Research Article

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Blow-up solutions with minimal mass for nonlinear Schrödinger equation with variable potential

<https://doi.org/10.1515/anona-2020-0185>

Received March 18, 2021; accepted April 7, 2021.

Abstract: This paper studies the mass-critical variable coefficient nonlinear Schrödinger equation. We first get the existence of the ground state by solving a minimization problem. Then we prove a compactness result by the variational characterization of the ground state solutions. In addition, we construct the blow-up solutions at the minimal mass threshold and further prove the uniqueness result on the minimal mass blow-up solutions which are pseudo-conformal transformation of the ground states.

Keywords: variable coefficient nonlinear Schrödinger equation; minimal mass blow-up solutions; variational characterization; ground state; compactness

MSC: 35Q55; 35B44

1 Introduction

In this paper, we study the behavior of blowup solutions for the variable coefficient nonlinear Schrödinger equation

$$i\varphi_t + \Delta\varphi + c|x|^{-2}\varphi + |x|^{-b}|\varphi|^{p-1}\varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D, \quad (1.1)$$

and

$$\varphi(0, x) = \varphi_0, \quad x \in \mathbb{R}^D, \quad (1.2)$$

in the case $p = 1 + \frac{4-2b}{D}$. Here and hereafter, $D \geq 3$ is the space dimension, $0 < b < 2$, $\varphi = \varphi(t, x) : [0, T) \times \mathbb{R}^D \rightarrow \mathbb{C}$ is a complex value wave function with $0 < T \leq \infty$, $i = \sqrt{-1}$, Δ is the Laplace operator and $c \in (0, c^*)$, where $c^* = \frac{(D-2)^2}{4}$ is the best constant in Hardy's inequality:

$$c^* \int_{\mathbb{R}^D} |x|^{-2} |u|^2 dx \leq \int_{\mathbb{R}^D} |\nabla u|^2 dx. \quad (1.3)$$

More precisely, we say that $\varphi(\cdot)$ is a solution of the Cauchy problem (1.1)–(1.2) on $[0, T)$ if for all $t \in [0, T)$,

$$\varphi(t) = S(t)\varphi + i \int_0^t S(t-s) \{ |x|^{-b} |\varphi(s)|^{\frac{4-2b}{D}} \varphi(s) \} ds,$$

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where $S(\cdot)$ is the group with infinitesimal generator $i(\Delta + c|x|^{-2})$.

From [2, 6, 15], we know that the Cauchy problem (1.1)–(1.2) is locally well-posed in $H^1(\mathbb{R}^D)$: there exists a unique solution $\varphi(t, x)$ of the Cauchy problem (1.1)–(1.2) in $C([0, T]; H^1(\mathbb{R}^D))$ for some $T \in (0, \infty]$ (maximal existence time), and for all $t \in [0, T)$, we have the following alternatives:

$$T = +\infty,$$

or else

$$T < \infty \text{ and } \lim_{t \rightarrow T^-} \|\varphi(t)\|_{H^1} = +\infty (\text{blowup}),$$

where $\|\cdot\|_{H^1}$ is the usual norm on H^1 , and H^1 is $H^1(\mathbb{R}^D)$. Furthermore, the unique solution $\varphi(t, x)$ satisfies the following two conservation laws: for all $t \in [0, T)$,

$$\int_{\mathbb{R}^D} |\varphi(t, x)|^2 dx = \int_{\mathbb{R}^D} |\varphi_0(x)|^2 dx, \quad (1.4)$$

and

$$E(\varphi(t)) = \int_{\mathbb{R}^D} |\nabla \varphi(t)|^2 dx - \int_{\mathbb{R}^D} c|x|^{-2} |\varphi(t)|^2 dx - \frac{D}{D+2-b} \int_{\mathbb{R}^D} |x|^{-b} |\varphi(t)|^{2+\frac{4-2b}{D}} dx = E(\varphi_0). \quad (1.5)$$

Also, the L^2 -norm of $\varphi(t, x)$ is invariant under the transformation

$$\varphi \rightarrow \varphi_\lambda(t, x) = \lambda^{D/2} \varphi(\lambda^2 t, \lambda x) \text{ for } \lambda > 0.$$

In other words, $\|\varphi_\lambda(t)\|_{L^2} = \|\varphi(t)\|_{L^2}$ and the Cauchy problem (1.1)–(1.2) is called L^2 -critical.

Eq.(1.1) is also called inhomogeneous nonlinear Schrödinger equation with inverse-square potential. Let us now recall some known results about the nonlinear Schrödinger equation. For the classical nonlinear Schrödinger equation, i.e. $c = b = 0$,

$$i\varphi_t + \Delta \varphi + |\varphi|^{\frac{4}{D}} \varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D. \quad (1.6)$$

Let $\varphi(t, x) = e^{it} u(x)$ be a standing wave solution of (1.6). Then $u(x)$ satisfies the following time-independent Schrödinger equation

$$\Delta u - u + |u|^{\frac{4}{D}} u = 0, \quad u \in H^1. \quad (1.7)$$

It is well known that (1.7) possesses a unique radial positive solution $Q(x)$ (see [2]). Moreover, the Cauchy problem (1.6)–(1.2) has no blow-up solution in the class $\{\varphi \in H^1 \mid \|\varphi\|_{L^2} < \|Q\|_{L^2}\}$ (see [19]), while in the class $\{\varphi \in H^1 \mid \|\varphi\|_{L^2} = \|Q\|_{L^2}\}$, there exists a unique blowup solution

$$\tilde{Q}(x, t) = \frac{1}{|t|^{D/2}} e^{\frac{i|x|^2}{4t} - \frac{i}{t}} Q\left(\frac{x}{t}\right)$$

up to the invariances of the equation (see [10, 13, 14, 20]).

For the nonlinear Schrödinger equation with inverse-square potential, i.e. $b = 0, c \neq 0$,

$$i\varphi_t + \Delta \varphi + c|x|^{-2} \varphi + |\varphi|^{\frac{4}{D}} \varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D. \quad (1.8)$$

Let $\varphi(t, x) = e^{it} u(x)$ be a standing wave solution of (1.8). Then $u(x)$ satisfies the following time-independent Schrödinger equation

$$\Delta u + c|x|^{-2} u - u + |u|^{\frac{4}{D}} u = 0, \quad u \in H^1. \quad (1.9)$$

By solving a variational problem, Csobo and Genoud [5] proved that (1.9) has radial positive solutions to exist. Mukherjee, Nam and Nguyem [18] showed the uniqueness of radial positive solution of (1.9). In addition, Csobo and Genoud [5] proved that all H^1 -solutions of Eq.(1.8) are global if $\|\varphi\|_{L^2} < \|U\|_{L^2}$ and by using the pseudo-conformal transformation, they constructed the minimal mass blow-up solutions defined as

$$\varphi(t, x) = e^{i\gamma_0} e^{i\frac{\lambda_0^2}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} \left(\frac{\lambda_0}{T-t}\right)^{D/2} U\left(\frac{\lambda_0 x}{T-t}\right),$$

such that $\|\varphi_0\|_{L^2} = \|U\|_{L^2}$, where $T \in \mathbb{R}$, $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$, U is the unique positive radial solution of (1.9).

