Shuai Zhou, Zhisu Liu, and Jianjun Zhang*

Groundstates for Choquard type equations with weighted potentials and Hardy–Littlewood–Sobolev lower critical exponent

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Abstract: We are concerned with a class of Choquard type equations with weighted potentials and Hardy–Littlewood–Sobolev lower critical exponent

$$-\Delta u + V(x)u = \left(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \right) Q(x)|u|^{\frac{\alpha}{N}-1}u, \quad x \in \mathbb{R}^N.$$

By using variational approaches, we investigate the existence of groundstates relying on the asymptotic behaviour of weighted potentials at infinity. Moreover, non-existence of non-trivial solutions is also considered. In particular, we give a partial answer to some open questions raised in [D. Cassani, J. Van Schaftingen and J. J. Zhang, Groundstates for Choquard type equations with Hardy–Littlewood–Sobolev lower critical exponent, *Proceedings of the Royal Society of Edinburgh, Section A Mathematics*, **150**(2020), 1377–1400].

Keywords: Ground states, Choquard equation, Hardy–Littlewood–Sobolev inequality, critical growth

MSC: 35B33, 35J61

1 Introduction and main results

In this paper, we are concerned with the following class of nonlocal equations

$$-\Delta u + V(x)u = (I_\alpha * F(x, u)) f(x, u), \quad x \in \mathbb{R}^N, \quad (1.1)$$

where $N \geq 3$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and I_α is the Riesz potential given for each $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{where } A_\alpha = \frac{\Gamma((N-\alpha)/2)}{\Gamma(\alpha/2)\pi^{N/2}2^\alpha} \quad \text{and } \alpha \in (0, N).$$

Here Γ is the Euler gamma function and F is the primitive function of $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ with respect to u and satisfies $F(0) = 0$. In the literature, problem (1.1) is known as Choquard’s type equation. Set $F(x, u) = |u|^p/p$ and $V \equiv a$, problem (1.1) becomes

$$-\Delta u + au = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N. \quad (1.2)$$

Shuai Zhou, College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, PR China, E-mail: 916164424@qq.com

Zhisu Liu, Center for Mathematical Sciences, China University of Geosciences, Wuhan, Hubei, 430074, PR China, E-mail: liuzhisu@cug.edu.cn

***Corresponding Author: Jianjun Zhang**, College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, PR China, E-mail: zhangjianjun09@tsinghua.org.cn

For $N = 3$, $\alpha = 2$ and $p = 2$, (1.2) reduces to

$$-\Delta u + au = (I_2 * u^2)u, \quad x \in \mathbb{R}^3. \quad (1.3)$$

It seems that such equations appear first in the seminal work of S.I. Pekar '54 [23], modeling the quantum Polaron and later were introduced by Choquard to study steady states of the one component plasma approximation in the context of Hartree-Fock theory [11]. Problem (1.1) has a variational structure, in the sense that H^1 -solutions to (1.1) turn out to be critical points of the energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \int_{\mathbb{R}^N} [I_\alpha * F(x, u)] F(x, u) dx.$$

Due to the presence of convolution, problem (1.1) is nonlocal. In contrast with local problems, Choquard type equations carry some extra difficulty due to the nonlocal nature. By using a rearrangement approach, E. Lieb in [12] proved existence and uniqueness of positive solutions to (1.3). Subsequently, multiplicity results for (1.3) were obtained by P.L. Lions [14, 15] via the variational methods. Initiated by the papers of E. Lieb [12] and P.L. Lions [14, 15], Choquard equations have attracted a considerable attention in the past decades. We refer to [21] for a survey.

In [19], V. Moroz and J. Van Schaftingen established existence of ground state solutions to (1.2). Thanks to a Pohožev identity, they show that (1.2) admits a nontrivial solution in $H^1(\mathbb{R}^N)$ if and only if

$$\frac{N + \alpha}{N} < p < \frac{N + \alpha}{N - 2}. \quad (1.4)$$

The endpoints $2_{\alpha,*} := \frac{N+\alpha}{N}$ and $2_\alpha^* := \frac{N+\alpha}{N-2}$ are sometimes called lower and upper Hardy-Littlewood-Sobolev critical exponents respectively in the sense of the Hardy-Littlewood-Sobolev inequality (see Lemma 2.1 below). Later, V. Moroz and J. Van Schaftingen [22] considered the autonomous form of Choquard equation (1.1)

$$-\Delta u + u = (I_\alpha * F(u)) f(u), \quad x \in \mathbb{R}^N \quad (1.5)$$

and investigated existence, symmetry and regularity of groundstates to problem (1.5) under almost necessary conditions on the nonlinearity F in the spirit of Berestycki and Lions. For the upper critical case, F. Gao and M. Yang [9] establish existence and nonexistence of solutions to the following Brezis-Nirenberg type problem of Choquard equation in bounded domains $\Omega \subset \mathbb{R}^N (N \geq 3)$

$$-\Delta u = \left(\int_{\Omega} \frac{|u(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}} dy \right) |u|^{2_\alpha^*-2} u + \lambda u, \quad u \in H_0^1(\Omega).$$

By using the penalization argument introduced by J. Byeon and L. Jeanjean [4], D. Cassani and J. Zhang [6] investigated singularly perturbed problems related to equation (1.5) involving upper critical exponent

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{-\alpha} (I_\alpha * F(u)) f(u), \quad x \in \mathbb{R}^N$$

and obtained existence of single peak solutions around local minimal points of the potential V . With the help of the concentration compactness principle in the Choquard-type setting, S. Liang, P. Pucci and B. Zhang [16] established multiplicity results for Choquard-Kirchhoff type equations with Hardy-Littlewood-Sobolev critical exponents

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u = \alpha k(x) |u|^{q-2} u + \beta \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}} dy \right) |u|^{2_\alpha^*-2} u, \quad x \in \mathbb{R}^N$$

Compared with the upper critical case, the lower critical case has been less considered. Combining variational arguments with the concentration-compactness principle [13], V. Moroz and J. Van Schaftingen [20] considered Choquard equations with a purely lower critical nonlinearity

$$-\Delta u + V(x)u = \left(I_\alpha * |u|^{\frac{N+\alpha}{N}} \right) |u|^{\frac{\alpha}{N}-1} u, \quad x \in \mathbb{R}^N \quad (1.6)$$

and established a sufficient condition on existence of groundstates to problem (1.6). Subsequently, D. Cassani, J. Van Schaftingen and J. Zhang [7] investigated existence and nonexistence of groundstates to problem (1.6) and give a partial answer to some open questions raised in [7]. By variational methods, J. Van Schaftingen and J. Xia [25] proved existence of ground state solutions to Choquard equations with lower critical exponent and a subcritical perturbation. In [24], J. Seok considered problem (1.5) with both upper and lower critical exponents and obtained existence of nontrivial solutions in the higher dimensional case. For the related results on the Choquard equations with upper critical growth in the fractional setting and for the planar Choquard equations, we refer to [1, 2, 5, 8, 17, 26] and references therein.

We point out that due to the presence of the lower critical exponent $2_{\alpha,*}$, the problem has a lack of compactness. Similarly to Sobolev critical problems, a Brezis-Nirenberg argument can be adopted to recover compactness. Actually, by imposing some suitable conditions on N , α and nonlinearities, one can get a candidate minimax value below a threshold, under which the compactness condition holds. In [7], compared with the high dimensional case $N \geq 3$, dimension $N = 3$ becomes more tough. Precisely, in [7] to recover compactness in the three dimensional case, one sufficient condition is established on α as follows:

$$\frac{3}{2} < \alpha < 3 \quad (1.7)$$

Moreover, a natural question is whether such restriction is necessary or not for the existence of groundstates.

In this paper, we consider the following class of equations

$$-\Delta u + V(x)u = \left(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \right) Q(x)|u|^{\frac{\alpha}{N}-1}u, \quad x \in \mathbb{R}^N \quad (1.8)$$

and show that condition (1.7) can be replaced by demanding in the presence of weights a suitable asymptotic behavior at infinity. In the following, we perform the variational method to study existence and nonexistence of ground states to (1.8).

