Research Article

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Blow-up solutions with minimal mass for nonlinear Schrödinger equation with variable potential

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Abstract: This paper studies the mass-critical variable coefficient nonlinear Schrödinger equation. We first get the existence of the ground state by solving a minimization problem. Then we prove a compactness result by the variational characterization of the ground state solutions. In addition, we construct the blow-up solutions at the minimal mass threshold and further prove the uniqueness result on the minimal mass blow-up solutions which are pseudo-conformal transformation of the ground states.

Keywords: variable coefficient nonlinear Schrödinger equation; minimal mass blow-up solutions; variational characterization; ground state; compactness

MSC: 35Q55; 35B44

1 Introduction

In this paper, we study the behavior of blowup solutions for the variable coefficient nonlinear Schrödinger equation

\[ \imath \phi_t + \Delta \phi + c|x|^{-2} \phi + |x|^{-b} |\phi|^{p-1} \phi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D, \]

and

\[ \phi(0, x) = \varphi_0, \quad x \in \mathbb{R}^D, \]

in the case \( p = 1 + \frac{4-2b}{D} \). Here and hereafter, \( D \geq 3 \) is the space dimension, \( 0 < b < 2 \), \( \varphi = \varphi(t, x) : [0, T) \times \mathbb{R}^D \rightarrow C \) is a complex value wave function with \( 0 < T \leq \infty \), \( i = \sqrt{-1} \), \( \Delta \) is the Laplace operator and \( c \in (0, c^*) \), where \( c^* = \frac{(D-2)^2}{4} \) is the best constant in Hardy’s inequality:

\[ c^* \int_{\mathbb{R}^D} |x|^{-2} |u|^2 \, dx \leq \int_{\mathbb{R}^D} |\nabla u|^2 \, dx. \] (1.3)

More precisely, we say that \( \varphi(\cdot) \) is a solution of the Cauchy problem (1.1)–(1.2) on \([0, T)\) if for all \( t \in [0, T) \),

\[ \varphi(t) = S(t) \varphi + \imath \int_0^t S(t-s) (|x|^{-b} |\varphi(s)|^{\frac{4-2b}{2-b}} \varphi(s)) \, ds, \]

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where $S(\cdot)$ is the group with infinitesimal generator $i(\Delta + c|x|^{-2})$.

From [2, 6, 15], we know that the Cauchy problem (1.1)-(1.2) is locally well-posed in $H^1(\mathbb{R}^D)$: there exists a unique solution $\varphi(t, x)$ of the Cauchy problem (1.1)-(1.2) in $C([0, T); H^1(\mathbb{R}^D))$ for some $T \in (0, \infty)$ (maximal existence time), and for all $t \in [0, T)$, we have the following alternatives:

$$T = +\infty,$$

or else

$$T < \infty \quad \text{and} \quad \lim_{t \to T^-} \|\varphi(t)\|_{H^1} = +\infty \quad \text{(blowup)},$$

where $\| \cdot \|_{H^1}$ is the usual norm on $H^1$, and $H^1$ is $H^1(\mathbb{R}^D)$. Furthermore, the unique solution $\varphi(t, x)$ satisfies the following two conservation laws: for all $t \in [0, T)$,

$$\int_{\mathbb{R}^D} |\varphi(t, x)|^2 \, dx = \int_{\mathbb{R}^D} |\varphi_0(x)|^2 \, dx, \quad (1A)$$

and

$$E(\varphi(t)) = \int_{\mathbb{R}^D} |\nabla \varphi(t)|^2 \, dx - \int_{\mathbb{R}^D} c|x|^{-2} |\varphi(t)|^2 \, dx - \frac{D}{2} \int_{\mathbb{R}^D} |x|^{-b} |\varphi(t)|^{2 + \frac{2b}{D-b}} \, dx = E(\varphi_0). \quad (1.5)$$

Also, the $L^2$-norm of $\varphi(t, x)$ is invariant under the transformation

$$\varphi \to \varphi_\lambda(t, x) = \lambda^{D/2} \varphi(\lambda^2 t, \lambda x) \quad \text{for} \lambda > 0.$$ 

In other words, $\|\varphi_\lambda(t)\|_{L^2} = \|\varphi(t)\|_{L^2}$ and the Cauchy problem (1.1)-(1.2) is called $L^2$-critical.

Eq.(1.1) is also called inhomogeneous nonlinear Schrödinger equation with inverse-square potential. Let us now recall some known results about the nonlinear Schrödinger equation. For the classical nonlinear Schrödinger equation, i.e. $c = b = 0$,

$$i\varphi_t + \Delta \varphi + |\varphi|^2 \varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D. \quad (1.6)$$

Let $\varphi(t, x) = e^{it} u(x)$ be a standing wave solution of (1.6). Then $u(x)$ satisfies the following time-independent Schrödinger equation

$$\Delta u - u + |u|^2 u = 0, \quad u \in H^1. \quad (1.7)$$

It is well known that (1.7) possesses a unique radial positive solution $Q(x)$ (see [2]). Moreover, the Cauchy problem (1.6)-(1.2) has no blow-up solution in the class $\{\varphi \in H^1 | \|\varphi\|_{L^2} < \|Q\|_{L^2}\}$ (see [19]), while in the class $\{\varphi \in H^1 | \|\varphi\|_{L^2} = \|Q\|_{L^2}\}$, there exists a unique blowup solution

$$\tilde{Q}(x, t) = \frac{1}{|t|^{D/2}} e^{i\frac{\|\varphi\|_{L^2}^2}{\|Q\|_{L^2}^2} t} Q \left( \frac{x}{t} \right).$$

up to the invariances of the equation (see [10, 13, 14, 20]).

For the nonlinear Schrödinger equation with inverse-square potential, i.e. $b = 0, c \neq 0$,

$$i\varphi_t + \Delta \varphi + c|x|^{-2} \varphi + |\varphi|^2 \varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D. \quad (1.8)$$

Let $\varphi(t, x) = e^{it} u(x)$ be a standing wave solution of (1.8). Then $u(x)$ satisfies the following time-independent Schrödinger equation

$$\Delta u + c|x|^{-2} u - u + |u|^2 u = 0, \quad u \in H^1. \quad (1.9)$$

By solving a variational problem, Csobo and Genoud [5] proved that (1.9) has radial positive solutions to exist. Mukherjee, Nam and Nguyen [18] showed the uniqueness of radial positive solution of (1.9). In addition, Csobo and Genoud [5] proved that all $H^1$-solutions of Eq.(1.8) are global if $\|\varphi\|_{L^2} < \|U\|_{L^2}$ and by using the pseudo-conformal transformation, they constructed the minimal mass blow-up solutions defined as

$$\varphi(t, x) = e^{i\gamma_0} e^{-\frac{i\|\varphi\|_{L^2}^2}{\|Q\|_{L^2}^2} t} \left( \frac{\Lambda_0}{T-t} \right)^{D/2} U \left( \frac{\Lambda_0 x}{T-t} \right),$$

where $\gamma_0$ is an arbitrary constant.
such that $\|\varphi_0\|_{L^2} = \|U\|_{L^2}$, where $T \in \mathbb{R}$, $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$, $U$ is the unique positive radial solution of (1.9).