For the inhomogeneous nonlinear Schrödinger equation, i.e. $c = 0$, $b \neq 0$,

$$i\varphi_t + \Delta\varphi + |x|^{-b}|\varphi|^{\frac{4-2b}{b}}\varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D. \quad (1.10)$$

Let $\varphi(t, x) = e^{it}u(x)$ be a standing wave solution of (1.10). Then $u(x)$ satisfies the following time-independent Schrödinger equation

$$\Delta u - u + |x|^{-b}|u|^{\frac{4-2b}{b}}u = 0, \quad u \in H^1. \quad (1.11)$$

By solving a variational problem, Genoud [9] proved that (1.11) has positive radial solutions to exist and the sharp condition for global existence of H^1 -solutions was established, also, the solutions with critical mass which blow-up in finite time were proved. By Yanagida [21] and Genoud [8], the positive radial solution of (1.11) is unique. Furthermore, Combet and Genoud [4, 9] constructed a 3-parameter family of critical mass solutions of (1.10) which blow-up in finite time, defined by

$$\varphi(t) = e^{i\gamma_0} e^{i\frac{\lambda_0^2}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} \left(\frac{\lambda_0}{T-t} \right)^{D/2} \psi \left(\frac{\lambda_0 x}{T-t} \right),$$

with $\|\varphi_0\|_{L^2} = \|\psi\|_{L^2}$, where $T \in \mathbb{R}$, $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$, ψ is the unique positive radial solution of (1.11). We also refer to Chen and Tang [3], Zhang and Ahmed [23] for related contributions on the nonlinear Schrödinger equation.

Motivated by the above studies, it is of interest to find the uniqueness of the minimal mass blow-up solutions for the Cauchy problem (1.1)-(1.2). In order to solve this problem, we need a notion of a ground state. So we consider the periodic solutions of equation of the form $\varphi(t, x) = e^{it}u(x)$, then $u(x)$ solves the nonlinear elliptic equation

$$\Delta u + c|x|^{-2}u - u + |x|^{-b}|u|^{\frac{4-2b}{b}}u = 0, \quad u \in H^1. \quad (1.12)$$

In section 2, we will prove the existence in H^1 of a positive radial solution of Eq.(1.12) by using Weinstein's argument [19] and that the set of solutions of (1.12) has a "minimal" element in L^2 called ground state. However, for $c > 0$, we are not aware of any uniqueness result of (1.12) on $\mathbb{R}^D$. According to [8], [18] and [21], we can assume that (1.12) has a unique positive radial solution, which we will denote by W throughout the paper. In terms of the existence of ground state, we first show the global well-posedness condition for the Cauchy problem (1.1)-(1.2). Then, we shall exhibit blow-up solutions at the mass $\|\varphi_0\|_{L^2} = \|W\|_{L^2}$ by applying the pseudo-conformal transformation to the standing waves $e^{it}W(x)$, which is thus the minimal mass where blowup can occur.

Now we state our main result.

Theorem 1.1. (Determination of minimal blow-up solutions) *Let initial data $\varphi_0 \in H^1$ such that the solution $\varphi(t, x)$ of the Cauchy problem (1.1)-(1.2) blows up in finite time $T > 0$. Moreover, assume that $\|\varphi_0\|_{L^2} = \|W\|_{L^2}$. Then there exist $\gamma_0 \in \mathbb{R}$ and $\lambda_0 > 0$ such that for all $t \in [0, T)$,*

$$\varphi(t, x) := \varphi_{T, \lambda_0, \gamma_0} = \left(\frac{\lambda_0}{T-t} \right)^{D/2} e^{i\gamma_0} e^{i\frac{\lambda_0^2}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} W \left(\frac{\lambda_0 x}{T-t} \right),$$

where $W(x)$ is the ground state of Eq.(1.12).

One can ask if there are other minimal mass blow-up solutions? That is, assume that

- $\varphi(t, x)$ blows up in finite time T , and
- $\|\varphi(t, x)\|_{L^2} = \|W\|_{L^2}$ for all t .

Is there $\gamma_0 \in \mathbb{R}$ and $\lambda_0 > 0$ such that

$$\tilde{\varphi}(t, x) = \varphi(t, x)?$$

The aim of this paper is to prove this uniqueness result on the minimal mass blow-up solutions. The method is mainly based on a compactness result.

The plan of this paper is as follows. In section 2, we prove the existence of ground state by solving a minimization problem. In section 3, we prove a compactness result, crucial in the proof of Theorem 1.1. In section 4, we construct the pseudo-conformal transformation and then give the proof of Theorem 1.1.

2 Ground State

In this section, we prove the existence of a positive radial solution $W \in H^1$ for (1.12). Such a solution is obtained by variational approach and called ground state. We first state the following result.

Lemma 2.1. For $D \geq 3$, $0 < c < c^* = (\frac{D-2}{2})^2$, $0 < b < 2$ and $u \in H^1 \setminus \{0\}$, define the functional

$$J(u) := \left(\int_{\mathbb{R}^D} (|\nabla u|^2 dx - c|x|^{-2}|u|^2) dx \right) \left(\int_{\mathbb{R}^D} |u|^2 dx \right)^{\frac{2-b}{D}} / \int_{\mathbb{R}^D} |x|^{-b}|u|^{2+\frac{4-2b}{D}} dx, \quad (2.1)$$

and the variational problem

$$d := \inf_{u \in H^1 \setminus \{0\}} J(u). \quad (2.2)$$

Then one has the followings:

- (i). There exists a positive radial function $v^*(x) \in H^1$ such that the variational problem (1.12) is attained at v^* .
- (ii). Let $v^* = [(1 + \frac{2-b}{D})d]^{-\frac{D}{4-2b}} \lambda^{\frac{D}{2}} W(\lambda x)$ with $\lambda = \sqrt{\frac{2-b}{D}}$. Then $W(x)$ is the positive radial solution of (1.12).
- (iii). Assume that the positive radial solution $W(x)$ of the nonlinear elliptic equation (1.12) is unique. Then one has the sharp interpolation inequality for $u \in H^1(\mathbb{R}^D)$ with $D \geq 3$,

$$\int_{\mathbb{R}^D} |x|^{-b}|u|^{2+\frac{4-2b}{D}} dx \leq \frac{D+2-b}{D} \left(\int_{\mathbb{R}^D} |W|^2 dx \right)^{-\frac{2-b}{D}} \left[\int_{\mathbb{R}^D} (|\nabla u|^2 dx - c|x|^{-2}|u|^2) dx \right] \left(\int_{\mathbb{R}^D} |u|^2 dx \right)^{\frac{2-b}{D}}. \quad (2.3)$$

Proof. From Hardy inequality (1.3), one has that for $u \in H^1(\mathbb{R}^D)$,

$$\left(1 - \frac{c}{c^*}\right) \int_{\mathbb{R}^D} |\nabla u|^2 dx \leq \int_{\mathbb{R}^D} (|\nabla u|^2 - c|x|^{-2}|u|^2) dx \leq \int_{\mathbb{R}^D} |\nabla u|^2 dx. \quad (2.4)$$

By (2.1), $J(u)$ is well-defined in $H^1 \setminus \{0\}$, and $J(u) \geq 0$. Furthermore, $J(u)$ is invariant under the scaling $u(x) \rightarrow u^{\mu, \lambda}(x) := \mu u(\lambda x)$ ($\mu, \lambda > 0$), that is

$$J(u^{\mu, \lambda}) = J(u) \text{ for } \mu > 0, \lambda > 0. \quad (2.5)$$

Let u^* be the Schwarz symmetrization of u . By [11, 12], one has that

$$\int_{\mathbb{R}^D} |x|^{-2}|u^*|^2 dx \geq \int_{\mathbb{R}^D} |x|^{-2}|u|^2 dx, \quad (2.6)$$

$$\int_{\mathbb{R}^D} |x|^{-b}|u^*|^{2+\frac{4-2b}{D}} dx > \int_{\mathbb{R}^D} |x|^{-b}|u|^{2+\frac{4-2b}{D}} dx, \quad (2.7)$$

$$\int_{\mathbb{R}^D} |\nabla u^*|^2 dx \leq \int_{\mathbb{R}^D} |\nabla u|^2 dx, \quad \int_{\mathbb{R}^D} |u^*|^2 dx = \int_{\mathbb{R}^D} |u|^2 dx. \quad (2.8)$$

