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Variation inequalities for rough singular integrals and their commutators on Morrey spaces and Besov spaces

<https://doi.org/10.1515/anona-2020-0187>

Received January 3, 2021; accepted April 21, 2021.

Abstract: This paper is devoted to investigating the boundedness, continuity and compactness for variation operators of singular integrals and their commutators on Morrey spaces and Besov spaces. More precisely, we establish the boundedness for the variation operators of singular integrals with rough kernels $\Omega \in L^q(S^{n-1})$ ($q > 1$) and their commutators on Morrey spaces as well as the compactness for the above commutators on Lebesgue spaces and Morrey spaces. In addition, we present a criterion on the boundedness and continuity for a class of variation operators of singular integrals and their commutators on Besov spaces. As applications, we obtain the boundedness and continuity for the variation operators of Hilbert transform, Hermit Riesz transform, Riesz transforms and rough singular integrals as well as their commutators on Besov spaces.

Keywords: Variation operator, Calderón-Zygmund singular integral, commutator, Morrey space, Besov space

MSC: Primary 42B20, 42B25.

1 Introduction

An active topic of current research is the investigation on the variational inequalities for various operators. The first work was due to Lépingle [21] in 1976 when he established the variational inequality for general martingales (see [31] for a simple proof). Lépingle's result was later used by Bourgain [2] to establish similar variational estimates for the ergodic averages. Since then, Bourgain's work has inaugurated a new research direction in ergodic theory and harmonic analysis. We can consult [2, 16, 17] for the ergodic averages, [24, 25] for the differential operators, [3, 13] for the Hilbert transform, [8, 13] for the Riesz transforms, [4, 5, 9, 18] for the singular integrals with rough kernels, [6, 24, 25, 35, 36] for the Calderón-Zygmund singular integrals and their commutators as well as [26, 27] for the discrete singular integral operators. Recently, Liu and Cui [22] established the boundedness and compactness for variation operators of Calderón-Zygmund singular integrals and their commutators on weighted Morrey spaces and Sobolev spaces. Based on this, we are interested in two types of results:

- Boundedness and compactness properties for variation operators of singular integrals with rough kernels and their commutators on Morrey spaces.
- Boundedness and continuity properties for variation operators of Calderón-Zygmund singular integrals and their commutators on Besov spaces.

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These contents are the main motivations of this work. It should be pointed out that this is the first work focusing on the boundedness and compactness for variation operators of singular integrals with rough kernels and their commutators on Morrey spaces as well as the boundedness and continuity of variation operators of Calderón-Zygmund singular integrals and their commutators on Besov spaces.

1.1 Background

Let $\mathcal{T} = \{T_\epsilon\}_{\epsilon>0}$ be a family of bounded operators satisfying

$$\lim_{\epsilon \rightarrow 0} T_\epsilon f(x) = Tf(x)$$

almost everywhere for a certain class of functions f . For $\rho > 2$, the ρ -variation operator of $\mathcal{T}$ is defined by

$$\mathcal{V}_\rho(\mathcal{T})(f)(x) = \sup_{\{\epsilon_i\} \searrow 0} \left(\sum_{i=1}^{\infty} |T_{\epsilon_i} f(x) - T_{\epsilon_{i+1}} f(x)|^\rho \right)^{1/\rho},$$

where the supremum runs over all sequences $\{\epsilon_i\}$ of positive numbers decreasing to zero.

Let $K(\cdot, \cdot)$ be a kernel defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$, we consider the following operator of Calderón-Zygmund type

$$T_K(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \quad \text{for all } x \notin \text{supp} f. \quad (1.1)$$

Formally, the operator T_K can be rewritten as

$$T_K(f)(x) = \lim_{\epsilon \rightarrow 0^+} T_{K, \epsilon}(f)(x),$$

where $T_{K, \epsilon}$ is the truncated singular integral operator, i.e.

$$T_{K, \epsilon}(f)(x) = \int_{|x-y|>\epsilon} K(x, y) f(y) dy.$$

The commutator of T_K with a suitable function b is defined as

$$T_{K, b}(f)(x) := [b, T_K](f)(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) K(x, y) f(y) dy = \lim_{\epsilon \rightarrow 0^+} T_{K, b, \epsilon}(f)(x),$$

where

$$T_{K, b, \epsilon}(f)(x) := \int_{|x-y|>\epsilon} (b(x) - b(y)) K(x, y) f(y) dy.$$

Denote $T_{K, b}^1 = T_{K, b}$. The iterated commutator $T_{K, b}^m$ with $m \geq 2$ is defined by

$$\begin{aligned} T_{K, b}^m(f)(x) &:= [b, T_{K, b}^{m-1}](f)(x) \\ &= \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x, y) f(y) dy = \lim_{\epsilon \rightarrow 0^+} T_{K, b, \epsilon}^m(f)(x), \end{aligned} \quad (1.2)$$

where

$$T_{K, b, \epsilon}^m(f)(x) := \int_{|x-y|>\epsilon} (b(x) - b(y))^m K(x, y) f(y) dy.$$

The variation operators of Calderón-Zygmund singular integrals and their commutators can be defined as follows:

Definition 1.1. (Variation operators for singular integrals and their commutators). Let $\mathcal{T}_K = \{T_{K, \epsilon}\}_{\epsilon>0}$ and $\mathcal{T}_{K, b}^m = \{T_{K, b, \epsilon}^m\}_{\epsilon>0}$ with $m \geq 1$. For $\rho > 2$, the ρ -variation operator of $\mathcal{T}_K$ is defined by

$$\mathcal{V}_\rho(\mathcal{T}_K)(f)(x) := \sup_{\epsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| \int_{\epsilon_{i+1} < |x-y| \leq \epsilon_i} K(x, y) f(y) dy \right|^\rho \right)^{1/\rho}. \quad (1.3)$$

Analogously, the ρ -variation operator of $\mathcal{T}_{K,b}^m$ can be given as

$$\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)(x) := \sup_{\varepsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| \int_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i} (b(x) - b(y))^m K(x, y) f(y) dy \right|^\rho \right)^{1/\rho}, \quad (1.4)$$

where the above sup is taken over all sequences $\{\varepsilon_i\}$ decreasing to zero. Clearly, $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m) = \mathcal{V}_\rho(\mathcal{T}_K)$ for $m = 0$. For convenience, we set $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m) = \mathcal{V}_\rho(\mathcal{T}_{K,b})$ for $m = 1$.

The operators defined in (1.1) and (1.2) have some classical models, which are listed as follows:

- When $n = 1$ and $K(x, y) = \frac{1}{x-y}$, then T_K (resp., $T_{K,b}^m$) is the (resp., the m -th order commutator of) Hilbert transform. We denote $\mathcal{T}_K = \mathcal{H}$ and $\mathcal{T}_{K,b}^m = \mathcal{H}_b^m$ for $m \geq 1$.
- When $n = 1$ and $K(x, y) = \mathfrak{R}^\pm(x, y)$, where $\mathfrak{R}^\pm(x, y)$ is a Hermit Riesz kernel whose expressions can be found in [32], then T_K (resp., $T_{K,b}^m$) is (resp., the m -th order commutator of) Hermit Riesz transform. We denote $\mathcal{T}_K = \mathcal{R}^\pm$ and $\mathcal{T}_{K,b}^m = \mathcal{R}_{\pm,b}^m$ for $m \geq 1$.
- When $n \geq 2$ and $K(x, y) = R_j(x, y)$, where $R_j(x, y) := \Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}} \frac{x_j - y_j}{|x-y|^{n+1}}$ for $1 \leq j \leq n$, then T_K (resp., $T_{K,b}^m$) is (resp., the m -th order commutator of) Riesz transform. We denote $\mathcal{T}_K = \mathcal{R}_j$ and $\mathcal{T}_{K,b}^m = \mathcal{R}_{j,b}^m$ for $m \geq 1$.
- When $n \geq 2$ and $K(x, y) = \frac{\Omega(x-y)}{|x-y|^n}$, where $\Omega \in L^1(S^{n-1})$ is homogeneous of zero and satisfies

$$\int_{S^{n-1}} \Omega(\theta) d\sigma(\theta) = 0. \quad (1.5)$$

Then T_K (resp., $T_{K,b}^m$) is just the usual (resp., the m -th order commutator of) singular integral operator with rough kernel Ω . We denote $\mathcal{T}_K = \mathcal{T}_\Omega$ and $\mathcal{T}_{K,b}^m = \mathcal{T}_{\Omega,b}^m$ for $m \geq 1$.

- When T_K is bounded on $L^2(\mathbb{R}^n)$ and the kernel K is a standard Calderón-Zygmund kernel, which satisfies the size condition

$$|K(x, y)| \leq \frac{A}{|x-y|^n}, \quad \text{for } x \neq y; \quad (1.6)$$

and the regularity conditions for some $\delta > 0$

$$|K(x, y) - K(z, y)| \leq \frac{A|x-z|^\delta}{|x-y|^{n+\delta}}, \quad \text{for } |x-y| > 2|x-z|; \quad (1.7)$$

$$|K(y, x) - K(y, z)| \leq \frac{A|x-z|^\delta}{|x-y|^{n+\delta}}, \quad \text{for } |x-y| > 2|x-z|. \quad (1.8)$$

Then T_K (resp., $T_{K,b}^m$) is the (resp., the m -th order commutator of) standard Calderón-Zygmund singular integral operator on $\mathbb{R}^n$.

Throughout this paper, we always assume that $\rho > 2$ since the ρ -variation in the case $\rho \leq 2$ is often not bounded (see [1, 2]). Let us attribute the developments of the variation operators for singular integrals to two stages.

Stage 1 ($n = 1$). The variation operators for singular integrals were first studied by Campbell et al. [3] who showed that $\mathcal{V}_\rho(\mathcal{H})$ is of type (p, p) for $1 < p < \infty$ and of weak type $(1, 1)$. The above result was later extended to weighted version in [8, 13]. Same conclusions hold for $\mathcal{V}_\rho(\mathcal{R}^\pm)$ (see [8, Theorem A]). A general result was given by Liu and Wu [23] who proved that $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R})$, provided that $n = 1$ and $\mathcal{V}_\rho(\mathcal{T}_K)$ is of type (p_0, p_0) for some $p_0 \in (1, \infty)$ and K satisfies the conditions (1.6)-(1.8).

Stage 2 ($n \geq 1$). In 2002, Campbell et al. [4] first established the $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) bounds for $\mathcal{V}_\rho(\mathcal{T}_\Omega)$, provided that $\Omega \in L \log^+ L(S^{n-1})$. This result was essentially improved by Ding et al. [9] to the case $\Omega \in H^1(S^{n-1})$ since $L \log^+ L(S^{n-1}) \subsetneq H^1(S^{n-1})$, which is a proper inclusion. The weighted result for $\mathcal{V}_\rho(\mathcal{T}_\Omega)$ was first considered by Ma et al. [25] who proved that $\mathcal{V}_\rho(\mathcal{T}_\Omega)$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, provided that $\Omega \in \text{Lip}_\alpha(S^{n-1})$ for $\alpha > 0$. Later on, the above result was improved by Chen et al. [5] to the case $\Omega \in L^q(S^{n-1})$ for some $q > 1$. In [13], Gillespie and Torrea studied the variation operators for Riesz transforms and showed that $\mathcal{V}_\rho(\mathcal{R}_j)$ is bounded on $L^p(|x|^\alpha)$ for $1 < p < \infty$ and $-1 < \alpha < p - 1$. Recently, Zhang and Wu

[35] extended the above result to general A_p weight. Particularly, Ma et al. [25] proved that $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on $L^p(w)$ for all $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$ if K satisfies (1.6)-(1.8) and the following priori estimate:

$$\|\mathcal{V}_\rho(\mathcal{T}_K)(f)\|_{L^{p_0}(\mathbb{R}^n)} \lesssim_{n,p_0} \|f\|_{L^{p_0}(\mathbb{R}^n)}, \quad (1.9)$$

for some $p_0 \in (1, \infty)$.