For the inhomogeneous nonlinear Schrödinger equation, i.e. $c = 0$, $b \neq 0$,
\begin{equation}
\label{1.10}
i\varphi_t + \Delta \varphi + |x|^{-b}|\varphi|^{\frac{4+b}{2}} \varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D.
\end{equation}

Let $\varphi(t, x) = e^{it}u(x)$ be a standing wave solution of (1.10). Then $u(x)$ satisfies the following time-independent Schrödinger equation
\begin{equation}
\label{1.11}
\Delta u - u + |x|^{-b}|u|^{\frac{4+b}{2}} u = 0, \quad u \in H^1.
\end{equation}

By solving a variational problem, Genoud [9] proved that (1.11) has positive radial solutions to exist and the sharp condition for global existence of $H^1$-solutions was established, also, the solutions with critical mass which blow-up in finite time were proved. By Yanagida [21] and Genoud [8], the positive radial solution of (1.11) is unique. Furthermore, Combet and Genoud [4, 9] constructed a 3-parameter family of critical mass solutions of (1.10) which blow-up in finite time, defined by
\begin{equation}
\label{1.12}
\varphi(t) = e^{i\gamma_0} e^{i\frac{\lambda_0 t}{T-t}} e^{-\frac{|x|^2}{4\pi T}} \left( \frac{\lambda_0}{T-t} \right)^{D/2} \psi \left( \frac{\lambda_0 x}{T-t} \right),
\end{equation}

with $\|\varphi_0\|_{L^2} = \|\psi\|_{L^2}$, where $T \in \mathbb{R}$, $\lambda_0 > 0$, $\gamma_0 \in \mathbb{R}$, $\psi$ is the unique positive radial solution of (1.11). We also refer to Chen and Tang [3], Zhang and Ahmed [23] for related contributions on the nonlinear Schrödinger equation.

Motivated by the above studies, it is of interest to find the uniqueness of the minimal mass blow-up solutions for the Cauchy problem (1.1)-(1.2). In order to solve this problem, we need a notion of a ground state. So we consider the periodic solutions of equation of the form $\varphi(t, x) = e^{it}u(x)$, then $u(x)$ solves the nonlinear elliptic equation
\begin{equation}
\label{1.13}
\Delta u + c|\varphi|^{\frac{4+b}{2}} u - |x|^{-b}|u|^{\frac{4+b}{2}} u = 0, \quad u \in H^1.
\end{equation}

In section 2, we will prove the existence in $H^1$ of a positive radial solution of Eq.(1.12) by using Weinstein’s argument [19] and that the set of solutions of (1.12) has a “minimal” element in $L^2$ called ground state. However, for $c > 0$, we are not aware of any uniqueness result of (1.12) on $\mathbb{R}^D$. According to [8], [18] and [21], we can assume that (1.12) has a unique positive radial solution, which we will denote by $W$ throughout the paper. In terms of the existence of ground state, we first show the global well-posedness condition for the Cauchy problem (1.1)-(1.2). Then, we shall exhibit blow-up solutions at the mass $\|\varphi_0\|_{L^2} = \|W\|_{L^2}$ by applying the pseudo-conformal transformation to the standing waves $e^{it}W(x)$, which is thus the minimal mass where blowup can occur.

Now we state our main result.

**Theorem 1.1.** (Determination of minimal blow-up solutions) Let initial data $\varphi_0 \in H^1$ such that the solution $\varphi(t, x)$ of the Cauchy problem (1.1)-(1.2) blows up in finite time $T > 0$. Moreover, assume that $\|\varphi_0\|_{L^2} = \|W\|_{L^2}$.

Then there exist $\gamma_0 \in \mathbb{R}$ and $\lambda_0 > 0$ such that for all $t \in [0, T)$,
\begin{equation}
\varphi(t, x) = \varphi_{T, \lambda_0, \gamma_0} = e^{i\gamma_0} e^{i\frac{\lambda_0 t}{T-t}} e^{-\frac{|x|^2}{4\pi T}} W \left( \frac{\lambda_0 x}{T-t} \right),
\end{equation}

where $W(x)$ is the ground state of Eq.(1.12).

One can ask if there are other minimal mass blow-up solutions? That is, assume that
- $\varphi(t, x)$ blows up in finite time $T$, and
- $\|\varphi(t, x)\|_{L^2} = \|W\|_{L^2}$ for all $t$.

Is there $\gamma_0 \in \mathbb{R}$ and $\lambda_0 > 0$ such that
\begin{equation}
\tilde{\varphi}(t, x) = \varphi(t, x)?
\end{equation}

The aim of this paper is to prove this uniqueness result on the minimal mass blow-up solutions. The method is mainly based on a compactness result.
The plan of this paper is as follows. In section 2, we prove the existence of ground state by solving a minimization problem. In section 3, we prove a compactness result, crucial in the proof of Theorem 1.1. In section 4, we construct the pseudo-conformal transformation and then give the proof of Theorem 1.1.

2 Ground State

In this section, we prove the existence of a positive radial solution \( W \in H^1 \) for (1.12). Such a solution is obtained by variational approach and called ground state. We first state the following result.

**Lemma 2.1.** For \( D \geq 3 \), \( 0 < c < c^* = (\frac{D-2}{2})^2 \), \( 0 < b < 2 \) and \( u \in H^1 \setminus \{0\} \), define the functional

\[
J(u) := \left( \int_{\mathbb{R}^D} (|\nabla u|^2 dx - c|u|^{-2}|u|^2) dx \right) \left( \int_{\mathbb{R}^D} |u|^2 dx \right)^{\frac{2-b}{2}} \left( \int_{\mathbb{R}^D} |x|^{-b} |u|^{2+\frac{4-2b}{D}} dx \right)^{-\frac{b}{2+\frac{4-2b}{D}}},
\]

and the variational problem

\[
d := \inf_{u \in H^1 \setminus \{0\}} J(u).
\]

Then one has the followings:

(i). There exists a positive radial function \( v^* (x) \in H^1 \) such that the variational problem (1.12) is attained at \( v^* \).

(ii). Let \( v^* = [(1 + \frac{2-b}{D})dx]^{-\frac{b}{2+\frac{4-2b}{D}}} W(\lambda x) \) with \( \lambda = \sqrt{\frac{2-b}{D}} \). Then \( W(x) \) is the positive radial solution of (1.12).