By (2.2), we can choose a minimizing sequence $\{u_n\} \subset H^1 \setminus \{0\}$ such that $\lim_{n \rightarrow \infty} J(u_n) = d$. Then from (2.1), (2.2), (2.6), (2.7), (2.8), one has

$$\lim_{n \rightarrow \infty} J(u_n^*) = d, \quad (2.9)$$

where u_n^* denotes the Schwarz symmetrization of u_n for any $n \in \mathbb{N}$. Now we choose

$$\lambda_n = \left[\int_{\mathbb{R}^D} |u_n^*|^2 dx \int_{\mathbb{R}^D} (|\nabla u_n^*|^2 - c|x|^{-2}|u_n^*|^2) dx \right]^{1/2},$$

$$\mu_n = \left(\int_{\mathbb{R}^D} |u_n^*|^2 dx \right)^{\frac{D-2}{4}} \left[\int_{\mathbb{R}^D} (|\nabla u_n^*|^2 - c|x|^{-2}|u_n^*|^2) dx \right]^{\frac{D}{4}},$$

and take $v_n = |(u_n^*)^{\mu_n, \lambda_n}|$. By (2.5) and (2.9), we have that

$$v_n \geq 0, \quad v_n(x) = v_n(|x|), \quad (2.10)$$

$$\int_{\mathbb{R}^D} |v_n|^2 dx = 1, \quad \int_{\mathbb{R}^D} (|\nabla v_n|^2 - c|x|^{-2}|v_n|^2) dx = 1, \quad (2.11)$$

$$\lim_{n \rightarrow \infty} J(v_n) = d. \quad (2.12)$$

Thus $\{v_n\}$ is bounded in H^1 . Therefore, up to subsequence, we can suppose that $\{v_n\}$ has a weak limit $v^* \in H^1$. By weak lower semi-continuity (see [17]), we obtain that

$$\int_{\mathbb{R}^D} |v^*|^2 dx \leq 1, \quad \int_{\mathbb{R}^D} (|\nabla v^*|^2 - c|x|^{-2}|v^*|^2) dx \leq 1. \quad (2.13)$$

It follows from the weakly sequence continuous of the functional $\int_{\mathbb{R}^D} |x|^{-b}|v^*|^{2+\frac{4-2b}{D}} dx$ (see [6]) that

$$d \leq J(v^*) \leq \frac{1}{\int_{\mathbb{R}^D} |x|^{-b}|v^*|^{2+\frac{4-2b}{D}} dx} = \lim_{n \rightarrow \infty} J(v_n) = d. \quad (2.14)$$

It yields that

$$\int_{\mathbb{R}^D} |v^*|^2 dx = 1, \quad \int_{\mathbb{R}^D} (|\nabla v^*|^2 - c|x|^{-2}|v^*|^2) dx = 1. \quad (2.15)$$

It follows that

$$\lim_{n \rightarrow \infty} v_n = v^* \text{ in } H^1. \quad (2.16)$$

Thus we get the proof of (i). Now we prove (ii).

Since v^* is the minimizer of (2.2), v^* must satisfy the Euler-Lagrange equation

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} J(v^* + \varepsilon\eta) = 0, \text{ for all } \eta \in C_0^\infty(\mathbb{R}^D). \quad (2.17)$$

By (2.14), (2.15) it follows that

$$\Delta v^* + c|x|^{-2}v^* - \frac{2-b}{D}v^* + \left(1 + \frac{2-b}{D}\right) d|x|^{-b}(v^*)^{1+\frac{4-2b}{D}} = 0. \quad (2.18)$$

Let $v^* = [(1 + \frac{2-b}{D})d]^{-\frac{D}{4-2b}} \lambda^{\frac{D}{2}} W(\lambda x)$ with $\lambda = \sqrt{\frac{2-b}{D}}$. Then from (2.18), $W(x)$ is the positive radial solution of (1.12). This completes the proof of (ii). Then we prove (iii) in the following.

Since $v^* = [(1 + \frac{2-b}{D})d]^{-\frac{D}{4-2b}} \lambda^{\frac{D}{2}} W(\lambda x)$ and $\lambda = \sqrt{\frac{2-b}{D}}$, by (2.15) it follows that

$$d = \left(1 + \frac{2-b}{D}\right)^{-1} \left(\int_{\mathbb{R}^D} |W|^2 dx \right)^{\frac{2-b}{D}}. \quad (2.19)$$

Thus if we assume that (1.12) has a unique positive radial solution, then for (ii) and (2.2), we get (2.3).

This completes the proof.

Remark 2.1. Let ξ be the minimizer of $J(u)$, then we define a set $\mathcal{A}$ as follows:

$$\mathcal{A} = \left\{ V(x) : \xi(x) = \left[\left(1 + \frac{2-b}{D} \right) d \right]^{-\frac{D}{4-2b}} \lambda^{\frac{D}{2}} V(\lambda x) \text{ with } \lambda = \sqrt{\frac{2-b}{D}} \right\}. \quad (2.20)$$

It is obvious that $W \in \mathcal{A}$, then $\|W\|_{L^2}^2 = [(1 + \frac{2-b}{D})d]^{\frac{D}{2-b}}$. Moreover, W satisfying the elliptic equation (1.12). Therefore, we call $W \in \mathcal{A}$ ground state of Eq.(1.12).

In the following, we show that the ground state solutions play an important role in the global well-posedness for the Cauchy problem (1.1)–(1.2).

Corollary 2.1. Let $c \in (0, c^*)$, if $\varphi_0 \in H^1$ satisfies

$$\|\varphi_0\|_{L^2} < \|W\|_{L^2}, \quad (2.21)$$

then the corresponding solution of the Cauchy problem (1.1)–(1.2) is global, where $W \in \mathcal{A}$ is the ground state of (1.12).

Proof. By the local well-posedness theory, we only need to show that $\int_{\mathbb{R}^D} (|\nabla \varphi|^2 - c|x|^{-2}|\varphi|^2)dx$ remains bounded. From the conservation laws of mass and energy,

$$E(\varphi_0) = \int_{\mathbb{R}^D} (|\nabla \varphi|^2 - c|x|^{-2}|\varphi|^2)dx - \frac{D}{D+2-b} \int_{\mathbb{R}^D} |x|^{-b}|\varphi|^{2+\frac{4-2b}{D}}dx.$$

Hence, by (2.3),

$$E(\varphi_0) \geq \int_{\mathbb{R}^D} (|\nabla \varphi|^2 - c|x|^{-2}|\varphi|^2)dx \left[1 - \left(\frac{\|\varphi_0\|_{L^2}}{\|W\|_{L^2}} \right)^{\frac{4-2b}{D}} \right].$$

If

$$\|\varphi_0\|_{L^2} < \|W\|_{L^2},$$

then $\int_{\mathbb{R}^D} (|\nabla \varphi|^2 - c|x|^{-2}|\varphi|^2)dx$ is bounded and so the solution is global. This completes the proof.

3 Compactness

In this section, we will establish a compactness result which is important in the proof of Theorem 1.1. For this aim, we introduce the notion of the variational characterization of ground state solutions.

Theorem 3.1. Let W be the positive radial minimizer of $J(u)$ and $v \in H^1$ satisfy

$$\|v\|_{L^2} = \|W\|_{L^2} \text{ and } E(v) = 0. \quad (3.1)$$

Then there exist $\lambda_0 > 0$ and $\gamma_0 \in \mathbb{R}$ such that $v(x) = e^{i\gamma_0} \lambda_0^{D/2} W(\lambda_0 x)$.