The variation operator for the commutators was first studied by Liu and Wu [23] who showed that $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R})$, provided that $m \geq 1$, $b \in \text{BMO}(\mathbb{R})$ and K satisfies (1.6)-(1.9). As applications, they obtained the $L^p(w)$ bounds for $\mathcal{V}_\rho(\mathcal{T}_b^m)$ and $\mathcal{V}_\rho(\mathcal{T}_{\pm,b}^m)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R})$ if $b \in \text{BMO}(\mathbb{R})$. Recently, Liu and Cui [22] extended the above results to the general case $n \geq 1$. For the commutator of rough singular integral, Chen et al. [6] proved that $\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)$ is bounded on $L^p(w)$ for $1 < p < \infty$ if $\Omega \in L^q(S^{n-1})$ for some $q > 1$ satisfying (1.5), $m = 1$, $b \in \text{BMO}(\mathbb{R})$ and one of the following conditions holds: (a) $q' \leq p < \infty$, $p \neq 1$ and $w \in A_{p/q'}(\mathbb{R}^n)$; (b) $1 < p \leq q$, $p \neq \infty$ and $w^{-\frac{1}{p-1}} \in A_{p'/q'}(\mathbb{R}^n)$. Actually, applying the above result, the method in the proof of [6, Theorem 1.1] and induction arguments as in getting [11, Theorem 1], one can conclude that $\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)$ is bounded on $L^p(w)$ for $1 < p < \infty$ if $\Omega \in L^q(S^{n-1})$ for some $q > 1$ satisfying (1.5), $m \geq 1$, $b \in \text{BMO}(\mathbb{R})$ and one of the following conditions holds: (a) $q' \leq p < \infty$, $p \neq 1$ and $w \in A_{p/q'}(\mathbb{R}^n)$; (b) $1 < p \leq q$, $p \neq \infty$ and $w^{-\frac{1}{p-1}} \in A_{p'/q'}(\mathbb{R}^n)$. These conclusions together with the main result of [5] imply the following result.

Theorem A. Let $m \in \mathbb{N}$, $\rho > 2$, $b \in \text{BMO}(\mathbb{R}^n)$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$ satisfying (1.5). Then, for $1 < p < \infty$, we have

$$\|\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,\rho,p,m} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p(\mathbb{R}^n).$$

Recently, Guo et al. [14] first studied the compactness for $\mathcal{V}_\rho(\mathcal{T}_{K,b})$ on $L^p(w)$. They proved that $\mathcal{V}_\rho(\mathcal{T}_{K,b})$ is a compact operator on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, provided that $b \in \text{CMO}(\mathbb{R}^n)$ and K satisfies (1.6)-(1.9). Here $\text{CMO}(\mathbb{R}^n)$ is the closure of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in the $\text{BMO}(\mathbb{R}^n)$ topology, which coincides with the space of functions of vanishing mean oscillation. Very recently, Liu and Cui [22] showed that $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$ is a compact operator on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, provided that $m \geq 1$, $b \in \text{CMO}(\mathbb{R}^n)$ and K satisfies (1.6)-(1.9).

1.2 Boundedness and compactness on Morrey spaces

As a natural extension of the classical Lebesgue spaces, the Morrey spaces play key roles in partial differential equations and harmonic analysis. Let us recall one definition.

Definition 1.2. (Weighted Morrey spaces) ([19]). Let $1 \leq p < \infty$ and $0 \leq \beta < 1$. For a weight w defined on $\mathbb{R}^n$, the weighted Morrey space $M^{p,\beta}(w)$ is defined by

$$M^{p,\beta}(w) := \{f \in L_{\text{loc}}^p(w) : \|f\|_{M^{p,\beta}(w)} < \infty\},$$

where

$$\|f\|_{M^{p,\beta}(w)} := \sup_{B \text{ balls in } \mathbb{R}^n} \left(\frac{1}{w(B)^\beta} \int_B |f(x)|^p w(x) dx \right)^{1/p},$$

where the supremum is taken over all balls in $\mathbb{R}^n$. Particularly, the $M^{p,\beta}(w)$ is just the classical weighted Lebesgue space $L^p(w)$ when $\beta = 0$.

When $w \equiv 1$, $M^{p,\beta}(w)$ reduces to the classical Morrey space $M^{p,\beta}(\mathbb{R}^n)$, which was first introduced by Morrey [28] to study the local behavior of solutions to second order elliptic partial differential equations. The weighted Morrey spaces $M^{p,\beta}(w)$ were originally introduced by Komori and Shirai [19] who established the bounds for the Hardy-Littlewood maximal operator, fractional integral operator and the Calderón-Zygmund

singular integral operator on $M^{p,\beta}(w)$. Later on, more and more scholars have devoted to investigating the boundedness of various operators on $M^{p,\beta}(\mathbb{R}^n)$ (cf. e.g. [12],[29],[30]).

The boundedness of variation operators of singular integrals and their commutators on Morrey spaces was first studied by Zhang and Wu [36] who proved that $\mathcal{V}_\rho(\mathcal{I}_K)$ is bounded on $M^{p,\beta}(w)$ for $0 < \beta < 1$, $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, provided that $n = 1$ and K satisfies (1.6)-(1.9). Very recently, Liu and Cui [22] studied the boundedness and compactness for variation operators of Calderón-Zygmund singular integrals and their commutators on weighted Morrey spaces. To be more precise, they showed that $\mathcal{V}_\rho(\mathcal{I}_{K,b}^m)$ is bounded on $M^{p,\beta}(w)$ for $0 < \beta < 1$, $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, provided that $m \in \mathbb{N}$, $b \in \text{BMO}(\mathbb{R}^n)$ and K satisfies (1.6)-(1.9). They also proved that $\mathcal{V}_\rho(\mathcal{I}_{K,b}^m)$ is a compact operator on $M^{p,\beta}(w)$ for $0 < \beta < 1$, $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, provided that $m \geq 1$, $b \in \text{CMO}(\mathbb{R}^n)$ and K satisfies (1.6)-(1.9).

Particularly, Liu and Cui [22] obtained the following result.

Theorem B ([22]). *Let $\Omega \in \text{Lip}_\alpha(S^{n-1})$ for some $\alpha > 0$ and Ω satisfy (1.5). Let $\rho > 2$, $1 < p < \infty$ and $0 \leq \beta < 1$. Then*

(i) *If $m \in \mathbb{N}$ and $b \in \text{BMO}(\mathbb{R}^n)$, then*

$$\|\mathcal{V}_\rho(\mathcal{I}_{\Omega,b}^m)(f)\|_{M^{p,\beta}(\mathbb{R}^n)} \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}, \quad \forall f \in M^{p,\beta}(\mathbb{R}^n).$$

(ii) *If $m \geq 1$ and $b \in \text{CMO}(\mathbb{R}^n)$, then $\mathcal{V}_\rho(\mathcal{I}_{\Omega,b}^m)$ is a compact operator on $M^{p,\beta}(\mathbb{R}^n)$.*

It is well known that

$$\text{Lip}_\alpha(S^{n-1}) \subsetneq L^q(S^{n-1}), \quad \forall \alpha > 0, \quad 1 < q \leq \infty. \quad (1.10)$$

Note that the above inclusion relationship is proper.

Based on Theorem B and (1.10), a natural question is the following

Question 1.3. Does Theorem B hold under the condition that $\Omega \in L^q(S^{n-1})$ for some $q > 1$?

This is one of the main motivations of this work. In this paper, we shall establish the following results.

Theorem 1.1. *Let $m \in \mathbb{N}$, $\rho > 2$, $0 \leq \beta < 1$ and $1 < p < \infty$. Assume that $b \in \text{BMO}(\mathbb{R}^n)$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$ satisfying (1.5). Then*

$$\|\mathcal{V}_\rho(\mathcal{I}_{\Omega,b}^m)(f)\|_{M^{p,\beta}(\mathbb{R}^n)} \lesssim_{n,p,\beta,m} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}, \quad \forall f \in M^{p,\beta}(\mathbb{R}^n).$$

Theorem 1.2. *Let $m \geq 1$, $\rho > 2$, $1 < p < \infty$ and $0 \leq \beta < 1$. Let $\Omega \in L^q(S^{n-1})$ for some $q > 1$ and satisfy (1.5). For $r \geq 1$, define*

$$F(r) := \int_0^1 \frac{w_r(\delta)}{\delta} (1 + |\log \delta|) d\delta < \infty. \quad (1.11)$$

Here $w_r(\delta)$ denotes the integral modulus of continuity of order r of Ω defined by

$$w_r(\delta) := \sup_{\|\rho\| < \delta} \left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^r d\sigma(x') \right)^{1/r}$$

and ρ is a rotation in $\mathbb{R}^n$ and $\|\rho\| := \sup_{x' \in S^{n-1}} |\rho x' - x'|$. Assume that $b \in \text{CMO}(\mathbb{R}^n)$ and $F(1) < \infty$, then the operator $\mathcal{V}_\rho(\mathcal{I}_{\Omega,b}^m)$ is a compact operator on $M^{p,\beta}(\mathbb{R}^n)$.

Remark 1.4. (i) The condition (1.11) was firstly introduced by Chen et al. [7] who proved $T_{\Omega,b}$ is a compact operator on $M^{p,\beta}(\mathbb{R}^n)$ for $1 < p < \infty$ and $0 < \beta < 1$, provided that Ω satisfies (1.5) and $\Omega \in L^q(S^{n-1})$ for some $q > 1/(1 - \beta)$ satisfying $F(q) < \infty$. Note that $F(r_1) \leq CF(r_2)$ for some $C > 0$ if $r_2 \geq r_1 \geq 1$.

- (ii) The condition (1.11) is strictly weaker than the condition $\Omega \in \text{Lip}_\alpha(S^{n-1})$ with some $\alpha > 0$. Thus, Theorems 1.1 and 1.2 essentially improve the conclusions of Theorem B.
- (iii) When $\beta = 0$, Theorems 1.1 and 1.2 imply the boundedness and compactness of $\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)$ on the Lebesgue spaces $L^p(\mathbb{R}^n)$.
- (iv) Theorem 1.1 for the case $0 < \beta < 1$ is new, even in the special case $m = 0$.
- (v) Theorem 1.2 is new, even in the special case $\beta = 0$ and $m = 1$.

Based on the above, some driving questions are the following

Question 1.5. Do the conclusions in Theorems 1.1 and 1.2 hold under the condition that $\Omega \in L \log^+ L(S^{n-1})$ or $\Omega \in H^1(S^{n-1})$ or $\Omega \in \mathcal{F}_\alpha(S^{n-1})$ for some $\alpha > 0$?

1.3 Boundedness and continuity on Besov spaces

The second motivation of this work is to investigate the boundedness and continuity for variation operators of singular integrals and their commutators. For $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ ($p \neq \infty$), we denote by $\dot{B}_s^{p,q}(\mathbb{R}^n)$ (resp., $B_s^{p,q}(\mathbb{R}^n)$) the homogeneous (resp., inhomogeneous) Besov spaces. It is well known that

$$\|f\|_{B_s^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}, \quad \text{for } s > 0, 1 < p, q < \infty, \quad (1.12)$$

In [23], Liu and Wu established the following criterion on the boundedness and continuity of a class of sublinear operators on Besov spaces.

Proposition 1.3. ([23]). *Let T be a sublinear operator. Assume that $T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$. If T satisfies*

$$|\Delta_\zeta(Tf)(x)| \leq |T(\Delta_\zeta(f))(x)| \quad (1.13)$$

for any $x, \zeta \in \mathbb{R}^n$. Here $\Delta_\zeta(f)$ is the difference of f for an arbitrary function f defined on $\mathbb{R}^n$ and $\zeta \in \mathfrak{R}_n$, i.e., $\Delta_\zeta(f)(x) = f_\zeta(x) - f(x)$ and $f_\zeta(x) = f(x + \zeta)$. Then T is bounded on $\dot{B}_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < q < \infty$. Specially, if T also satisfies the following

$$|Tf - Tg| \leq |T(f - g)| \quad (1.14)$$

for arbitrary functions f, g defined on $\mathbb{R}^n$. Then T is continuous from $B_s^{p,q}(\mathbb{R}^n)$ to $\dot{B}_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < q < \infty$.

Note that the operator $\mathcal{V}_\rho(\mathcal{T}_K)$ is sublinearity and commutes with translations, i.e. $\mathcal{V}_\rho(\mathcal{T}_K)(f)(x + h) = \mathcal{V}_\rho(\mathcal{T}_K)(f_h)(x)$ when $K(x, y) = K(x - y)$. One can easily check that $\mathcal{V}_\rho(\mathcal{T}_K)$ satisfies (1.13) and (1.14). Applying Proposition 1.3, we have the following result.