(iii). Assume that the positive radial solution \( W(x) \) of the nonlinear elliptic equation (1.12) is unique. Then one has the sharp interpolation inequality for \( u \in H^1 (\mathbb{R}^D) \) with \( D \geq 3 \),

\[
\int_{\mathbb{R}^D} |x|^{-b} |u|^{2+\frac{4-2b}{D}} dx \leq \frac{D+2-b}{D} \left( \int_{\mathbb{R}^D} |W|^2 dx \right)^{-\frac{b}{2+\frac{4-2b}{D}}} \left( \int_{\mathbb{R}^D} (|\nabla u|^2 dx - c|u|^{-2}|u|^2) dx \right) \left( \int_{\mathbb{R}^D} |u|^2 dx \right)^{\frac{2-b}{2}}.
\]

**Proof.** From Hardy inequality (1.3), one has that for \( u \in H^1 (\mathbb{R}^D) \),

\[
(1 - \frac{c}{c^*}) \int_{\mathbb{R}^D} |\nabla u|^2 dx \leq \int_{\mathbb{R}^D} (|\nabla u|^2 - c|u|^{-2}|u|^2) dx \leq \int_{\mathbb{R}^D} |\nabla u|^2 dx.
\]

By (2.1), \( J(u) \) is well-defined in \( H^1 \setminus \{0\} \), and \( J(u) \geq 0 \). Furthermore, \( J(u) \) is invariant under the scaling \( u(x) \rightarrow u^{\mu,\lambda}(x) := \mu u(\lambda x) (\mu, \lambda > 0) \), that is

\[
J(u^{\mu,\lambda}) = J(u) \quad \text{for} \quad \mu > 0, \lambda > 0.
\]

Let \( u^* \) be the Schwarz symmetrization of \( u \). By [11, 12], one has that

\[
\int_{\mathbb{R}^D} |x|^{-2} |u^*|^2 dx \geq \int_{\mathbb{R}^D} |x|^{-2} |u|^2 dx,
\]

\[
\int_{\mathbb{R}^D} |x|^{-b} |u^*|^{2+\frac{4-2b}{D}} dx > \int_{\mathbb{R}^D} |x|^{-b} |u|^{2+\frac{4-2b}{D}} dx,
\]

\[
\int_{\mathbb{R}^D} |\nabla u^*|^2 dx \leq \int_{\mathbb{R}^D} |\nabla u|^2 dx, \quad \int_{\mathbb{R}^D} u^*^2 dx = \int_{\mathbb{R}^D} u^2 dx.
\]

By (2.2), we can choose a minimizing sequence \( \{u_n\} \subset H^1 \setminus \{0\} \) such that \( \lim_{n \to \infty} J(u_n) = d \). Then from (2.1), (2.2), (2.6), (2.7), (2.8), one has

\[
\lim_{n \to \infty} J(u_n^*) = d,
\]
where \( u_n^* \) denotes the Schwarz symmetrization of \( u_n \) for any \( n \in \mathbb{N} \). Now we choose

\[
\lambda_n = \left[ \frac{\int_{\mathbb{R}^D} |u_n^*|^2 \, dx}{\int_{\mathbb{R}^D} (|\nabla u_n^*|^2 - c|x|^{-2} |u_n^*|^2) \, dx} \right]^{1/2},
\]

\[
\mu_n = \left( \frac{\int_{\mathbb{R}^D} |u_n^*|^2 \, dx}{\int_{\mathbb{R}^D} (|\nabla u_n^*|^2 - c|x|^{-2} |u_n^*|^2) \, dx} \right)^{1/2},
\]

and take \( v_n = (u_n^*)^{\mu_n, \lambda_n} \). By (2.5) and (2.9), we have that

\[
v_n \geq 0, \quad v_n(x) = v_n(|x|), \tag{2.10}
\]

\[
\int_{\mathbb{R}^D} |v_n|^2 \, dx = 1, \quad \int_{\mathbb{R}^D} (|\nabla v_n|^2 - c|x|^{-2} |v_n|^2) \, dx = 1, \tag{2.11}
\]

\[
\lim_{n \to \infty} J(v_n) = d. \tag{2.12}
\]

Thus \( \{v_n\} \) is bounded in \( H^1 \). Therefore, up to subsequence, we can suppose that \( \{v_n\} \) has a weak limit \( v^* \in H^1 \). By weak lower semi-continuity (see [17]), we obtain that

\[
\int_{\mathbb{R}^D} |v^*|^2 \, dx \leq 1, \quad \int_{\mathbb{R}^D} (|\nabla v|^2 - c|x|^{-2} |v|^2) \, dx \leq 1. \tag{2.13}
\]

It follows from the weakly sequence continuous of the functional \( \int_{\mathbb{R}^D} |x|^{-b} |v^*|^{2+\frac{4b}{D}} \, dx \) (see [6]) that

\[
d \leq J(v^*) \leq \frac{1}{\int_{\mathbb{R}^D} |x|^{-b} |v^*|^{2+\frac{4b}{D}} \, dx} \lim_{n \to \infty} J(v_n) = d. \tag{2.14}
\]

It yields that

\[
\int_{\mathbb{R}^D} |v^*|^2 \, dx = 1, \quad \int_{\mathbb{R}^D} (|\nabla v|^2 - c|x|^{-2} |v|^2) \, dx = 1. \tag{2.15}
\]

It follows that

\[
\lim_{n \to \infty} v_n = v^* \text{ in } H^1. \tag{2.16}
\]

Thus we get the proof of (i). Now we prove (ii).

Since \( v^* \) is the minimizer of (2.2), \( v^* \) must satisfy the Euler-Lagrange equation

\[
\frac{d}{dx} J(v^* + \epsilon \eta) = 0, \text{ for all } \eta \in C_0^\infty(\mathbb{R}^D). \tag{2.17}
\]

By (2.14), (2.15) it follows that

\[
\Delta v^* + c|x|^{-2} v^* - \frac{2 - b}{D} v^* + \left( 1 + \frac{2 - b}{D} \right) d|x|^{-b} (v^*)^{1+\frac{4b}{D}} = 0. \tag{2.18}
\]

Let \( v^* = [(1 + \frac{2 - b}{D}) d]^{-\frac{D}{2b}} \lambda^{\frac{D}{2b}} W(\lambda x) \) with \( \lambda = \sqrt{\frac{2 - b}{D}} \). Then from (2.18), \( W(x) \) is the positive radial solution of (1.12). This completes the proof of (ii). Then we prove (iii) in the following.

Since \( v^* = [(1 + \frac{2 - b}{D}) d]^{-\frac{D}{2b}} \lambda^{\frac{D}{2b}} W(\lambda x) \) and \( \lambda = \sqrt{\frac{2 - b}{D}} \), by (2.15) it follows that

\[
d = \left( 1 + \frac{2 - b}{D} \right)^{-1} \left( \int_{\mathbb{R}^D} |W|^2 \, dx \right)^{\frac{-b}{2b}}. \tag{2.19}
\]

Thus if we assume that (1.12) has a unique positive radial solution, then for (ii) and (2.2), we get (2.3).

This completes the proof.
Remark 2.1. Let \( \xi \) be the minimizer of \( f(u) \), then we define a set \( \mathcal{A} \) as follows:

\[
\mathcal{A} = \left\{ V(x) : \xi(x) = \left[ (1 + 2 - \frac{b}{D}) d \right]^{-\frac{D}{2}} \lambda^b V(\lambda x) \text{ with } \lambda = \sqrt{\frac{2 - b}{D}} \right\}.
\]

It is obvious that \( W \in \mathcal{A} \), then \( \|W\|_{L^2}^2 = [(1 + \frac{2-b}{D})d]^{\frac{D}{2}} \). Moreover, \( W \) satisfying the elliptic equation (1.12). Therefore, we call \( W \in \mathcal{A} \) ground state of Eq. (1.12).

In the following, we show that the ground state solutions play an important role in the global well-posedness for the Cauchy problem (1.1)–(1.2).

Corollary 2.1. Let \( c \in (0, c^*) \), if \( \varphi_0 \in H^1 \) satisfies

\[
\|\varphi_0\|_{L^2} < \|W\|_{L^2},
\]

then the corresponding solution of the Cauchy problem (1.1)–(1.2) is global, where \( W \in \mathcal{A} \) is the ground state of (1.12).