Proof. It follows from Lemma 2.1 and (3.1) that v is a minimizer of $J(u)$. Since $\|\nabla(|v|)\|_{L^2} \leq \|\nabla v\|_{L^2}$, then $|v|$ is also a minimizer. Furthermore, any positive minimizer is radial thanks to a result of Hajaiej [11]. Indeed, suppose v_0 is a positive minimizer that is not radial, and consider its Schwarz symmetrization v_0^* . On the other hand,

$$\|\nabla v_0^*\|_{L^2} \leq \|\nabla v_0\|_{L^2}, \quad \|v_0^*\|_{L^2} \leq \|v_0\|_{L^2},$$

then we get $J(v_0^*) < J(v_0)$, a contradiction. We deduce that $|v|$ is radial. Furthermore, the Euler-Lagrange equation (2.17) expressing the fact that $|v|$ is a minimizer reads

$$\Delta(|v|) + c|x|^{-2}(|v|) - \frac{2-b}{D} \frac{\int_{\mathbb{R}^D} (|\nabla v|^2 - c|x|^{-2}|v|^2)dx}{\int_{\mathbb{R}^D} |v|^2 dx} |v| + |x|^{-b}|v|^{1+\frac{4-2b}{D}} = 0.$$

It now follows by the scaling properties of this elliptic equation, there exists $W \in \mathcal{A}$ such that

$$|v(x)| = \lambda_0^{D/2} W(\lambda_0 x), \text{ where } \lambda_0 = \sqrt{\frac{2-b}{D}} \sqrt{\frac{\int_{\mathbb{R}^D} (|\nabla v|^2 - c|x|^{-2}|v|^2) dx}{\int_{\mathbb{R}^D} |v|^2 dx}}.$$

It only remains to show that w defined by $w(x) = \frac{v(x)}{|v(x)|}$ is constant on $\mathbb{R}^D$. Differentiating $|w|^2 = 1$ yields $\operatorname{Re}(\bar{w} \nabla w) = 0$, thus

$$|\nabla v|^2 = |\nabla(|v|)|^2 + |v|^2 |\nabla w|^2 + 2|v| \nabla(|v|) \cdot \operatorname{Re}(\bar{w} \nabla w),$$

and

$$\|\nabla v\|_{L^2}^2 = \|\nabla(|v|)\|_{L^2}^2 + \int_{\mathbb{R}^D} |v|^2 |\nabla w|^2 dx.$$

If $|\nabla w| \neq 0$, then $J(|v|) < J(v)$, which is a contradiction. Therefore, w is constant, which completes the proof. Then we review the following result, for the proofs we refer readers to [7].

Lemma 3.1. *Let $0 < b < 2$, $D \geq 3$ and $1 < p < 1 + \frac{4-2b}{D-2}$. Then there exists a constant $C = C(D, b, p) > 0$ such that*

$$\int_{\mathbb{R}^D} |x|^{-b} \|\varphi\|^{p-1} - |v|^{p-1} \|u\| \|\phi\| dx \leq C \{ \|\varphi\|^{p-1} - |v|^{p-1} \|u\|_{L^\alpha} \|\phi\|_{L^\beta} + \|\varphi\|^{p-1} - |v|^{p-1} \|u\|_{L^\eta} \|\phi\|_{L^{p+1}} \}$$

for all $\varphi, v, u, \phi \in H^1$, where

$$(p-1)\alpha = \beta \in \left(\frac{D(p+1)}{D-b}, 2^* \right) \text{ and } (p-1)\eta = p+1.$$

In addition, we also need the following concentration-compactness Lemma which can be proved with a minor modification to the proof of Proposition 1.7.6 in [2].

Lemma 3.2. *Let $\{v_n\}_{n \in \mathbb{N}} \in H^1$ satisfy*

$$\lim_{n \rightarrow \infty} \|v_n\|_{L^2} = M < +\infty \text{ and } \sup_{n \in \mathbb{N}} H(v_n) < +\infty.$$

Then there exists a subsequence $\{v_{n_k}\}_{k \in \mathbb{N}}$ which satisfies one of the following properties:

(V) $\|v_{n_k}\|_{L^q} \rightarrow 0$ as $k \rightarrow \infty$ for all $q \in (2, 2^*)$.

(D) *There are sequences $w_k, z_k \in H^1$ and a constant $\alpha \in (0, 1)$ such that:*

(1) $\operatorname{dist}(\operatorname{supp}(w_k), \operatorname{supp}(z_k)) \rightarrow \infty$;

(2) $\sup_{k \in \mathbb{N}} (\|w_k\|_{H^1} + \|z_k\|_{H^1}) < \infty$ for all $k \in \mathbb{N}$;

(3) $\|w_k\|_{L^2} \rightarrow \alpha M$ and $\|z_k\|_{L^2} \rightarrow (1-\alpha)M$ as $k \rightarrow \infty$, for some $\alpha \in (0, 1)$;

(4) $\lim_{k \rightarrow \infty} \left| \int_{\mathbb{R}^D} |v_{n_k}|^q dx - \int_{\mathbb{R}^D} |w_k|^q dx - \int_{\mathbb{R}^D} |z_k|^q dx \right| = 0$, for all $q \in [2, 2^*)$;

(5) $\liminf_{k \rightarrow \infty} \{H(v_{n_k}) - H(w_k) - H(z_k)\} \geq 0$.

(C) *There exist $v \in H^1$ and a sequence $\{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^D$ such that*

$$v_{n_k}(\cdot - y_k) \rightarrow v \text{ in } L^q(\mathbb{R}^D), \forall q \in [2, 2^*).$$

Here and subsequently, for abbreviation, $H(\varphi)$ stands for the Hardy functional defined by $\int (|\nabla \varphi|^2 - c|x|^{-2}|\varphi|^2) dx$.

We are now show the main result of this section.

Theorem 3.2. *Consider a sequence $\{v_n\}_{n \in \mathbb{N}} \subset H^1$ satisfying*

$$\lim_{n \rightarrow \infty} \|v_n\|_{L^2} = \|W\|_{L^2}, \quad 0 < \limsup_{n \in \mathbb{N}} H(v_n) < +\infty, \quad \limsup_{n \rightarrow \infty} E(v_n) \leq 0. \quad (3.2)$$

Then there exists a subsequence $\{v_{n_k}\}_{k \in \mathbb{N}}$ and $\gamma_0 \in \mathbb{R}$ such that

$$\lim_{k \rightarrow \infty} \|v_{n_k} - e^{i\gamma_0} W\|_{H^1} = 0, \quad (3.3)$$

where $W \in \mathcal{A}$ is the ground state of (1.12).

Proof. The behavior of the sequence $\{v_n\}$ is constrained by the concentration-compactness Lemma. The proof will be divided into three steps: we first show that property (C) holds in Lemma 3.2 by ruling out (V) and (D). And then prove the sequence $\{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^D$ in (C) is bounded. We thus obtain the desired conclusion by using the variational characteristic of ground states.