The associated functional with the Choquard equation (1.8) is given for any function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\begin{aligned} I(u) = & \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \\ & - \frac{N}{2(N+\alpha)} \int_{\mathbb{R}^N} \left(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \right) Q(x)|u|^{\frac{N+\alpha}{N}} dx \end{aligned}$$

We assume that V and Q satisfy

$$(V_1) \quad \inf_{x \in \mathbb{R}^N} V(x) > 0, \quad \lim_{|x| \rightarrow \infty} V(x) = 1;$$

$$(V_2) \quad \text{There exists } \mu \in \mathbb{R} \text{ such that}$$

$$\lim_{|x| \rightarrow \infty} (1 - V(x))|x|^2 = \mu;$$

$$(Q_1) \quad \inf_{x \in \mathbb{R}^N} Q(x) > 0, \quad \lim_{|x| \rightarrow \infty} Q(x) = 1;$$

$$(Q_2) \quad \text{There exist } \beta \geq 0 \text{ and } v_\beta \in \mathbb{R} \text{ such that}$$

$$\lim_{|x| \rightarrow \infty} (Q(x) - 1)|x|^\beta = v_\beta.$$

Our main results are the following:

Theorem 1.1. Assume (V_1) , (V_2) , (Q_1) , (Q_2) hold, then (1.8) admits a positive ground state solution, provided one of the following conditions holds

$$(i) \quad \beta = 0, \quad \mu > \frac{N^2(N-2)}{4(N+1)}, \quad \inf_{x \in \mathbb{R}^N} Q(x) \geq 1;$$

$$(ii) \quad 0 < \beta < 2, \quad v_\beta > 0;$$

$$(iii) \quad \beta > 2, \quad \mu > \frac{N^2(N-2)}{4(N+1)}.$$

Combining a Pohožaev identity (see Proposition 2.1 below) with Hardy's inequality, we have the following non-existence result for problem (1.8).

Theorem 1.2. Assume $V, Q \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$(VQ) \quad \begin{cases} \sup_{x \in \mathbb{R}^N} |x|^2 \langle x, \nabla V(x) \rangle < \frac{(N-2)^2}{2}, \\ \inf_{x \in \mathbb{R}^N} \langle x, \nabla Q(x) \rangle \geq 0, \end{cases}$$

then (1.8) admits in $H^1(\mathbb{R}^N) \cap W_{loc}^{2,2}(\mathbb{R}^N)$ only trivial solution.

Remark 1.1. The restriction $\inf_{x \in \mathbb{R}^N} V(x) > 0$ is used to guarantee c_* (see below) is finite. It can be replaced by a weaker condition $\inf \sigma(-\Delta + V) > 0$.

Remark 1.2. In the case $\beta = 2$, Theorem 1.1 is still valid if $\mu > \mu_*$, where

$$\mu_* = \frac{N^2(N-2)}{4(N+1)} - \frac{2Nv_\beta}{(N+\alpha)c_\infty} \left(\int_{\mathbb{R}^N} \frac{|U(x)|^2}{|x|^2} dx \right)^{-1} \int_{\mathbb{R}^N} \left(I_\alpha * |U|^{\frac{N+\alpha}{N}} \right) \left[|x|^{-2} |U|^{\frac{N+\alpha}{N}} \right] dx.$$

Here c_∞ and U are given in Section 2.

As a special case, for the external Schrödinger potential $V_{\mu,v} : \mathbb{R}^N \rightarrow \mathbb{R}$

$$V_{\mu,v}(x) = 1 - \frac{\mu}{v^2 + |x|^2}, \quad \text{for } \mu \in \mathbb{R}, v > 0 \text{ and } x \in \mathbb{R}^N$$

and the weighted potential $Q_\beta : \mathbb{R}^N \rightarrow \mathbb{R}$

$$Q_\beta(x) = 1 + \frac{v_\beta}{1 + |x|^\beta}, \quad \text{for } v_\beta \in \mathbb{R}, \beta \geq 0 \text{ and } x \in \mathbb{R}^N,$$

problem (1.8) reduces to the following

$$-\Delta u + V_{\mu,v}(x)u = \left(I_\alpha * [Q_\beta(x)|u|^{\frac{N+\alpha}{N}}] \right) Q_\beta(x)|u|^{\frac{\alpha}{N}-1}u, \quad x \in \mathbb{R}^N. \quad (1.9)$$

Denote by μ^v the best constant of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, (v^2 + |x|^2)^{-1} dx)$, that is,

$$\mu^v := \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2}{\int_{\mathbb{R}^N} \frac{|u(x)|^2}{v^2 + |x|^2} dx}.$$

It has been shown in [7] that $\mu^v > \frac{(N-2)^2}{4} + v^2$.

Corollary 1.1. Problem (1.9) admits a positive ground state solution, provided one of the following conditions holds

- (i) $\beta = 0$, $\frac{N^2(N-2)}{4(N+1)} < \mu < \mu^v$, $v_\beta = 0$;
- (ii) $0 < \beta < 2$, $\mu < \mu^v$, $v_\beta > 0$;
- (iii) $\beta > 2$, $\frac{N^2(N-2)}{4(N+1)} < \mu < \mu^v$, $v_\beta > -1$

and has no non-trivial solutions if $\mu < \frac{(N-2)^2}{4}$ and $v_\beta \leq 0$.

In the case $\mu > \mu^v$, the operator $-\Delta + V_{\mu,v}$ is not positively definite and the problem becomes more complicated. By the linking theorem, we have the following result.

Theorem 1.3. Assume that either $N \geq 4$ or $N = 3$, $\ker(-\Delta + V_{\mu,v}) = \{0\}$. If $\mu > \max\left\{\mu^v, \frac{N^2(N-2)}{4(N+1)}\right\}$, then (1.8) admits a ground state solution (necessarily sign changing) provided when $N = 3$, one of the followings holds

- (i) $\beta = 0$, $v_\beta = 0$, $\frac{3}{2} < \alpha < 3$;
- (ii) $0 < \beta < \min\left\{\frac{9}{6-\alpha}, 2\right\}$, $v_\beta > 0$
- (iii) $\beta > 2$, $v_\beta > -1$, $\frac{3}{2} < \alpha < 3$;

and when $N \geq 4$, one of the followings holds

- (iv) $\beta = 0$, $v_\beta = 0$;
- (v) $0 < \beta < 2$, $v_\beta > 0$;
- (vi) $\beta > 2$, $v_\beta > -1$.

Notations.

- $\|u\|_p := \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p}$ for $u \in L^p(\mathbb{R}^N)$, $p \in [1, \infty)$.
- $\|u\| = \left(\|\nabla u\|_2^2 + \|u\|_2^2\right)^{\frac{1}{2}}$, $u \in H^1(\mathbb{R}^N)$.

2 Proof of Theorem 1.1-1.2

Before proving Theorem 1.1, let us introduce some preliminary results. First, the following Hardy–Littlewood–Sobolev inequality will be frequently used in the sequel.

Lemma 2.1 (Hardy–Littlewood–Sobolev inequality [10, Theorem 4.3]). Let $s, r > 1$ and $0 < \alpha < N$ with $1/s + 1/r = 1 + \alpha/N$, $f \in L^s(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$, then there exists a positive constant $C(s, N, \alpha)$ (independent of f, g) such that

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x) |x-y|^{\alpha-N} g(y) dx dy \right| \leq C(s, N, \alpha) \|f\|_s \|g\|_r.$$

In particular, if $s = r = 2N/(N + \alpha)$, the sharp constant is given by

$$\mathcal{C}_\alpha := \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\alpha/2)}{\Gamma((N+\alpha)/2)} \left[\frac{\Gamma(N/2)}{\Gamma(N)} \right]^{-\alpha/N}.$$

Due to the presence of the lower critical exponent $\frac{N+\alpha}{N}$, the compactness fails in general. In fact, the convolution term enjoys the invariance of dilation, that is, for any $u \in L^2(\mathbb{R}^N)$ and $t > 0$, one has

$$G_\infty(u_t) = G_\infty(u), \quad u_t(\cdot) = t^{\frac{N}{2}} u(t\cdot),$$

where

$$G_\infty(u) = \int_{\mathbb{R}^N} \left(I_\alpha \star |u|^{\frac{N+\alpha}{N}} \right) |u|^{\frac{N+\alpha}{N}} dx.$$

To recover the compactness, the following Brezis–Lieb type lemma plays a crucial role in the decomposition of the maximization sequence for c_* given below.

