigma(T) \leq t \leq T} \|\theta - 1\|_{L^\infty} \leq \sup_{0 \leq t \leq T} \sigma^2 (\|\theta - 1\|_{L^2}^{\frac{1}{4}} \|\nabla^2 \theta\|_{L^2}^{\frac{3}{4}}) \leq C E_0^{\frac{1}{24}} \leq \frac{1}{2},$$

provided $E_0 \leq (2C)^{-24}$. Thus

$$\frac{1}{2} \leq \theta \leq \frac{3}{2}, \quad t \in (\sigma(T), T). \quad (2.18)$$

Applying (2.7) and (2.18), we have

$$\begin{aligned} & \sup_{\sigma(T) \leq t \leq T} \frac{1}{2} \int (\rho |\mathbf{u}|^2 + |\mathbf{B}|^2 + \rho(\theta - 1)^2) dx + \int_{\sigma(T)}^T \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{B}|^2 + |\nabla \theta|^2) dx dt \\ & \leq C E_0. \end{aligned} \quad (2.19)$$

Then, the combination of (2.17) with (2.19) yields

$$\sup_{0 \leq t \leq T} \int (|\mathbf{u}|^2 + |\mathbf{B}|^2 + (\theta - 1)^2) dx + \int_0^T \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{B}|^2 + |\nabla \theta|^2) dx dt \leq E_0^{\frac{1}{6}},$$

provided $E_0 \leq \min\{(2C)^{-\frac{1}{24}}, (\frac{\kappa}{C(M)})^{12}\}$. Thus, we complete the proof of Lemma 2.5. $\square$

The estimates of $\|\nabla \mathbf{B}\|_{L^2}$ and $\|\nabla^2 \mathbf{B}\|_{L^2}$ will be given by following Lemma.

Lemma 2.6. *Under the assumptions of Proposition 2.1, it holds that*

$$\sup_{0 \leq t \leq T} \|\nabla \mathbf{B}\|_{L^2}^2 + \nu \int_0^T \|\nabla^2 \mathbf{B}\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{6}} + \|\nabla \mathbf{B}_0\|_{L^2}^2, \quad (2.20)$$

$$\sup_{0 \leq t \leq T} \sigma^2 \|\nabla \mathbf{B}\|_{L^2}^2 + \int_0^T \sigma^2 \|\nabla^2 \mathbf{B}\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{6}}, \quad (2.21)$$

$$\sup_{0 \leq t \leq T} \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \nu \int_0^T \|\nabla^3 \mathbf{B}\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{12}} + \|\nabla^2 \mathbf{B}_0\|_{L^2}^2, \quad (2.22)$$

$$\sup_{0 \leq t \leq T} \sigma^4 \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \int_0^T \sigma^4 \|\nabla^3 \mathbf{B}\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{12}}. \quad (2.23)$$

Proof: Applying ∇ to (2.1)₄ and multiplying by $\nabla \mathbf{B}$, then integrating it over $\mathbb{R}^3$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{B}\|_{L^2}^2 + \nu \|\nabla^2 \mathbf{B}\|_{L^2}^2 \\ & = \int \nabla \mathbf{B} \cdot \nabla (-\mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \operatorname{div} \mathbf{u}) dx - \epsilon \int \nabla \mathbf{B} \cdot \nabla \operatorname{curl} \left(\frac{(\operatorname{curl} \mathbf{B}) \times \mathbf{B}}{\rho} \right) dx \\ & := I_1 + I_2. \end{aligned} \quad (2.24)$$

Using the integration by parts and with the help of Hölder, Sobolev and Young inequalities, I_1 and I_2 can be bounded as

$$\begin{aligned} I_1 &\leq C\|\nabla^2 \mathbf{B}\|_{L^2}(\|\mathbf{u}\|_{L^6}\|\nabla \mathbf{B}\|_{L^3} + 2\|\nabla \mathbf{u}\|_{L^2}\|\mathbf{B}\|_{L^\infty}) \\ &\leq C\|\nabla \mathbf{u}\|_{L^2}\|\nabla^2 \mathbf{B}\|_{L^2}(\|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{1}{2}}\|\nabla \mathbf{B}\|_{L^2}^{\frac{1}{2}} + \|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{3}{4}}\|\nabla \mathbf{B}\|_{L^2}^{\frac{1}{4}}) \\ &\leq \frac{\nu}{4}\|\nabla^2 \mathbf{B}\|_{L^2}^2 + C(\|\nabla \mathbf{u}\|_{L^2}^4 + C\|\nabla \mathbf{u}\|_{L^2}^8)\|\nabla \mathbf{B}\|_{L^2}^2 \\ &\leq \frac{\nu}{4}\|\nabla^2 \mathbf{B}\|_{L^2}^2 + C(M)\|\nabla \mathbf{B}\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} I_2 &= \epsilon \int \nabla(\operatorname{curl} \mathbf{B}) \cdot \nabla \left(\frac{(\operatorname{curl} \mathbf{B}) \times \mathbf{B}}{\rho} \right) dx \\ &= \epsilon \int \nabla(\operatorname{curl} \mathbf{B}) \cdot (\operatorname{curl} \mathbf{B}) \times \nabla \left(\frac{\mathbf{B}}{\rho} \right) dx \\ &\leq C\|\nabla^2 \mathbf{B}\|_{L^2}(\|\rho^{-1}\|_{L^\infty}\|\nabla \mathbf{B}\|_{L^4}^2 + \|\rho^{-2}\|_{L^\infty}\|\mathbf{B}\|_{L^\infty}\|\nabla \mathbf{B}\|_{L^3}\|\nabla \rho\|_{L^6}) \\ &\leq C\|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{5}{2}}\|\nabla \mathbf{B}\|_{L^2}^{\frac{1}{2}} + C\|\nabla^2 \mathbf{B}\|_{L^2}\|\nabla \mathbf{B}\|_{L^2}^{\frac{1}{2}}\|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{1}{2}}\|\mathbf{B}\|_{L^2}^{\frac{3}{4}}\|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{1}{4}}\|\nabla^2 \rho\|_{L^2} \\ &\leq \frac{\nu}{4}\|\nabla^2 \mathbf{B}\|_{L^2}^2 + C(M)(\|\nabla^2 \mathbf{B}\|_{L^2}^4\|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^4\|\mathbf{B}\|_{L^2}^6\|\nabla^2 \rho\|_{L^2}^8) \\ &\leq \frac{\nu}{4}\|\nabla^2 \mathbf{B}\|_{L^2}^2 + C(M)\|\nabla \mathbf{B}\|_{L^2}^2. \end{aligned}$$

Putting the bounds of I_1 and I_2 into (2.24), and then applying (2.4) and (2.8), it leads to

$$\frac{d}{dt}\|\nabla \mathbf{B}\|_{L^2}^2 + \nu\|\nabla^2 \mathbf{B}\|_{L^2}^2 \leq C(M)\|\nabla \mathbf{B}\|_{L^2}^2. \quad (2.25)$$

Integrating (2.25) over $(0, T)$ and by (2.10), we obtain

$$\sup_{0 \leq t \leq T} \|\nabla \mathbf{B}\|_{L^2}^2 + \nu \int_0^T \|\nabla^2 \mathbf{B}\|_{L^2}^2 dt \leq C(M)E_0^{\frac{1}{6}} + \|\nabla \mathbf{B}_0\|_{L^2}^2. \quad (2.26)$$

Multiplying (2.25) by σ^2 , then, integrating it over $(0, T)$ and by (2.10), we have

$$\sup_{0 \leq t \leq T} \sigma^2 \|\nabla \mathbf{B}\|_{L^2}^2 + \nu \int_0^T \sigma^2 \|\nabla^2 \mathbf{B}\|_{L^2}^2 dt \leq (C(M) + 2\sigma \cdot \sigma') \int_0^T \|\nabla \mathbf{B}\|_{L^2}^2 dt \leq C(M)E_0^{\frac{1}{6}}. \quad (2.27)$$

By (2.26) and (2.27), we complete the proof of (2.20) and (2.21).