Step 1: Compactness. Let $v = 0$ and $\varphi = u = \phi = v_{n_k}$ in Lemma 3.1, there exists $\beta \in (\frac{D(p+1)}{D-b}, 2^*)$ such that

$$\int_{\mathbb{R}^D} |x|^{-b} |v_{n_k}|^{p+1} dx \leq C(\|v_{n_k}\|_{L^\beta}^{p+1} + \|v_{n_k}\|_{L^{p+1}}^{p+1}).$$

Since $\beta, p+1 \in (2, 2^*)$, (V) would imply that $\int_{\mathbb{R}^D} |x|^{-b} |v_{n_k}|^{p+1} dx \rightarrow 0$ and so

$$\limsup_{k \rightarrow \infty} E(v_{n_k}) = \limsup_{k \rightarrow \infty} \{H(v_{n_k}) - \frac{2}{p+1} \int_{\mathbb{R}^D} |v_{n_k}|^{p+1} dx\} = \limsup_{k \rightarrow \infty} H(v_{n_k}) > 0, \quad (3.4)$$

which contradicts (3.2).

Now suppose that (D) holds. It is worth pointing out that

$$\lim_{k \rightarrow \infty} \left| \int_{\mathbb{R}^D} |x|^{-b} |v_{n_k}|^{p+1} dx - \int_{\mathbb{R}^D} |x|^{-b} |w_k|^{p+1} dx - \int_{\mathbb{R}^D} |x|^{-b} |z_k|^{p+1} dx \right| = 0. \quad (3.5)$$

It then follows from property (D)(5) in Lemma 3.2 that

$$\limsup_{k \rightarrow \infty} [E(w_k) + E(z_k)] \leq \liminf_{k \rightarrow \infty} H(v_{n_k}) - \frac{2}{p+1} \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^D} |x|^{-b} |v_{n_k}|^{p+1} dx \leq \limsup_{k \rightarrow \infty} E(v_{n_k}) \leq 0. \quad (3.6)$$

On the other hand, $E(w_k) \geq 0$, $E(z_k) \geq 0$ for sufficiently large k by using inequality (2.3) and property (D)(3) of Lemma 3.2, thus

$$E(w_k) \rightarrow 0, \quad E(z_k) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

which implies that

$$H(w_k) \rightarrow 0, \quad H(z_k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Therefore,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^D} |x|^{-b} |v_{n_k}|^{p+1} dx = \lim_{k \rightarrow \infty} \left(\int_{\mathbb{R}^D} |x|^{-b} |w_k|^{p+1} dx + \int_{\mathbb{R}^D} |x|^{-b} |z_k|^{p+1} dx \right) = 0,$$

again, it contradicts (3.4). Thus, to rule out (D), we just have to prove (3.5) holds. For this aim, defining $\xi_k = v_{n_k} - w_k - z_k$, it follows from the construction of the sequence w_k and z_k in the proof of [2] that

$$|v_{n_k}|^{p+1} - |w_k|^{p+1} - |z_k|^{p+1} \leq C|v_{n_k}|^p |\xi_k|$$

and

$$\|\xi_k\|_{L^2} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Since $\|\xi_k\|_{L^2}$ is bounded by property (D)(5), applying the inequality (2.3), one can deduce that

$$\|\xi_k\|_{L^q} \rightarrow 0 \text{ as } k \rightarrow \infty, \quad \forall q \in [2, 2^*).$$

Hence, it follows from Lemma 3.1 that

$$\left| \int_{\mathbb{R}^D} |x|^{-b} |v_{n_k}|^{p+1} dx - \int_{\mathbb{R}^D} |x|^{-b} |w_k|^{p+1} dx - \int_{\mathbb{R}^D} |x|^{-b} |z_k|^{p+1} dx \right| \\ \leq C \int_{\mathbb{R}^D} |x|^{-b} |v_{n_k}|^p |\xi_k| dx \leq C (\|v_{n_k}\|_{L^\beta}^p \|\xi_k\|_{L^\beta} + \|v_{n_k}\|_{L^{p+1}}^p \|\xi_k\|_{L^{p+1}}) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

which proves (3.5). Therefore, according to Lemma 3.2, there exist $v \in H^1$ and a sequence $\{y_k\}_{k \in \mathbb{D}} \subset \mathbb{R}^D$ such that

$$v_{n_k}(\cdot - y_k) \rightarrow v \text{ in } L^q(\mathbb{R}^D), \forall q \in [2, 2^*). \quad (3.7)$$

Step 2: Localization. We will now show that $\{y_k\}$ is bounded in $\mathbb{R}^D$. Suppose that there exists a subsequence denoted still by $\{y_k\}$ such that $|y_k| \rightarrow \infty$ as $k \rightarrow \infty$. Note that $E(v_{n_k})$ can be written as

$$E(v_{n_k}) = H(v_{n_k}) - \frac{2}{p+1} \int_{\mathbb{R}^D} |x - y_k|^{-b} |v_{n_k}(x - y_k)|^{p+1} dx. \quad (3.8)$$

Then we will show that the second term in the right hand side of (3.8) goes to zero as $k \rightarrow \infty$, so that

$$E(v_{n_k}) \rightarrow H(v_{n_k}) > 0 \text{ as } k \rightarrow \infty, \quad (3.9)$$

which contradicts (3.2). We split the integral as

$$\int_{|x-y_k| < R} |x - y_k|^{-b} |v_{n_k}(x - y_k)|^{p+1} dx + \int_{|x-y_k| \geq R} |x - y_k|^{-b} |v_{n_k}(x - y_k)|^{p+1} dx \\ = I + II, \quad (3.10)$$

for some $R > 0$. We first observe that, by Holder's inequality,

$$I \leq \left(\int_{|x-y_k| < R} |x - y_k|^{-b\alpha} dx \right)^{1/\alpha} \left(\int_{|x-y_k| < R} |v_{n_k}(x - y_k)|^{(p+1)\beta} dx \right)^{1/\beta}, \quad (3.11)$$

where $\alpha, \beta \geq 1$ satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the first term in the right hand side of (3.11) is finite provided $\beta > \frac{D}{D-b}$. Indeed, it is possible to choose β satisfies that $\beta(p+1) \in (\frac{D(p+1)}{D-b}, 2^*)$ and it follows from (3.7) that

$$\int_{|x-y_k| < R} |v_{n_k}(x - y_k)|^{(p+1)\beta} dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

On the other hand, by the Sobolev embedding Theorem and the boundedness of $\{v_{n_k}(x - y_k)\}$ in H^1 , the second term of (3.10) can be estimated as

$$II \leq R^{-b} \int_{\mathbb{R}^D} |v_{n_k}(x - y_k)|^{p+1} dx \leq CR^{-b}.$$

Hence, II can be made arbitrarily small by choosing R large enough, uniformly in k . This completes the proof of (3.9), and we conclude that the sequence $\{y_k\} \subset \mathbb{R}^D$ is bounded.

Step 3: Conclusion. We shall suppose, without loss of generality, that $y_k \rightarrow y^*$ as $k \rightarrow \infty$, for some $y^* \in \mathbb{R}^D$. Hence,

$$v_{n_k} \rightarrow v^* = v(\cdot + y^*) \text{ in } L^q(\mathbb{R}^D), \forall q \in [2, 2^*).$$

Furthermore, we can also suppose that $v_{n_k} \rightharpoonup v^*$ weakly in H^1 . Since the Hardy functional H is weakly lower-continuous[17], $v \mapsto \int_{\mathbb{R}^D} |x|^{-b} |v|^{p+1} dx$ is weakly sequentially continuous [9] and $p+1 \in (2, 2^*)$, it follows

that

$$\begin{aligned} 0 \leq E(v^*) &= H(v^*) - \frac{2}{p+1} \int_{\mathbb{R}^D} |x|^{-b} |v^*|^{p+1} dx \leq \limsup_{k \rightarrow \infty} H(v_{n_k}) - \frac{2}{p+1} \int_{\mathbb{R}^D} |x|^{-b} |v^*|^{p+1} dx \\ &= \limsup_{k \rightarrow \infty} E(v_{n_k}) \leq 0. \end{aligned}$$

So from the inequality (2.3), one has that $E(v^*) \geq 0$, and so

$$E(v^*) = 0 \text{ and } H(v^*) = \lim_{k \rightarrow \infty} H(v_{n_k}). \quad (3.12)$$

Together with $\|v^*\|_{L^2} = \|W\|_{L^2}$ and the variational characteristic of the ground states, it implies that $v^* = e^{i\gamma_0} W$ for some $\gamma_0 \in \mathbb{R}$. Finally, we have $\|v_{n_k}\|_{H^1} \rightarrow \|W\|_{H^1} = \|v^*\|_{H^1}$, $\{v_{n_k}\}$ converges to v^* in H^1 , which concludes the proof.