For any $u \in H^1(\mathbb{R}^N)$, let

$$G(u) = \int_{\mathbb{R}^N} \left(I_\alpha \star [Q(x)|u|^{\frac{N+\alpha}{N}}] \right) Q(x)|u|^{\frac{N+\alpha}{N}} dx,$$

$$T(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx.$$

Lemma 2.2 (Brezis-Lieb type Lemma). Assume that (V_1) and (Q_1) hold and let $\{u_n\}_{n=1}^\infty$ be a bounded sequence in $L^2(\mathbb{R}^N)$ and for some $u \in L^2(\mathbb{R}^N)$, $u_n \rightarrow u$ strongly in $L^2_{loc}(\mathbb{R}^N)$ as $n \rightarrow \infty$, then, up to a subsequence, there holds

$$\lim_{n \rightarrow \infty} [G(u_n) - G(u) - G_\infty(u_n - u)] = 0.$$

Proof. Without loss of generality, we may assume that $u_n \rightarrow u$ almost everywhere on $\mathbb{R}^N$ as $n \rightarrow \infty$. Let $v_n = \sqrt{Q}u_n$, $v = \sqrt{Q}u$, it follows from [19, Lemma 2.4] that

$$\lim_{n \rightarrow \infty} [G(u_n) - G(u) - G(u_n - u)] = 0.$$

In the following, we show that

$$\lim_{n \rightarrow \infty} [G(u_n - u) - G_\infty(u_n - u)] = 0.$$

Let $w_n = u_n - u$,

$$\begin{aligned} G(u_n - u) &= \int_{\mathbb{R}^N} \left(I_\alpha * [Q(x)|w_n|^{\frac{N+\alpha}{N}}] \right) Q(x)|w_n|^{\frac{N+\alpha}{N}} dx \\ &= \int_{\mathbb{R}^N} \left(I_\alpha * [Q(x)|w_n|^{\frac{N+\alpha}{N}}] \right) (Q(x) - 1)|w_n|^{\frac{N+\alpha}{N}} dx \\ &\quad + \int_{\mathbb{R}^N} \left(I_\alpha * [(Q(x) - 1)|w_n|^{\frac{N+\alpha}{N}}] \right) |w_n|^{\frac{N+\alpha}{N}} dx + G_\infty(w_n) \\ &:= J_{1,n} + J_{2,n} + G_\infty(w_n). \end{aligned}$$

By virtue of Lemma 2.1, we have

$$J_{1,n} \leq C_\alpha \max_{x \in \mathbb{R}^N} Q(x) \left(\int_{\mathbb{R}^N} |w_n|^2 dx \right)^{\frac{N+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |Q(x) - 1|^{\frac{2N}{N+\alpha}} |w_n|^2 dx \right)^{\frac{N+\alpha}{2N}}.$$

Since $Q(x) \rightarrow 1$ as $|x| \rightarrow \infty$ and $w_n \rightarrow 0$ strongly in $L^2_{loc}(\mathbb{R}^N)$ as $n \rightarrow \infty$, one can get that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |Q(x) - 1|^{\frac{2N}{N+\alpha}} |w_n|^2 dx = 0,$$

which implies that $J_{1,n} \rightarrow 0$ as $n \rightarrow \infty$. Similarly, we have $J_{2,n} \rightarrow 0$ as $n \rightarrow \infty$. The proof is complete. $\square$

Set

$$c_\infty := \sup \left\{ G_\infty(u) : \int_{\mathbb{R}^N} |u|^2 dx = 1, u \in L^2(\mathbb{R}^N) \right\}$$

and

$$c_* := \sup \left\{ G(u) : T(u) = 1, u \in H^1(\mathbb{R}^N) \right\}.$$

Then by Lemma 2.1, $0 < c_*$, $c_\infty < \infty$ and it follows from [10, Theorem 4.3] that c_∞ can be achieved by the family of functions

$$U(x) = C\lambda^{\frac{N}{2}} (\lambda^2 + |x|^2)^{-\frac{N}{2}},$$

for some fixed $C > 0$ and $\lambda \in \mathbb{R}^+$ as parameters.

Lemma 2.3 (Compactness). If $c_* > c_\infty$, then c_* can be achieved.

Proof. For any minimization sequence $\{u_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N)$ of c_* , i. e, $G(u_n) \rightarrow c_*$, $n \rightarrow \infty$ with

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)|u_n|^2) dx = 1,$$

without loss of generality, we assume that u_n is non-negative for all n and for some $u_0 \in H^1(\mathbb{R}^N)$, $u_n \rightarrow u_0 \geq 0$ weakly in $H^1(\mathbb{R}^N)$, strongly in $L^2_{loc}(\mathbb{R}^N)$ and a. e. on $\mathbb{R}^N$ as $n \rightarrow \infty$. Thanks to Lemma 2.2,

$$c_* = G(u_0) + G_\infty(u_n - u_0) + o_n(1), \quad (2.1)$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$. Moreover, set $w_n = u_n - u_0$, we have

$$1 = \int_{\mathbb{R}^N} (|\nabla u_0|^2 + V(x)|u_0|^2) dx + \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2) dx + o_n(1). \quad (2.2)$$

On the other hand, by the definitions of c_* and c_∞ , it is easy to know that

$$G(u) \leq c_* \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \right)^{\frac{N+\alpha}{N}}, \quad \text{for any } u \in H^1(\mathbb{R}^N)$$

and

$$G_\infty(u) \leq c_\infty \left(\int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{N+\alpha}{N}}, \quad \text{for any } u \in L^2(\mathbb{R}^N).$$

Then by (2.1) and (2.2),

$$\begin{aligned} c_* &\leq c_* \left(\int_{\mathbb{R}^N} (|\nabla u_0|^2 + V(x)|u_0|^2) dx \right)^{\frac{N+\alpha}{N}} + c_\infty \left(\int_{\mathbb{R}^N} |w_n|^2 dx \right)^{\frac{N+\alpha}{N}} + o_n(1) \\ &\leq c_* \left(\int_{\mathbb{R}^N} (|\nabla u_0|^2 + V(x)|u_0|^2) dx \right)^{\frac{N+\alpha}{N}} + c_\infty \left(\int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2) dx \right)^{\frac{N+\alpha}{N}} + o_n(1) \\ &\leq c_* \int_{\mathbb{R}^N} (|\nabla u_0|^2 + V(x)|u_0|^2) dx + c_\infty \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2) dx + o_n(1). \end{aligned}$$

If $c_* > c_\infty$, we claim that $w_n \rightarrow 0$ strongly in $H^1(\mathbb{R}^N)$ as $n \rightarrow \infty$. Otherwise, we have

$$c_* < c_* \int_{\mathbb{R}^N} (|\nabla u_0|^2 + V(x)|u_0|^2) dx + c_* \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2) dx + o_n(1) = c_* + o_n(1),$$

which is a contradiction. So $w_n \rightarrow 0$ strongly in $H^1(\mathbb{R}^N)$ as $n \rightarrow \infty$ and then $u_n \rightarrow u_0$ strongly in $H^1(\mathbb{R}^N)$ as $n \rightarrow \infty$. This implies that u_0 is a non-negative maximizer of c_* . The proof is complete. $\square$

In the following, we give a lower bound estimate for c_* . For any $\varepsilon > 0$, set

$$u_\varepsilon(x) = \varepsilon^{\frac{N}{2}} U(\varepsilon x),$$

where $U(x)$ is a maximizer of c_∞ and given above with $\lambda = 1$. Following [20], we have

Lemma 2.4 (Energy estimate). *Assume (V_1) , (V_2) , (Q_1) , (Q_2) hold, then $c_* > c_\infty$ if one of the following conditions holds*

- (i) $\beta = 0$, $\mu > \frac{N^2(N-2)}{4(N+1)}$, $\inf_{x \in \mathbb{R}^N} Q(x) \geq 1$;
- (ii) $0 < \beta < 2$, $v_\beta > 0$;
- (iii) $\beta > 2$, $\mu > \frac{N^2(N-2)}{4(N+1)}$.