Applying ∇^2 to (2.1)₄ and multiplying it by $\nabla^2 \mathbf{B}$, then integrating it over $\mathbb{R}^3$, we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \nu \|\nabla^3 \mathbf{B}\|_{L^2}^2 \\ &= \int \nabla^2 \mathbf{B} \cdot \nabla^2 (-\mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \operatorname{div} \mathbf{u}) dx - \epsilon \int \nabla^2 \mathbf{B} \cdot \nabla^2 \operatorname{curl} \left(\frac{(\operatorname{curl} \mathbf{B}) \times \mathbf{B}}{\rho} \right) dx \\ &:= II_1 + II_2. \end{aligned} \quad (2.28)$$

Using the integration by parts and with the help of Hölder, Sobolev and Young inequalities, by (2.4) and (2.10), II_1 and II_2 can be bounded as

$$\begin{aligned} II_1 &\leq C\|\nabla^3 \mathbf{B}\|_{L^2}(\|\mathbf{u}\|_{L^6}\|\nabla^2 \mathbf{B}\|_{L^3} + \|\nabla \mathbf{u}\|_{L^3}\|\nabla \mathbf{B}\|_{L^6} + \|\nabla^2 \mathbf{u}\|_{L^2}\|\mathbf{B}\|_{L^\infty}) \\ &\leq \frac{\nu}{4}\|\nabla^2 \mathbf{B}\|_{L^2}^2 + C(M)(E_0^{-\frac{1}{12}}\|\nabla \mathbf{u}\|_{L^2}^2 + E_0^{\frac{1}{4}}\|\nabla^2 \mathbf{u}\|_{L^2}^2 + E_0^{\frac{1}{12}}\|\nabla^2 \mathbf{B}\|_{L^2}^2) \\ &\leq \frac{\nu}{4}\|\nabla^2 \mathbf{B}\|_{L^2}^2 + C(M)\|\nabla \mathbf{B}\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned}
 II_2 &= \epsilon \int \nabla^2(\operatorname{curl} \mathbf{B}) \cdot \nabla^2 \left(\frac{(\operatorname{curl} \mathbf{B}) \times \mathbf{B}}{\rho} \right) dx \\
 &= \epsilon \int \nabla^2(\operatorname{curl} \mathbf{B}) \cdot (\operatorname{curl} \mathbf{B}) \times \nabla^2 \left(\frac{\mathbf{B}}{\rho} \right) dx \\
 &\leq C \int |\nabla^3 \mathbf{B} \cdot \nabla \mathbf{B}| (|\nabla^2 \mathbf{B}| + |\nabla \mathbf{B}| |\nabla \rho| + |\mathbf{B}| |\nabla^2 \rho|) \\
 &\leq C \|\nabla^3 \mathbf{B}\|_{L^2} (\|\rho^{-1}\|_{L^\infty} \|\nabla \mathbf{B}\|_{L^4}^2 + \|\rho^{-2}\|_{L^\infty} \|\mathbf{B}\|_{L^\infty} \|\nabla \mathbf{B}\|_{L^3} \|\nabla \rho\|_{L^6}) dx \\
 &\leq C \|\nabla^3 \mathbf{B}\|_{L^2} (\|\nabla \mathbf{B}\|_{L^4} \|\nabla^2 \mathbf{B}\|_{L^4} + \|\nabla \mathbf{B}\|_{L^\infty} \|\nabla \mathbf{B}\|_{L^3} \|\nabla \rho\|_{L^6} + \|\mathbf{B}\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}) \\
 &\leq C(M) \|\nabla^3 \mathbf{B}\|_{L^2}^{\frac{7}{4}} (\|\nabla \mathbf{B}\|_{L^2} \|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{1}{4}} + \|\nabla \mathbf{B}\|_{L^2}^{\frac{3}{4}} \|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \rho\|_{L^2}) \\
 &\leq \frac{\nu}{4} \|\nabla^2 \mathbf{B}\|_{L^2}^2 + C(M) \|\nabla \mathbf{B}\|_{L^2}^2.
 \end{aligned}$$

Inserting the estimates of II_1 and II_2 into (2.28), and then applying (2.4) and (2.8), it implies

$$\frac{d}{dt} \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \nu \|\nabla^3 \mathbf{B}\|_{L^2}^2 \leq C(M) \|\nabla \mathbf{B}\|_{L^2}^2. \quad (2.29)$$

Integrating (2.29) over $(0, T)$ and by (2.10), we obtain

$$\sup_{0 \leq t \leq T} \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \nu \int_0^T \|\nabla^3 \mathbf{B}\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{12}} + \|\nabla \mathbf{B}_0\|_{L^2}^2. \quad (2.30)$$

Multiplying (2.29) by σ^4 , then, integrating it over $(0, T)$ and using (2.10) and (2.21), we have

$$\begin{aligned}
 &\sup_{0 \leq t \leq T} \sigma^4 \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \nu \int_0^T \sigma^4 \|\nabla^3 \mathbf{B}\|_{L^2}^2 dt \\
 &\leq 4\sigma^3 \cdot \sigma' \int_0^T \|\nabla^2 \mathbf{B}\|_{L^2}^2 dt + C(M) \int_0^T \|\nabla \mathbf{B}\|_{L^2}^2 dt \\
 &\leq C(M) E_0^{\frac{1}{12}}.
 \end{aligned} \quad (2.31)$$

Thus, by (2.26), (2.27), (2.30) and (2.31), we complete the proof of this lemma. $\square$

For $\|\nabla \rho\|_{L^2}$, we want to establish two different estimates in the following lemma.

Lemma 2.7. *Under the assumptions of Proposition 2.1, it holds that*

$$\int_0^T \|\nabla \rho\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{12}} \left(\int_0^T \|\nabla^2 \rho\|_{L^2}^2 dt + 1 \right), \quad (2.32)$$

$$\int_0^T \|\nabla \rho\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{6}} + C(M) \int_0^T \|\nabla^2 \mathbf{u}\|_{L^2}^2 dt. \quad (2.33)$$

Proof: Multiplying (2.1)₂ by $\frac{\nabla \rho}{\rho}$ and integrating the resulting equality over $\mathbb{R}^3$, one has

$$\begin{aligned} & \int \rho^{-1} |\nabla \rho|^2 dx \\ &= - \int \left(\nabla \rho \cdot \mathbf{u}_t - \nabla \rho \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u}] + \frac{1}{\rho} \nabla \rho \cdot (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}) \right. \\ & \quad \left. + \frac{1}{\rho} \nabla \rho \cdot ((\operatorname{curl} \mathbf{B}) \times \mathbf{B}) \right) dx - \int \left(\frac{\theta - 1}{\rho} |\nabla \rho|^2 + \nabla \theta \cdot \nabla \rho \right) dx \\ &= - \frac{d}{dt} \int \mathbf{u} \cdot \nabla \rho dx + \int \operatorname{div} \mathbf{u} \cdot \operatorname{div}(\rho \mathbf{u}) dx \\ & \quad - \int \nabla \rho \cdot \mathbf{u} \cdot \nabla \mathbf{u} dx + \int \frac{1}{\rho} \nabla \rho \cdot (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}) dx \\ & \quad + \int \frac{1}{\rho} \nabla \rho \cdot ((\operatorname{curl} \mathbf{B}) \times \mathbf{B}) dx - \int \left(\frac{\theta - 1}{\rho} |\nabla \rho|^2 + \nabla \theta \cdot \nabla \rho \right) dx \\ &= - \frac{d}{dt} \int \mathbf{u} \cdot \nabla \rho dx + \sum_{i=1}^5 J_i. \end{aligned} \quad (2.34)$$

By (2.4) and (2.8), with the help of Hölder, Young and Sobolev inequalities, we obtain

$$J_1 + J_4 \leq C(M) \|\nabla \mathbf{u}\|_{L^2}^2 + C \|\nabla \rho\|_{L^3} \|\nabla \mathbf{B}\|_{L^2} \|\mathbf{B}\|_{L^6} \leq C(M) (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2).$$