Proposition 1.4. *Let $p > 2$ and $\mathcal{V}_\rho(\mathcal{T}_K)$ be given as in (1.3). Assume that $K(x, y) = K(x - y)$ and $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on $L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$. Then $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded and continuous on $B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < q < \infty$.*

As applications of Proposition 1.4, the following results are valid.

Corollary 1.5. *Let $p > 2$ and $\mathcal{V}_\rho(\mathcal{T}_K)$ be given as in (1.3). Assume that $K(x, y) = K(x - y)$ and K satisfies the conditions (1.6)-(1.9). Then $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded and continuous on $B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$, $1 < p < \infty$ and $1 < q < \infty$.*

Corollary 1.6. *Let $p > 2$ and one of the following conditions hold:*

- (i) $n = 1$ and $\mathcal{T} = \mathcal{H}$;
- (ii) $n = 1$ and $\mathcal{T} = \mathcal{R}^\pm$;

- (iii) $n \geq 2$, $\mathcal{T} = \mathcal{R}_j$, $1 \leq j \leq n$;
 (iv) $n \geq 2$ and $\mathcal{T} = \mathcal{T}_\Omega$, where $\Omega \in H^1(S^{n-1})$ or $\Omega \in \bigcap_{\alpha > 2} \mathcal{F}_\alpha(S^{n-1})$. Here $\mathcal{F}_\alpha(S^{n-1})$ for $\alpha > 0$ denotes the set of all integrable functions over S^{n-1} which satisfy

$$\sup_{\xi' \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \left(\log^+ \frac{1}{|\xi' \cdot y'|} \right)^\alpha d\sigma(y') < \infty.$$

Then the operator $\mathcal{V}_\rho(\mathcal{T})$ is bounded and continuous on $B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$, $1 < p < \infty$ and $1 < q < \infty$.

Remark 1.6. (i) It should be pointed out that Corollary 1.5 follows from Theorem 1 in [25] and Proposition 1.4.

- (ii) The corresponding results in Corollary 1.6 for the cases (i)-(iii) follow from the known bounds for the corresponding operators and Proposition 1.4. It was shown in [9] (see Theorem 1.2 and Corollary 1.6 in [9]) that $\mathcal{V}_\rho(\mathcal{T}_\Omega)$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$ under the condition that $\Omega \in H^1(S^{n-1})$ or $\Omega \in \bigcap_{\alpha > 2} \mathcal{F}_\alpha(S^{n-1})$. This together with Proposition 1.4 yields the conclusion of Corollary 1.6 for case (iv).
 (iii) We remark that the space $\mathcal{F}_\alpha(S^{n-1})$ was introduced by Grafakos and Stefanov [15] in the study of L^p boundedness of singular integral operator with rough kernels. Clearly, $\bigcup_{q > 1} L^q(S^{n-1}) \subsetneq \mathcal{F}_\alpha(S^{n-1})$ for any $\alpha > 0$. Moreover, the examples in [15] show that

$$\bigcap_{\alpha > 1} \mathcal{F}_\alpha(S^{n-1}) \not\subseteq H^1(S^{n-1}) \not\subseteq \bigcup_{\alpha > 1} \mathcal{F}_\alpha(S^{n-1}).$$

It should be pointed out that the operator $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$ does not satisfy the condition (1.13), even in the special case $m = 1$ and $K(x, y) = K(x - y)$, which makes that Proposition 1.3 does not apply for $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$. Therefore, it is natural to ask the following

Question 1.7. Is the operator $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$ bounded and continuous on $B_s^{p,q}(\mathbb{R}^n)$ for some $0 < s < 1$ and $1 < p, q < \infty$ when $m \geq 1$?

In this paper we shall present a positive answer to this question, which is another one of main motivations. Before presenting the rest of main results, let us introduce the following definition.

Definition 1.8. (Lipschitz space). The homogeneous Lipschitz space $\dot{\Lambda}(\mathbb{R}^n)$ is given by

$$\dot{\Lambda}(\mathbb{R}^n) := \{f : \mathbb{R}^n \rightarrow \mathbb{C} \text{ continuous} : \|f\|_{\dot{\Lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{\Lambda}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \sup_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f(x+h) - f(x)|}{|h|} < \infty.$$

The inhomogeneous Lipschitz space $\Lambda(\mathbb{R}^n)$ is defined by

$$\Lambda(\mathbb{R}^n) := \{f : \mathbb{R}^n \rightarrow \mathbb{C} \text{ continuous} : \|f\|_{\Lambda(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{\dot{\Lambda}(\mathbb{R}^n)} < \infty\}.$$

The rest of main results can be formulated as follows:

Proposition 1.7. Let $\rho > 2$, $m \geq 1$ and $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$ be given as in (1.4). Assume that $b \in \Lambda(\mathbb{R}^n)$, $K(x, y) = K(x - y)$ and $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on $L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$. Then $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$ is bounded and continuous on $B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < q < \infty$. Particularly,

$$\|\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)\|_{B_s^{p,q}(\mathbb{R}^n)} \leq C \|b\|_{\Lambda(\mathbb{R}^n)}^m \|f\|_{B_s^{p,q}(\mathbb{R}^n)}, \quad \forall f \in B_s^{p,q}(\mathbb{R}^n). \quad (1.15)$$

As some applications of Proposition 1.7, we obtain

Corollary 1.8. Let $\rho > 2$, $m \geq 1$ and $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$ be given as in (1.4). Assume that $b \in \Lambda(\mathbb{R}^n)$, $K(x, y) = K(x - y)$ and K satisfies the conditions (1.6)-(1.9). Then $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$ is bounded and continuous on $B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$, $1 < p < \infty$ and $1 < q < \infty$. Particularly,

$$\|\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)\|_{B_s^{p,q}(\mathbb{R}^n)} \leq C \|b\|_{\Lambda(\mathbb{R}^n)}^m \|f\|_{B_s^{p,q}(\mathbb{R}^n)}, \quad \forall f \in B_s^{p,q}(\mathbb{R}^n).$$

Corollary 1.9. Let $m \geq 1$, $\rho > 2$, $b \in \Lambda(\mathbb{R}^n)$ and one of the following conditions hold:

- (i) $n = 1$ and $\mathcal{T} = \mathcal{T}_b^m$;
- (ii) $n = 1$ and $\mathcal{T} = \mathcal{T}_{\pm,b}^m$;
- (iii) $\mathcal{T} = \mathcal{T}_{j,b}^m$, $1 \leq j \leq n$;
- (iv) $\mathcal{T} = \mathcal{T}_{\Omega,b}^m$, where $\Omega \in H^1(S^{n-1})$ or $\Omega \in \bigcap_{\alpha>2} \mathcal{F}_\alpha(S^{n-1})$.

Then the operator $\mathcal{V}_\rho(\mathcal{T})$ is bounded and continuous on $B_s^{p,q}(\mathbb{R}^n)$ for $0 < s < 1$, $1 < p < \infty$ and $1 < q < \infty$. Moreover,

$$\|\mathcal{V}_\rho(\mathcal{T})(f)\|_{B_s^{p,q}(\mathbb{R}^n)} \leq C \|b\|_{\Lambda(\mathbb{R}^n)}^m \|f\|_{B_s^{p,q}(\mathbb{R}^n)}, \quad \forall f \in B_s^{p,q}(\mathbb{R}^n).$$

Remark 1.9. (i) Corollary 1.8 follows from Theorem A and Proposition 1.7.

- (ii) The corresponding results in Corollary 1.9 for the cases (i)-(iii) follow from the known bounds for the corresponding operators and Proposition 1.7. It was shown in [9] (see Theorem 1.2 and Corollary 1.6 in [9]) that $\mathcal{V}_\rho(\mathcal{T}_\Omega)$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$ under the condition that $\Omega \in H^1(S^{n-1})$ or $\Omega \in \bigcap_{\alpha>2} \mathcal{F}_\alpha(S^{n-1})$. This together with Proposition 1.7 yields the conclusions of Corollary 1.9 for case (iv).

Some interesting questions can be formulated as follows:

Question 1.10. Do the corresponding results in Propositions 1.4 and 1.7 and Corollaries 1.5 and 1.8 hold when $K(x, y) \neq K(x - y)$?

1.4 Outline of this paper and some notations

The rest of this paper is organized as follows. Section 2 is devoted to presenting the proof of Theorem 1.1. The proof of Theorem 1.2 will be given in Section 3. Finally, we shall prove Proposition 1.7 in Section 4. We would like to remark that the proofs of Theorems 1.1 and 1.2 are motivated by the methods from [7]. The proof of Proposition 1.7 is based on some known arguments from [22, 23]. However, some new techniques are needed to be explored.

Throughout this paper, for any $p \in (1, \infty)$, we let p' denote the dual exponent to p defined as $1/p + 1/p' = 1$. For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r . For $t > 0$ and $B := B(x, r)$ with $x \in \mathbb{R}^n$ and $r > 0$, we denote $tB = B(x, tr)$. We end this section by presenting an useful inequality:

$$\left(\sum_{i=1}^{\infty} \left| \int_{\varepsilon_{i+1} < f(x,y) \leq \varepsilon_i} F(x, y) dy \right|^\rho \right)^{1/\rho} \leq \int_{\mathbb{R}^n} |F(x, y)| dy, \quad (1.16)$$

for all $x \in \mathbb{R}^n$, any arbitrary functions F and f defined on $\mathbb{R}^n \times \mathbb{R}^n$, where $\rho > 1$ and $\{\varepsilon_i\}$ is an increasing or decreasing sequence of positive numbers.

2 Proof of Theorem 1.1

In this section we shall prove Theorem 1.1. At first, let us introduce some notations and lemmas, which are the main ingredients of our proof. For $\Omega \in L^1(S^{n-1})$, the maximal operator with rough kernel Ω is defined by

$$M_{\Omega}f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\leq r} |\Omega(y')f(x-y)|dy.$$

The following lemma was proved by Chen et al. [7].

Lemma 2.1. ([7]). *Let $0 < \beta < 1$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$ satisfying (1.5). Then for $1 < p < \infty$, there exists an $\epsilon > 0$ such that for any $k \in \mathbb{N}$ and $f \in M^{p,\beta}(\mathbb{R}^n)$,*

$$\int_{B(t,r)} |M_{\Omega}f_k(x)|^p dx \lesssim_{p,\beta,n} 2^{-k\epsilon} r^{n\beta} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}^p,$$

where $B := B(t, r)$ is an arbitrary fixed ball and $f_k = f\chi_{2^{k+1}B \setminus 2^k B}$.

Motivated by the idea in the proof of Theorem 1.8 in [7], we have the following result.

Proposition 2.2. *Let $0 < \beta < 1$, $m \in \mathbb{N}$, $b \in \text{BMO}(\mathbb{R}^n)$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$ satisfying (1.5). Let T_b be a linear or sublinear operator satisfying*

$$|T_b f(x)| \leq C_1 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |(b(x) - b(y))^m f(y)| dy, \quad (2.1)$$

where $C_1 > 0$. If there exist $p \in (1, \infty)$ and $C_2 > 0$ such that T_b satisfies

$$\|T_b f\|_{L^p(\mathbb{R}^n)} \leq C_2 \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p(\mathbb{R}^n). \quad (2.2)$$

Then we have

$$\|T_b f\|_{M^{p,\beta}(\mathbb{R}^n)} \lesssim_{m,n,p,\beta,\Omega,C_1,C_2} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}, \quad \forall f \in M^{p,\beta}(\mathbb{R}^n). \quad (2.3)$$

Proof. Proposition 2.2 for the case $m = 0$ was proved by Chen et al. in [7] (see Theorem 1.8 in [7]). We shall prove the case $m \geq 1$ by adopting the method as in the proof of Theorem 1.8 in [7]. Let $B = B(x_0, r)$, where $x_0 \in \mathbb{R}^n$ and $r > 0$. To prove (2.3), it suffices to show that

$$\left(\frac{1}{|B|^\beta} \int_B |T_b f(x)|^p dx \right)^{1/p} \lesssim_{m,n,p,\beta,\Omega,C_1,C_2} C \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}, \quad (2.4)$$

where $C > 0$ is independent of x_0, r and b .