Proof. Observe that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\varepsilon|^{\frac{N+\alpha}{N}}) |u_\varepsilon|^{\frac{N+\alpha}{N}} = \int_{\mathbb{R}^N} (I_\alpha * |U|^{\frac{N+\alpha}{N}}) |U|^{\frac{N+\alpha}{N}} = c_\infty,$$

$$\int_{\mathbb{R}^N} |u_\varepsilon|^2 = \int_{\mathbb{R}^N} |U|^2 = 1, \quad \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 = \varepsilon^2 \int_{\mathbb{R}^N} |\nabla U|^2 < +\infty.$$

Let

$$m_\varepsilon := \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 + V(x)|u_\varepsilon|^2,$$

then $m_\varepsilon = 1 + \varepsilon^2 \mathcal{J}_\mu(\varepsilon)$, where

$$\mathcal{J}_\mu(\varepsilon) = \varepsilon^{-2} \int_{\mathbb{R}^N} \left[|\nabla u_\varepsilon(x)|^2 + (V(x) - 1)|u_\varepsilon(x)|^2 \right] dx.$$

Since

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 &= \frac{N^2(N-2)}{4(N+1)} \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^2}{|x|^2} dx, \\ \mathcal{J}_\mu(\varepsilon) &= \varepsilon^{-2} \int_{\mathbb{R}^N} \left[|\nabla u_\varepsilon(x)|^2 + (V(x) - 1)|u_\varepsilon(x)|^2 \right] dx. \end{aligned}$$

By Lebesgue's monotone convergence theorem, we obtain

$$\begin{aligned} \mathcal{J}_\mu(\varepsilon) &= \int_{\mathbb{R}^N} \left[\frac{N^2(N-2)}{4(N+1)} + \varepsilon^{-2}(V(\varepsilon^{-1}x) - 1)|x|^2 \right] \frac{|U(x)|^2}{|x|^2} dx \\ &\rightarrow \left[\frac{N^2(N-2)}{4(N+1)} - \mu \right] \int_{\mathbb{R}^N} \frac{|U(x)|^2}{|x|^2} dx, \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

□

Let

$$a_\mu := \left[\frac{N^2(N-2)}{4(N+1)} - \mu \right] \int_{\mathbb{R}^N} \frac{|U(x)|^2}{|x|^2} dx,$$

then we get that

$$m_\varepsilon = 1 + a_\mu \varepsilon^2 + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0$$

and

$$a_\mu \begin{cases} > 0, & \text{if } \mu < \frac{N^2(N-2)}{4(N+1)}; \\ = 0, & \text{if } \mu = \frac{N^2(N-2)}{4(N+1)}; \\ < 0, & \text{if } \mu > \frac{N^2(N-2)}{4(N+1)}. \end{cases}$$

Let $v_\varepsilon := \frac{u_\varepsilon}{\sqrt{m_\varepsilon}}$, then

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + V(x)|v_\varepsilon|^2 = 1$$

and $c_\star \geq G(v_\varepsilon)$. In the following, we show that $G(v_\varepsilon) > c_\infty$ for $\varepsilon > 0$ small. In fact,

$$\begin{aligned} m_\varepsilon^{\frac{N+a}{N}} G(v_\varepsilon) &= \int_{\mathbb{R}^N} \left(I_\alpha \star [Q(x)|u_\varepsilon|^{\frac{N+a}{N}}] \right) Q(x)|u_\varepsilon|^{\frac{N+a}{N}} dx \\ &= \int_{\mathbb{R}^N} \left(I_\alpha \star [Q(\varepsilon^{-1}x)|U|^{\frac{N+a}{N}}] \right) Q(\varepsilon^{-1}x)|U|^{\frac{N+a}{N}} dx \\ &= \int_{\mathbb{R}^N} \left(I_\alpha \star [Q(\varepsilon^{-1}x)|U|^{\frac{N+a}{N}}] \right) [Q(\varepsilon^{-1}x) - 1]|U|^{\frac{N+a}{N}} dx \\ &\quad + \int_{\mathbb{R}^N} \left(I_\alpha \star [(Q(\varepsilon^{-1}x) - 1)|U|^{\frac{N+a}{N}}] \right) |U|^{\frac{N+a}{N}} dx + G_\infty(U). \end{aligned}$$

By virtue of Lemma 2.1, it is easy to check that

$$\int_{\mathbb{R}^N} \left(I_\alpha \star [|x|^{-\beta} |U|^{\frac{N+\alpha}{N}}] \right) |U|^{\frac{N+\alpha}{N}} dx < \infty, \text{ if } \beta \in [0, N).$$

It follows from the Lebesgue's monotone convergence theorem that, if $\beta \in [0, N)$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \int_{\mathbb{R}^N} \left(I_\alpha \star [(Q(\varepsilon^{-1}x) - 1)|U|^{\frac{N+\alpha}{N}}] \right) |U|^{\frac{N+\alpha}{N}} dx \\ &= v_\beta \int_{\mathbb{R}^N} \left(I_\alpha \star [|x|^{-\beta} |U|^{\frac{N+\alpha}{N}}] \right) |U|^{\frac{N+\alpha}{N}} dx. \end{aligned}$$

Similarly, if $\beta \in [0, N)$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{-\beta} \int_{\mathbb{R}^N} \left(I_\alpha \star [Q(\varepsilon^{-1}x)|U|^{\frac{N+\alpha}{N}}] \right) [Q(\varepsilon^{-1}x) - 1]|U|^{\frac{N+\alpha}{N}} dx \\ &= v_\beta \int_{\mathbb{R}^N} \left(I_\alpha \star |U|^{\frac{N+\alpha}{N}} \right) [|x|^{-\beta} |U|^{\frac{N+\alpha}{N}}] dx. \end{aligned}$$

Let

$$b_{\beta, \alpha} := 2 \int_{\mathbb{R}^N} \left(I_\alpha \star |U|^{\frac{N+\alpha}{N}} \right) [|x|^{-\beta} |U|^{\frac{N+\alpha}{N}}] dx > 0,$$

then, for $\beta \in [0, N)$, we have, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} G(v_\varepsilon) &= m_\varepsilon^{-\frac{N+\alpha}{N}} (c_\infty + b_{\beta, \alpha} v_\beta \varepsilon^\beta + o(\varepsilon^\beta)) \\ &= (1 + a_\mu \varepsilon^2 + o(\varepsilon^2))^{-\frac{N+\alpha}{N}} (c_\infty + b_{\beta, \alpha} v_\beta \varepsilon^\beta + o(\varepsilon^\beta)) \\ &= \left(1 - \frac{N+\alpha}{N} a_\mu \varepsilon^2 + o(\varepsilon^2) \right) (c_\infty + b_{\beta, \alpha} v_\beta \varepsilon^\beta + o(\varepsilon^\beta)) \\ &= c_\infty + b_{\beta, \alpha} v_\beta \varepsilon^\beta - \frac{N+\alpha}{N} c_\infty a_\mu \varepsilon^2 + o(\varepsilon^\beta) + o(\varepsilon^2). \end{aligned}$$

which implies that if $2 < \beta < N$,

$$G(v_\varepsilon) = c_\infty - \frac{N+\alpha}{N} c_\infty a_\mu \varepsilon^2 + o(\varepsilon^2), \text{ as } \varepsilon \rightarrow 0. \quad (2.3)$$

Moreover, by (Q_1) , if $\beta = 0$ and in addition $\inf_{x \in \mathbb{R}^N} Q(x) \geq 1$, we know $v_\beta = 0$ and

$$G(v_\varepsilon) \geq m_\varepsilon^{-\frac{N+\alpha}{N}} c_\infty \geq c_\infty - \frac{N+\alpha}{N} c_\infty a_\mu \varepsilon^2 + o(\varepsilon^2), \text{ as } \varepsilon \rightarrow 0. \quad (2.4)$$

It follows from (2.3) and (2.4) that $G(v_\varepsilon) > c_\infty$ for $\varepsilon > 0$ small if

$$\mu > \frac{N^2(N-2)}{4(N+1)}, \beta = 0 \text{ or } 2 < \beta < N.$$

For $\beta \geq N$, by (Q_2) we have for some $C > 0$,

$$|x|^2 |Q(x) - 1| \leq \frac{C}{|x|^{\beta-2}}, \quad x \in \mathbb{R} \setminus \{0\}.$$