Similarly, we give the estimates for J_5 as follows,

$$\begin{aligned} J_5 &\leq \|\theta - 1\|_{L^6} \|\nabla \rho\|_{L^2} \|\nabla \rho\|_{L^3} + C \|\nabla \theta\|_{L^2} \|\nabla \rho\|_{L^2} \\ &\leq \delta_1 \|\nabla \rho\|_{L^2}^2 + C(M) \|\nabla \theta\|_{L^2}^2. \end{aligned}$$

Particularly, we established the different estimates for $J_2 + J_3$,

$$\begin{aligned} J_2 + J_3 &\leq C(\|\rho^{-1} \nabla^2 \rho\|_{L^2} + \|\nabla \rho\|_{L^2}^2 \rho^{-2}) \|\nabla \mathbf{u}\|_{L^2} + C \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{u}\|_{L^3} \|\nabla \rho\|_{L^6} \\ &\leq C(\|\nabla^2 \rho\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} + \|\nabla \rho\|_{L^4} \|\nabla \rho\|_{L^4} \|\nabla \mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{u}\|_{L^2}^{\frac{3}{2}} \|\nabla^2 \rho\|_{L^2}) \\ &\leq C(M) (E_0^{\frac{1}{12}} \|\nabla^2 \rho\|_{L^2}^2 + E_0^{-\frac{1}{12}} \|\nabla \mathbf{u}\|_{L^2}^2 + E_0^{\frac{1}{12}} \|\nabla \rho\|_{L^2}^2) \end{aligned}$$

or

$$\begin{aligned} J_2 + J_3 &\leq C \|\rho^{-1} \nabla \rho\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^2} + C \|\nabla \mathbf{u}\|_{L^6} \|\mathbf{u}\|_{L^3} \|\nabla \rho\|_{L^2} \\ &\leq \delta_1 \|\nabla \rho\|_{L^2}^2 + C(M) (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2). \end{aligned}$$

Substituting the estimates of J_i ($i = 1, \dots, 5$) into (2.34), it is clear that

$$\|\nabla \rho\|_{L^2}^2 \leq -C \frac{d}{dt} \int \mathbf{u} \cdot \nabla \rho dx + C(M) E_0^{-\frac{1}{12}} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + E_0^{\frac{1}{12}} \|\nabla^2 \rho\|_{L^2}^2 \quad (2.35)$$

or

$$\|\nabla \rho\|_{L^2}^2 \leq -C \frac{d}{dt} \int \mathbf{u} \cdot \nabla \rho dx + C(M) (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + C(M) \|\nabla^2 \mathbf{u}\|_{L^2}^2. \quad (2.36)$$

Integrating (2.35) and (2.36) from 0 to T , and then applying (2.8) and (2.10), we obtain (2.32) and (2.33), respectively. $\square$

The following lemma is established to estimate $A_2(T)$.

Lemma 2.8. *Under the assumptions of Proposition 2.1, it holds that*

$$\sup_{0 \leq t \leq T} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) + C_1 \int_0^T \|\nabla^2 \mathbf{u}\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{6}} + (\|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\nabla \rho_0\|_{L^2}^2), \quad (2.37)$$

$$\sup_{0 \leq t \leq T} \sigma^2 (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) + \int_0^T \sigma^2 \|\nabla^2 \mathbf{u}\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{2}} \left(\int_0^T \|\nabla^2 \rho\|_{L^2}^2 dt + 1 \right), \quad (2.38)$$

where C_1 is a positive constant depending on μ and λ .

Proof: Using ∇ to (2.1)₁ and (2.1)₂, multiplying the resulting equations by $\nabla \rho$ and $\nabla \mathbf{u}$, respectively, then summing up and integrating it over $\mathbb{R}^3$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) + \int \frac{1}{\rho} \nabla^2 \mathbf{u} \cdot (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}) dx \\ &= \int \nabla \rho \cdot \nabla \operatorname{div}((\rho - 1) \mathbf{u}) dx + \int \nabla \mathbf{u} \cdot \nabla \left(\frac{1}{\rho} \nabla(\rho \theta) - \nabla \rho \right) dx \\ & \quad + \int \left(\nabla^2 \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \frac{1}{\rho} \nabla^2 \mathbf{u} \cdot ((\operatorname{curl} \mathbf{B}) \times \mathbf{B}) \right) dx \\ &= \sum_{i=1}^3 K_i. \end{aligned} \quad (2.39)$$

By (2.4) and (2.8), using Hölder, Young and Sobolev inequalities, let us show the estimates of K_i ($i = 1, 2, 3$), respectively.

$$\begin{aligned} K_1 &= \int \left(\frac{1}{2} |\nabla \rho|^2 \nabla \mathbf{u} + (\rho - 1) \nabla \rho \cdot \operatorname{div} \mathbf{u} \right) dx \\ &\leq C \|\nabla \rho\|_{L^2} \|\nabla \rho\|_{L^3} \|\nabla \mathbf{u}\|_{L^6} + \|\nabla \rho\|_{L^2} \|\rho - 1\|_{L^3} \|\nabla \mathbf{u}\|_{L^6} \\ &\leq C (\|\rho - 1\|_{L^2}^{\frac{1}{2}} \|\nabla \rho\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \rho\|_{L^2}^{\frac{3}{2}} + \|\rho - 1\|_{L^2} \|\nabla \rho\|_{L^2}^3) + \delta_1 \|\nabla^2 \mathbf{u}\|_{L^2}^2 \\ &\leq \delta_1 \|\nabla^2 \mathbf{u}\|_{L^2}^2 + C(M) E_0^{\frac{1}{4}} \|\nabla \rho\|_{L^2}^2, \end{aligned} \quad (2.40)$$

similarly,

$$\begin{aligned} K_2 &= \int \nabla^2 \mathbf{u} \cdot \left(\frac{1}{\rho} \nabla(\rho \theta) - \nabla \rho \right) dx \\ &\leq C \|\nabla^2 \mathbf{u}\|_{L^2} (\|\theta - 1\|_{L^6} \|\nabla \rho\|_{L^3} + \|\rho - 1\|_{L^4} \|\nabla \rho\|_{L^4} + \|\nabla \theta\|_{L^2}) \\ &\leq \delta_1 \|\nabla^2 \mathbf{u}\|_{L^2}^2 + C(M) \|\nabla \theta\|_{L^2}^2 + C(M) E_0^{\frac{1}{4}} \|\nabla \rho\|_{L^2}^2 \end{aligned} \quad (2.41)$$

and

$$\begin{aligned} K_3 &\leq C \|\nabla^2 \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^4} \|\mathbf{u}\|_{L^4} + C \|\nabla^2 \mathbf{u}\|_{L^2} \|\nabla \mathbf{B}\|_{L^4} \|\mathbf{B}\|_{L^4} \\ &\leq C \|\nabla^2 \mathbf{u}\|_{L^2}^{\frac{7}{4}} \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{u}\|_{L^2}^{\frac{1}{4}} + C \|\nabla^2 \mathbf{u}\|_{L^2} \|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{3}{4}} \|\nabla \mathbf{B}\|_{L^2} \|\mathbf{B}\|_{L^2}^{\frac{1}{4}} \\ &\leq \delta_1 \|\nabla^2 \mathbf{u}\|_{L^2}^2 + C (\|\nabla \mathbf{u}\|_{L^2}^8 \|\mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 \|\mathbf{B}\|_{L^2}^{\frac{1}{2}}) \\ &\leq \delta_1 \|\nabla^2 \mathbf{u}\|_{L^2}^2 + C(M) E_0^{\frac{1}{4}} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2). \end{aligned} \quad (2.42)$$

Substituting (2.40)-(2.42) into (2.39), one has

$$\begin{aligned} & \frac{d}{dt} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) + 2C_1 \|\nabla^2 \mathbf{u}\|_{L^2}^2 \\ & \leq C(M) E_0^{\frac{1}{4}} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) + C(M) \|\nabla \theta\|_{L^2}^2. \end{aligned} \quad (2.43)$$

Integrating (2.43) from 0 to T , applying (2.10) and (2.33), we arrive at (2.37).