4 Classification

In this section, we construct finite time blow-up solutions with minimal mass $\|W\|_{L^2}$ to the Cauchy problem (1.1)–(1.2). Furthermore, we prove the uniqueness of finite time blow-up solutions at the minimal mass threshold. We first show that the equation is invariant under the pseudo-conformal transformation, which is defined as follows.

Lemma 4.1. *Let φ be a global solution of the Cauchy problem (1.1)–(1.2). Then, for all $T \in \mathbb{R}$, the function*

$$\varphi_T(t, x) = \frac{e^{-i\frac{|x|^2}{4(T-t)}}}{(T-t)^{D/2}} \varphi\left(\frac{1}{T-t}, \frac{x}{T-t}\right)$$

is a solution of (1.1) and (1.2) on $(-\infty, T)$, with $\|\varphi_T\|_{L^2} = \|\varphi\|_{L^2}$.

Proof. By a direct calculation we have

$$\partial_t \varphi_T(t, x) = \frac{e^{-i\frac{|x|^2}{4(T-t)}}}{(T-t)^{D/2+2}} \left[-i\frac{|x|^2}{4} \varphi + \partial_t \varphi + \frac{D}{2} \varphi(T-t) + x \cdot \nabla \varphi \right] \left(\frac{1}{T-t}, \frac{x}{T-t} \right),$$

and

$$\Delta \varphi_T = \frac{e^{-i\frac{|x|^2}{4(T-t)}}}{(T-t)^{D/2+2}} \left[-i\frac{|x|^2}{4} \varphi - \frac{D}{2} \varphi(T-t) - ix \cdot \nabla \varphi + \Delta \varphi \right] \left(\frac{1}{T-t}, \frac{x}{T-t} \right)$$

for the derivatives. For the nonlinear term we find

$$|x|^{-b} |\varphi_T|^{\frac{4-2b}{D}} \varphi_T = \frac{e^{-i\frac{|x|^2}{4(T-t)}}}{(T-t)^{D/2+2}} \left| \frac{x}{T-t} \right|^{-b} |\varphi|^{\frac{4-2b}{D}} \left(\frac{1}{T-t}, \frac{x}{T-t} \right),$$

and for the potential term

$$|x|^{-2} \varphi_T = \frac{e^{-i\frac{|x|^2}{4(T-t)}}}{(T-t)^{D/2+2}} \left| \frac{x}{T-t} \right|^{-2} \varphi \left(\frac{1}{T-t}, \frac{x}{T-t} \right).$$

Then it follows that

$$\begin{aligned} &i\partial_t \varphi_T + \Delta \varphi_T + c|x|^{-2} \varphi_T + |x|^{-b} |\varphi_T|^{\frac{4-2b}{D}} \varphi_T \\ &= \frac{e^{-i\frac{|x|^2}{4(T-t)}}}{(T-t)^{D/2+2}} \left[i\partial_t \varphi + \Delta \varphi + c|x|^{-2} \varphi + |x|^{-b} |\varphi|^{\frac{4-2b}{D}} \varphi \right] \left(\frac{1}{T-t}, \frac{x}{T-t} \right) = 0, \end{aligned}$$

which proves the Lemma.

Remark 4.1. From Lemma 4.1, we claim that any solution u of (1.12) satisfies $\|u\|_{L^2} \geq \|W\|_{L^2}$. Otherwise, applying Lemma 4.1 with $u(t, x) = e^{it}\phi(x)$, one could construct a finite time blow-up solution below the minimal mass threshold, which would contradict the global well-posedness result Corollary 2.1.

We now construct a 3-parameter family solutions of the Cauchy problem (1.1)–(1.2) which blow-up in finite time by using the pseudo-conformal transformation and the symmetries of the equation.

Proposition 4.1. For all $T \in \mathbb{R}$, $\lambda_0 > 0$ and $\gamma_0 \in \mathbb{R}$, then the function $\varphi_{T, \lambda_0, \gamma_0}$, defined by

$$\varphi_{T, \lambda_0, \gamma_0}(t, x) = e^{i\gamma_0} e^{i\frac{\lambda_0^2}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} \left(\frac{\lambda_0}{T-t}\right)^{D/2} W\left(\frac{\lambda_0 x}{T-t}\right) \quad (4.1)$$

is a minimal mass solution of the Cauchy problem (1.1)–(1.2) defined on $(-\infty, T)$, and which blows up with speed

$$\|\nabla \varphi_{T, \lambda_0, \gamma_0}(t, x)\|_{L^2} \sim \frac{C}{T-t}, \text{ as } t \rightarrow T, \text{ for } C > 0,$$

where $W \in \mathcal{A}$ is the ground state of (1.12).

Proof. The proof may be proved by applying Lemma 4.1 to the global solution

$$\varphi_{\lambda_0, \gamma_0}(t, x) = e^{i\gamma_0} e^{i\lambda_0^2 t} \lambda_0^{D/2} W(\lambda_0 x),$$

which can be obtained under the scaling and phase symmetries of the standing wave $\varphi(t, x) = e^{it}W(x)$.

Remark 4.2. Note that the blow-up solutions of the family showed in Proposition 4.1 can be retrieved from the solution

$$\varphi_{0,1,0}(t, x) = e^{i\frac{|x|^2}{4t}} e^{-\frac{i}{t}} \frac{1}{|t|^{D/2}} W\left(-\frac{x}{t}\right),$$

defined on $(-\infty, 0)$ and which blows up at $t = 0$ with speed

$$\|\nabla \varphi_{0,1,0}\|_{L^2} \sim \frac{C}{|t|} \text{ as } t \uparrow 0, \text{ for } C > 0.$$

Indeed, all the solutions $\varphi_{T, \lambda_0, \gamma_0}$ are equal to $\varphi_{0,1,0}$, up to the symmetries. In other words, if we apply the changes $\varphi(t, x) \rightarrow \lambda_0^{-D/2} \varphi(\lambda_0^{-2}t, \lambda_0^{-1}x)$, $\varphi(t, x) \rightarrow \varphi(t-T, x)$ and finally $\varphi(t, x) \rightarrow e^{i\gamma_0} \varphi(t, x)$ to $\varphi_{0,1,0}$, we obtain $\varphi_{T, \lambda_0, \gamma_0}$.

In order to prove our main result, we also need to deduce information about $\varphi(t)$ for $t < T$ from the blow-up solution at $t = T$. Using a result of Banica[1], we will show that $\varphi(t) \in H^1$ and $x\varphi \in L^2(\mathbb{R}^D)$ for all $t < T$. For $\varphi \in H^1$, $\theta \in C_0^\infty(\mathbb{R}^D)$ and $\eta \in \mathbb{R}$, we have that $\nabla(\varphi e^{i\eta\theta}) = (\nabla\varphi + i\eta\nabla\theta)\varphi e^{i\eta\theta}$, and so

$$|\nabla(\varphi e^{i\eta\theta})|^2 = |\nabla\varphi|^2 + 2\eta\nabla\theta \cdot \text{Im}(\overline{\varphi}\nabla\varphi) + \eta^2|\nabla\theta|^2|\varphi|^2.$$

By a direct calculation we have

$$E(\varphi e^{i\eta\theta}) = E(\varphi) + 2\eta \int_{\mathbb{R}^D} \nabla\theta \cdot \text{Im}(\overline{\varphi}\nabla\varphi) dx + \eta^2 \int_{\mathbb{R}^D} |\nabla\theta|^2 |\varphi|^2 dx. \quad (4.2)$$

Then the refined Cauchy-Schwarz estimate for critical mass functions can be proved.