Decompose f as $f = f\chi_{2B} + f\chi_{(2B)^c}$. By Minkowski's inequality and the sublinearity of T_b , one gets

$$\begin{aligned} & \left(\frac{1}{|B|^\beta} \int_B |T_b f(x)|^p dx \right)^{1/p} \\ & \leq \left(\frac{1}{|B|^\beta} \int_B |T_b(f\chi_{2B})(x)|^p dx \right)^{1/p} + \left(\frac{1}{|B|^\beta} \int_B |T_b(f\chi_{(2B)^c})(x)|^p dx \right)^{1/p} \\ & =: I_1 + I_2. \end{aligned} \quad (2.5)$$

From the assumption (2.2) we see that

$$\begin{aligned} I_1 & \leq C_2 \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \left(\frac{1}{|B|^\beta} \int_{2B} |f(x)|^p dx \right)^{1/p} \\ & \leq 2^{n\beta/p} C_2 \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \left(\frac{1}{|2B|^\beta} \int_{2B} |f(x)|^p dx \right)^{1/p} \\ & \leq 2^{n\beta/p} C_2 \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}. \end{aligned} \quad (2.6)$$

Next we estimate I_2 . Fix $x \in B$. We get from (2.1) that

$$\begin{aligned} T_b(f\chi_{(2B)^c})(x) &\leq C_1 \int_{(2B)^c} \frac{|\Omega(x-z)|}{|x-z|^n} |b(x) - b(z)|^m |f(z)| dz \\ &= C_1 \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|\Omega(x-z)|}{|x-z|^n} |b(x) - b(z)|^m |f(z)| dz. \end{aligned} \quad (2.7)$$

Fix $k \geq 1$. Note that

$$\begin{aligned} 2^{k+2}r &\geq (2^{k+1} + 1)r \geq |z - x_0| + |x - x_0| \\ &\geq |x - z| \geq |z - x_0| - |x - x_0| \geq (2^k - 1)r \geq 2^{k-1}r, \end{aligned} \quad (2.8)$$

when $z \in 2^{k+1}B \setminus 2^k B$. By (2.8), we have

$$\begin{aligned} &\int_{2^{k+1}B \setminus 2^k B} \frac{|\Omega(x-z)|}{|x-z|^n} |b(x) - b(z)|^m |f(z)| dz \\ &\leq (2^{k-1}r)^{-n} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)| |b(x) - b(z)|^m |f(z)| dz \\ &\lesssim_{n,m} \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)| |b(z) - b_{2^{k+1}B}|^m |f(z)| dz \right. \\ &\quad \left. + |b(x) - b_{2^{k+1}B}|^m \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)| |f(z)| dz \right) \\ &\lesssim_{n,m} (J_{k,1}(x) + J_{k,2}(x)). \end{aligned} \quad (2.9)$$

Let $f_k = f\chi_{2^{k+1}B \setminus 2^k B}$ and $s \in (1, \min\{p, q\})$. By Hölder's inequality and the well-known property for $\|b\|_{\text{BMO}(\mathbb{R}^n)}$, one has

$$\begin{aligned} J_{k,1}(x) &\leq \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} (|\Omega(x-z)| |f_k(z)|)^s dz \right)^{1/s} \\ &\quad \times \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |b(z) - b_{2^{k+1}B}|^{ms'} dz \right)^{1/s'} \\ &\lesssim_{m,n,s} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\ &\quad \times \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} (|\Omega(x-z)| |f_k(z)|)^s dz \right)^{1/s}. \end{aligned} \quad (2.10)$$

On the other hand, by (2.8) and a change of variable, we have

$$\begin{aligned} &\left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} (|\Omega(x-z)| |f_k(z)|)^s dz \right)^{1/s} \\ &\leq \left(\frac{1}{|2^{k+1}B|} \int_{2^{k-1}r \leq |x-z| \leq 2^{k+2}r} (|\Omega(x-z)| |f_k(z)|)^s dz \right)^{1/s} \\ &\lesssim_n \left(\frac{1}{(2^{k+1}r)^n} \int_{2^{k-1}r \leq |z| \leq 2^{k+2}r} (|\Omega(z)| |f_k(x-z)|)^s dz \right)^{1/s} \\ &\lesssim_{n,s} (M_{\Omega^s}(f_k)^s)^{1/s}(x). \end{aligned} \quad (2.11)$$

In light of (2.10) and (2.11) we would have

$$J_{k,1}(x) \lesssim_{m,n,s} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m (M_{\Omega^s}(f_k)^s)^{1/s}(x). \quad (2.12)$$

Note that $\Omega^s \in L^{q/s}(S^{n-1})$ and $q/s > 1$. By Lemma 2.1 and (2.12), there exists a constant $\epsilon_1 > 0$ independent of B such that

$$\begin{aligned} &\left(\frac{1}{|B|^\beta} \int_B |J_{k,1}(x)|^p dx \right)^{1/p} \\ &\lesssim_{m,n,s} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \left(\frac{1}{|B|^\beta} \int_B (M_{\Omega^s}(f_k)^s)^{p/s}(x) dx \right)^{1/p} \\ &\lesssim_{m,n,s} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \left(\frac{1}{|B|^\beta} 2^{-k\epsilon_1} r^{n\beta} \|f^s\|_{M^{p/s,\beta}(\mathbb{R}^n)}^{p/s} \right)^{1/p} \\ &\lesssim_{m,n,s} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m 2^{-k\epsilon_1/p} \|f^s\|_{M^{p/s,\beta}(\mathbb{R}^n)}^{1/s} \\ &\lesssim_{m,n,s,p,\beta} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m 2^{-k\epsilon_1/p} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}, \end{aligned} \quad (2.13)$$

where in the last of inequality (2.13) we have used the fact that $\|f^S\|_{M^{p/s,\beta}(\mathbb{R}^n)}^{1/s} = \|f\|_{M^{p,\beta}(\mathbb{R}^n)}$ and $p/s > 1$. Then we get from (2.13) that

$$\begin{aligned} & \sum_{k=1}^{\infty} \left(\frac{1}{|B|^\beta} \int_B |J_{k,1}(x)|^p dx \right)^{1/p} \\ & \lesssim_{m,n,s,p,\beta} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \sum_{k=1}^{\infty} 2^{-k\epsilon_1/p} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \\ & \lesssim_{m,n,s,p,\beta} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}. \end{aligned} \quad (2.14)$$

It remains to estimate $J_{k,2}(x)$. Let us consider two cases:

(Case 1: $p \geq q'$). In this case we have that $\Omega \in L^{p'}(S^{n-1})$. By Hölder's inequality, a change of variable and (2.8), we get

$$\begin{aligned} & \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)| |f(z)| dz \\ & \leq \left(\int_{2^{k+1}B \setminus 2^k B} |f(z)|^p dz \right)^{1/p} \left(\int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)|^{p'} dz \right)^{1/p'} \\ & \leq \left(\int_{2^{k+1}B} |f(z)|^p dz \right)^{1/p} \left(\int_{2^{k-1}r \leq |x-z| \leq 2^{k+2}r} |\Omega(x-z)|^{p'} dz \right)^{1/p'} \\ & \leq |2^{k+1}B|^{\beta/p} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \left(\int_{2^{k-1}r \leq |z| \leq 2^{k+2}r} |\Omega(z)|^{p'} dz \right)^{1/p'} \\ & \leq |2^{k+1}B|^{\beta/p} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \left(\int_{2^{k-1}r}^{2^{k+2}r} t^{n-1} dt \int_{S^{n-1}} |\Omega(\theta)|^{p'} d\sigma(\theta) \right)^{1/p'} \\ & \lesssim_{n,p} \|\Omega\|_{L^{p'}(S^{n-1})} |2^{k+1}B|^{\beta/p+1/p'} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}. \end{aligned}$$

It follows that

$$J_{k,2}(x) \lesssim_{n,p} \|\Omega\|_{L^{p'}(S^{n-1})} |2^{k+1}B|^{(\beta-1)/p} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} |b(x) - b_{2^{k+1}B}|^m.$$

Then we have

$$\begin{aligned} & \left(\frac{1}{|B|^\beta} \int_B |J_{k,2}(x)|^p dx \right)^{1/p} \\ & \lesssim_{n,p,\Omega} |2^{k+1}B|^{(\beta-1)/p} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \left(\frac{1}{|B|^\beta} \int_B |b(x) - b_{2^{k+1}B}|^{mp} dx \right)^{1/p}. \end{aligned} \quad (2.15)$$

By the property of $\|b\|_{\text{BMO}(\mathbb{R}^n)}$, we have

$$\begin{aligned} & \int_B |b(x) - b_{2^{k+1}B}|^{mp} dx \\ & \leq 2^{mp-1} \left(\int_B |b(x) - b_B|^{mp} dx + |b_B - b_{2^{k+1}B}|^{mp} |B| \right) \\ & \lesssim_{m,p,n} (\|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp} |B| + (k+1)^{mp} \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp} |B|) \\ & \lesssim_{m,p,n} k^{mp} \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mp} |B|. \end{aligned} \quad (2.16)$$

Combining (2.16) with (2.15) implies that

$$\begin{aligned} & \left(\frac{1}{|B|^\beta} \int_B |J_{k,2}(x)|^p dx \right)^{1/p} \\ & \lesssim_{m,n,p} |2^{k+1}B|^{(\beta-1)/p} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} |B|^{(1-\beta)/p} k^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\ & \lesssim_{m,n,p} \frac{k^m}{2^{(k+1)n(1-\beta)/p}} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}. \end{aligned} \quad (2.17)$$

Note that $\beta \in (0, 1)$. From (2.17) we see that

$$\begin{aligned} & \sum_{k=1}^{\infty} \left(\frac{1}{|B|^\beta} \int_B |J_{k,2}(x)|^p dx \right)^{1/p} \\ & \lesssim_{m,n,p} \sum_{k=1}^{\infty} \frac{k^m}{2^{(k+1)n(1-\beta)/p}} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \\ & \lesssim_{m,n,p,\beta} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}. \end{aligned} \quad (2.18)$$

(Case 2: $p < q'$). We can choose $u > 1$ and $s \in (1/q, 1)$ such that $1/(pu) + 1/q < 1$ and $1/(pu) + 1/(qs) = 1$. It is clear that $pu' > qs$. By Hölder's inequality, one has

$$\begin{aligned}
 & \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)| |f(z)| dz \\
 &= \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)| |f_k(z)|^{1/u'} |f(z)|^{1/u} dz \\
 &\leq \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)|^{qs} |f_k(z)|^{qs/u'} dz \right)^{1/(qs)} \\
 &\quad \times \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |f(z)|^p dz \right)^{1/(pu)} \\
 &\leq |2^{k+1}B|^{(\beta-1)/(pu)} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}^{1/u} \\
 &\quad \times \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)|^{qs} |f_k(z)|^{qs/u'} dz \right)^{1/(qs)}.
 \end{aligned} \tag{2.19}$$

By the arguments similar to those used to derive (2.11), one gets

$$\left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-z)|^{qs} |f_k(z)|^{qs/u'} dz \right)^{1/(qs)} \lesssim_{n,s,q,u} (M_{\Omega^{sq}}(f_k)^{qs/u'})^{1/(qs)}(x).$$

Then we get from (2.19) that

$$\begin{aligned}
 J_{k,2}(x) &\lesssim_{n,s,q,u} |2^{k+1}B|^{(\beta-1)/(pu)} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}^{1/u} \\
 &\quad \times |b(x) - b_{2^{k+1}B}|^m (M_{\Omega^{sq}}(f_k)^{sq/u'})^{1/(qs)}(x).
 \end{aligned} \tag{2.20}$$