Noting that

$$\int_{\mathbb{R}^N} \left(I_\alpha \star [|x|^{-2} |U|^{\frac{N+\alpha}{N}}] \right) |U|^{\frac{N+\alpha}{N}} dx < \infty,$$

due to the Lebesgue's monotone convergence theorem, if $\beta \geq N$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{\mathbb{R}^N} \left(I_\alpha \star [(Q(\varepsilon^{-1}x) - 1)|U|^{\frac{N+\alpha}{N}}] \right) |U|^{\frac{N+\alpha}{N}} dx = 0.$$

Similarly, if $\beta \geq N$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{\mathbb{R}^N} \left(I_\alpha \star [Q(\varepsilon^{-1}x)|U|^{\frac{N+\alpha}{N}}] \right) [Q(\varepsilon^{-1}x) - 1]|U|^{\frac{N+\alpha}{N}} dx = 0.$$

This yields that, for $\beta \geq N$, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} G(v_\varepsilon) &= m_\varepsilon^{-\frac{N+\alpha}{N}} (c_\infty + o(\varepsilon^2)) \\ &= (1 + a_\mu \varepsilon^2 + o(\varepsilon^2))^{-\frac{N+\alpha}{N}} (c_\infty + o(\varepsilon^2)) \\ &= \left(1 - \frac{N+\alpha}{N} a_\mu \varepsilon^2 + o(\varepsilon^2) \right) (c_\infty + o(\varepsilon^2)) \\ &= c_\infty - \frac{N+\alpha}{N} c_\infty a_\mu \varepsilon^2 + o(\varepsilon^2), \end{aligned}$$

As a consequence, $G(v_\varepsilon) > c_\infty$ for $\varepsilon > 0$ small if

$$\mu > \frac{N^2(N-2)}{4(N+1)}, \quad \beta \geq N.$$

Finally, if $0 < \beta < 2$, then

$$G(v_\varepsilon) = c_\infty + b_{\beta,\alpha} v_\beta \varepsilon^\beta + o(\varepsilon^\beta), \quad \text{as } \varepsilon \rightarrow 0,$$

which implies that $G(v_\varepsilon) > c_\infty$ for $\varepsilon > 0$ small and any $\mu \in \mathbb{R}$, $v_\beta > 0$. The proof is complete.

Proof of Theorem 1.1 completed

Proof. As an immediate consequence of Lemma 2.3 and 2.4, there exists $u_\star \in H^1(\mathbb{R}^N)$ such that $G(u_\star) = c_\star$ and

$$\int_{\mathbb{R}^N} (|\nabla u_\star|^2 + V(x)|u_\star|^2) dx = 1.$$

By the Lagrange multiplier theorem, there holds that for some $\kappa \in \mathbb{R}$ such that $G'(u_\star) = \kappa T'(u)$ in $H^{-1}(\mathbb{R}^N)$, that is, in the weak sense, $u_\star$ satisfies

$$\kappa (-\Delta u + V(x)u) = \frac{N+\alpha}{N} (I_\alpha \star [Q(x)|u|^{\frac{N+\alpha}{N}}]) Q(x)|u|^{\frac{\alpha}{N}-1} u, \quad x \in \mathbb{R}^N.$$

Obviously, $\kappa = \frac{N+\alpha}{N} c_\star > 0$ and by virtue of the maximum principle, $u_\star$ is positive. To remove the multiplier, let

$$u_\theta(\cdot) = \theta u_\star(\cdot), \quad \theta = c_\star^{-\frac{N}{2\alpha}},$$

then u_θ is a weak solution of problem (1.8) and

$$\begin{aligned} I(u_\theta) &= \frac{1}{2} T(u_\theta) - \frac{N}{2(N+\alpha)} G(u_\theta) \\ &= \frac{1}{2} \theta^2 T(u_\star) - \frac{N}{2(N+\alpha)} \theta^{\frac{2(N+\alpha)}{N}} G(u_\star) \\ &= \frac{1}{2} \theta^2 - \frac{N}{2(N+\alpha)} \theta^{\frac{2(N+\alpha)}{N}} c_\star = \frac{\alpha}{2(N+\alpha)} c_\star^{-\frac{N}{\alpha}} > 0. \end{aligned}$$

Finally, we show that u_θ is a ground state solution of problem (1.8). In fact, for any nontrivial solution u of problem (1.8), we can see that $T(u) = G(u) > 0$ and

$$I(u) = \frac{\alpha}{2(N+\alpha)} G(u).$$

Let

$$\tau = \sqrt{\frac{1}{T(u)}} > 0, \quad u_\tau(\cdot) = \tau u(\cdot),$$

then $T(u_\tau) = 1$ and

$$c_* \geq G(u_\tau) = \tau^{\frac{2(N+\alpha)}{N}} G(u).$$

It follows that

$$G(u) \leq c_* [T(u)]^{\frac{N+\alpha}{N}} = c_* [G(u)]^{\frac{N+\alpha}{N}}$$

and then

$$I(u) \geq \frac{\alpha}{2(N+\alpha)} c_*^{-\frac{N}{\alpha}}.$$

The proof is complete. $\square$

Proposition 2.1 (Pohožăev Identity). *If $u \in H^1(\mathbb{R}^N)$ is a solution of problem (1.8), then the following Pohožăev identity holds*

$$\begin{aligned} & \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} [NV(x) + \langle x, \nabla V(x) \rangle] u^2 dx \\ &= \frac{N}{2} \int_{\mathbb{R}^N} \left(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \right) Q(x)|u|^{\frac{N+\alpha}{N}} dx \\ & \quad + \frac{N}{N+\alpha} \int_{\mathbb{R}^N} \left(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \right) \langle x, \nabla Q(x) \rangle |u|^{\frac{N+\alpha}{N}} dx. \end{aligned}$$

Proof. The proof is similar to [20, Proposition 11] and [22, Theorem 3]. We omit the details here. $\square$

Completion of Proof of Theorem 1.2

Proof. For any solution $u \in H^1(\mathbb{R}^N)$ of problem (1.8), using u as a test function, we have

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx = \int_{\mathbb{R}^N} \left(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \right) Q(x)|u|^{\frac{N+\alpha}{N}} dx$$

Thanks to Proposition 2.1,

$$\begin{aligned} & \int_{\mathbb{R}^N} \left(-|\nabla u|^2 + \frac{1}{2} \langle x, \nabla V(x) \rangle u^2 \right) dx \\ &= \frac{N}{N+\alpha} \int_{\mathbb{R}^N} \left(I_\alpha * [Q(x)|u|^{\frac{N+\alpha}{N}}] \right) \langle x, \nabla Q(x) \rangle |u|^{\frac{N+\alpha}{N}} dx. \end{aligned}$$

Then by (VQ) and Hardy's inequality, if u is nontrivial,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx < \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

which is a contradiction. The proof is complete. $\square$

3 Proof of Theorem 1.3

3.1 Eigenvalues and eigenfunctions.

Consider the eigenvalue problem

$$-\Delta u + u = \frac{\lambda}{v^2 + |x|^2} u, \quad u \in H^1(\mathbb{R}^N). \quad (3.1)$$

It was proven in [7] that problem (3.1) admits a sequence of eigenvalues $\{\lambda_n\}$ with finite multiplicity and associated eigenfunctions $\{\varphi_n\}$, such that

$$0 < \mu^v = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \leq \dots \rightarrow +\infty, \quad n \rightarrow \infty$$

and for any $i, j \in \mathbb{N}$ and $i \neq j$,

$$\int_{\mathbb{R}^N} \nabla \varphi_i \nabla \varphi_j + \varphi_i \varphi_j = 0, \quad \int_{\mathbb{R}^N} \frac{1}{v^2 + |x|^2} |\varphi_i|^2 dx = 1.$$

Moreover, for some $C_n, \delta_n > 0$,

$$|\varphi_n(x)| + |\nabla \varphi_n(x)| \leq C_n \exp(-\delta_n |x|), \quad x \in \mathbb{R}^N.$$

For any $n \in \mathbb{N}$, we have the orthogonal decomposition $H^1(\mathbb{R}^N) = E^- \oplus E^+$, where

$$E^- = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}, \quad E^+ = \overline{\text{span}\{\varphi_{n+1}, \varphi_{n+2}, \dots\}}.$$

3.2 Energy estimates.

For any $\varepsilon > 0$, set

$$u_\varepsilon(x) = \varepsilon^{\frac{N}{2}} U(\varepsilon x),$$

where U is given in Section 2 above for $\lambda = v$. Similarly, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\varepsilon|^{\frac{N+\alpha}{N}}) |u_\varepsilon|^{\frac{N+\alpha}{N}} = c_\infty$$

and

$$m_\varepsilon := \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 + V_{\mu,v}(x) |u_\varepsilon|^2 = 1 + a_{\mu,v} \varepsilon^2 + o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0$$

where

$$a_{\mu,v} := C v^{-2} \left[\frac{N^2(N-2)}{4(N+1)} - \mu \right] \int_{\mathbb{R}^N} \frac{1}{|x|^2(1+|x|^2)^N} dx$$

and $C > 0$ is independent of ε, μ, v .