Proof. By the local well-posedness theory, we only need to show that \( \int_{\mathbb{R}^D}(|\nabla \varphi|^2 - c|x|^{-2}(|\varphi|^2))dx \) remains bounded. From the conservation laws of mass and energy,

\[
E(\varphi_0) = \int_{\mathbb{R}^D}(|\nabla \varphi|^2 - c|x|^{-2}(|\varphi|^2))dx - \frac{D}{D+2-b} \int_{\mathbb{R}^D} |x|^{-b} |\varphi|^{2+\frac{4}{D-b}} dx.
\]

Hence, by (2.3),

\[
E(\varphi_0) \geq \int_{\mathbb{R}^D}(|\nabla \varphi|^2 - c|x|^{-2}(|\varphi|^2))dx \left[ 1 - \left( \frac{\|\varphi_0\|_{L^2}}{\|W\|_{L^2}} \right)^\frac{4}{D-b} \right].
\]

If

\[
\|\varphi_0\|_{L^2} < \|W\|_{L^2},
\]

then \( \int_{\mathbb{R}^D}(|\nabla \varphi|^2 - c|x|^{-2}(|\varphi|^2))dx \) is bounded and so the solution is global. This completes the proof.

3 Compactness

In this section, we will establish a compactness result which is important in the proof of Theorem 1.1. For this aim, we introduce the notion of the variational characterization of ground state solutions.

Theorem 3.1. Let \( W \) be the positive radial minimizer of \( f(u) \) and \( \nu \in H^1 \) satisfy

\[
\|\nu\|_{L^2} = \|W\|_{L^2} \text{ and } E(\nu) = 0.
\]

Then there exist \( \lambda_0 > 0 \) and \( \gamma_0 \in \mathbb{R} \) such that \( \nu(x) = e^{i\gamma_0} \lambda_0^{D/2} W(\lambda_0 x) \).

Proof. It follows from Lemma 2.1 and (3.1) that \( \nu \) is a minimizer of \( f(u) \). Since \( \|\nabla(|\nu|)\|_{L^2} \leq \|\nabla \psi\|_{L^2} \), then \( |\nu| \) is also a minimizer. Furthermore, any positive minimizer is radial thanks to a result of Hajaiej [11]. Indeed, suppose \( \nu_0 \) is a positive minimizer that is not radial, and consider its Schwarz symmetrization \( \nu_0^* \). On the other hand,

\[
\|\nabla \nu_0^*\|_{L^2} \leq \|\nabla \nu_0\|_{L^2}, \quad \|\nu_0^*\|_{L^2} \leq \|\nu_0\|_{L^2},
\]

then we get \( J(\nu_0^*) < J(\nu_0) \), a contradiction. We deduce that \( |\nu| \) is radial. Furthermore, the Euler-Lagrange equation (2.17) expressing the fact that \( |\nu| \) is a minimizer reads

\[
\Delta(|\nu|) + c|x|^{-4}(|\nu|) - \frac{2 - b}{D} \int_{\mathbb{R}^D}(|\nabla \nu|^2 - c|x|^{-2}(|\nu|^2))dx \frac{\|\nu\|_{L^2}}{d} |\nu| + |x|^{-b} |\nu|^{1+\frac{4}{D-b}} = 0.
\]
It now follows by the scaling properties of this elliptic equation, there exists \( W \in \mathcal{A} \) such that
\[
|v(x)| = \lambda_0^{D/2} W(\lambda_0 x), \quad \text{where } \lambda_0 = \sqrt{\frac{2 - b}{D} \left(\frac{\int_{\mathbb{R}^D} (|\nabla v|^2 - c|x|^{-2}|v|^2) \, dx}{\int_{\mathbb{R}^D} |v|^2 \, dx}\right)}.
\]

It only remains to show that \( w \) defined by \( w(x) = \frac{\sqrt{\int_{\mathbb{R}^D} |\nabla v|^2 + 2|v|^2 |\nabla w|^2 \, dx}}{\int_{\mathbb{R}^D} |\nabla w|^2 \, dx} \) is constant on \( \mathbb{R}^D \). Differentiating \( |w|^2 = 1 \) yields
\[
\operatorname{Re}(\overline{w} \nabla w) = 0,
\]
thus
\[
|\nabla v|^2 = |\nabla (|v|)|^2 + |v|^2 |\nabla w|^2 + 2|v| \nabla (|v|) \cdot \operatorname{Re}(\overline{w} \nabla w),
\]
and
\[
\| \nabla v \|_{L^2}^2 = \| \nabla (|v|) \|_{L^2}^2 + \int_{\mathbb{R}^D} |v|^2 |\nabla w|^2 \, dx.
\]

If \( |\nabla w| \neq 0 \), then \( f(|v|) < f(v) \), which is a contradiction. Therefore, \( w \) is constant, which completes the proof.

Then we review the following result, for the proofs we refer readers to [7].

**Lemma 3.1.** Let \( 0 < b < 2, D \geq 3 \) and \( 1 < p < 1 + \frac{4 - 2b}{D-2} \). Then there exists a constant \( C = C(D, b, p) > 0 \) such that
\[
\int_{\mathbb{R}^D} |x|^{-b} |\phi|^{p-1} - |\phi|^{p-1} ||u||_{L^p} |\phi| \, dx \leq C \{ ||\phi||^{p-1} \| u \|_{L^p} ||\phi||_{L^p} + ||\phi||^{p-1} \| u \|_{L^{p;1}} \| \phi \|_{L^{p;1}} \}
\]
for all \( \phi, v, u, \phi \in H^1 \), where
\[
(p-1)\alpha = \beta \in \left( \frac{D(p+1)}{D-b}, 2^* \right) \quad \text{and} \quad (p-1)\eta = p + 1.
\]

In addition, we also need the following concentration-compactness Lemma which can be proved with a minor modification to the proof of Proposition 1.7.6 in [2].

**Lemma 3.2.** Let \( \{v_n\}_{n \in \mathbb{N}} \in H^1 \) satisfy
\[
\lim_{n \to \infty} \| v_n \|_{L^2} = M < +\infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} H(v_n) < +\infty.
\]

Then there exists a subsequence \( \{v_{n_k}\}_{k \in \mathbb{N}} \) which satisfies one of the following properties:

(V) \( \|v_{n_k}\|_{L^r} \to 0 \) as \( k \to \infty \) for all \( q \in (2, 2^*) \).

(D) There are sequences \( w_k, z_k \in H^1 \) and a constant \( \alpha \in (0, 1) \) such that:

1. \( \text{dist}(\text{supp}(w_k), \text{supp}(z_k)) \to \infty \);
2. \( \sup_{k \in \mathbb{N}}(\| w_k \|_{H^1} + \| z_k \|_{H^1}) < \infty \) for all \( k \in \mathbb{N} \);
3. \( \| w_k \|_{L^2} \to aM \) and \( \| z_k \|_{L^2} \to (1 - a)M \) as \( k \to \infty \), for some \( \alpha \in (0, 1) \);
4. \( \lim_{k \to \infty} \left| \int_{\mathbb{R}^D} |v_{n_k}|^q \, dx - \int_{\mathbb{R}^D} |w_k|^q \, dx - \int_{\mathbb{R}^D} |z_k|^q \, dx \right| = 0 \), for all \( q \in [2, 2^*) \);
5. \( \liminf_{k \to \infty} (H(v_{n_k}) - H(w_k) - H(z_k)) \geq 0 \).