Next, multiplying (2.43) by σ^2 and integrating the resulting inequality from 0 to T , applying (2.10) and (2.32), we can obtain (2.38) as follows,

$$\sigma^2 (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) + \int_0^T \sigma^2 \|\nabla^2 \mathbf{u}\|_{L^2}^2 dt$$

$$\begin{aligned}
&\leq C \int_0^T \sigma \sigma' (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) dt + C(M) \int_0^T \sigma^2 \|\nabla \theta\|_{L^2}^2 dt \\
&\quad C(M) E_0^{\frac{1}{4}} \int_0^T \sigma^2 (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) dt \\
&\leq C(M) E_0^{\frac{1}{12}} \left(\int_0^T \|\nabla^2 \rho\|_{L^2}^2 dt + 1 \right).
\end{aligned}$$

This completes the proof of this lemma. $\square$

Lemma 2.9. *Under the assumptions of Proposition 2.1, it holds that*

$$\sup_{0 \leq t \leq T} \|\nabla \theta\|_{L^2}^2 + C_2 \int_0^T (\|\theta_t\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2) dt \leq C(M) E_0^{\frac{1}{12}} + \|\nabla \theta_0\|_{L^2}^2, \quad (2.44)$$

$$\sup_{0 \leq t \leq T} \sigma^2 \|\nabla \theta\|_{L^2}^2 + \int_0^T \sigma^2 \int (|\theta_t|^2 + |\nabla^2 \theta|^2) dx dt \leq C(M) E_0^{\frac{1}{12}}, \quad (2.45)$$

where C_2 is a positive constant depending on κ .

Proof: It follows from (2.1)₃, (2.4), Hölder and Sobolev inequalities that

$$\begin{aligned}
&\kappa \frac{d}{dt} \int |\nabla \theta|^2 dx + \int \left(\rho |\theta_t|^2 + \frac{\kappa^2}{\rho} |\Delta \theta|^2 \right) dx \\
&= \int (\rho \theta_t - \kappa \Delta \theta) \left(\theta_t - \frac{\kappa}{\rho} \Delta \theta \right) dx \\
&\leq C \int (|\mathbf{u}|^2 |\nabla \theta|^2 + |\theta|^2 |\operatorname{div} \mathbf{u}|^2 + |\nabla \mathbf{u}|^4 + |\nabla \mathbf{B}|^4) dx \\
&\leq (\|\mathbf{u}\|_{L^\infty}^2 \|\nabla \theta\|_{L^2}^2 + \|\theta - 1\|_{L^6}^2 \|\operatorname{div} \mathbf{u}\|_{L^6} \|\operatorname{div} \mathbf{u}\|_{L^2} + 2 \|\theta - 1\|_{L^6} \|\operatorname{div} \mathbf{u}\|_{L^3} \|\operatorname{div} \mathbf{u}\|_{L^2} \\
&\quad + \|\operatorname{div} \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{B}\|_{L^6} \|\nabla \mathbf{B}\|_{L^6} \|\nabla \mathbf{B}\|_{L^6} \|\nabla \mathbf{B}\|_{L^2}) \\
&\leq C(M) (\|\nabla \theta\|_{L^2}^2 + \|\nabla \theta\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} + \|\nabla^2 \mathbf{B}\|_{L^2} \|\nabla \mathbf{B}\|_{L^2}).
\end{aligned}$$

By above inequality and Young inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt} \int |\nabla \theta|^2 dx + C_2 \int (|\theta_t|^2 + |\nabla^2 \theta|^2) dx \\
&\leq C(M) E_0^{-\frac{1}{12}} (\|\nabla \theta\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2) + E_0^{\frac{1}{12}} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2)
\end{aligned} \quad (2.46)$$

and

$$\begin{aligned}
&\frac{d}{dt} \int |\nabla \theta|^2 dx + C_2 \int (|\theta_t|^2 + |\nabla^2 \theta|^2) dx \\
&\leq C(M) E_0^{-\frac{1}{24}} (\|\nabla \theta\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2) + E_0^{\frac{1}{24}} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2).
\end{aligned} \quad (2.47)$$

Integrating (2.46) from 0 to T , applying (2.10) and (2.37), (2.44) holds.

Multiplying (2.47) by σ^2 and integrating the resulting inequality from 0 to T , applying (2.10) and (2.37), we obtain (2.45) as follows,

$$\begin{aligned}
&\sigma^2 \int |\nabla \theta|^2 dx + \int_0^T \sigma^2 \int (|\theta_t|^2 + |\nabla^2 \theta|^2) dx \\
&\leq C \int_0^T \sigma \sigma' \int |\nabla \theta|^2 dx dt + C(M) E_0^{-\frac{1}{24}} \int_0^T (\|\nabla \theta\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2) dt
\end{aligned}$$

$$\begin{aligned}
 & + E_0^{\frac{1}{24}} \int_0^T (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2) dt \\
 & \leq C(M) E_0^{\frac{1}{12}}.
 \end{aligned}$$

Thus, we complete the proof of this lemma. $\square$

By the bound of $\|\nabla \theta\|_{L^2}^2 + \int_0^T \|\nabla^2 \theta\|_{L^2}^2 dt$, we give the following lemma to estimate $A_3(T)$.

Lemma 2.10. *Under the assumptions of Proposition 2.1, it holds that*

$$\int_0^T \|\nabla^2 \rho\|_{L^2}^2 dt \leq C(M), \quad (2.48)$$

$$\int_0^T \|\nabla \rho\|_{L^2}^2 dt \leq C(M) E_0^{\frac{1}{12}}, \quad (2.49)$$

$$\sup_{0 \leq t \leq T} \sigma^2 (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) + \int_0^T \sigma^2 \|\nabla^2 \mathbf{u}\|_{L^2}^2 d\tau \leq C(M) E_0^{\frac{1}{12}}, \quad (2.50)$$

$$\begin{aligned}
 & \sup_{0 \leq t \leq T} (C_3 \|\nabla \theta\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2) + C_4 \int_0^T (\|\nabla^3 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2) dt \\
 & \leq C(M) E_0^{\frac{1}{30}} + C_3 \|\nabla \theta_0\|_{L^2}^2 + \|\nabla^2 \mathbf{u}_0\|_{L^2}^2 + \|\nabla^2 \rho_0\|_{L^2}^2,
 \end{aligned} \quad (2.51)$$

where C_3 and C_4 are positive constants depending on some known constants μ, λ, k, v .

Proof: Multiplying (2.1)₂ by $\frac{\nabla^3 \rho}{\rho}$ and integrating the resulting equality over $\mathbb{R}^3$, then using integration by parts, one has

$$\begin{aligned}
 & \int |\nabla^2 \rho|^2 dx \\
 & = - \int \nabla^2 \rho \cdot \nabla \mathbf{u}_t - \int \nabla^2 \rho \cdot \left(\nabla \left(\frac{1}{\rho} \nabla(\rho \theta) \right) - \nabla^2 \rho \right) dx - \nabla^2 \rho \cdot \nabla (\mathbf{u} \cdot \nabla \mathbf{u}) \\
 & \quad + \nabla^2 \rho \cdot \nabla \left(\frac{1}{\rho} (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}) \right) + \nabla^2 \rho \cdot \nabla ((\operatorname{curl} \mathbf{B}) \times \mathbf{B}) dx \\
 & = - \frac{d}{dt} \int \nabla \mathbf{u} \cdot \nabla^2 \rho dx + \int \operatorname{div} \nabla \mathbf{u} \cdot \nabla \operatorname{div}(\rho \mathbf{u}) dx \\
 & \quad + \int \nabla^2 \rho \cdot \nabla (\mathbf{u} \cdot \nabla \mathbf{u}) dx + \int \nabla^2 \rho \cdot \nabla \left(\frac{1}{\rho} (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}) \right) dx \\
 & \quad + \int \nabla^2 \rho \cdot \nabla \left(\frac{1}{\rho} (\operatorname{curl} \mathbf{B}) \times \mathbf{B} \right) dx - \int \left(\nabla^2 \rho \cdot \nabla \left(\frac{1}{\rho} \nabla(\rho \theta) \right) - |\nabla^2 \rho|^2 \right) dx \\
 & = - \frac{d}{dt} \int \nabla \mathbf{u} \cdot \nabla^2 \rho dx + \sum_{i=1}^5 J_i.
 \end{aligned} \quad (2.52)$$