Note that $\Omega^{qs} \in L^{1/s}(S^{n-1})$ and $pu'/(qs) > 1$. By Hölder's inequality with exponents u and u' , Lemma 2.1 and (2.20), there exists a constant $\epsilon_2 > 0$ independent of B and k such that

$$\begin{aligned}
 & \left(\int_B |J_{k,2}(x)|^p dx \right)^{1/p} \\
 &\lesssim_{n,s,q,u} |2^{k+1}B|^{(\beta-1)/(pu)} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}^{1/u} \\
 &\quad \times \left(\int_B |b(x) - b_{2^{k+1}B}|^{mp} (M_{\Omega^{sq}}(f_k)^{sq/u'})^{p/(qs)}(x) dx \right)^{1/p} \\
 &\lesssim_{n,s,q,u} |2^{k+1}B|^{(\beta-1)/(pu)} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}^{1/u} \left(\int_B |b(x) - b_{2^{k+1}B}|^{mpu} dx \right)^{1/(pu)} \\
 &\quad \times \left(\int_B (M_{\Omega^{sq}}(f_k)^{sq/u'})^{pu'/(qs)}(x) dx \right)^{1/(pu')} \\
 &\lesssim_{n,s,q,u} |2^{k+1}B|^{(\beta-1)/(pu)} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}^{1/u} \left(\int_B |b(x) - b_{2^{k+1}B}|^{mpu} dx \right)^{1/(pu)} \\
 &\quad \times \left(2^{-k\epsilon_2} r^{n\beta} \|f^{sq/u'}\|_{M^{pu'/(qs),\beta}(\mathbb{R}^n)}^{pu'/(qs)} \right)^{1/(pu)} \\
 &\lesssim_{n,s,q,u} |2^{k+1}B|^{(\beta-1)/(pu)} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}^{1/u} 2^{-k\epsilon_2/(pu')} |B|^{\beta/(pu')} \|f^{sq/u'}\|_{M^{pu'/(qs),\beta}(\mathbb{R}^n)}^{1/(qs)} \\
 &\quad \times \left(\int_B |b(x) - b_{2^{k+1}B}|^{mpu} dx \right)^{1/(pu)}.
 \end{aligned}$$

Note that

$$\|f^{sq/u'}\|_{M^{pu'/(qs),\beta}(\mathbb{R}^n)}^{1/(qs)} = \|f\|_{M^{p,\beta}(\mathbb{R}^n)}^{1/u'}.$$

By the arguments similar to those used in getting (2.16), we have

$$\int_B |b(x) - b_{2^{k+1}B}|^{mpu} dx \lesssim_{m,p,u} k^{mpu} \|b\|_{\text{BMO}(\mathbb{R}^n)}^{mpu} |B|.$$

Therefore, we have

$$\begin{aligned}
 & \left(\frac{1}{|B|^\beta} \int_B |J_{k,2}(x)|^p dx \right)^{1/p} \\
 &\lesssim_{m,n,s,p,q,u} |2^{k+1}B|^{(\beta-1)/(pu)} |B|^{-\beta/p} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} 2^{-k\epsilon_2/(pu')} \\
 &\quad \times |B|^{\beta/(pu')} k^m \|b\|_{\text{BMO}(\mathbb{R}^n)}^m |B|^{1/(pu)} \\
 &\lesssim_{m,n,s,p,q,u} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m 2^{-k\epsilon_2/(pu')} \frac{k^m}{2^{(k+1)n(1-\beta)/(pu)}}.
 \end{aligned}$$

It follows that

$$\begin{aligned} & \sum_{k=1}^{\infty} \left(\frac{1}{|B|^{\beta}} \int_B |J_{k,2}(x)|^p dx \right)^{1/p} \\ & \lesssim_{m,n,s,p,q,u} \sum_{k=1}^{\infty} 2^{-k\epsilon_2/(pu')} \frac{k^m}{2^{(k+1)n(1-\beta)/(pu)}} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \\ & \lesssim_{m,n,s,p,q,u} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}, \end{aligned} \quad (2.21)$$

since $\beta \in [0, 1)$.

Hence, by (2.7), (2.9), (2.14), (2.18), (2.21) and Minkowski's inequality, we get

$$\begin{aligned} I_2 & \lesssim_{n,m,C_1} \sum_{k=1}^{\infty} \left(\frac{1}{|B|^{\beta}} \int_B |J_{k,1}(x)|^p dx \right)^{1/p} \\ & \quad + \sum_{k=1}^{\infty} \left(\frac{1}{|B|^{\beta}} \int_B |J_{k,2}(x)|^p dx \right)^{1/p} \\ & \lesssim_{m,n,s,p,q,\beta,C_1} \|b\|_{\text{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(\mathbb{R}^n)}. \end{aligned} \quad (2.22)$$

Then (2.4) follows from (2.5), (2.6) and (2.22). This completes the proof of Proposition 2.2. $\square$

We now proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. By (1.16) and the definition of $\mathcal{V}_{\rho}(\mathcal{T}_{\Omega,b}^m)$, it is not difficult to see that $\mathcal{V}_{\rho}(\mathcal{T}_{\Omega,b}^m)$ satisfies (2.1). Applying Proposition 2.2 and Theorem A, we can get the desired conclusions of Theorem 1.1. $\square$

3 Proof of Theorem 1.2

3.1 Preliminaries

To prove Theorem 1.2, we need the following characterization that a subset in $M^{p,\beta}(\mathbb{R}^n)$ is a strongly pre-compact set.

Proposition 3.1. *Let $1 < p < \infty$ and $0 \leq \beta < 1$. Then a subset $\mathfrak{F}$ of $M^{p,\beta}(\mathbb{R}^n)$ is strongly pre-compact set in $M^{p,\beta}(\mathbb{R}^n)$ if $\mathfrak{F}$ satisfies the following conditions:*

(i) $\mathfrak{F}$ is bounded, i.e.

$$\sup_{f \in \mathfrak{F}} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} < \infty.$$

(ii) $\mathfrak{F}$ uniformly vanishes as infinity, i.e.

$$\lim_{N \rightarrow +\infty} \|f\chi_{E_N}\|_{M^{p,\beta}(\mathbb{R}^n)} = 0, \text{ uniformly for all } f \in \mathfrak{F},$$

where $E_N = \{x \in \mathbb{R}^n; |x| > N\}$.

(iii) $\mathfrak{F}$ is uniformly translation continuous, i.e.

$$\lim_{r \rightarrow 0} \sup_{h \in B(0,r)} \|f(\cdot + h) - f(\cdot)\|_{M^{p,\beta}(\mathbb{R}^n)} = 0, \text{ uniformly for all } f \in \mathfrak{F}.$$

Remark 3.1. When $\beta = 0$, Proposition 3.1 is just the classical Fréchet-Kolmogorov theorem. When $\beta \in (0, 1)$, Proposition 3.1 was proved by Chen. et al in [7].

The following result follows from [10].

Lemma 3.2. ([10]). *Let $0 \leq \beta < n$ and $\Omega \in L^1(S^{n-1})$ satisfying (1.5). Then for $R > 0$, there exists a constant $C > 0$ independent of R such that for $x \in \mathbb{R}^n$ with $|x| < R/2$,*

$$\int_{R < |y| < 2R} \left| \frac{\Omega(y-x)}{|y-x|^{n-\beta}} - \frac{\Omega(y)}{|y|^{n-\beta}} \right| dy \leq CR^\beta \left(\frac{|x|}{R} + \int_{|x|/(2R)}^{|x|/R} \frac{w(\delta)}{\delta} d\delta \right).$$

Here $w(\delta)$ is given as in Theorem 1.2.

3.2 Proof of Theorem 1.2

We divide the proof of Theorem 1.2 into four steps:

Step 1. Reduction via smooth truncated techniques. We shall adopt the truncated techniques followed from [20] to prove Theorem 1.2. Let $\varphi \in C^\infty([0, \infty))$ satisfy that $0 \leq \varphi \leq 1$, $\varphi(t) \equiv 1$ if $t \in [0, 1]$ and $\varphi(t) \equiv 0$ if $t \in [2, \infty)$. For any $\eta > 0$, we define the function Ω_η by

$$\Omega_\eta(z) = \Omega(z) \left(1 - \varphi\left(\frac{2}{\eta}|z|\right) \right). \quad (3.1)$$

It is clear that $\Omega_\eta \in L^q(S^{n-1})$ and Ω_η satisfies (1.5). In what follows, let us fix $b \in \text{CMO}(\mathbb{R}^n)$, $1 < p < \infty$ and $0 \leq \beta < 1$. We shall prove that there exists a constant $C > 0$ independent of η such that

$$\|\mathcal{V}_\rho(\mathcal{T}_{\Omega_\eta, b}^m)(f) - \mathcal{V}_\rho(\mathcal{T}_{\Omega, b}^m)(f)\|_{M^{p, \beta}(\mathbb{R}^n)} \leq C\eta \|f\|_{M^{p, \beta}(\mathbb{R}^n)}, \quad \forall f \in M^{p, \beta}(\mathbb{R}^n). \quad (3.2)$$

By (1.16), we have

$$\begin{aligned} & |\mathcal{V}_\rho(\mathcal{T}_{\Omega_\eta, b}^m)(f)(x) - \mathcal{V}_\rho(\mathcal{T}_{\Omega, b}^m)(f)(x)| \\ & \leq \sup_{\epsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| \int_{\epsilon_{i+1} < |x-z| \leq \epsilon_i} \left(\frac{\Omega_\eta(x-z)}{|x-z|^{n-\beta}} - \frac{\Omega(x-z)}{|x-z|^{n-\beta}} \right) \right. \right. \\ & \quad \left. \left. \times (b(x) - b(z))^m f(z) dz \right|^{\rho} \right)^{1/\rho} \\ & \leq \int_{\mathbb{R}^n} \left| \frac{\Omega_\eta(x-z)}{|x-z|^{n-\beta}} - \frac{\Omega(x-z)}{|x-z|^{n-\beta}} \right| |(b(x) - b(z))^m f(z)| dz \\ & = \int_{\mathbb{R}^n} |(b(x) - b(z))^m f(z)| \frac{|\Omega(x-z)|}{|x-z|^{n-\beta}} \varphi\left(\frac{2}{\eta}|x-z|\right) dz \\ & \leq (|b(x)| + \|b\|_{L^\infty(\mathbb{R}^n)})^{m-1} \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \int_{|x-z| \leq \eta} \frac{|\Omega(x-z)f(z)|}{|x-z|^{n-1}} dz. \end{aligned} \quad (3.3)$$

Note that

$$\begin{aligned} & \int_{|x-z| \leq \eta} \frac{|\Omega(x-z)f(z)|}{|x-z|^{n-1}} dz \\ & = \int_{|z| \leq \eta} \frac{|\Omega(z)f(x-z)|}{|z|^{n-1}} dz \\ & \leq \sum_{k=0}^{\infty} \int_{2^{-(k+1)}\eta \leq |z| \leq 2^{-k}\eta} \frac{|\Omega(z)f(x-z)|}{|z|^{n-1}} dz \\ & \leq \sum_{k=0}^{\infty} 2^{-k} \eta \frac{1}{(2^{-(k+1)}\eta)^n} \int_{|z| \leq 2^{-k}\eta} |\Omega(z)f(x-z)| dy \\ & \lesssim_n \eta M_\Omega f(x). \end{aligned}$$

This together with (3.3) leads to

$$|\mathcal{V}_\rho(\mathcal{T}_{\Omega_\eta, b}^m)(f)(x) - \mathcal{V}_\rho(\mathcal{T}_{\Omega, b}^m)(f)(x)| \lesssim_n (|b(x)| + \|b\|_{L^\infty(\mathbb{R}^n)})^{m-1} \eta M_\Omega f(x). \quad (3.4)$$

Note that

$$M_\Omega f(x) \leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy.$$

Combining this with the $L^p(\mathbb{R}^n)$ bounds for M_Ω with $1 < p < \infty$ and Proposition 2.2 yields that

$$\|M_\Omega f\|_{M^{p,\beta}(\mathbb{R}^n)} \lesssim_{n,p,\beta,\Omega,b} \|f\|_{M^{p,\beta}(\mathbb{R}^n)}, \quad \forall f \in M^{p,\beta}(\mathbb{R}^n). \quad (3.5)$$

Combining (3.5) with (3.4) leads to (3.2).

By (3.2) and [34, p. 278, Theorem (iii)], to prove the compactness for $\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)$, it suffices to prove the compactness for $\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)$ when $\eta > 0$ is small enough. For $\beta > 0$, let

$$\mathcal{F} := \{\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f) : \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \leq 1\}.$$

To prove the compactness for $\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)$, it is enough to show that the set $\mathcal{F}$ satisfies the conditions (i)-(iii) of Proposition 3.1 when $\eta > 0$ is small enough.

Step 2. A verification for condition (i) of Proposition 3.1. Let $\eta \in (0, 1)$. By Theorem 1.1, (3.4) and (3.5), we have

$$\begin{aligned} & \|\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)\|_{M^{p,\beta}(\mathbb{R}^n)} \\ & \leq \|\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f) - \mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)\|_{M^{p,\beta}(\mathbb{R}^n)} + \|\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)\|_{M^{p,\beta}(\mathbb{R}^n)} \\ & \leq C\|f\|_{M^{p,\beta}(\mathbb{R}^n)} \leq C, \end{aligned}$$

when $\|f\|_{M^{p,\beta}(\mathbb{R}^n)} \leq 1$. This yields that $\mathcal{F}$ satisfies condition (i) of Proposition 3.1.