(C) There exist \( v \in H^1 \) and a sequence \( \{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^D \) such that
\[
v_{n_k}(\cdot - y_k) \to v \quad \text{in} \quad L^q(\mathbb{R}^D), \quad q \in [2, 2^*).
\]

Here and subsequently, for abbreviation, \( H(\phi) \) stands for the Hardy functional defined by \( \int (|\nabla v|^2 - c|x|^{-2}|v|^2) \, dx \).

We are now show the main result of this section.

**Theorem 3.2.** Consider a sequence \( \{v_n\}_{n \in \mathbb{N}} \subset H^1 \) satisfying
\[
\lim_{n \to \infty} \| v_n \|_{L^2} = \| W \|_{L^2}, \quad 0 < \limsup_{n \in \mathbb{N}} H(v_n) < +\infty, \quad \limsup_{n \to \infty} E(v_n) \leq 0.
\]
Then there exists a subsequence \( \{v_n\}_{k \in \mathbb{N}} \) and \( \gamma_0 \in \mathbb{R} \) such that
\[
\lim_{k \to \infty} \|v_n - e^{i \gamma_0} W\|_{H^1} = 0,
\]
where \( W \in \mathcal{A} \) is the ground state of (1.12).

**Proof.** The behavior of the sequence \( \{v_n\} \) is constrained by the concentration-compactness Lemma. The proof will be divided into three steps: we first show that property (C) holds in Lemma 3.2 by ruling out (V) and (D). And then prove the sequence \( \{v_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^D \) in (C) is bounded. We thus obtain the desired conclusion by using the variational characteristic of ground states.

**Step 1: Compactness.** Let \( v = 0 \) and \( \varphi = u = \phi = v_n \) in Lemma 3.1, there exists \( \beta \in (\frac{D(p+1)}{b}, 2^*) \) such that
\[
\int_{\mathbb{R}^D} |x|^{-b} |v_n|^{p+1} dx \leq C \left( \|v_n\|_{L^6}^{p+1} + \|v_n\|_{L^{p+1}}^{p+1} \right).
\]

Since \( \beta, p + 1 \in (2, 2^*) \), (V) would imply that \( \int_{\mathbb{R}^D} |x|^{-b} |v_n|^{p+1} dx \to 0 \) and so
\[
\lim_{k \to \infty} \sup_{k \to \infty} E(v_n) = \lim \sup_{k \to \infty} \left( H(v_n) - \frac{2}{p+1} \int_{\mathbb{R}^D} |v_n|^{p+1} dx \right) \leq \lim \sup_{k \to \infty} H(v_n) > 0,
\]
which contradicts (3.2).

Now suppose that (D) holds. It is worth pointing out that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^D} |x|^{-b} |v_n|^{p+1} dx = \int_{\mathbb{R}^D} |x|^{-b} |w_k|^{p+1} dx - \int_{\mathbb{R}^D} |x|^{-b} |z_k|^{p+1} dx = 0.
\]

It then follows from property (D)(5) in Lemma 3.2 that
\[
\lim_{k \to \infty} \sup_{k \to \infty} \left| E(w_k) + E(z_k) \right| \leq \lim \inf_{k \to \infty} H(v_n) - \frac{2}{p+1} \lim \inf_{k \to \infty} \int_{\mathbb{R}^D} |x|^{-b} |v_n|^{p+1} dx \leq \lim \sup_{k \to \infty} E(v_n) \leq 0.
\]

On the other hand, \( E(w_k) \geq 0, E(z_k) \geq 0 \) for sufficiently large \( k \) by using inequality (2.3) and property (D)(3) of Lemma 3.2, thus
\[
E(w_k) \to 0, \ E(z_k) \to 0 \text{ as } k \to \infty,
\]
which implies that
\[
H(w_k) \to 0, \ H(z_k) \to 0 \text{ as } k \to \infty.
\]

Therefore,
\[
\lim_{k \to \infty} \int_{\mathbb{R}^D} |x|^{-b} |v_n|^{p+1} dx = \lim_{k \to \infty} \int_{\mathbb{R}^D} |x|^{-b} |w_k|^{p+1} dx + \int_{\mathbb{R}^D} |x|^{-b} |z_k|^{p+1} dx = 0,
\]
again, it contradicts (3.4). Thus, to rule out (D), we just have to prove (3.5) holds. For this aim, defining \( \xi_k = v_n - w_k - z_k \), it follows from the construction of the sequence \( w_k \) and \( z_k \) in the proof of [2] that
\[
\|v_n\|_{L^p}^{p+1} - \|w_k\|_{L^p}^{p+1} - \|z_k\|_{L^p}^{p+1} \leq C \|v_n\|_{L^p} \|\xi_k\|
\]
and
\[
\|\xi_k\|_{L^2} \to 0 \text{ as } k \to \infty.
\]
Since \( \|\xi_k\|_{L^q} \) is bounded by property (D)(5), applying the inequality (2.3), one can deduce that
\[
\|\xi_k\|_{L^q} \to 0 \text{ as } k \to \infty, \ \forall \ q \in [2, 2^*).
which contradicts (3.2). We split the integral as
\[
\int_{\mathbb{R}^D} |x|^{-b} |v_{n_k}|^{p+1} \, dx - \int_{\mathbb{R}^D} |x|^{-b} |w_k|^{p+1} \, dx - \int_{\mathbb{R}^D} |x|^{-b} |z_k|^{p+1} \, dx
\]
\[
\leq C \int_{\mathbb{R}^D} |x|^{-b} |v_{n_k}|^p |\xi_k| \, dx \leq C(\|v_{n_k}\|_{L^p}^p \|\xi_k\|_{L^\infty} + \|v_{n_k}\|_{L^{p-1}}^p \|\xi_k\|_{L^{p-1}}) \to 0 \text{ as } k \to \infty,
\]
which proves (3.5). Therefore, according to Lemma 3.2, there exist \( v \in H^1 \) and a sequence \( \{v_k\}_{k \in \mathbb{D}} \subset \mathbb{R}^D \) such that
\[
v_n_k (\cdot - y_k) \to v \text{ in } L^q(\mathbb{R}^D), \forall q \in [2, 2').
\]
Step 2: Localization. We will now show that \( y_k \) is bounded in \( \mathbb{R}^D \). Suppose that there exists a subsequence denoted still by \( \{y_k\} \) such that \( |y_k| \to \infty \) as \( k \to \infty \). Note that \( E(v_{n_k}) \) can be written as
\[
E(v_{n_k}) = H(v_{n_k}) - \frac{2}{p+1} \int_{\mathbb{R}^D} |x - y_k|^{-b} |v_{n_k}(x - y_k)|^{p+1} \, dx.
\]
Then we will show that the second term in the right hand side of (3.8) goes to zero as \( k \to \infty \), so that
\[
E(v_{n_k}) \to H(v) > 0 \text{ as } k \to \infty,
\]
which contradicts (3.2). We split the integral as
\[
\int_{|x - y_k| < R} |x - y_k|^{-b} |v_{n_k}(x - y_k)|^{p+1} \, dx + \int_{|x - y_k| \geq R} |x - y_k|^{-b} |v_{n_k}(x - y_k)|^{p+1} \, dx
\]
\[
= I + II,
\]
for some \( R > 0 \). We first observe that, by Holder’s inequality,
\[
I \leq \left( \int_{|x - y_k| < R} |x - y_k|^{-\beta a} \, dx \right)^{1/a} \left( \int_{|x - y_k| < R} |v_{n_k}(x - y_k)|^{(p+1)\beta} \, dx \right)^{1/\beta},
\]
where \( \alpha, \beta \geq 1 \) satisfy \( \frac{1}{a} + \frac{1}{\beta} = 1 \). Then the first term in the right hand side of (3.11) is finite provided \( \beta > \frac{D}{D-B} \).
Indeed, it is possible to choose \( \beta \) satisfies that \( \beta(p + 1) \in \left( \frac{D(p+1)}{D-B}, 2' \right) \) and it follows from (3.7) that
\[
\int_{|x - y_k| < R} |v_{n_k}(x - y_k)|^{(p+1)\beta} \, dx \to 0 \text{ as } k \to \infty.
\]
On the other hand, by the Sobolev embedding Theorem and the boundedness of \( \{v_{n_k}(x - y_k)\} \) in \( H^1 \), the second term of (3.10) can be estimated as
\[
II \leq R^{-b} \int_{\mathbb{R}^D} |v_{n_k}(x - y_k)|^{p+1} \, dx \leq CR^{-b}.
\]
Hence, II can be made arbitrarily small by choosing \( R \) large enough, uniformly in \( k \). This completes the proof of (3.9), and we conclude that the sequence \( \{y_k\} \subset \mathbb{R}^D \) is bounded.
Step 3: Conclusion. We shall suppose, without loss of generality, that \( y_k \to y^* \) as \( k \to \infty \), for some \( y^* \in \mathbb{R}^D \).
Hence,
\[
v_{n_k} \to v^* = v(\cdot + y^*) \text{ in } L^q(\mathbb{R}^D), \forall q \in [2, 2').
\]
Furthermore, we can also suppose that \( v_{n_k} \to v^* \) weakly in \( H^1 \). Since the Hardy functional \( H \) is weakly lower-continuous[17], \( v \mapsto \int_{\mathbb{R}^D} |x|^{-b} |v|^{p+1} \, dx \) is weakly sequentially continuous [9] and \( p + 1 \in (2, 2') \), it follows
that
\[ 0 \leq E(\nu^*) = H(\nu^*) - \frac{2}{p+1} \int_{\mathbb{R}^d} |x|^{-b} |\nu^*|^{p+1} \, dx \leq \limsup_{k \to \infty} H(\nu_{n_k}) - \frac{2}{p+1} \int_{\mathbb{R}^d} |x|^{-b} |\nu^*|^{p+1} \, dx \]
\[ = \limsup_{k \to \infty} E(\nu_{n_k}) \leq 0. \]