By (2.4) and using Hölder, Sobolev and Young inequalities, we can bound $\sum_{1 \leq i \leq 4} J_i$ as

$$\sum_{1 \leq i \leq 4} J_i \leq \frac{1}{4} \|\nabla^2 \rho\|_{L^2}^2 + C(M) (\|\nabla^3 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2).$$

For J_5 , it follows from (2.4), (2.44), Hölder, Sobolev and Young inequalities that

$$\begin{aligned} J_5 &\leq C \|\nabla^2 \rho\|_{L^2} (\|\nabla^2 \theta\|_{L^2} + \|\theta - 1\|_{L^\infty} \|\nabla^2 \rho\|_{L^2} \\ &\quad + \|1 - \rho\|_{L^\infty} \|\nabla^2 \rho\|_{L^2} + \|\nabla \rho\|_{L^3} \|\nabla \theta\|_{L^6} + \|\nabla \rho\|_{L^3} \|\nabla \rho\|_{L^6}) \\ &\leq \frac{1}{4} \|\nabla^2 \rho\|_{L^2}^2 + C(M) (\|\nabla^2 \theta\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2). \end{aligned}$$

Substituting the estimates of J_i ($i = 1, \dots, 5$) into (2.52), we get

$$\begin{aligned} \|\nabla^2 \rho\|_{L^2}^2 &\leq -C \frac{d}{dt} \int \nabla \mathbf{u} \cdot \nabla^2 \rho dx \\ &\quad + C(M) (\|\nabla \rho\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^3 \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2). \end{aligned} \quad (2.53)$$

Applying ∇^2 to (2.1)₁ and (2.1)₂, multiplying the resulting equations by $\nabla^2 \rho$ and $\nabla^2 \mathbf{u}$, then summing up and integrating it over $\mathbb{R}^3$, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2) + \int \frac{1}{\rho} \nabla^3 \mathbf{u} \cdot \nabla (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}) dx \\ &= \int \nabla^2 \rho \cdot \nabla^2 \operatorname{div} ((\rho - 1) \mathbf{u}) dx - \int \nabla^2 \mathbf{u} \cdot \nabla^2 \left(\frac{1}{\rho} \nabla (\rho \theta) - \nabla \rho \right) dx \\ &\quad + \int \left(\nabla^3 \mathbf{u} \cdot \nabla (\mathbf{u} \cdot \nabla \mathbf{u}) - \nabla^3 \mathbf{u} \cdot \nabla \left(\frac{1}{\rho} (\operatorname{curl} \mathbf{B} \times \mathbf{B}) \right) \right) dx \\ &\quad + \int \frac{\nabla \rho}{\rho^2} \cdot \nabla^3 \mathbf{u} \cdot (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}) dx \\ &= \sum_{i=1}^4 F_i. \end{aligned} \quad (2.54)$$

Now, let us bound F_i ($i = 1, 2, 3, 4$) by (2.4), (2.8), (2.10) and Hölder, Sobolev and Young inequalities.

$$\begin{aligned} F_1 &= \int \nabla^2 \rho \cdot \nabla^3 \rho \cdot \mathbf{u} + |\nabla^2 \rho|^2 \nabla \mathbf{u} + \nabla^2 \rho \cdot \nabla^2 ((\rho - 1) \operatorname{div} \mathbf{u}) dx \\ &\leq C (\|\nabla^2 \rho\|_{L^2} \|\nabla^2 \rho\|_{L^2} \|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla^2 \rho\|_{L^2} \|\nabla \rho\|_{L^3} \|\nabla^2 \mathbf{u}\|_{L^6} \\ &\quad + \|\nabla^2 \rho\|_{L^2} \|\rho - 1\|_{L^\infty} \|\nabla^3 \mathbf{u}\|_{L^2}) \\ &\leq C (\|\nabla^2 \rho\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{u}\|_{L^2}^{\frac{3}{4}} \|\nabla^3 \mathbf{u}\|_{L^2}^{\frac{3}{4}} + \|\nabla \rho\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \rho\|_{L^2}^{\frac{3}{2}} \|\nabla^3 \mathbf{u}\|_{L^2} \\ &\quad + \|\nabla^2 \rho\|_{L^2} \|\rho - 1\|_{L^\infty}^{\frac{1}{4}} \|\nabla^2 \rho\|_{L^2}^{\frac{3}{4}} \|\nabla^3 \mathbf{u}\|_{L^2}) \\ &\leq \delta_1 \|\nabla^3 \mathbf{u}\|_{L^2}^2 + C (\|\nabla^2 \rho\|_{L^2}^{\frac{16}{5}} \|\nabla \mathbf{u}\|_{L^2}^{\frac{2}{5}} + \|\nabla \rho\|_{L^2} \|\nabla^2 \rho\|_{L^2}^3 + \|\rho - 1\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \rho\|_{L^2}^{\frac{7}{2}}) \\ &\leq \delta_1 \|\nabla^3 \mathbf{u}\|_{L^2}^2 + E_0^{\frac{1}{30}} \|\nabla^2 \rho\|_{L^2}^2 + C(M) (E_0^{-\frac{1}{24}} \|\nabla \rho\|_{L^2}^2 + E_0^{-\frac{2}{15}} \|\nabla \mathbf{u}\|_{L^2}^2). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} F_2 &\leq \|\nabla^3 \mathbf{u}\|_{L^2}^2 \|\nabla^2 \theta\|_{L^2}^2 + \|\rho^{-1}\|_{L^\infty} \|\nabla^3 \mathbf{u}\|_{L^2}^2 (\|\theta - 1\|_{L^\infty} \|\nabla \rho\|_{L^2} \\ &\quad + \|\rho - 1\|_{L^\infty} \|\nabla^2 \rho\|_{L^2} + \|\nabla^2 \rho\|_{L^2} \|\nabla \theta\|_{L^3} + \|\theta - 1\|_{L^2} \|\nabla \rho\|_{L^6}^2) \\ &\leq 2\delta_1 \|\nabla^3 \mathbf{u}\|_{L^2}^2 + \frac{1}{4\delta_1} \|\nabla^2 \theta\|_{L^2}^2 + C (\|\theta - 1\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \theta\|_{L^2}^{\frac{3}{2}} \|\nabla \rho\|_{L^2}^2 \\ &\quad + \|\rho - 1\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \rho\|_{L^2}^{\frac{7}{2}} + \|\nabla^2 \rho\|_{L^2}^2 \|\nabla \theta\|_{L^2} \|\nabla^2 \theta\|_{L^2} + \|\nabla \theta\|_{L^2} \|\nabla^2 \rho\|_{L^2}^2) \\ &\leq 2\delta_1 \|\nabla^3 \mathbf{u}\|_{L^2}^2 + \frac{1}{2\delta_1} \|\nabla^2 \theta\|_{L^2}^2 + C(M) (E_0^{\frac{1}{24}} \|\nabla \rho\|_{L^2}^2 + E_0^{\frac{1}{12}} \|\nabla^2 \rho\|_{L^2}^2 + E_0^{-\frac{1}{12}} \|\nabla \theta\|_{L^2}^2) \end{aligned}$$

and

$$\begin{aligned} F_3 + F_4 &\leq C \|\nabla^3 \mathbf{u}\|_{L^2} (\|\nabla^2 \mathbf{u}\|_{L^4} \|\mathbf{u}\|_{L^4} + \|\nabla \mathbf{u}\|_{L^4} \|\nabla \mathbf{u}\|_{L^4} + \|\nabla \rho\|_{L^6} \|\nabla \mathbf{B}\|_{L^6} \|\mathbf{B}\|_{L^6} \\ &\quad + \|\nabla^2 \mathbf{B}\|_{L^4} \|\mathbf{B}\|_{L^4} + \|\nabla \mathbf{B}\|_{L^4} \|\nabla \mathbf{B}\|_{L^4} + \|\nabla \rho\|_{L^4} \|\nabla^2 \mathbf{u}\|_{L^4}) \\ &\leq \delta_1 \|\nabla^3 \mathbf{u}\|_{L^2}^2 + C(M) E_0^{-\frac{1}{12}} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2) \\ &\quad + C(M) \|\nabla \rho\|_{L^2}^2 + C(M) E_0^{\frac{1}{12}} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2). \end{aligned}$$