Step 3. A verification for condition (ii) of Proposition 3.1. Assume that $b \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and is supported in a ball $B = B(0, r)$. Fix $f \in M^{p,\beta}(\mathbb{R}^n)$ with $\|f\|_{M^{p,\beta}(\mathbb{R}^n)} \leq 1$ and $E_N := \{x \in \mathbb{R}^n : |x| > N\}$ with $N \geq \max\{nr, 1\}$. Note that $b(x) = 0$ when $x \in E_N$ since $N \geq nr$. By (1.16) and a change of variable, we have

$$\begin{aligned} \mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(x) & \leq \int_{\mathbb{R}^n} |(b(x) - b(z))^m f(z)| \frac{|\Omega_\eta(x - z)|}{|x - z|^n} dz \\ & \lesssim_n \|b\|_{L^\infty(\mathbb{R}^n)}^m \int_{|x-z| \leq r} \frac{|\Omega(z)|}{|z|^n} |f(x - z)| dz, \end{aligned} \quad (3.6)$$

when $x \in E_N$.

For a fixed ball $\tilde{B} = \tilde{B}(x_0, t)$. We consider two cases:

(1) ($q < p$). By Hölder's inequality and (3.6), we have

$$\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(x) \lesssim_{n,m,b,r} \left(\int_{|x-z| \leq r} \frac{|\Omega(z)|^q}{|z|^{qn}} |f(x - z)|^q dz \right)^{1/q}.$$

By Minkowski's inequality, we have

$$\begin{aligned} & \left(\frac{1}{|\tilde{B}|^\beta} \int_{\tilde{B}} |\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(x)| \chi_{E_N}(x) |^p dx \right)^{1/p} \\ & \lesssim_{n,m,b,r} |\tilde{B}|^{-\beta/p} \left(\int_{\tilde{B} \cap E_N} \left(\int_{|x-z| \leq r} \frac{|\Omega(z)|^q}{|z|^{qn}} |f(x - z)|^q dz \right)^{p/q} dx \right)^{1/p} \\ & \lesssim_{n,m,b,r} \left(\int_{|z| \geq (N-1)r} \frac{|\Omega(z)|^q}{|z|^{qn}} \left(\frac{1}{|\tilde{B}|^\beta} \int_{\tilde{B} \cap E_N} |f(x - z)|^p dx \right)^{q/p} dz \right)^{1/q} \\ & \lesssim_{n,m,b,r} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \left(\int_{|z| \geq (N-1)r} \frac{|\Omega(z)|^q}{|z|^{qn}} dz \right)^{1/q} \\ & \lesssim_{n,m,b,r,\Omega,q} ((q-1)n)^{1/q} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} ((N-1)r)^{-(q-1)n/q}. \end{aligned} \quad (3.7)$$

(2) ($q \geq p$). By Hölder's inequality and (3.6), we get

$$\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(x) \lesssim_{n,m,b,r} \left(\int_{|x-z| \leq r} \frac{|\Omega(z)|^p}{|z|^{pn}} |f(x - z)|^p dz \right)^{1/p}.$$

Then by Fubini's theorem, we get

$$\begin{aligned}
 & \left(\frac{1}{|\tilde{B}|^\beta} \int_{\tilde{B}} |\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(x) \chi_{E_N}(x)|^p dx \right)^{1/p} \\
 & \lesssim_{n,m,b,r} |\tilde{B}|^{-\beta/p} \left(\int_{\tilde{B} \cap E_N} \int_{|x-z| \leq r} \frac{|\Omega(z)|^p}{|z|^{pn}} |f(x-z)|^p dz dx \right)^{1/p} \\
 & \lesssim_{n,m,b,r} \left(\int_{|z| \geq (N-1)r} \frac{|\Omega(z)|^p}{|z|^{pn}} \frac{1}{|\tilde{B}|^\beta} \int_{\tilde{B} \cap E_N} |f(x-z)|^p dx dz \right)^{1/p} \\
 & \lesssim_{n,m,b,r} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \left(\int_{|z| \geq (N-1)r} \frac{|\Omega(z)|^p}{|z|^{pn}} dz \right)^{1/p} \\
 & \lesssim_{n,m,b,r,p} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} ((N-1)r)^{-(p-1)n/p}.
 \end{aligned} \tag{3.8}$$

Combining (3.8) with (3.7) implies that $\mathcal{F}$ satisfies condition (ii) of Proposition 3.1.

Step 4. A verification for condition (iii) of Proposition 3.1. We want to show that

$$\lim_{|h| \rightarrow 0} \|\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(\cdot + h) - \mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(\cdot)\|_{M^{p,\beta}(\mathbb{R}^n)} = 0 \tag{3.9}$$

for a fixed $\eta \in (0, 1)$.

In what follows, we set $|h| < \frac{\eta}{8}$ and $\eta \in (0, 1)$. By the definition of $\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)$, we have

$$\begin{aligned}
 & |\mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(x+h) - \mathcal{V}_\rho(\mathcal{T}_{\Omega,b}^m)(f)(x)| \\
 & \leq \sup_{\varepsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| \int_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i} (b(x+h) - b(y))^m \frac{\Omega_\eta(x+h-y)}{|x+h-y|^n} f(y) dy \right. \right. \\
 & \quad \left. \left. - \int_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i} (b(x) - b(y))^m \frac{\Omega_\eta(x-y)}{|x-y|^n} f(y) dy \right|^\rho \right)^{1/\rho} \\
 & \leq \sup_{\varepsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| \int_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i} ((b(x+h) - b(y))^m - (b(x) - b(y))^m) \right. \right. \\
 & \quad \left. \left. \times \frac{\Omega_\eta(x-y)}{|x-y|^n} f(y) dy \right|^\rho \right)^{1/\rho} \\
 & \quad + \sup_{\varepsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| \int_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i} (b(x+h) - b(y))^m \right. \right. \\
 & \quad \left. \left. \times \left(\frac{\Omega_\eta(x+h-y)}{|x+h-y|^n} - \frac{\Omega_\eta(x-y)}{|x-y|^n} \right) f(y) dy \right|^\rho \right)^{1/\rho} \\
 & \quad + \sup_{\varepsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| \int_{\mathbb{R}^n} (b(x+h) - b(y))^m \frac{\Omega_\eta(x+h-y)}{|x+h-y|^n} f(y) \right. \right. \\
 & \quad \left. \left. \times (\chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) - \chi_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i}(y)) dy \right|^\rho \right)^{1/\rho} \\
 & =: I_1 f(x) + I_2 f(x) + I_3 f(x).
 \end{aligned} \tag{3.10}$$

For $I_1 f(x)$. Note that

$$(b(x+h) - b(y))^m - (b(x) - b(y))^m = \sum_{j=0}^{m-1} c_m^j (b(x+h) - b(x))^{m-j} (b(x) - b(y))^j$$

and

$$(b(x) - b(y))^j = \sum_{\mu=0}^j c_j^\mu b(x)^{j-\mu} (-1)^\mu b(y)^\mu, \quad \forall 0 \leq j \leq m-1,$$

where $c_N^r = \frac{N!}{r!(N-r)!}$ for any $r, N \in \mathbb{N}$ with $r \leq N$. Therefore, we have

$$\begin{aligned}
 I_1 f(x) & \leq \sum_{j=0}^{m-1} c_m^j \|\nabla b\|_{L^\infty(\mathbb{R}^n)}^{m-j} |h|^{m-j} \sum_{\mu=0}^j c_j^\mu |b(x)|^{j-\mu} \\
 & \quad \times \sup_{\varepsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| \int_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i} \frac{\Omega_\eta(x-y)}{|x-y|^n} b(y)^\mu f(y) dy \right|^\rho \right)^{1/\rho}.
 \end{aligned} \tag{3.11}$$

For a decreasing sequence $\{\varepsilon_i\}_{i \geq 1}$ of positive numbers converging to 0, we set $N(\{\varepsilon_i\}) := \max\{i \geq 1 : \varepsilon_i \geq \eta\}$. Note that $\Omega_\eta(x - y) = 0$ when $|x - y| \leq \frac{\eta}{2}$ and $\Omega_\eta(x - y) = \Omega(x - y)$ when $|x - y| \geq \eta$. By (1.16), we have that, for $0 \leq j \leq m - 1$ and $0 \leq \mu \leq j$,

$$\begin{aligned} & \left(\sum_{i=1}^{\infty} \left| \int_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i} \frac{\Omega_\eta(x-y)}{|x-y|^n} b^\mu(y) f(y) dy \right|^\rho \right)^{1/\rho} \\ & \leq \left(\sum_{i=1}^{N(\{\varepsilon_i\})-1} \left| \int_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i} \frac{\Omega(x-y)}{|x-y|^n} b^\mu(y) f(y) dy \right|^\rho \right)^{1/\rho} \\ & \quad + \left(\sum_{i=N(\{\varepsilon_i\})}^{\infty} \left| \int_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i} \frac{\Omega_\eta(x-y)}{|x-y|^n} \chi_{\frac{\eta}{2} < |x-y| \leq \eta}(y) b^\mu(y) f(y) dy \right|^\rho \right)^{1/\rho} \\ & \leq \mathcal{V}_\rho(\mathcal{T}_\Omega)(b^\mu f)(x) + \|b\|_{L^\infty(\mathbb{R}^n)}^\mu \int_{\frac{\eta}{2} < |x-y| \leq \eta} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy \\ & \leq \mathcal{V}_\rho(\mathcal{T}_\Omega)(b^\mu f)(x) + 2^n \omega_n \|b\|_{L^\infty(\mathbb{R}^n)}^\mu M_\Omega f(x). \end{aligned}$$

This together with (3.11) yields that

$$I_1 f(x) \lesssim_{m,n,b} |h| \sum_{j=0}^m |b(x)|^j \left(\sum_{\mu=0}^{m-1} \mathcal{V}_\rho(\mathcal{T}_\Omega)(b^\mu f)(x) + M_\Omega f(x) \right). \quad (3.12)$$

By Theorem 1.1 and Minkowski's inequality, we get from (3.5) and (3.12) that

$$\begin{aligned} & \|I_1 f\|_{M^{p,\beta}(\mathbb{R}^n)} \\ & \lesssim_{m,n,b} |h| \left(\sum_{\mu=0}^{m-1} \|\mathcal{V}_\rho(\mathcal{T}_\Omega)(b^\mu f)\|_{M^{p,\beta}(\mathbb{R}^n)} + \|M_\Omega f\|_{M^{p,\beta}(\mathbb{R}^n)} \right) \\ & \lesssim_{m,n,b,p,\beta} |h|. \end{aligned} \quad (3.13)$$

For $I_2 f(x)$. Let $|h| < \frac{\eta}{8} e^{-1/\eta}$, then we have $\Omega_\eta(x + h - y) = \Omega_\eta(x - y) = 0$ when $|x - y| \leq \frac{\eta}{4}$. Moreover, $|x - y| > 2|h|$ when $|x - y| > \frac{\eta}{4}$. We get by (1.16) and a change of variables that

$$\begin{aligned} & I_2 f(x) \\ & \leq \sup_{\varepsilon_i \searrow 0} \left(\sum_{i=1}^{\infty} \left| \int_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i} (b(x+h) - b(y))^m \right. \right. \\ & \quad \times \left. \left(\frac{\Omega_\eta(x+h-y)}{|x+h-y|^n} - \frac{\Omega_\eta(x-y)}{|x-y|^n} \right) f(y) \chi_{|x-y| > \frac{\eta}{4}}(y) dy \right|^\rho \right)^{1/\rho} \\ & \leq \int_{\mathbb{R}^n} \left| \frac{\Omega_\eta(x+h-y)}{|x+h-y|^n} - \frac{\Omega_\eta(x-y)}{|x-y|^n} \right| \\ & \quad \times |(b(x+h) - b(y))^m f(y)| \chi_{|x-y| > \frac{\eta}{4}}(y) dy \\ & \leq (|b(x+h)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m \int_{|z| > \frac{\eta}{4}} \left| \frac{\Omega_\eta(z+h)}{|z+h|^n} - \frac{\Omega_\eta(z)}{|z|^n} \right| |f(z)| dz. \end{aligned} \quad (3.14)$$