From now on, we assume

$$\mu > \max \left\{ \frac{N^2(N-2)}{4(N+1)}, \mu^v \right\}$$

and for some $n \in \mathbb{N}$, we assume $\mu \in [\lambda_n, \lambda_{n+1})$ when $N \geq 4$ or $\mu \in (\lambda_n, \lambda_{n+1})$ when $N = 3$. Define

$$\hat{E}(v_\varepsilon) := \{w \in H^1(\mathbb{R}^N) : w = tv_\varepsilon + v, \quad t \geq 0, \quad v \in E^-\}$$

where $v_\varepsilon := \frac{u_\varepsilon}{\sqrt{m_\varepsilon}}$ and the energy functional

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_{\mu,v}(x) |u|^2) dx \\ &\quad - \frac{N}{2(N+\alpha)} \int_{\mathbb{R}^N} \left(I_\alpha * [Q_\beta(x) |u|^{\frac{N+\alpha}{N}}] \right) Q_\beta(x) |u|^{\frac{N+\alpha}{N}} dx. \end{aligned}$$

Lemma 3.1. *For $\varepsilon > 0$ small enough, there holds that*

$$\sup_{w \in \hat{E}(v_\varepsilon)} J(w) < \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}},$$

provided when $N = 3$, one of the followings holds

- (i) $\beta = 0$, $v_\beta = 0$, $\frac{3}{2} < \alpha < 3$;
 (ii) $0 < \beta < \min\{\frac{2}{6-\alpha}, 2\}$, $v_\beta > 0$
 (iii) $\beta > 2$, $v_\beta > -1$, $\frac{3}{2} < \alpha < 3$;

and when $N \geq 4$, one of the followings holds

- (iv) $\beta = 0$, $v_\beta = 0$;
 (v) $0 < \beta < 2$, $v_\beta > 0$;
 (vi) $\beta > 2$, $v_\beta > -1$.

Proof. For any $v \in E^-$ and $t > 0$,

$$\begin{aligned} J(tv_\varepsilon + v) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + V_{\mu,v}|v_\varepsilon|^2 + t \int_{\mathbb{R}^N} \nabla v_\varepsilon \cdot \nabla v + V_{\mu,v}v_\varepsilon v + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + V_{\mu,v}|v|^2 \\ &\quad - \frac{N}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha \star [Q_\beta(x)|tv_\varepsilon + v|^{\frac{N+\alpha}{N}}])Q_\beta(x)|tv_\varepsilon + v|^{\frac{N+\alpha}{N}}. \end{aligned} \quad (3.2)$$

Noting that $v \in E^-$, $\int_{\mathbb{R}^N} |\nabla v|^2 + V_{\mu,v}|v|^2 \leq 0$ and then

$$\begin{aligned} J(tv_\varepsilon + v) &\leq \frac{t^2}{2} + t \int_{\mathbb{R}^N} \nabla v_\varepsilon \cdot \nabla v + V_{\mu,v}v_\varepsilon v \\ &\quad - \frac{N}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha \star [Q_\beta(x)|tv_\varepsilon + v|^{\frac{N+\alpha}{N}}])Q_\beta(x)|tv_\varepsilon + v|^{\frac{N+\alpha}{N}}. \end{aligned} \quad (3.3)$$

Since $|\nabla v_\varepsilon| \leq C_1 \varepsilon^{\frac{N}{2}+1}$ and $|\nabla v| \in L^r(\mathbb{R}^N)$ for any $r > 0$,

$$\left| \int_{\mathbb{R}^N} \nabla v_\varepsilon \cdot \nabla v + V_{\mu,v}v_\varepsilon v \right| \leq \|\nabla v_\varepsilon\|_\infty \|\nabla v\|_{L^1} + (1+\mu)\|v_\varepsilon\|_\infty \|v\|_1.$$

Recalling that it is proven in [18] that

$$\|u\|_* := \left[\int_{\mathbb{R}^N} (I_\alpha \star |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} \right]^{\frac{N}{2(N+\alpha)}}, \quad u \in L^2(\mathbb{R}^N),$$

is a norm in $L^2(\mathbb{R}^N)$. Thanks to the fact that $\dim(E^-) = n$ and all the norms are equivalent in E^- , we get that

$$\left| \int_{\mathbb{R}^N} \nabla v_\varepsilon \cdot \nabla v + V_{\mu,v}v_\varepsilon v \right| \leq C_2 \varepsilon^{\frac{N}{2}} \|v\|. \quad (3.4)$$

In the following, we estimate the convolution term

$$G(tv_\varepsilon + v) := \int_{\mathbb{R}^N} (I_\alpha \star [Q_\beta(x)|tv_\varepsilon + v|^{\frac{N+\alpha}{N}}])Q_\beta(x)|tv_\varepsilon + v|^{\frac{N+\alpha}{N}}.$$

By [7, Lemma 3.5], for any $x \in \mathbb{R}^N$, $\varepsilon > 0$, $v \in E^-$ and $t > 0$,

$$\left| |tv_\varepsilon + v|^{\frac{N+\alpha}{N}} - |tv_\varepsilon|^{\frac{N+\alpha}{N}} - |v|^{\frac{N+\alpha}{N}} \right| \leq C_3 |tv_\varepsilon| |v|^{\frac{\alpha}{N}}, \quad (3.5)$$

where $C_3 = 2^{(N-\alpha)/N}(N+\alpha)/N$. It follows that

$$\int_{\mathbb{R}^N} (I_\alpha \star [Q_\beta(x)|tv_\varepsilon + v|^{\frac{N+\alpha}{N}}])Q_\beta(x)|tv_\varepsilon + v|^{\frac{N+\alpha}{N}}$$

$$\begin{aligned}
&\geq t^{\frac{2(N+\alpha)}{N}} \int_{\mathbb{R}^N} (I_\alpha * [Q_\beta(x)|v_\varepsilon|^{\frac{N+\alpha}{N}}]) Q_\beta(x)|v_\varepsilon|^{\frac{N+\alpha}{N}} \\
&\quad + 2t^{\frac{N+\alpha}{N}} \int_{\mathbb{R}^N} (I_\alpha * [Q_\beta(x)|v_\varepsilon|^{\frac{N+\alpha}{N}}]) Q_\beta(x)|v|^{\frac{N+\alpha}{N}} \\
&\quad + \int_{\mathbb{R}^N} (I_\alpha * [Q_\beta(x)|v|^{\frac{N+\alpha}{N}}]) Q_\beta(x)|v|^{\frac{N+\alpha}{N}} \\
&\quad - 2C_3 \int_{\mathbb{R}^N} (I_\alpha * [Q_\beta(x)(|tv_\varepsilon|^{\frac{N+\alpha}{N}} + |v|^{\frac{N+\alpha}{N}})]) Q_\beta(x)|tv_\varepsilon||v|^{\frac{\alpha}{N}} \\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.6}$$

Obviously, $J_2 \geq 0$. So combining (3.3)-(3.6), we have

$$J(tv_\varepsilon + v) \leq \frac{1}{2}t^2 + C_2t\varepsilon^{\frac{N}{2}}\|v\| - \frac{N}{2(N+\alpha)}(J_1 + J_3 + J_4). \tag{3.7}$$

Similarly as above, as $\varepsilon \rightarrow 0$,

$$t^{-\frac{2(N+\alpha)}{N}}J_1 \geq \begin{cases} c_\infty - \frac{N+\alpha}{N}c_\infty a_{\mu,v}\varepsilon^2 + o(\varepsilon^2), & \text{if } \beta = 0, v_\beta = 0; \\ c_\infty + b_{\beta,\alpha}v_\beta\varepsilon^\beta + o(\varepsilon^\beta), & \text{if } 0 < \beta < 2, v_\beta > 0; \\ c_\infty - \frac{N+\alpha}{N}c_\infty a_{\mu,v}\varepsilon^2 + o(\varepsilon^2), & \text{if } \beta > 2, v_\beta > -1. \end{cases}$$