Substituting the estimates of F_i ($i = 1, \dots, 4$) into (2.54), yields

$$\begin{aligned} & \frac{d}{dt} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2) + C_5 \|\nabla^3 \mathbf{u}\|_{L^2}^2 \\ & \leq C(M) E_0^{-\frac{2}{15}} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + C(M) E_0^{-\frac{1}{24}} \|\nabla \rho\|_{L^2}^2 \\ & \quad + E_0^{\frac{1}{30}} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2) + \frac{1}{2\delta_1} \|\nabla^2 \theta\|_{L^2}^2. \end{aligned} \quad (2.55)$$

Multiplying (2.46) by $(C_2 \delta_1)^{-1}$, then adding up it and (2.55), substituting (2.35) and (2.53) into the resulting inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{C_2 \delta_1} \|\nabla \theta\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2 \right) + C_5 \|\nabla^3 \mathbf{u}\|_{L^2}^2 + \frac{1}{2\delta_1} \|\nabla^2 \theta\|_{L^2}^2 \\ & \leq -C E_0^{-\frac{1}{24}} \frac{d}{dt} \int \mathbf{u} \cdot \nabla \rho dx + C(M) E_0^{-\frac{2}{15}} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) \\ & \quad - C E_0^{\frac{1}{30}} \left(\frac{d}{dt} \int \nabla \mathbf{u} \cdot \nabla^2 \rho dx + \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 \right). \end{aligned}$$

Integrating above inequality over $(0, T)$, it follows from (2.4), (2.8), (2.10), (2.26) and (2.37) that

$$\begin{aligned} & \sup_{0 \leq t \leq T} (C_3 \|\nabla \theta\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2) + C_4 \int_0^T (\|\nabla^3 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2) dt \\ & \leq C(M) E_0^{\frac{1}{30}} + C_3 \|\nabla \theta_0\|_{L^2}^2 + \|\nabla^2 \mathbf{u}_0\|_{L^2}^2 + \|\nabla^2 \rho_0\|_{L^2}^2, \end{aligned}$$

where $C_3 = \frac{1}{C_2 \delta_1}$ and $C_4 = \min\{C_5, \frac{1}{2\delta_1}\}$. Thus, we complete the proof of (2.51).

Integrating (2.53) from 0 to T , by (2.10), (2.26), (2.37), (2.45) and (2.51), we obtain (2.48). Substituting (2.48) into (2.35) and (2.38), we get (2.49) and (2.50), respectively, which completes the proof of this lemma. $\square$

Next, the following lemma is needed to bound $A_4(T)$.

Lemma 2.11. *Under the assumptions of Proposition 2.1, it holds that*

$$\int_0^T \sigma^4 (\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2) dt \leq C(M) E_0^{\frac{1}{12}}, \quad (2.56)$$

$$\int_0^T (\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2) dt \leq C(M). \quad (2.57)$$

Proof: Applying ∇ to (2.1)₂, squaring both sides of resulting equation, by (2.4) and (2.8), using Hölder, Sobolev and Young inequalities, we have

$$\begin{aligned} & \int |\nabla \mathbf{u}_t - \mu \nabla \Delta \mathbf{u} - (\mu + \lambda) \nabla^2 \operatorname{div} \mathbf{u}|^2 dx \\ & \leq \int \left(\left| \nabla \left(\frac{1}{\rho} \right) \cdot \nabla^2 \mathbf{u} \right|^2 + \left| \left(\frac{\rho - 1}{\rho} \right) \cdot \nabla^3 \mathbf{u} \right|^2 + |\nabla^2(\rho \theta)|^2 + |\nabla(\rho \mathbf{u} \cdot \nabla \mathbf{u})|^2 \right. \\ & \quad \left. + |\nabla((\operatorname{curl} \mathbf{B}) \times \mathbf{B})|^2 \right) dx \\ & \leq C(M) (\|\nabla \rho\|_{L^6}^2 \|\nabla^2 \mathbf{u}\|_{L^6} \|\nabla^2 \mathbf{u}\|_{L^2} + \|\rho - 1\|_{L^\infty} \|\nabla^3 \mathbf{u}\|_{L^2}^2 + \|\theta - 1\|_{L^\infty}^2 \|\nabla^2 \rho\|_{L^2}^2 \\ & \quad + \|\theta - 1\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 \rho\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 \|\nabla \theta\|_{L^6}^2 \|\nabla \theta\|_{L^6}^2 \\ & \quad + \|\rho\|_{L^\infty}^2 \|\nabla^2 \theta\|_{L^2}^2 + \|\rho\|_{L^\infty}^2 \|\nabla \mathbf{u}\|_{L^4}^4 + \|\mathbf{u}\|_{L^\infty}^2 \|\nabla \rho\|_{L^2} \|\nabla \rho\|_{L^6} \|\nabla \mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^6} \\ & \quad + \|\rho\|_{L^\infty}^2 \|\mathbf{u}\|_{L^\infty}^2 \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\mathbf{B}\|_{L^\infty}^2 \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^4}^4) \\ & \leq \delta_1 \|\nabla^3 \mathbf{u}\|_{L^2}^2 + C(M) (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2). \end{aligned} \quad (2.58)$$

Similarly, it follows from (2.1)₄, (2.4), (2.8), Hölder, Sobolev and Young inequalities that

$$\begin{aligned}
 & \nu \frac{d}{dt} \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2 + \nu^2 \|\nabla^3 \mathbf{B}\|_{L^2}^2 \\
 & \leq C(M) \left(\|\nabla \operatorname{curl} \left(\frac{\operatorname{curl} \mathbf{B} \times \mathbf{B}}{\rho} \right)\|_{L^2}^2 + \|\nabla \operatorname{curl}(\mathbf{u} \times \mathbf{B})\|_{L^2}^2 \right) \\
 & \leq C(M) (\|\mathbf{B}\|_{L^\infty}^2 \|\nabla^3 \mathbf{B}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^3}^2 \|\nabla^2 \mathbf{B}\|_{L^6}^2 + \|\nabla^2 \mathbf{B}\|_{L^6}^2 \|\mathbf{B}\|_{L^6}^2 \|\nabla \rho\|_{L^6}^2 \\
 & \quad + \|\nabla \mathbf{B}\|_{L^6}^2 \|\nabla \mathbf{B}\|_{L^6}^2 \|\nabla \rho\|_{L^6}^2 + \|\nabla \mathbf{B}\|_{L^6}^2 \|\mathbf{B}\|_{L^6}^2 \|\nabla \rho\|_{L^6}^2 + \|\nabla \mathbf{B}\|_{L^6}^2 \|\mathbf{B}\|_{L^6}^2 \|\nabla^2 \rho\|_{L^6}^2 \\
 & \quad + \|\nabla^2 \mathbf{B}\|_{L^2}^2 \|\mathbf{u}\|_{L^\infty}^2 + \|\nabla \mathbf{u}\|_{L^3}^2 \|\nabla \mathbf{B}\|_{L^6}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 \|\mathbf{B}\|_{L^\infty}^2) \\
 & \leq C(M) (\|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla^3 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2).
 \end{aligned} \tag{2.59}$$