Fix a ball $B = B(x_0, r)$. By Minkowski's inequality, one has

$$\begin{aligned} & \left(\frac{1}{|B|^\beta} \int_B |I_2 f(x)|^p dx \right)^{1/p} \\ & \leq 2^m \|b\|_{L^\infty(\mathbb{R}^n)}^m |B|^{-\beta/p} \\ & \quad \times \left(\int_B \left(\int_{|z| > \frac{\eta}{4}} \left| \frac{\Omega_\eta(z+h)}{|z+h|^n} - \frac{\Omega_\eta(z)}{|z|^n} \right| |f(x-z)| dz \right)^p dx \right)^{1/p} \\ & \lesssim_{m,b} |B|^{-\beta/p} \int_{|z| > \frac{\eta}{4}} \left(\int_B |f(x-z)|^p dx \right)^{1/p} \left| \frac{\Omega_\eta(z+h)}{|z+h|^n} - \frac{\Omega_\eta(z)}{|z|^n} \right| dz \\ & \lesssim_{m,b} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \int_{|z| > \frac{\eta}{4}} \left| \frac{\Omega_\eta(z+h)}{|z+h|^n} - \frac{\Omega_\eta(z)}{|z|^n} \right| dz. \end{aligned} \quad (3.15)$$

Invoking Lemma 3.2, we obtain

$$\begin{aligned}
 & \int_{|z| > \frac{\eta}{4}} \left| \frac{\Omega_\eta(z+h)}{|z+h|^n} - \frac{\Omega_\eta(z)}{|z|^n} \right| dz \\
 & \leq \sum_{k=0}^{\infty} \int_{2^{k-2}\eta < |z| \leq 2^{k-1}\eta} \left| \frac{\Omega_\eta(z+h)}{|z+h|^n} - \frac{\Omega_\eta(z)}{|z|^n} \right| dz \\
 & \leq C \sum_{k=0}^{\infty} \left(\frac{|h|}{2^{k-2}\eta} + \int_{|h|/(2^{k-1}\eta)}^{|h|/(2^{k-2}\eta)} \frac{w(\delta)}{\delta} d\delta \right) \\
 & \leq C e^{-1/\eta} + C \sum_{k=0}^{\infty} \frac{1}{1+k+\eta^{-1}} \int_{|h|/(2^{k-1}\eta)}^{|h|/(2^{k-2}\eta)} \frac{w(\delta)}{\delta} (1+|\log \delta|) d\delta \\
 & \leq C e^{-1/\eta} + C \eta \int_0^1 \frac{w(\delta)}{\delta} (1+|\log \delta|) d\delta \\
 & \leq C \eta (1+F(1)).
 \end{aligned}$$

This together with (3.15) implies that

$$\|I_2 f\|_{MP, \beta(\mathbb{R}^n)} \leq C \eta (1+F(1)) \|f\|_{MP, \beta(\mathbb{R}^n)}. \quad (3.16)$$

Finally, we estimate $I_3 f$. Let $1 < s < \min\{p, q\}$. Note that

$$\chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) - \chi_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i}(y) \neq 0$$

if and only if at least one of the following statements holds:

- (a) $\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i$, $|x-y| \leq \varepsilon_{i+1}$;
- (b) $\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i$, $|x-y| > \varepsilon_i$;
- (c) $\varepsilon_{i+1} < |x-y| \leq \varepsilon_i$, $|x+h-y| \leq \varepsilon_{i+1}$;
- (d) $\varepsilon_{i+1} < |x-y| \leq \varepsilon_i$, $|x+h-y| > \varepsilon_i$.

In case (a) we have that $|x+h-y| \leq |x-y| + |h| \leq \varepsilon_{i+1} + |h|$; In case (b) we have that $|x-y| \leq |x-y+h| + |h| \leq \varepsilon_i + |h|$; In case (c) we have that $|x-y| \leq |x-y+h| + |h| \leq \varepsilon_{i+1} + |h|$; In case (d) we have that $|x+h-y| \leq |x-y| + |h| \leq \varepsilon_i + |h|$. Then we have

$$\begin{aligned}
 & \left| \int_{\mathbb{R}^n} \frac{\Omega_\eta(x+h-y)}{|x+h-y|^n} (b(x+h) - b(y))^m f(y) \right. \\
 & \quad \times (\chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) - \chi_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i}(y)) dy \Big| \\
 & \leq (|b(x+h)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m \left(\int_{|x+h-y| \geq \frac{\eta}{2}} \frac{|\Omega_\eta(x+h-y)|}{|x+h-y|^n} |f(y)| \right. \\
 & \quad \times \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_{i+1} + |h|}(y) dy \\
 & \quad + \int_{|x+h-y| \geq \frac{\eta}{2}} \frac{|\Omega_\eta(x+h-y)|}{|x+h-y|^n} |f(y)| \\
 & \quad \times \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) \chi_{\varepsilon_i < |x-y| \leq \varepsilon_i + |h|}(y) dy \\
 & \quad + \int_{|x+h-y| \geq \frac{\eta}{2}} \frac{|\Omega_\eta(x+h-y)|}{|x+h-y|^n} |f(y)| \\
 & \quad \times \chi_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i}(y) \chi_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_{i+1} + |h|}(y) dy \\
 & \quad \left. + \int_{|x+h-y| \geq \frac{\eta}{2}} \frac{|\Omega_\eta(x+h-y)|}{|x+h-y|^n} |f(y)| \right. \\
 & \quad \times \chi_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i}(y) \chi_{\varepsilon_i < |x+h-y| \leq \varepsilon_i + |h|}(y) dy \Big) \\
 & =: (|b(x+h)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m \sum_{j=1}^4 I_{3,j} f(x).
 \end{aligned} \quad (3.17)$$

We now estimate $I_{3,j} f$, respectively.

For $I_{3,1}f$. In case (a) we have that $|x-y| \geq |x+h-y|-|h| \geq \frac{\eta}{2} - \frac{\eta}{8} > \frac{\eta}{4} > 2|h|$ and $|x+h-y| \geq |x-y|-|h| \geq \frac{1}{2}|x-y|$ when $|x+h-y| > \frac{\eta}{2}$. By Hölder's inequality, we obtain

$$\begin{aligned} I_{3,1}f(x) &\leq \left(\int_{|x+h-y| \geq \frac{\eta}{2}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x+h-y|^{ns}} \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) dy \right)^{1/s} \\ &\quad \times \left(\int_{\mathbb{R}^n} \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_{i+1}+|h|}(y) dy \right)^{1/s'} \\ &\lesssim_{n,s} \left(\int_{|x+h-y| \geq \frac{\eta}{2}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x+h-y|^{ns}} \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) dy \right)^{1/s} \\ &\quad \times ((\varepsilon_{i+1} + |h|)^n - \varepsilon_{i+1}^n)^{1/s'} \\ &\lesssim_{n,s} \left(\int_{|x+h-y| \geq \frac{\eta}{2}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x+h-y|^{ns}} \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) dy \right)^{1/s} \\ &\quad \times ((\varepsilon_{i+1} + |h|)^{n-1} |h|)^{1/s'}. \end{aligned}$$

Note that $\varepsilon_{i+1} + |h| \leq \frac{5}{4}|x+h-y|$ and $|x+h-y| \geq \frac{1}{2}|x-y|$ when $|x+h-y| > \frac{\eta}{2}$. Then we have

$$\begin{aligned} I_{3,1}f(x) &\lesssim_{n,s} |h|^{1/s'} \\ &\quad \times \left(\int_{|x-y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x-y|^{n+s-1}} \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) dy \right)^{1/s}. \end{aligned} \quad (3.18)$$

For $I_{3,2}f$. In case (b) we have that $|x-y| \geq \varepsilon_i \geq |x+h-y| \geq \frac{\eta}{2} > 4|h|$ and $|x+h-y| \geq |x-y|-|h| \geq \frac{|x-y|}{2}$ when $|x+h-y| > \frac{\eta}{2}$. By the arguments similar to those used to derive (3.18),

$$\begin{aligned} I_{3,2}f(x) &\lesssim_{n,s} |h|^{1/s'} \\ &\quad \times \left(\int_{|x-y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x-y|^{n+s-1}} \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) dy \right)^{1/s}. \end{aligned} \quad (3.19)$$

For $I_{3,3}f$. In case (c) we have that $|x-y| > \varepsilon_{i+1} \geq |x+h-y| \geq \frac{\eta}{2} > 4|h|$ and $|x+h-y| \geq |x-y|-|h| \geq \frac{|x-y|}{2}$ when $|x+h-y| > \frac{\eta}{2}$. The arguments similar to (3.18) will give that

$$\begin{aligned} I_{3,3}f(x) &\lesssim_{n,s} |h|^{1/s'} \\ &\quad \times \left(\int_{|x-y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x-y|^{n+s-1}} \chi_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i}(y) dy \right)^{1/s}. \end{aligned} \quad (3.20)$$

For $I_{3,4}f$. In case (d) we have that $\varepsilon_i \geq |x-y| \geq |x+h-y|-|h| \geq \frac{\eta}{2} - \frac{\eta}{8} \geq \frac{\eta}{4} > 2|h|$ and $|x+h-y| \geq |x-y|-|h| \geq \frac{|x-y|}{2}$ when $|x+h-y| > \frac{\eta}{2}$. Similar arguments to those in getting (3.18) may give that

$$I_{3,4}f(x) \lesssim_{n,s} |h|^{1/s'} \left(\int_{|x-y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x-y|^{n+s-1}} \chi_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i}(y) dy \right)^{1/s}. \quad (3.21)$$

By (3.17)-(3.21), (1.16) and a change of variable, we have

$$\begin{aligned} I_3f(x) &\lesssim_{n,s} |h|^{1/s'} (|b(x+h)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m \\ &\quad \times \left(\sup_{\varepsilon_i \geq 0} \sum_{i=1}^{\infty} \left(\int_{|x-y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x-y|^{n+s-1}} \right. \right. \\ &\quad \times \chi_{\varepsilon_{i+1} < |x+h-y| \leq \varepsilon_i}(y) dy \Big)^{\rho/s} \Big)^{1/\rho} \\ &\quad + \sup_{\varepsilon_i \geq 0} \left(\sum_{i=1}^{\infty} \left(\int_{|x-y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x-y|^{n+s-1}} \right. \right. \\ &\quad \times \chi_{\varepsilon_{i+1} < |x-y| \leq \varepsilon_i}(y) dy \Big)^{\rho/s} \Big)^{1/\rho} \\ &\lesssim_{n,s} (|b(x+h)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m |h|^{1/s'} \\ &\quad \times \left(\int_{|x-y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(x+h-y)f(y)|^s}{|x-y|^{n+s-1}} dy \right)^{1/s} \\ &\lesssim_{n,s} (|b(x+h)| + \|b\|_{L^\infty(\mathbb{R}^n)})^m |h|^{1/s'} \\ &\quad \times \left(\int_{|y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(y+h)f(x-y)|^s}{|y|^{n+s-1}} dy \right)^{1/s}. \end{aligned} \quad (3.22)$$

Fix a ball B . By (3.22) and Minkowski's inequality, one gets

$$\begin{aligned}
 & \left(\frac{1}{|B|^\beta} \int_B |I_3 f(x)|^p dx \right)^{1/p} \\
 & \lesssim_{n,s} \|b\|_{L^\infty(\mathbb{R}^n)}^m |h|^{1/s'} |B|^{-\beta/p} \\
 & \quad \times \left(\int_B \left(\int_{|y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(y+h)f(x-y)|^s}{|y|^{n+s-1}} dy \right)^{p/s} dx \right)^{1/p} \\
 & \lesssim_{m,n,b,s} |h|^{1/s'} \left(\int_{|y| \geq \frac{\eta}{4}} \left(\frac{1}{|B|^\beta} \int_B |f(x-y)|^p dx \right)^{s/p} \frac{|\Omega_\eta(y+h)|^s}{|y|^{n+s-1}} dy \right)^{1/s} \\
 & \lesssim_{m,n,b,s} |h|^{1/s'} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \left(\int_{|y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(y+h)|^s}{|y|^{n+s-1}} dy \right)^{1/s}.
 \end{aligned} \tag{3.23}$$