Lemma 4.2. Let $\varphi \in H^1$ be a function such that $\|\varphi\|_{L^2} = \|W\|_{L^2}$. Then for all $\theta \in C_0^\infty(\mathbb{R}^D)$, one has

$$\left| \int_{\mathbb{R}^D} \nabla\theta \cdot \text{Im}(\overline{\varphi}\nabla\varphi) dx \right| \leq \sqrt{E(\varphi)} \left(\int_{\mathbb{R}^D} |\nabla\theta|^2 |\varphi|^2 dx \right)^{1/2}.$$

Proof. For all $\eta \in \mathbb{R}$, we now have $\|\varphi e^{i\eta\theta}\|_{L^2} = \|\varphi\|_{L^2} = \|W\|_{L^2}$, so $E(\varphi e^{i\eta\theta}) \geq 0$ and $E(\varphi) \geq 0$ by (2.3). The result follows from the quadratic polynomial expression (4.2) in η of $E(\varphi e^{i\eta\theta})$, which thus must have a non-positive discriminant. This completes the proof.

In the reminder of this section we prove the main result.

Proof of Theorem 1.1. Let φ be a solution of the Cauchy problem (1.1)–(1.2) such that $\|\varphi\|_{L^2} = \|W\|_{L^2}$ and which blows up in finite time. The proof then falls into several steps.

Step 1. Let $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ be a sequence of times such that $t_n \rightarrow T$ as $n \rightarrow \infty$. We set

$$\varphi_n = \varphi(t_n), \quad \lambda_n = \sqrt{\frac{H(W)}{H(\varphi_n)}}, \quad v_n(x) = \lambda_n^{D/2} \varphi_n(\lambda_n x). \quad (4.3)$$

We note that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, and

$$\|v_n\|_{L^2} = \|\varphi_n\|_{L^2} = \|\varphi_0\|_{L^2} = \|W\|_{L^2}, \quad H(v_n) = \lambda_n^2 H(\varphi_n) = H(W).$$

On the other hand, by conservation of the energy,

$$\begin{aligned} E(v_n) &= H(v_n) - \frac{D}{D+2-b} \int_{\mathbb{R}^D} |x_n|^{-b} |v_n|^{2+\frac{4-2b}{D}} dx \\ &= \lambda_n^2 H(\varphi_n) - \frac{D}{D+2-b} \lambda_n^2 \int_{\mathbb{R}^D} |x_n|^{-b} |\varphi_n|^{2+\frac{4-2b}{D}} dx \\ &= \lambda_n^2 E(\varphi_n) = \lambda_n^2 E(\varphi_0) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence, by Theorem 3.2, there exist $\gamma_0 \in \mathbb{R}$ such that, up to extracting a subsequence of $\{v_n\}$, we have

$$\lim_{n \rightarrow +\infty} \|v_n - e^{i\gamma_0} W\|_{H^1} = 0. \quad (4.4)$$

Step 2. Using the variational arguments, we prove that blowup's profile is a Dirac function, i.e. in the sense of distributions,

$$|\varphi_n|^2 \rightarrow \|W\|_{L^2} \delta_0.$$

Indeed, by scaling (4.3), we observe that for $u \in C_0^\infty(\mathbb{R}^D)$,

$$\int_{\mathbb{R}^D} |v_n(x)|^2 u(\lambda_n x) dx = \int_{\mathbb{R}^D} |\varphi_n(x)|^2 u(x) dx,$$

and

$$\int_{\mathbb{R}^D} |v_n(x)|^2 u(\lambda_n x) dx = \int_{\mathbb{R}^D} (|v_n(x)|^2 - |W|^2) u(\lambda_n x) dx + \int_{\mathbb{R}^D} |W(x)|^2 u(0) dx + \int_{\mathbb{R}^D} |W|^2 [u(\lambda_n x) - u(0)] dx.$$

Thus, we obtain

$$\left| \int_{\mathbb{R}^D} |\varphi_n(x)|^2 u(x) dx - \|W\|_{L^2}^2 u(0) \right| \leq \|u\|_\infty \int_{\mathbb{R}^D} (|v_n(x)|^2 - |W(x)|^2) dx + \int_{\mathbb{R}^D} |W(x)|^2 |u(\lambda_n x) - u(0)| dx.$$

(4.4) implies that $|v_n|^2$ converges to $|W|^2$ strongly in $L^1(\mathbb{R}^D)$, so the first integral converges to 0 as $n \rightarrow +\infty$. Since $\lambda_n \rightarrow 0$, then the second integral also converges to 0 as $n \rightarrow +\infty$ by the dominated convergence theorem, thus

$$\int_{\mathbb{R}^D} |\varphi_n(x)|^2 u(x) dx \rightarrow \|W\|_{L^2}^2 u(0). \quad (4.5)$$

Step 3. We claim that $\varphi(t) \in H^1$ and $x\varphi \in L^2(\mathbb{R}^D)$ for all $t \in [0, T)$. Let ϕ a nonnegative radial $C_0^\infty(\mathbb{R}^D)$ function such that $\phi(x) = |x|^2$ for $|x| \leq 1$ and $|\nabla \phi(x)|^2 \leq C\phi(x)$. For every $R > 0$, we define $\phi_R(x) = R^2 \phi(x/R)$, and for all $t \in [0, T)$,

$$\phi_R(x) = R^2 \phi(x/R), \quad J_R(t) = \int_{\mathbb{R}^D} \phi_R(x) |\varphi(t, x)|^2 dx.$$

By a direct calculation (also see [2, 22]), we have

$$J'_R(t) = 2\operatorname{Re} \int_{\mathbb{R}^D} \phi_R \bar{\varphi} \varphi_t dx = 2 \int_{\mathbb{R}^D} \nabla \phi_R \cdot \operatorname{Im}(\bar{\varphi} \nabla \varphi) dx.$$

Since $\|\varphi\|_{L^2} = \|W\|_{L^2}$ and $|\nabla \phi_R|^2 \leq C\phi_R$, we can apply Lemma 4.2 to get

$$|J'_R(t)| \leq 2\sqrt{E(\varphi)} \left(\int_{\mathbb{R}^D} |\nabla \phi_R|^2 |\varphi|^2 dx \right)^{1/2} \leq C\sqrt{E(\varphi_0)}\sqrt{J_R(t)}.$$