Due to the equivalence of norms in E^- , $J_3 \geq C_4\|v\|^{\frac{2(N+\alpha)}{N}}$. Thanks to the Hardy–Littlewood–Sobolev inequality and $Q_\beta \in L^\infty(\mathbb{R}^N)$,

$$|J_4| \leq C_5t(\|tv_\varepsilon\|_{L^2} + \|v\|_{L^2})^{\frac{N+\alpha}{N}} \left(\int_{\mathbb{R}^N} |v_\varepsilon|^{\frac{2N}{N+\alpha}} |v|^{\frac{2\alpha}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}}.$$

Then noting that $|v_\varepsilon(x)| \leq C_6\varepsilon^{\frac{N}{2}}$, $x \in \mathbb{R}^N$ and $E^- \subset L^{\frac{2\alpha}{N+\alpha}}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |v_\varepsilon|^{\frac{2N}{N+\alpha}} |v|^{\frac{2\alpha}{N+\alpha}} \leq C_7\varepsilon^{\frac{N^2}{N+\alpha}} \int_{\mathbb{R}^N} |v|^{\frac{2\alpha}{N+\alpha}}.$$

Thanks to $\dim(E^-) < \infty$, it is easy to check that

$$\int_{\mathbb{R}^N} |v|^{\frac{2\alpha}{N+\alpha}} \leq C_8\|v\|^{\frac{2\alpha}{N+\alpha}}, \text{ for all } v \in E^-,$$

which implies that

$$|J_4| \leq C_9(t^{\frac{N+\alpha}{N}}\|v\|^{\frac{\alpha}{N}} + \|v\|^{\frac{N+2\alpha}{N}})t\varepsilon^{\frac{N}{2}}.$$

Let $C_{10} = \frac{NC_4}{2(N+\alpha)}$ (independent of v), by Young's inequality the following hold:

$$C_2t\varepsilon^{\frac{N}{2}}\|v\| \leq \frac{C_{10}}{3}\|v\|^{\frac{2(N+\alpha)}{N}} + C_{11}t^{\frac{2(N+\alpha)}{N+2\alpha}}\varepsilon^{\frac{N(N+\alpha)}{N+2\alpha}}, \tag{3.8}$$

$$C_9\|v\|^{\frac{\alpha}{N}}t^{\frac{2N+\alpha}{N}}\varepsilon^{\frac{N}{2}} \leq \frac{C_{10}}{3}\|v\|^{\frac{2(N+\alpha)}{N}} + C_{12}t^{\frac{2(N+\alpha)}{N}}\varepsilon^{\frac{N(N+\alpha)}{2N+\alpha}}, \tag{3.9}$$

$$C_9\|v\|^{\frac{N+2\alpha}{N}}t\varepsilon^{\frac{N}{2}} \leq \frac{C_{10}}{3}\|v\|^{\frac{2(N+\alpha)}{N}} + C_{13}t^{\frac{2(N+\alpha)}{N}}\varepsilon^{N+\alpha}, \tag{3.10}$$

from which we obtain

$$|J_4| \leq \frac{2C_{10}}{3}\|v\|^{\frac{2(N+\alpha)}{N}} + C_{12}t^{\frac{2(N+\alpha)}{N}}\varepsilon^{\frac{N(N+\alpha)}{2N+\alpha}} + C_{13}t^{\frac{2(N+\alpha)}{N}}\varepsilon^{N+\alpha}.$$

Then it follows from (3.7)-(3.8) that

$$\begin{aligned}
J(tv_\varepsilon + v) &\leq \frac{1}{2}t^2 - \frac{N}{2(N+\alpha)}J_1 + C_{11}t^{\frac{2(N+\alpha)}{N+2\alpha}}\varepsilon^{\frac{N(N+\alpha)}{N+2\alpha}} \\
&\quad + \frac{N}{2(N+\alpha)}t^{\frac{2(N+\alpha)}{N}} \left(C_{12}\varepsilon^{\frac{N(N+\alpha)}{2N+\alpha}} + C_{13}\varepsilon^{N+\alpha} \right).
\end{aligned} \tag{3.11}$$

In particular, (3.11) implies that there exist $0 < t_* < t^*$ (independent of ε) such that, for all $v \in E^-$, $\varepsilon > 0$ small,

$$J(tv_\varepsilon + v) \leq \frac{\alpha}{4(N+\alpha)} c_\infty^{-\frac{N}{\alpha}}, \quad t \in (0, t_*) \cup (t^*, \infty). \quad (3.12)$$

If $\mu \in (\lambda_n, \lambda_{n+1})$, there exists a constant C_{14} such that

$$\int_{\mathbb{R}^N} |\nabla v|^2 + V_{\mu,v} \leq -C_{14} \|v\|^2, \quad \text{for all } v \in E^-.$$

Similarly as above, by (3.2) we have in place of (3.7) that

$$J(tv_\varepsilon + v) \leq \frac{1}{2} t^2 - \frac{C_{14}}{2} \|v\|^2 + C_2 t \varepsilon^{\frac{N}{2}} \|v\| - \frac{N}{2(N+\alpha)} (J_1 + J_3 + J_4).$$

We now use again estimates (3.8) and (3.10) and (3.9) replaced by the following

$$C_9 \|v\|^{\frac{\alpha}{N}} t^{\frac{2N+\alpha}{N}} \varepsilon^{\frac{N}{2}} \leq C_{14} \|v\|^2 + C_{15} t^{\frac{2(2N+\alpha)}{2N-\alpha}} \varepsilon^{\frac{N^2}{2N-\alpha}},$$

to get the following estimate

$$\begin{aligned} J(tv_\varepsilon + v) \leq & \frac{1}{2} t^2 - \frac{N}{2(N+\alpha)} J_1 + C_{11} t^{\frac{2(N+\alpha)}{N+2\alpha}} \varepsilon^{\frac{N(N+\alpha)}{N+2\alpha}} \\ & + \frac{N}{2(N+\alpha)} \left(C_{15} t^{\frac{2(2N+\alpha)}{2N-\alpha}} \varepsilon^{\frac{N^2}{2N-\alpha}} + C_{13} t^{\frac{2(N+\alpha)}{N}} \varepsilon^{N+\alpha} \right). \end{aligned} \quad (3.13)$$

Case 1. $\beta = v_\beta = 0$ or $\beta > 2$, $v_\beta \geq -1$.

If $N \geq 4$, then $\frac{N(N+\alpha)}{2N+\alpha} > 2$. Moreover, it follows from (3.11) that, for all $v \in E^-$, as $\varepsilon \rightarrow 0$, there holds that

$$\begin{aligned} J(tv_\varepsilon + v) \leq & \frac{1}{2} t^2 - \frac{N}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N}} \left(c_\infty - \frac{N+\alpha}{N} c_\infty a_{\mu,v} \varepsilon^2 - C_{12} \varepsilon^{\frac{N(N+\alpha)}{2N+\alpha}} \right. \\ & \left. + o(\varepsilon^2) + o(\varepsilon^{\frac{N(N+\alpha)}{2N+\alpha}}) \right), \quad t \in [t_*, t^*], \end{aligned}$$

which implies that

$$J(tv_\varepsilon + v) \leq \frac{1}{2} t^2 - \frac{N}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N}} \left(c_\infty - \frac{N+\alpha}{N} c_\infty a_{\mu,v} \varepsilon^2 + o(\varepsilon^2) \right), \quad t \in [t_*, t^*].$$

Noting that

$$\max_{t \geq 0} g(t) = g(c_\infty^{-\frac{N}{2\alpha}}) = \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}},$$

where

$$g(t) = \frac{1}{2} t^2 - \frac{N}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N}} c_\infty.$$

So thanks to (3.12) and $a_{\mu,v} < 0$ when $\mu > \frac{N^2(N-2)}{4(N+1)}$, we have as $\varepsilon > 0$ small,

$$\sup_{w \in \hat{E}(v_\varepsilon)} J(w) \leq \max_{t \geq 0} g(t) + C_{16} c_\infty a_{\mu,v} \varepsilon^2 + o(\varepsilon^2) < \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}},$$

where C_{16} is independent of ε .