Note that $|y| \geq |y+h| - |h| \geq \frac{3}{8}|y+h|$ when $|h| \leq \frac{\eta}{8}$ and $|y+h| \geq \frac{\eta}{2}$. By a change of variable and Hölder's inequality, we have

$$\begin{aligned}
 & \int_{|y| \geq \frac{\eta}{4}} \frac{|\Omega_\eta(y+h)|^s}{|y|^{n+s-1}} dy \\
 & \lesssim_{n,s} \int_{|y+h| \geq \frac{\eta}{2}} \frac{|\Omega_\eta(y+h)|^s}{|y+h|^{n+s-1}} dy \\
 & \lesssim_{n,s} \int_{|z| \geq \frac{\eta}{2}} \frac{|\Omega(z)|^s}{|z|^{n+s-1}} dz \\
 & \lesssim_{n,s} \int_{\eta/2}^\infty r^{-s} dr \int_{S^{n-1}} |\Omega(\theta)|^s d\sigma(\theta) \\
 & \lesssim_{n,s} \eta^{1-s} \|\Omega\|_{L^s(S^{n-1})}^s \lesssim_{n,s} \|\Omega\|_{L^q(S^{n-1})}^s \eta^{1-s}.
 \end{aligned}$$

This together with (3.23) leads to

$$\begin{aligned}
 \|I_3 f\|_{M^{p,\beta}(\mathbb{R}^n)} & \lesssim_{m,n,b,s} \|\Omega\|_{L^q(S^{n-1})} |h|^{1/s'} \|f\|_{M^{p,\beta}(\mathbb{R}^n)} \eta^{-1/s'} \\
 & \lesssim_{m,n,b,s} \left(\frac{|h|}{\eta} \right)^{1/s'}.
 \end{aligned} \tag{3.24}$$

It follows from (3.10), (3.13), (3.16) and (3.24) that

$$\|\mathcal{V}_\rho(\mathcal{T}_{\Omega_\eta,b}^m f)(\cdot+h) - \mathcal{V}_\rho(\mathcal{T}_{\Omega_\eta,b}^m f)(\cdot)\|_{M^{p,\beta}(\mathbb{R}^n)} \lesssim_{m,n,b,s,p,q,\beta} |h| + \left(\frac{|h|}{\eta} \right)^{1/s'}.$$

This yields (3.9) and finishes the proof of Theorem 1.2. $\square$

4 Proof of Proposition 1.7

This section is devoted to proving Proposition 1.7. The proof will be divided into two parts:

Step 1: Proof of the boundedness part. It was proved by Yabuta [33] that if $0 < s < 1$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \sim \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta} f(x)|^p dx d\zeta \right)^{q/p} \right)^{1/q}. \tag{4.1}$$

It was shown in [22] (see [22, (5.16)]) that

$$\begin{aligned}
 |\Delta_h((\mathcal{V}_\rho(\mathcal{T}_{K,b}^m f))(x))| & \leq \sum_{l=0}^m c_m^l |b_h^{m-l}(x)| |\mathcal{V}_\rho(\mathcal{T}_K)(b_h^l \Delta_h f)(x)| \\
 & \quad + \sum_{l=1}^m c_m^l \sum_{\ell=0}^l c_l^\ell |\Delta_h b(x)|^\ell \sum_{\mu=0}^{m-l} c_{m-l}^\mu \\
 & \quad \times |b^{m-l-\mu}(x)| |\mathcal{V}_\rho(\mathcal{T}_K)(b^\mu (\Delta_h b)^{l-\ell} f)(x)| \\
 & =: \mathcal{G}(f)(h, x)
 \end{aligned} \tag{4.2}$$

for any $x, h \in \mathbb{R}^n$. By (4.1), (4.2), Minkowski's inequality and the property of b , we have

$$\begin{aligned}
 & \|\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \\
 & \leq C \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f))(x)|^p dx d\zeta \right)^{q/p} \right)^{1/q} \\
 & \leq C \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\mathcal{G}(f)(2^{-k}\zeta, x)|^p dx d\zeta \right)^{q/p} \right)^{1/q} \\
 & \leq C \|b\|_{L^\infty(\mathbb{R}^n)}^m \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} (\mathcal{V}_\rho(\mathcal{T}_K)(\Delta_{2^{-k}\zeta}f)(x))^p dx d\zeta \right)^{q/p} \right)^{1/q} \\
 & \quad + C \|b\|_{L^\infty(\mathbb{R}^n)}^{m-1} \\
 & \quad \times \left(\left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} (\mathcal{V}_\rho(\mathcal{T}_K)(\Delta_{2^{-k}\zeta}bf)(x))^p dx d\zeta \right)^{q/p} \right)^{1/q} \right. \\
 & \quad \left. + \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}b(x)\mathcal{V}_\rho(\mathcal{T}_K)(f)(x)|^p dx d\zeta \right)^{q/p} \right)^{1/q} \right). \tag{4.3}
 \end{aligned}$$

Note that $0 < s < 1$. By the L^p bounds for $\mathcal{V}_\rho(\mathcal{T}_K)$ and (4.1), we have

$$\begin{aligned}
 & \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} (\mathcal{V}_\rho(\mathcal{T}_K)(\Delta_{2^{-k}\zeta}f)(x))^p dx d\zeta \right)^{q/p} \right)^{1/q} \\
 & \leq C \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}f(x)|^p dx d\zeta \right)^{q/p} \right)^{1/q} \\
 & \leq C \|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}. \tag{4.4}
 \end{aligned}$$

By the L^p bounds for $\mathcal{V}_\rho(\mathcal{T}_K)$, (4.1) and the property of b , we get

$$\begin{aligned}
 & \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} (\mathcal{V}_\rho(\mathcal{T}_K)(\Delta_{2^{-k}\zeta}bf)(x))^p dx d\zeta \right)^{q/p} \right)^{1/q} \\
 & \leq C \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}b(x)f(x)|^p dx d\zeta \right)^{q/p} \right)^{1/q} \\
 & \leq C \|f\|_{L^p(\mathbb{R}^n)} \left(\|b\|_{A(\mathbb{R}^n)}^q \sum_{k=1}^{\infty} 2^{-kq(1-s)} + \|b\|_{L^\infty(\mathbb{R}^n)}^q \sum_{k=-\infty}^0 2^{ksq} \right)^{1/q} \\
 & \leq C \|b\|_{A(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \\
 & \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}b(x)\mathcal{V}_\rho(\mathcal{T}_K)(f)(x)|^p dx d\zeta \right)^{q/p} \right)^{1/q} \\
 & \leq C \left(\sum_{k \in \mathbb{Z}} 2^{ksq} (\|b\|_{A(\mathbb{R}^n)}^q 2^{-kq} \chi_{[1,\infty)}(k) + \|b\|_{L^\infty(\mathbb{R}^n)}^q \chi_{(-\infty,0]}(k)) \right. \\
 & \quad \left. \times \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |f(x)|^p dx d\zeta \right)^{q/p} \right)^{1/q} \\
 & \leq C \|b\|_{A(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \left(\sum_{k=1}^{\infty} 2^{-kq(1-s)} + \sum_{k=-\infty}^0 2^{ksq} \right)^{1/q} \\
 & \leq C \|b\|_{A(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \tag{4.5}
 \end{aligned}$$

It follows from (4.3)-(4.6) and (1.12) that

$$\|\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \leq C \|b\|_{A(\mathbb{R}^n)}^m \|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}. \tag{4.7}$$

On the other hand, it was shown in [22] (see [22, (5.11)]) that

$$\|\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)\|_{L^p(\mathbb{R}^n)} \leq C \|b\|_{L^\infty(\mathbb{R}^n)}^m \|f\|_{L^p(\mathbb{R}^n)}. \tag{4.8}$$

Then we get from (4.7), (4.8) and (1.12) that

$$\|\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \leq C \|b\|_{A(\mathbb{R}^n)}^m \|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}.$$

This proves the boundedness part.

Step 2: Proof of the continuity part. Let $0 < s < 1$, $1 < q < \infty$ and $f_j \rightarrow f$ in $B_s^{p,q}(\mathbb{R}^n)$ as $j \rightarrow \infty$. From (1.12) we see that $f_j \rightarrow f$ in $\dot{B}_s^{p,q}(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$ as $j \rightarrow \infty$. By the sublinearity of $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$ and (4.8), we have that $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) \rightarrow \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)$ in $L^p(\mathbb{R}^n)$ as $j \rightarrow \infty$. Hence, to get the continuity of $\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)$, it is enough to conclude that

$$\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) \rightarrow \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f) \text{ in } \dot{B}_s^{p,q}(\mathbb{R}^n) \text{ as } j \rightarrow \infty. \quad (4.9)$$

Next we shall prove (4.9) by contradiction. Without loss of generality we may assume that there exists $c > 0$ such that

$$\|\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) - \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} > c$$

for every j . Note that

$$\begin{aligned} & \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) - \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f))(x)|^p dx d\zeta \right)^{1/p} \\ & \leq 2|\mathfrak{R}_n|^{1/p} \|\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) - \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

This yields that

$$\begin{aligned} & \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) - \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f))(x)|^p dx d\zeta \right)^{1/p} \\ & \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned} \quad (4.10)$$

On the other hand, by (4.2) and the sublinearity of $\mathcal{V}_\rho(\mathcal{T}_K)$, we have that

$$\begin{aligned} & |\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) - \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f))(x)| \\ & \leq |\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j))(x)| + |\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f))(x)| \\ & \leq \mathcal{G}(f_j)(2^{-k}\zeta, x) + \mathcal{G}(f)(2^{-k}\zeta, x) \\ & \leq \mathcal{G}(f_j - f)(2^{-k}\zeta, x) + 2\mathcal{G}(2^{-k}\zeta, x). \end{aligned} \quad (4.11)$$

By (4.3)-(4.6), we get

$$\begin{aligned} & \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} (\mathcal{G}(f)(2^{-k}\zeta, x))^p dx d\zeta \right)^{q/p} \right)^{1/q} \\ & \leq C \|b\|_{\Lambda(\mathbb{R}^n)}^m \|f\|_{B_s^{p,q}(\mathbb{R}^n)}. \end{aligned} \quad (4.12)$$

It follows that

$$\begin{aligned} & \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} (\mathcal{G}(f_j - f)(2^{-k}\zeta, x))^p dx d\zeta \right)^{q/p} \right)^{1/q} \\ & \leq C \|b\|_{\Lambda(\mathbb{R}^n)}^m \|f_j - f\|_{B_s^{p,q}(\mathbb{R}^n)} \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Thus, one can extract a subsequence such that

$$\sum_{j=1}^{\infty} \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} (\mathcal{G}(f_j - f)(2^{-k}\zeta, x))^p dx d\zeta \right)^{q/p} \right)^{1/q} < \infty. \quad (4.13)$$

For $(k, \zeta, x) \in \mathbb{Z} \times \mathfrak{R}_n \times \mathbb{R}^n$, we set

$$\Gamma(k, \zeta, x) := \sum_{j=1}^{\infty} \mathcal{G}(f_j - f)(2^{-k}\zeta, x) + 2\mathcal{G}(2^{-k}\zeta, x).$$

By (4.11), we have

$$|\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) - \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f))(x)| \leq \Gamma(k, \zeta, x), \quad (4.14)$$

for $(k, \zeta, x) \in \mathbb{Z} \times \mathfrak{R}_n \times \mathbb{R}^n$. By (4.12), (4.13) and Minkowski's inequality, we get

$$\left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} (\Gamma(k, \zeta, x))^p dx d\zeta \right)^{q/p} \right)^{1/q} < \infty. \quad (4.15)$$

It follows from (4.10), (4.14), (4.15) and the dominated convergence theorem that

$$\left(\sum_{k \in \mathbb{Z}} 2^{ksq} \left(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) - \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f))(x)|^p dx d\zeta \right)^{q/p} \right)^{1/q} \rightarrow 0 \text{ as } j \rightarrow \infty,$$

which together with (4.1) leads to

$$\|\mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f_j) - \mathcal{V}_\rho(\mathcal{T}_{K,b}^m)(f)\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

This is a contradiction and completes the proof of Proposition 1.7.

Acknowledgement. The authors want to express their sincere thanks to the referees for their valuable remarks and suggestions, which made this paper more readable.

This work was supported partly by NNSF of China (Grant No. 11701333).

Conflict of interest: Authors state no conflict of interest.

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