So from the inequality (2.3), one has that \( E(\nu^*) \geq 0 \), and so
\[ E(\nu^*) = 0 \quad \text{and} \quad H(\nu^*) = \lim_{k \to \infty} H(\nu_{n_k}). \quad (3.12) \]

Together with \( \|\nu^*\|_{L^2} = \|W\|_{L^2} \) and the variational characteristic of the ground states, it implies that \( \nu^* = e^{i\gamma_0} W \) for some \( \gamma_0 \in \mathbb{R} \). Finally, we have \( \|\nu_{n_k}\|_{H^1} \to \|\nu^*\|_{H^1} = \|\nu^*\|_{H^1} \), \( \{\nu_{n_k}\} \) converges to \( \nu^* \) in \( H^1 \), which concludes the proof.

### 4 Classification

In this section, we construct finite time blow-up solutions with minimal mass \( \|W\|_{L^2} \) to the Cauchy problem (1.1)–(1.2). Furthermore, we prove the uniqueness of finite time blow-up solutions at the minimal mass threshold. We first show that the equation is invariant under the pseudo-conformal transformation, which is defined as follows.

**Lemma 4.1.** Let \( \varphi \) be a global solution of the Cauchy problem (1.1)–(1.2). Then, for all \( T \in \mathbb{R} \), the function
\[ \varphi_T(t, x) = \frac{e^{-i|x|^2}}{(T-t)^{d/2}} \varphi \left( \frac{1}{T-t}, \frac{x}{T-t} \right) \]
is a solution of (1.1) and (1.2) on \( (-\infty, T) \), with \( \|\varphi_T\|_{L^2} = \|\varphi\|_{L^2} \).

**Proof.** By a direct calculation we have
\[ \partial_t \varphi_T(t, x) = \frac{e^{-i|x|^2}}{(T-t)^{d/2}} \left[ -i|\varphi|^2 + \partial_t \varphi + \frac{D}{2} \varphi(T-t) + x \cdot \nabla \varphi \right] \left( \frac{1}{T-t}, \frac{x}{T-t} \right), \]
and
\[ \Delta \varphi_T = \frac{e^{-i|x|^2}}{(T-t)^{d/2}} \left[ -i|\varphi|^2 - \frac{D}{2} \varphi(T-t) - i x \cdot \nabla \varphi + \Delta \varphi \right] \left( \frac{1}{T-t}, \frac{x}{T-t} \right) \]
for the derivatives. For the nonlinear term we find
\[ |x|^{-b} \varphi_T \|^{\frac{d+2b}{2}}_{\bar{r}} \| \varphi_T = \frac{e^{-i|x|^2}}{(T-t)^{d/2}} \left| x \right|^{-b} \varphi \left( \frac{1}{T-t}, \frac{x}{T-t} \right), \]
and for the potential term
\[ |x|^{-2} \varphi_T \| = \frac{e^{-i|x|^2}}{(T-t)^{d/2}} \left| x \right|^{-2} \varphi \left( \frac{1}{T-t}, \frac{x}{T-t} \right). \]

Then it follows that
\[ i \partial_t \varphi_T + \Delta \varphi_T + c |x|^{-2} \varphi_T + |x|^{-b} \varphi_T \|^{\frac{d+2b}{2}}_{\bar{r}} \varphi_T \]
\[ = \frac{e^{-i|x|^2}}{(T-t)^{d/2}} \left[ i \partial_t \varphi + \Delta \varphi + c |x|^{-2} \varphi + |x|^{-b} \varphi \|^{\frac{d+2b}{2}}_{\bar{r}} \varphi \right] \left( \frac{1}{T-t}, \frac{x}{T-t} \right) = 0, \]
which proves the Lemma.
Remark 4.1. From Lemma 4.1, we claim that any solution $u$ of (1.12) satisfies $\|u\|_{L^2} \geq \|W\|_{L^2}$. Otherwise, applying Lemma 4.1 with $u(t, x) = e^{it} \phi(x)$, one could construct a finite time blow-up solution below the minimal mass threshold, which would contradict the global well-posedness result Corollary 2.1.

We now construct a 3-parameter family solutions of the Cauchy problem (1.1)–(1.2) which blow-up in finite time by using the pseudo-conformal transformation and the symmetries of the equation.