By integration, we obtain that for $t \in [0, T)$,

$$|\sqrt{J_R(t)} - \sqrt{J_R(t_n)}| \leq C|t - t_n|.$$

Since $J_R(t_n) \rightarrow 0$ by (4.5). Thus, letting $t_n \rightarrow T$, we obtain that for all $t \in [0, T)$ and all $R > 0$,

$$J_R(t) \leq C(T - t)^2.$$

Since the right-hand side of the last expression of R , we obtain that by letting $R \rightarrow \infty$, for all $t \in [0, T)$,

$$\varphi(t) \in H^1, \quad x\varphi \in L^2(\mathbb{R}^D) \text{ and } 0 \leq J(t) \leq C(T - t)^2,$$

where $J(t) = \int_{\mathbb{R}^D} |x|^2 |\varphi|^2 dx$. From this estimate, we can extend by continuity $J(t)$ at $t = T$ by setting $J(T) = 0$, from which we also obtain $J'(T) = 0$. Moreover, since $\varphi(t) \in H^1$, $x\varphi \in L^2(\mathbb{R}^D)$ and φ is a solution of the Cauchy problem (1.1)–(1.2), we obtain $J''(t) = 8E(\varphi_0)$, which finally gives, for all $t \in [0, T)$,

$$J(t) = 4E(\varphi_0)(T - t)^2.$$

Letting $t = 0$, we find

$$J(0) = \int_{\mathbb{R}^D} |x|^2 |\varphi_0|^2 dx = 4E(\varphi_0)T^2 \text{ and } J'(0) = 4 \int_{\mathbb{R}^D} x \cdot \operatorname{Im}(\bar{\varphi}_0 \nabla \varphi_0) dx = -8E(\varphi_0)T.$$

Step 4. Determination of φ_0 and conclusion. We finally apply identity (4.2) to φ_0 and $\eta = \frac{|x|^2}{2T}$, with $\theta(x) = \frac{|x|^2}{2}$. Since $\nabla \theta(x) = x$, we obtain

$$\begin{aligned} E(\varphi_0 e^{i\frac{|x|^2}{4T}}) &= E(\varphi_0) + \frac{1}{T} \int_{\mathbb{R}^D} x \cdot \operatorname{Im}(\bar{\varphi}_0 \nabla \varphi_0) dx + \frac{1}{4T^2} \int_{\mathbb{R}^D} |x|^2 |\varphi_0|^2 dx \\ &= E(\varphi_0) + \frac{1}{T}(-2E(\varphi_0)T) + \frac{1}{4T^2}(4E(\varphi_0)T^2) \\ &= 0. \end{aligned}$$

Note that this calculation justifies, a posteriori, the application of (4.2) with the function $\theta(x) = \frac{|x|^2}{2} \notin C_0^\infty(\mathbb{R}^D)$. Hence, we have $\|\varphi_0 e^{i\frac{|x|^2}{4T}}\|_{L^2} = \|W\|_{L^2}$ and $E(\varphi_0 e^{i\frac{|x|^2}{4T}}) = 0$, and we deduce from the variational characteristic of ground state that there exist $\lambda_0 > 0$ and $\gamma_0 \in \mathbb{R}$ such that

$$\varphi_0 = e^{i\gamma_0} e^{-i\frac{|x|^2}{4T}} \lambda_0^{\frac{D}{2}} W(\lambda_0 x).$$

Finally, we use the pseudo-conformal transformation. We define $\tilde{\lambda}_0 = \lambda_0 T > 0$ and $\tilde{\gamma}_0 = \gamma_0 - \lambda_0^2 T \in \mathbb{R}$, and write φ_0 as

$$\varphi_0(x) = e^{i\tilde{\gamma}_0} e^{i\frac{\tilde{\lambda}_0^2}{T}} e^{-i\frac{|x|^2}{4T}} \left(\frac{\tilde{\lambda}_0}{T} \right)^{D/2} \tilde{W}\left(\frac{\tilde{\lambda}_0 x}{T} \right).$$

Thus, $\varphi_0(x) = \varphi_{T, \tilde{\lambda}_0, \tilde{\gamma}_0}(0)$, where $\varphi_{T, \tilde{\lambda}_0, \tilde{\gamma}_0}(0)$ is defined by (4.1), which concludes the proof.

Acknowledgements: This research was supported by the National Natural Science Foundation of China 11871138.

Conflict of interest: Authors state no conflict of interest.

References

- [1] V. Banica, Remarks on the blow-up for the nonlinear Schrödinger equation with critical mass on a plane domain, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 3 (2004), no. 5, 139-170.
- [2] T. Cazenave, *Semilinear Schrödinger Equations*, American Mathematical Society, 2003.
- [3] S. Chen, X. Tang, Berestycki-Lions conditions on ground state solutions for a nonlinear Schrödinger equation with variable potentials, *Adv. Nonlinear Anal.* 9 (2020), no. 1, 496-515.
- [4] V. Combet, F. Genoud, Classification of minimal mass blow-up solutions for an L^2 critical inhomogeneous NLS, *J. Evol. Equ.* 16 (2016), no. 2, 483-500.
- [5] E. Csobo, F. Genoud, Minimal mass blow-up solutions for the L^2 critical NLS with inverse-square potential, *Nonlinear Analysis* 168 (2018), 110-129.
- [6] F. Genoud, C.A. Stuart, Schrödinger equations with a spatially decaying nonlinearity: existence and stability of standing waves, *Discrete Contin. Dyn. Syst.* 21 (2008), no. 1, 137-186.
- [7] F. Genoud, Bifurcation and stability of travelling waves in self-focusing planar waveguided, *Adv. Nonlinear Stud.* 10 (2010), no. 2, 357-400.
- [8] F. Genoud, A uniqueness result for $\Delta u - \lambda u + V(|x|)u^p = 0$ on $\mathbb{R}^2$, *Adv. Nonlinear Stud.* 11 (2011), 483-491.
- [9] F. Genoud, An inhomogeneous, L^2 critical, nonlinear Schrödinger equation, *Z. Anal. Anwend.* 31(2012), no. 3, 283-290.
- [10] J. Ginibre, G. Velo, On a class of nonlinear Schrödinger equations I, The Cauchy problem, general case, *J. Funct. Anal.* 32 (1979), no. 1, 1-32.
- [11] H. Hajaiej, Cases of equality and strict inequality in the extend Hardy-Littlewood inequalities, *Proc. Roy. Soc. Edinburgh Sect. A* 135 (2005), no.3, 643-661.
- [12] E.H. Lieb, M. Loss, *Analysis*, American Mathematical Society, 2000.
- [13] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equation with critical power, *Duke Math. J.* 69 (1993), no. 2, 427-453.
- [14] F. Merle, On uniqueness and continuation properties after blowup time of self-similar solutions of the nonlinear Schrödinger equation with critical exponent and critical mass, *Comm. Pure Appl. Math.* 45 (1992), no. 2, 203-254.
- [15] N. Okazawa, T. Suzuki, T. Yokota, Energy methods for abstract nonlinear Schrödinger equations, *Evol. Equ. Control Theory* 1 (2012), no. 2, 337-354.
- [16] W.A. Strauss, Existence of solitary waves in higher dimensions, *Commun. Math. Phys.* 55 (1977), no. 2, 149-162.
- [17] E. Montefusco, Lower semicontinuity of functionals via the concentration-compactness principle, *J. Math. Anal. Appl.* 263 (2001), no. 1, 264-276.
- [18] D. Mukherjee, P.T. Nam, P.T. Nguyen, Uniqueness of ground state and minimal-mass blow-up solutions for focusing NLS with Hardy potential, 2019.
- [19] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates. *Commun. Math. Phys.* 87 (1983), no. 4, 567-576.
- [20] M.I. Weinstein, On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations, *Commun. Partial Differential Equations* 11 (1986), no. 5, 545-565.
- [21] E. Yanagida, Uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$ in $\mathbb{R}^N$, *Arch. Rational Mech. Anal.* 115 (1991), no. 3, 257-274.
- [22] J. Zhang, Stability of attractive Bose-Einstein condensates, *J. Statistical Physics* 101 (2000), no. 3, 731-746.
- [23] M. Zhang, M.S. Ahmed, Sharp conditions of global existence for nonlinear Schrödinger equation with a harmonic potential, *Adv. Nonlinear Anal.* 9 (2020), no. 1, 882-894.