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On the existence of multiple solutions for a partial discrete Dirichlet boundary value problem with mean curvature operator

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Abstract: A partial discrete Dirichlet boundary value problem involving mean curvature operator is concerned in this paper. Under proper assumptions on the nonlinear term, we obtain some feasible conditions on the existence of multiple solutions by the method of critical point theory. We further separately determine open intervals of the parameter to attain at least two positive solutions and an unbounded sequence of positive solutions with the help of the maximum principle.

Keywords: partial difference equations; boundary value problem; mean curvature operator; multiple solutions; critical point theory

MSC: 39A14; 39A70

1 Introduction

Let $\mathbb{Z}$ and $\mathbb{R}$ signify all integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$ and $\mathbb{Z}(a) = \{a, a+1, \dots\}$.

In this paper, we consider the following partial difference equation, that is (H_λ^f)

$$\Delta_1(\phi_c(\Delta_1 u(i-1, j))) + \Delta_2(\phi_c(\Delta_2 u(i, j-1))) + \lambda f((i, j), u(i, j)) = 0, \quad (i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n),$$

with boundary conditions

$$u(i, 0) = u(i, n+1) = 0, \quad i \in \mathbb{Z}(0, m+1),$$

$$u(0, j) = u(m+1, j) = 0, \quad j \in \mathbb{Z}(0, n+1),$$

where λ is a positive real parameter, m and n are two given positive integers, Δ_1 and Δ_2 are two first-order forward difference operators, respectively given by $\Delta_1 u(i, j) = u(i+1, j) - u(i, j)$ and $\Delta_2 u(i, j) = u(i, j+1) - u(i, j)$, ϕ_c is a ϕ -Laplacian operator (mean curvature operator [1]) defined by $\phi_c(s) = s/\sqrt{1+s^2}$, and for each $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$, $f((i, j), \cdot) \in C(\mathbb{R}, \mathbb{R})$.

As we all know, there are many excellent results on the study of the existence and multiplicity of solutions for differential equations [2–4]. In recent years, there are more and more researches on the subject of difference equations since it has been widely used as mathematical models in many fields [5–10]. Important methods generally used in the study of difference equations are upper and lower solution techniques and fixed point methods [11, 12]. In 2003, for the first time, Guo and Yu [13] considered periodic solutions and subharmonic solutions for a class of difference equations by using critical point theory. Since then, many researchers have studied difference equations via critical point theory so that a lot of excellent results have been

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obtained, such as existence results on homoclinic solutions [14–24], periodic solutions [25–27] and solutions for boundary value problems [28–38]. Particularly, due to many applications of partial difference equation models in lots of fields lately, there is of great interest in studying partial difference equations involving functions with two or more discrete variables. In this direction, some scholars have obtained some good results [39–42].

In [39], Heidarkhani and Imbesi discussed the following problem (E_λ^f)

$$\Delta_1^2 u(i-1, j) + \Delta_2^2 u(i, j-1) + \lambda f((i, j), u(i, j)) = 0, \quad (i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n),$$

with boundary conditions

$$u(i, 0) = u(i, n+1) = 0, \quad i \in \mathbb{Z}(1, m),$$

$$u(0, j) = u(m+1, j) = 0, \quad j \in \mathbb{Z}(1, n),$$

where Δ_1^2 and Δ_2^2 are two second-order forward difference operators given by $\Delta_1^2 u(i, j) = \Delta_1(\Delta_1 u(i, j))$ and $\Delta_2^2 u(i, j) = \Delta_2(\Delta_2 u(i, j))$, respectively. The two-dimensional discrete problem (E_λ^f) was considered as the one-dimensional nonlinear algebraic problem by transforming the matrix form into the one-dimensional vector form. Then by two critical point theorems, the existence of at least three solutions for (E_λ^f) was obtained. Imbesi and Bisci [40] further discussed that either an unbounded sequence of solutions or a pairwise distinct sequence of solutions for (E_λ^f) admitted when unbounded intervals of parameters were set by using critical point theory. Galewski and Orpel [41] considered the existence of solutions to the boundary value problem (E_λ^f) in terms of critical point theory and some monotonicity results.

In [42], we dealt with the existence of infinitely many solutions for a partial discrete Dirichlet boundary value problem with p -Laplacian, namely (S_λ^f)

$$\Delta_1(\phi_p(\Delta_1 u(i-1, j))) + \Delta_2(\phi_p(\Delta_2 u(i, j-1))) + \lambda f((i, j), u(i, j)) = 0, \quad (i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n),$$

with boundary conditions

$$u(i, 0) = u(i, n+1) = 0, \quad i \in \mathbb{Z}(0, m+1),$$

$$u(0, j) = u(m+1, j) = 0, \quad j \in \mathbb{Z}(0, n+1),$$

where ϕ_p is the p -Laplacian operator given by $\phi_p(s) = |s|^{p-2}s$, $1 < p < +\infty$ and $f((i, j), \cdot) \in C(\mathbb{R}, \mathbb{R})$ for all $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$. Based on critical point theory and the maximum principle, we focused on appropriate assumptions on the nonlinear term and open intervals of the parameter to respectively obtain the existence of at least two positive solutions and an unbounded sequence of positive solutions.