**Proposition 4.1.** For all $T \in \mathbb{R}, \lambda_0 > 0$ and $\gamma_0 \in \mathbb{R}$, then the function $\varphi_{T, \lambda_0, \gamma_0}$ defined by

$$\varphi_{T, \lambda_0, \gamma_0}(t, x) = e^{i \gamma_0 (x)} e^{i \frac{\lambda_0^2 t^2}{T - t}} \left( \frac{\lambda_0}{T - t} \right)^{D/2} W \left( \frac{\lambda_0 x}{T - t} \right)$$

(4.1)

is a minimal mass solution of the Cauchy problem (1.1)–(1.2) defined on $(-\infty, T)$, and which blows up with speed

$$\|\nabla \varphi_{T, \lambda_0, \gamma_0}(t, x)\|_{L^2} \sim \frac{C}{T - t}, \text{ as } t \to T, \text{ for } C > 0,$$

where $W \in A$ is the ground state of (1.12).

**Proof.** The proof may be proved by applying Lemma 4.1 to the global solution

$$\varphi_{\lambda_0, \gamma_0}(t, x) = e^{i \gamma_0 (x)} e^{i \frac{\lambda_0^2 t^2}{T}} \lambda_0^{D/2} W(\lambda_0 x),$$

which can be obtained under the scaling and phase symmetries of the standing wave $\varphi(t, x) = e^{it} W(x)$.

**Remark 4.2.** Note that the blow-up solutions of the family showed in Proposition 4.1 can be retrieved from the solution

$$\varphi_{0,1,0}(t, x) = e^{i \frac{\lambda_0^2 t^2}{T}} e^{-i \frac{1}{|t|} \lambda_0^{D/2} W(\frac{x}{T})},$$

defined on $(-\infty, 0)$ and which blows up at $t = 0$ with speed

$$\|\nabla \varphi_{0,1,0}\|_{L^2} \sim \frac{C}{|t|} \text{ as } t \uparrow 0, \text{ for } C > 0.$$

Indeed, all the solutions $\varphi_{T, \lambda_0, \gamma_0}$ are equal to $\varphi_{0,1,0}$, up to the symmetries. In other words, if we apply the changes $\varphi(t, x) \rightarrow \lambda_0^{D/2} \varphi(\lambda_0^2 t, \lambda_0^{-1} x)$, $\varphi(t, x) \rightarrow \varphi(t - T, x)$ and finally $\varphi(t, x) \rightarrow e^{i \gamma_0} \varphi(t, x)$ to $\varphi_{0,1,0}$, we obtain $\varphi_{T, \lambda_0, \gamma_0}$.

In order to prove our main result, we also need to deduce information about $\varphi(t)$ for $t < T$ from the blow-up solution at $t = T$. Using a result of Banica[1], we will show that $\varphi(t) \in H^1$ and $x \varphi \in L^2(\mathbb{R}^D)$ for all $t < T$. For $\varphi \in H^1, \theta \in C_0^\infty(\mathbb{R}^D)$ and $\eta \in \mathbb{R}$, we have that $\nabla(\varphi e^{i \theta}) = (\nabla \varphi + i \eta \nabla \theta) e^{i \theta}$, and so

$$|\nabla(\varphi e^{i \theta})|^2 = |\nabla \varphi|^2 + 2 \eta \nabla \theta \cdot \text{Im}(\overline{\varphi} \nabla \varphi) + \eta^2 |\nabla \theta|^2 |\varphi|^2.$$

By a direct calculation we have

$$E(\varphi e^{i \theta}) = E(\varphi) + 2 \eta \int_{\mathbb{R}^D} \nabla \theta \cdot \text{Im}(\overline{\varphi} \nabla \varphi) dx + \eta^2 \int_{\mathbb{R}^D} |\nabla \theta|^2 |\varphi|^2 dx. \quad (4.2)$$

Then the refined Cauchy-Schwarz estimate for critical mass functions can be proved.

**Lemma 4.2.** Let $\varphi \in H^1$ be a function such that $\|\varphi\|_{L^2} = \|W\|_{L^2}$. Then for all $\theta \in C_0^\infty(\mathbb{R}^D)$, one has

$$\left| \int_{\mathbb{R}^D} \nabla \theta \cdot \text{Im}(\overline{\varphi} \nabla \varphi) dx \right| \leq \sqrt{E(\varphi)} \left( \int_{\mathbb{R}^D} |\nabla \theta|^2 |\varphi|^2 dx \right)^{1/2}.$$
Proof. For all \( \eta \in \mathbb{R} \), we now have \( \| \varphi e^{i\eta t} \|_{L^2} = \| \varphi \|_{L^2} \). The result follows from the quadratic polynomial expression (4.2) in \( \eta \) of \( E(\varphi e^{i\eta t}) \), which thus must have a nonpositive discriminant. This completes the proof.

In the reminder of this section we prove the main result.

Proof of Theorem 1.1. Let \( \varphi \) be a solution of the Cauchy problem (1.1)–(1.2) such that \( \| \varphi \|_{L^2} = \| W \|_{L^2} \) and which blows up in finite time. The proof then falls into several steps.

Step 1. Let \( \{ t_n \}_{n \in \mathbb{N}} \subset \mathbb{R} \) be a sequence of times such that \( t_n \to T \) as \( n \to \infty \). We set

\[
\varphi_n = \varphi(t_n), \quad \lambda_n = \sqrt{\frac{H(W)}{H(\varphi_n)}}, \quad v_n(x) = \lambda_n^{D/2} \varphi_n(\lambda_n x).
\]

We note that \( \lambda_n \to \infty \) as \( n \to \infty \), and

\[
\| v_n \|_{L^2} = \| \varphi_n \|_{L^2} = \| \varphi_0 \|_{L^2} = \| W \|_{L^2}, \quad H(v_n) = \lambda_n^2 H(\varphi_n) = H(W).
\]

On the other hand, by conservation of the energy,

\[
E(v_n) = H(v_n) - \frac{D}{D + 2 - b} \int_{\mathbb{R}^D} |x_n|^{-b} |v_n|^{2 + \frac{2b}{D}} dx
= \lambda_n^2 H(\varphi_n) - \frac{D}{D + 2 - b} \lambda_n^2 \int_{\mathbb{R}^D} |x_n|^{-b} |\varphi_n|^{2 + \frac{2b}{D}} dx
= \lambda_n^2 E(\varphi_n) = \lambda_n^2 E(\varphi_0) \to 0 \text{ as } n \to \infty.
\]

Hence, by Theorem 3.2, there exist \( \gamma_0 \in \mathbb{R} \) such that, up to extracting a subsequence of \( \{ v_n \} \), we have

\[
\lim_{n \to +\infty} \| v_n - e^{i\gamma_0} W \|_{H^1} = 0. \tag{4.4}
\]

Step 2. Using the variational arguments, we prove that blowup's profile is a Dirac function, i.e. in the sense of distributions,

\[
|\varphi_n|^2 \to \| W \|_{L^2} \delta_0.
\]