Summing up (2.58) and (2.59), we have

$$\begin{aligned}
 & C_6 \frac{d}{dt} (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2) + (\|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla^3 \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2 + \|\nabla^3 \mathbf{B}\|_{L^2}^2) \\
 & \leq C(M) (\|\nabla^2 \rho\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla^3 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2),
 \end{aligned} \tag{2.60}$$

where C_6 is a positive constant depending on some known constants μ, λ, ν . Integrating (2.60) over $(0, T)$, using (2.10), (2.22), and (2.37), we obtain

$$\int_0^T (\|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2) dt \leq C(M). \tag{2.61}$$

Multiplying (2.60) by σ^4 and integrating it over $(0, T)$, by (2.10), (2.23), and (2.49), one has

$$\int_0^T \sigma^4 (\|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2) dt \leq C(M) E_0^{\frac{1}{12}}. \tag{2.62}$$

Similarly to (2.61) and (2.62), we can get (2.63) and (2.64).

$$\int_0^T (\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2) dt \leq C(M), \tag{2.63}$$

$$\int_0^T \sigma^4 (\|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2) dt \leq C(M) E_0^{\frac{1}{12}}. \tag{2.64}$$

Therefore, we complete the proof of this lemma by (2.61)-(2.64). $\square$

For $A_4(T)$, we have

Lemma 2.12. *Under the assumptions of Proposition 2.1, it holds that*

$$\sup_{0 \leq t \leq T} \sigma^4 \int (|\theta_t|^2 + |\nabla^2 \theta|^2) dx \leq E_0^{\frac{1}{32}}. \tag{2.65}$$

Proof: This lemma can be proved in the similar way of Lemma 3.12 in [14]. Here, we omit the proof of them here for brevity. $\square$

Thus, we are ready to show the proof of Proposition 2.1.

Proof of Proposition 2.1: By Lemma 2.4 and Lemma 2.5, we obtain

$$A_1(T) \leq CE_0 + E_0^{\frac{1}{6}} < 2E_0^{\frac{1}{6}},$$

provided $E_0 < C^{-\frac{6}{5}}$. By Lemma 2.6 and Lemma 2.8, we obtain

$$A_2(T) \leq C(M)E_0^{\frac{1}{6}} + M < \frac{3}{2}M,$$

provided $E_0 < \left(\frac{M}{2C(M)}\right)^6$. By Lemma 2.6 and 2.10, we obtain

$$A_3(T) \leq C(M)E_0^{\frac{1}{30}} + M < \frac{3}{2}M,$$

provided $E_0 < \left(\frac{M}{2C(M)}\right)^{30}$. $A_4(T) \leq E_0^{\frac{1}{32}}$ was given by using Lemma 2.12 directly. Hence, we conclude the proof of Proposition 2.1. $\square$

In addition, we also need to establish the estimates for $\nabla \rho_t$ and $\nabla^2 \theta$, which are independent of the time.

Lemma 2.13. *Under the assumption of Proposition 2.1, it holds that*

$$\|\nabla \rho_t\|_{L^2}^2 + \int_0^T \|\nabla \rho_t\|_{L^2}^2 dt \leq C(M), \quad (2.66)$$

$$\|\nabla^2 \theta\|_{L^2}^2 + \int_0^T (\|\nabla^3 \theta\|_{L^2}^2 + \|\nabla \theta_t\|_{L^2}^2) dt \leq C(M) + \|\nabla^2 \theta_0\|_{L^2}^2. \quad (2.67)$$

Proof: By (2.1)₁ and using (2.4), Hölder, Young and Sobolev inequalities, one can get

$$\begin{aligned} \|\nabla \rho_t\|_{L^2}^2 &\leq C \int (|\nabla^2 \rho|^2 |\mathbf{u}|^2 + |\nabla \rho|^2 |\nabla \mathbf{u}|^2 + |\rho|^2 |\nabla^2 \mathbf{u}|^2) dx \\ &\leq C(\|\mathbf{u}\|_{L^\infty}^2 \|\nabla^2 \rho\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^3} \|\nabla \mathbf{u}\|_{L^6} \|\nabla \rho\|_{L^6}^2 + \|\rho\|_{L^\infty}^2 \|\nabla^2 \mathbf{u}\|_{L^2}^2) \\ &\leq C(M) \|\nabla^2 \rho\|_{L^2}^2, \end{aligned}$$

which together with (2.48) yields (2.66). Similarly, it follows from (2.1)₃ that

$$\begin{aligned} &\kappa \frac{d}{dt} \int |\nabla^2 \theta|^2 dx + \int (|\nabla \theta_t|^2 + \kappa^2 |\nabla \Delta \theta|^2) dx \\ &\leq C \int \left(\left| \frac{\rho-1}{\rho} \right|^2 |\nabla^3 \theta|^2 + \left| \nabla \left(\frac{1}{\rho} \right) \right|^2 |\nabla^2 \theta|^2 + |\nabla(\mathbf{u} \cdot \nabla \theta)|^2 + |\nabla(\theta \cdot \operatorname{div} \mathbf{u})|^2 \right) dx \\ &\quad + C \int (|\nabla \mathbf{u}|^2 |\nabla^2 \mathbf{u}|^2 + |\nabla \mathbf{B}|^2 |\nabla^2 \mathbf{B}|^2) dx \\ &\leq (E_0^{\frac{1}{6}} + \delta_1) \|\nabla^3 \theta\|_{L^2}^2 + C(M) (\|\nabla^2 \theta\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{B}\|_{H^2}^2). \end{aligned}$$

Integrating above inequalities over $(0, T)$, and by virtue of Lemma 2.5, Lemma 2.6, Lemma 2.8 and Lemma 2.9, we complete the proof of this lemma. $\square$

2.3 Time-dependent higher-order estimates

In this subsection, we establish the higher-order estimates for the solution $(\rho_0, \mathbf{u}, \theta, \mathbf{B})$. Throughout this subsection, we always assume $E_0 \leq \delta \leq 1$, and we denote positive constant C_T depending on $\mu, \lambda, \kappa, \nu, M, \epsilon$ and T .

Lemma 2.14. *Under the conditions of Theorem 1.1, it holds that*

$$\begin{aligned} &\|\nabla^3 \rho\|_{L^2}^2 + \|\nabla^3 \mathbf{u}\|_{L^2}^2 + \|\nabla^3 \theta\|_{L^2}^2 + \|\nabla^3 \mathbf{B}\|_{L^2}^2 \\ &+ \int_0^T (\|\nabla^3 \rho\|_{L^2}^2 + \|\nabla^4 \mathbf{u}\|_{L^2}^2 + \|\nabla^4 \theta\|_{L^2}^2 + \|\nabla^4 \mathbf{B}\|_{L^2}^2) dt \leq C_T. \end{aligned} \quad (2.68)$$

Proof: Applying ∇^3 to (2.1)₁, multiplying it by $\nabla^3 \rho$, and integrating the resulting equation over $\mathbb{R}^3$, after integration by parts, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla^3 \rho\|_{L^2}^2 &= - \int \nabla^3 \operatorname{div}(\rho \mathbf{u}) \cdot \nabla^3 \rho dx \\ &\leq C(M) (\|\nabla^3 \rho\|_{L^2}^2 (\|\nabla^3 \mathbf{u}\|_{L^2}^2 + 1) + \|\nabla \mathbf{u}\|_{H^2}^2) + \delta_1 \|\nabla^4 \mathbf{u}\|_{L^2}^2. \end{aligned} \quad (2.69)$$