If $N = 3$, we have $\frac{N(N+\alpha)}{2N+\alpha} > 2$ and $\frac{N^2}{2N-\alpha} > 2$ if $\frac{3}{2} < \alpha < 3$. By (3.13), for all $v \in E^-$ and $t \in [t_*, t^*]$, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} J(tv_\varepsilon + v) \leq & \frac{1}{2} t^2 - \frac{N}{2(N+\alpha)} J_1 + C_{11} t^{\frac{2(N+\alpha)}{N+2\alpha}} \varepsilon^{\frac{N(N+\alpha)}{N+2\alpha}} \\ & + \frac{N}{2(N+\alpha)} \left(C_{15} t^{\frac{2(2N+\alpha)}{2N-\alpha}} \varepsilon^{\frac{N^2}{2N-\alpha}} + C_{13} \varepsilon^{N+\alpha} \right) t^{\frac{2(N+\alpha)}{N}} \\ = & \frac{1}{2} t^2 - \frac{N}{2(N+\alpha)} c_\infty t^{\frac{2(N+\alpha)}{N}} + \frac{1}{2} t_*^{\frac{2(N+\alpha)}{N}} c_\infty a_{\mu,v} \varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

Similarly as above, $\sup_{w \in \hat{E}(v_\varepsilon)} J(w) < \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}}$ for $\varepsilon > 0$ small.

Case 2. $0 < \beta < 2$, $v_\beta > 0$.

Similarly as above, if $N \geq 4$, for all $v \in E^-$, as $\varepsilon \rightarrow 0$, there holds that

$$J(tv_\varepsilon + v) \leq \frac{1}{2}t^2 - \frac{N}{2(N+\alpha)}t^{\frac{2(N+\alpha)}{N}} \left(c_\infty + b_{\beta,\alpha}v_\beta\varepsilon^\beta - C_{12}\varepsilon^{\frac{N(N+\alpha)}{2N+\alpha}} + o(\varepsilon^\beta) + o(\varepsilon^{\frac{N(N+\alpha)}{2N+\alpha}}) \right), \quad t \in [t_*, t^*].$$

Recalling that $\frac{N(N+\alpha)}{2N+\alpha} > 2$ when $N \geq 4$, which implies that

$$J(tv_\varepsilon + v) \leq \frac{1}{2}t^2 - \frac{N}{2(N+\alpha)}t^{\frac{2(N+\alpha)}{N}} \left(c_\infty + b_{\beta,\alpha}v_\beta\varepsilon^\beta + o(\varepsilon^\beta) \right), \quad t \in [t_*, t^*].$$

Similarly as above, we get that $\sup_{w \in \hat{E}(v_\varepsilon)} J(w) < \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}}$, for $\varepsilon > 0$ small.

If $N = 3$, similarly as above, for all $v \in E^-$ and $t \in [t_*, t^*]$, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} J(tv_\varepsilon + v) &\leq \frac{1}{2}t^2 - \frac{N}{2(N+\alpha)}t^{\frac{2(N+\alpha)}{N}} \left(c_\infty + b_{\beta,\alpha}v_\beta\varepsilon^\beta + o(\varepsilon^\beta) \right) \\ &\quad + \frac{N}{2(N+\alpha)}C_{15}t^{\frac{4N+2\alpha}{2N-\alpha}}\varepsilon^{\frac{N^2}{2N-\alpha}} \\ &= \frac{1}{2}t^2 - \frac{N}{2(N+\alpha)}c_\infty t^{\frac{2(N+\alpha)}{N}} - \frac{N}{2(N+\alpha)}t_*^{\frac{2(N+\alpha)}{N}}b_{\beta,\alpha}v_\beta\varepsilon^\beta \\ &\quad + \frac{N}{2(N+\alpha)}C_{15}t_*^{\frac{4N+2\alpha}{2N-\alpha}}\varepsilon^{\frac{N^2}{2N-\alpha}} + o(\varepsilon^\beta). \end{aligned}$$

It follows that if $\beta < \frac{N^2}{2N-\alpha}$, for all $v \in E^-$ and $t \in [t_*, t^*]$, as $\varepsilon \rightarrow 0$,

$$J(tv_\varepsilon + v) \leq \frac{1}{2}t^2 - \frac{N}{2(N+\alpha)}c_\infty t^{\frac{2(N+\alpha)}{N}} - \frac{N}{2(N+\alpha)}t_*^{\frac{2(N+\alpha)}{N}}b_{\beta,\alpha}v_\beta\varepsilon^\beta + o(\varepsilon^\beta).$$

Similarly as above, $\sup_{w \in \hat{E}(v_\varepsilon)} J(w) < \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}}$ for $\varepsilon > 0$ small. The proof is complete. $\square$

3.3 Palais-Smale condition.

Similarly to [7], we have the following compactness result, which plays a crucial role in finding nontrivial solutions of (1.8).

Lemma 3.2. *The functional J satisfies the Palais-Smale condition in $(-\infty, c)$ if $c < \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}}$. Namely, if $\{u_m\}_{m \in \mathbb{N}} \subset H^1(\mathbb{R}^N)$ satisfies*

$$J(u_m) \rightarrow c, \quad J'(u_m) \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^N), \text{ as } m \rightarrow \infty$$

then up to a subsequence, there exists $u \in H^1(\mathbb{R}^N)$ such that $u_m \rightarrow u$ strongly in $H^1(\mathbb{R}^N)$, as $m \rightarrow \infty$.

Proof of Lemma 3.2. The proof is similar to [7, Lemma 3.7]. For the sake of completeness, we give just a sketch. Let $\{u_m\}_{m \in \mathbb{N}} \subset H^1(\mathbb{R}^N)$ be a (P-S)_c sequence, namely

$$J(u_m) \rightarrow c < \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}}, \quad J'(u_m) \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^N), \text{ as } m \rightarrow \infty.$$

Then by a similar fashion, we know that the sequence $\{u_m\}_{m \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ and up to a subsequence, there exists $u \in H^1(\mathbb{R}^N)$ such that $u_m \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$, as $m \rightarrow \infty$. Let $v_m = u_m - u$, then by the weak convergence and a Brezis-Lieb type lemma [19, Lemma 2.4],

$$\begin{cases} c + o_m(1) = J(u) + \frac{1}{2}\|v_m\|^2 - \frac{N}{2(N+\alpha)}G_\infty(v_m), \\ o_m(1) = \langle J'(u), u \rangle + \|v_m\|^2 - G_\infty(v_m). \end{cases} \quad (3.14)$$

Since $J'(u) = 0$ in $H^{-1}(\mathbb{R}^N)$,

$$\begin{aligned} J(u) &= J(u) - \frac{N}{2(N+\alpha)} \langle J'(u), u \rangle \\ &= \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 + V_{\mu, \nu} |u|^2 \geq 0. \end{aligned}$$

Suppose $G_\infty(v_m) \rightarrow l \geq 0$, as $m \rightarrow \infty$, then by (3.14) $\lim_{m \rightarrow \infty} \|v_m\|^2 = l$. If $l > 0$, then

$$l + o_m(1) = G_\infty(v_m) \leq c_\infty \|v_m\|_2^{\frac{2(N+\alpha)}{N}} \leq c_\infty (l + o_m(1))^{\frac{N+\alpha}{N}},$$

which implies $l \geq c_\infty^{-\frac{N}{\alpha}}$. Then by (3.14),

$$c \geq \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}},$$

which is a contradiction. Therefore $l = 0$ and the proof is complete. $\square$

Proof of Theorem 1.3. Now, we are in position to prove Theorem 1.3. By virtue of [3, Theorem 2.4] due to P. Bartolo, V. Benci and D. Fortunato, similarly as that in [7], (1.8) admits at least one nontrivial solution $u \in H^1(\mathbb{R}^N)$ with $J(u) < \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}}$. Let

$$K := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : J'(u) = 0 \text{ in } H^{-1}(\mathbb{R}^N)\},$$

then $K \neq \emptyset$ and

$$m := \inf_{u \in K} J(u) < \frac{\alpha}{2(N+\alpha)} c_\infty^{-\frac{N}{\alpha}}.$$

Similarly to [7], $m > 0$ and there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $u_0 \in K$ and $J(u_0) = m$. The proof of Theorem 1.3 is complete. $\square$

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