Indeed, by scaling (4.3), we observe that for \( u \in C_0^\infty(\mathbb{R}^D) \),

\[
\int_{\mathbb{R}^D} |v_n(x)|^2 u(\lambda_n x) dx = \int_{\mathbb{R}^D} |\varphi_n(x)|^2 u(x) dx,
\]

and

\[
\int_{\mathbb{R}^D} |v_n(x)|^2 u(\lambda_n x) dx = \int_{\mathbb{R}^D} (\| v_n(x) \|^2 - |W|^2) u(\lambda_n x) dx + \int_{\mathbb{R}^D} |W(x)|^2 u(0) dx + \int_{\mathbb{R}^D} |W|^2 [u(\lambda_n x) - u(0)] dx.
\]

Thus, we obtain

\[
\left| \int_{\mathbb{R}^D} |\varphi_n(x)|^2 u(x) dx - \| W \|^2 u(0) \right| \leq \| u \|_{\infty} \int_{\mathbb{R}^D} \left| |v_n(x)|^2 - |W(x)|^2 \right| dx + \int_{\mathbb{R}^D} |W(x)|^2 |u(\lambda_n x) - u(0)| dx.
\]

(4.4) implies that \( |v_n|^2 \) converges to \( |W|^2 \) strongly in \( L^1(\mathbb{R}^D) \), so the first integral converges to 0 as \( n \to +\infty \). Since \( \lambda_n \to 0 \), then the second integral also converges to 0 as \( n \to +\infty \) by the dominated convergence theorem, thus

\[
\int_{\mathbb{R}^D} |\varphi_n|^2 u(x) dx \to \| W \|_{L^2} u(0). \tag{4.5}
\]

Step 3. We claim that \( \varphi(t) \in H^1 \) and \( x \varphi \in L^2(\mathbb{R}^D) \) for all \( t \in [0, T) \). Let \( \phi \) a nonnegative radial \( C_0^\infty(\mathbb{R}^D) \) function such that \( \phi(x) = |x|^2 \) for \( |x| \leq 1 \) and \( |\nabla \phi(x)|^2 \leq C \phi(x) \). For every \( R > 0 \), we define \( \phi_R(x) = R^2 \phi(x/R) \), and for all \( t \in [0, T) \),

\[
\phi_R(t) = R^2 \phi(x/R), \quad J_R(t) = \int_{\mathbb{R}^D} \phi_R(x)|\varphi(t, x)|^2 dx.
\]
By a direct calculation (also see [2, 22]), we have
\[ J_R(t) = 2 \text{Re} \int_{\mathbb{R}^D} \phi_R \overline{\varphi} \varphi_t \, dx = 2 \int_{\mathbb{R}^D} \nabla \phi_R \cdot \text{Im}(\overline{\varphi} \nabla \varphi) \, dx. \]

Since \( \| \varphi \|_{L^2} = \| W \|_{L^2} \) and \( |\nabla \phi_R|^2 \leq C \phi_R \), we can apply Lemma 4.2 to get
\[ |J_R(t)| \leq 2 \sqrt{E(\varphi)} \left( \int_{\mathbb{R}^D} |\nabla \phi_R|^2 |\varphi|^2 \, dx \right)^{1/2} \leq C \sqrt{E(\varphi_0)} \sqrt{J_R(t)}. \]

By integration, we obtain that for \( t \in [0, T] \),
\[ |\sqrt{J_R(t)} - \sqrt{J_R(t_n)}| \leq C|t - t_n|. \]

Since \( J_R(t_n) \to 0 \) by (4.5). Thus, letting \( t_n \to T \), we obtain that for all \( t \in [0, T] \) and all \( R > 0 \),
\[ J_R(t) \leq C(T - t)^2. \]

Since the right-hand side of the last expression of \( R \), we obtain that by letting \( R \to \infty \), for all \( t \in [0, T] \),
\[ \varphi(t) \in H^1, \quad x \varphi \in L^2(\mathbb{R}^D) \text{ and } 0 \leq J(t) \leq C(T - t)^2, \]

where \( J(t) = \int_{\mathbb{R}^D} |x|^2 |\varphi|^2 \, dx \). From this estimate, we can extend by continuity \( J(t) \) at \( t = T \) by setting \( J(T) = 0 \), from which we also obtain \( J'(T) = 0 \). Moreover, since \( \varphi(t) \in H^1, \quad x \varphi \in L^2(\mathbb{R}^D) \) and \( \varphi \) is a solution of the Cauchy problem (1.1)–(1.2), we obtain \( J(t) = 8E(\varphi_0) \), which finally gives, for all \( t \in [0, T] \),
\[ J(t) = 4E(\varphi_0)(T - t)^2. \]

Letting \( t = 0 \), we find
\[ J(0) = \int_{\mathbb{R}^D} |x|^2 |\varphi_0|^2 \, dx = 4E(\varphi_0)T^2 \quad \text{and} \quad J'(0) = 4 \int_{\mathbb{R}^D} x \cdot \text{Im}(\overline{\varphi_0} \nabla \varphi_0) \, dx = -8E(\varphi_0)T. \]

**Step 4. Determination of \( \varphi_0 \) and conclusion.** We finally apply identity (4.2) to \( \varphi_0 \) and \( \eta = \frac{1}{T} \), with \( \theta(x) = \frac{|x|^2}{T} \).

Since \( \nabla \theta(x) = x \), we obtain
\[ E(\varphi_0 e^{i|\varphi_0|^2}) = E(\varphi_0) + \frac{1}{T} \int_{\mathbb{R}^D} x \cdot \text{Im}(\overline{\varphi_0} \nabla \varphi_0) \, dx + \frac{1}{4T^2} \int_{\mathbb{R}^D} |x|^2 |\varphi_0|^2 \, dx \]
\[ = E(\varphi_0) + \frac{1}{T} (-2E(\varphi_0)T) + \frac{1}{4T^2} (4E(\varphi_0)T^2) \]
\[ = 0. \]

Note that this calculation justifies, a posteriori, the application of (4.2) with the function \( \theta(x) = \frac{|x|^2}{T} \notin C^\infty_0(\mathbb{R}^D) \).

Hence, we have \( \| \varphi_0 e^{i|\varphi_0|^2} \|_{L^2} = \| W \|_{L^2} \) and \( E(\varphi_0 e^{i|\varphi_0|^2}) = 0 \), and we deduce from the variational characteristic of ground state that there exist \( \lambda_0 > 0 \) and \( \gamma_0 \in \mathbb{R} \) such that
\[ \varphi_0 = e^{i\gamma_0} e^{-i|\varphi_0|^2} \lambda_0^2 W(\lambda_0 x). \]

Finally, we use the pseudo-conformal transformation. We define \( \widetilde{\lambda}_0 = \lambda_0 T > 0 \) and \( \tilde{\gamma}_0 = \gamma_0 - \lambda_0^2 T \in \mathbb{R} \), and write \( \varphi_0 \) as
\[ \varphi_0(x) = e^{i\gamma_0} e^{i\frac{T}{\lambda_0^2}} e^{-i|\varphi_0|^2} \left( \frac{\widetilde{\lambda}_0}{T} \right)^D W \left( \frac{\lambda_0 x}{T} \right). \]

Thus, \( \varphi_0(x) = \varphi_{T, \widetilde{\lambda}_0, \tilde{\gamma}_0} (0) \), where \( \varphi_{T, \widetilde{\lambda}_0, \tilde{\gamma}_0} (0) \) is defined by (4.1), which concludes the proof.

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References