Applying ∇^2 to (2.1)₂, squaring both sides of resulting equation, then integrating it over $\mathbb{R}^3$, we have

$$\begin{aligned} &\int |\nabla^2 \mathbf{u}_t - \mu \nabla^2 \Delta \mathbf{u} - (\mu + \lambda) \nabla^3 \operatorname{div} \mathbf{u}|^2 dx \\ &\leq \int \left(|\nabla^2 \left(\frac{1}{\rho} \right) \cdot \nabla^2 \mathbf{u}|^2 + |\nabla \left(\frac{1}{\rho} \right) \cdot \nabla^3 \mathbf{u}|^2 + \left| \left(\frac{\rho - 1}{\rho} \right) \nabla^4 \mathbf{u} \right|^2 \right) dx \\ &\quad + \int \left(|\nabla^2 \left(\frac{1}{\rho} \right) \nabla(\rho \theta)|^2 + |\nabla^2 (\mathbf{u} \cdot \nabla \mathbf{u})|^2 \right) dx + \int |\nabla^2 \left(\frac{1}{\rho} (\operatorname{curl} \mathbf{B}) \times \mathbf{B} \right)|^2 dx \\ &= G_1 + G_2 + G_3. \end{aligned} \quad (2.70)$$

It follows from (2.4), (2.8), Hölder, Sobolev and Young inequalities that

$$\begin{aligned} &G_1 + G_2 \\ &\leq C(M) (\|\nabla^2 \rho\|_{L^6} \|\nabla^2 \rho\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^6}^2 + \|\nabla \rho\|_{L^6}^4 \|\nabla^2 \mathbf{u}\|_{L^6}^2 \\ &\quad + \|\nabla \rho\|_{L^6} \|\nabla \rho\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^6}^2 + \|\rho - 1\|_{L^\infty} \|\nabla^4 \mathbf{u}\|_{L^2}^2 + \|\nabla \theta\|_{H^2}^2 + \|\nabla \mathbf{u}\|_{H^2}^2 + \|\nabla^3 \rho\|_{L^2}^2) \\ &\leq C(M) (\|\nabla \theta\|_{H^2}^2 + \|\nabla \mathbf{u}\|_{H^2}^2) + (\delta_1 + E_0^{\frac{1}{6}}) \|\nabla^4 \mathbf{u}\|_{L^2}^2 + C(M) \|\nabla^3 \rho\|_{L^2}^2. \end{aligned} \quad (2.71)$$

For G_3 , we have

$$\begin{aligned} G_3 &\leq C(M) (\|\nabla^2 \rho\|_{L^6}^2 \|\nabla \mathbf{B}\|_{L^6}^2 \|\mathbf{B}\|_{L^6}^2 + \|\nabla \rho\|_{L^6}^4) \|\nabla \mathbf{B}\|_{L^6}^2 \|\mathbf{B}\|_{L^\infty}^2 \\ &\quad + \|\nabla \rho\|_{L^6}^2 (\|\nabla^2 \mathbf{B}\|_{L^6}^2 \|\mathbf{B}\|_{L^6}^2 + \|\nabla \mathbf{B}\|_{L^6}^4) \\ &\quad + \|\nabla^3 \mathbf{B}\|_{L^2}^2 \|\mathbf{B}\|_{L^\infty}^2 + \|\nabla^2 \mathbf{B}\|_{L^6}^2 \|\nabla \mathbf{B}\|_{L^6} \|\nabla \mathbf{B}\|_{L^2}) \\ &\leq C(M) \|\nabla \mathbf{B}\|_{H^2}^2 + C(M) \|\nabla^3 \rho\|_{L^2}^2. \end{aligned} \quad (2.72)$$

Substituting (2.71)-(2.72) into (2.70), together with (2.69), it yields

$$\begin{aligned} &\frac{d}{dt} (\|\nabla^3 \rho\|_{L^2}^2 + \|\nabla^3 \mathbf{u}\|_{L^2}^2) + C \|\nabla^4 \mathbf{u}\|_{L^2}^2 \\ &\leq C(M) (\|\nabla \theta\|_{H^2}^2 + \|\nabla \mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{B}\|_{H^2}^2) + C(M) \|\nabla^3 \rho\|_{L^2}^2 (\|\nabla^3 \mathbf{u}\|_{L^2}^2 + 1). \end{aligned} \quad (2.73)$$

Similarly, for $\|\nabla^3 \mathbf{B}\|_{L^2}^2$, it follows from (2.1)₃, (2.1)₄, (2.4), (2.8), Hölder, Sobolev and Young inequalities that

$$\begin{aligned} &\nu \frac{d}{dt} \|\nabla^3 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}_t\|_{L^2}^2 + \nu^2 \|\nabla^4 \mathbf{B}\|_{L^2}^2 \\ &\leq C(M) \left(\|\nabla^2 \operatorname{curl} \left(\frac{\operatorname{curl} \mathbf{B} \times \mathbf{B}}{\rho} \right)\|_{L^2}^2 + \|\nabla^2 \operatorname{curl}(\mathbf{u} \times \mathbf{B})\|_{L^2}^2 \right) \\ &\leq \frac{\nu^2}{2} \|\nabla^4 \mathbf{B}\|_{L^2}^2 + C(M) (\|\nabla^3 \rho\|_{L^2}^2 (\|\nabla^3 \mathbf{B}\|_{L^2}^2 + 1) + \|\nabla \mathbf{B}\|_{H^2}^2 + \|\nabla \mathbf{u}\|_{H^2}^2). \end{aligned} \quad (2.74)$$

Adding (2.73) and (2.74) up, and then using Gronwall inequality, we obtain

$$\|\nabla^3 \rho\|_{L^2}^2 + \|\nabla^3 \mathbf{u}\|_{L^2}^2 + \|\nabla^3 \mathbf{B}\|_{L^2}^2 + \int_0^T (\|\nabla^4 \mathbf{u}\|_{L^2}^2 + \|\nabla^4 \mathbf{B}\|_{L^2}^2) dt \leq C_T. \quad (2.75)$$

Finally, similar to the proof of (2.75), we can bound

$$\|\nabla^3 \theta\| + \int_0^T \|\nabla^4 \theta\|_{L^2}^2 dt \leq C_T, \quad (2.76)$$

we omit the proof of it for brevity. Thus, we complete the proof of this lemma by (2.75) and (2.76). $\square$

3 Proof of Theorem 1.1

Applying Lemma 2.4–2.6, 2.8, 2.9 and 2.11–2.13, we obtain the following estimate,

$$\begin{aligned} & \|\rho - 1\|_{H^3}^2 + \|\mathbf{u}\|_{H^3}^2 + \|\theta - 1\|_{H^3}^2 + \|\mathbf{B}\|_{H^3}^2 \\ & + \int_0^T (\|\nabla \rho\|_{H^2}^2 + \|\nabla \mathbf{u}\|_{H^3}^2 + \|\nabla \theta\|_{H^3}^2 + \|\nabla \mathbf{B}\|_{H^3}^2) dt \leq C_T. \end{aligned} \quad (3.1)$$

With the help of the existence and uniqueness of local solutions which has been proved in [4] and all the a priori estimates above, using the standard continuum arguments, we extend the local solution to the global one.

Next, we investigate the large time behavior of solution. It follows from (2.10) and (2.33) that

$$\int_0^\infty (\|\nabla \rho\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2) dt \leq C(M). \quad (3.2)$$

Then, by Young inequality, (3.2), (2.45), (2.58) and (2.66), we obtain

$$\begin{aligned} & \int_0^\infty \left| \frac{d}{dt} (\|\nabla \rho_t\|_{L^2}^2 + \|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2 + \|\nabla \theta_t\|_{L^2}^2) \right| dt \\ & \leq \int_0^\infty (\|\nabla \rho\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2) dt \\ & + \int_0^\infty (\|\nabla \rho_t\|_{L^2}^2 + \|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2 + \|\nabla \theta_t\|_{L^2}^2) dt \\ & \leq C(M). \end{aligned} \quad (3.3)$$

Using (3.2) and (3.3), we have

$$\lim_{t \rightarrow \infty} (\|\nabla \rho\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2) = 0, \quad (3.4)$$

which combining

$$\|\rho - 1\|_{H^2}^2 + \|\mathbf{u}\|_{H^2}^2 + \|\theta - 1\|_{H^2}^2 + \|\mathbf{B}\|_{H^2}^2 \leq C(M)$$

and using Sobolev inequality, we arrive at (1.9). Thus, the proof of Theorem 1.1 is completed. $\square$

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