Note that the research on partial difference equations with ϕ_c -Laplacian is relatively rare. In this paper, we are interested in investigating infinitely many solutions of the two-dimensional discrete Dirichlet problem (H_λ^f) via critical point theory. Under appropriate assumptions on the nonlinear term, we give sufficient conditions to admit at least two positive solutions for (H_λ^f) by applying Theorem 2.2 of [39] and Theorem 3.3 of [43]. In addition, we carefully consider the existence of an unbounded sequence of solutions for (H_λ^f) by utilizing Theorem 2.1 of [44]. Moreover, due to the maximum principle, we determine specific open intervals of the parameter to admit an unbounded sequence of positive solutions.

We organize the rest of this work as follows. In Section 2, we first introduce some abstract multiple critical point theorems, and then set up the variational setting associated with (H_λ^f) . In Section 3, we prove the main results of (H_λ^f) by combining the variational method and the maximum principle. Finally, we present an easily testified example to demonstrate in the last section.

2 Preliminaries

Let X signify a finite dimensional real Banach space and let $I_\lambda : X \rightarrow \mathbb{R}$ be a function satisfying the following structure hypothesis:

(A) $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in X$, where $\Phi, \Psi : X \rightarrow \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, i.e., $\lim_{\|u\| \rightarrow \infty} \Phi(u) = +\infty$, and λ is a real positive parameter.

In this framework a finite dimensional variant of Theorem 3.3 of [43] (see also Corollary 3.1 and Remark 3.9 of [43]) is the following

Lemma 2.1. Assume that condition (A) and the following conditions hold:

(B) Φ is convex and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(C) for each $\lambda > 0$ and local minima $u_1, u_2 \in X$ of $I_\lambda := \Phi - \lambda\Psi$ with $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has $\inf_{0 \leq t \leq 1} \Psi(tu_1 + (1-t)u_2) \geq 0$.

Assume further that there are two positive constants ρ_1, ρ_2 and $\bar{r} \in X$ with $\rho_1 < \Phi(\bar{r}) < \frac{\rho_2}{2}$, such that

(a₁)

$$\frac{\sup_{u \in \Phi^{-1}(-\infty, \rho_1)} \Psi(u)}{\rho_1} < \frac{1}{2} \cdot \frac{\Psi(\bar{r})}{\Phi(\bar{r})};$$

(a₂)

$$\frac{\sup_{u \in \Phi^{-1}(-\infty, \rho_2)} \Psi(u)}{\rho_2} < \frac{1}{4} \cdot \frac{\Psi(\bar{r})}{\Phi(\bar{r})}.$$

Then, for all $\lambda \in \left(\frac{2\Phi(\bar{r})}{\Psi(\bar{r})}, \min \left\{ \frac{\rho_1}{\sup_{u \in \Phi^{-1}(-\infty, \rho_1)} \Psi(u)}, \frac{\rho_2/2}{\sup_{u \in \Phi^{-1}(-\infty, \rho_2)} \Psi(u)} \right\} \right)$, the functional I_λ has at least three distinct critical points which lie in $\Phi^{-1}(-\infty, \rho_2)$.

We recall the following lemma (Theorem 2.2 of [39]).

Lemma 2.2. Assume that condition (A) and the following conditions hold:

(D) Φ is convex and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(E) for each $\lambda > 0$ and local minima $u_1, u_2 \in X$ of $I_\lambda := \Phi - \lambda\Psi$ with $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has $\inf_{0 \leq t \leq 1} \Psi(tu_1 + (1-t)u_2) \geq 0$.

Assume further that there are two positive constants ρ_1, ρ_2 and $\bar{r} \in X$ with $2\rho_1 < \Phi(\bar{r}) < \frac{\rho_2}{2}$, such that

(a₃)

$$\frac{\sup_{u \in \Phi^{-1}(-\infty, \rho_1)} \Psi(u)}{\rho_1} < \frac{2}{3} \cdot \frac{\Psi(\bar{r})}{\Phi(\bar{r})};$$

(a₄)

$$\frac{\sup_{u \in \Phi^{-1}(-\infty, \rho_2)} \Psi(u)}{\rho_2} < \frac{1}{3} \cdot \frac{\Psi(\bar{r})}{\Phi(\bar{r})}.$$

Then, for all $\lambda \in \left(\frac{3}{2} \cdot \frac{\Phi(\bar{r})}{\Psi(\bar{r})}, \min \left\{ \frac{\rho_1}{\sup_{u \in \Phi^{-1}(-\infty, \rho_1)} \Psi(u)}, \frac{\rho_2/2}{\sup_{u \in \Phi^{-1}(-\infty, \rho_2)} \Psi(u)} \right\} \right)$, the functional I_λ has at least three distinct critical points which lie in $\Phi^{-1}(-\infty, \rho_2)$.

For each $v, v_1, v_2 > \inf_X \Phi$ with $v_2 > v_1$ and $v_3 > 0$, set

$$\begin{aligned} \varphi(v) &= \inf_{u \in \Phi^{-1}(-\infty, v)} \frac{\left(\sup_{u \in \Phi^{-1}(-\infty, v)} \Psi(u) \right) - \Psi(u)}{v - \Phi(u)}, \\ \beta(v_1, v_2) &= \inf_{u \in \Phi^{-1}(-\infty, v_1)} \sup_{\omega \in \Phi^{-1}[v_1, v_2]} \frac{\Psi(\omega) - \Psi(u)}{\Phi(\omega) - \Phi(u)}, \\ \gamma(v_2, v_3) &= \frac{\sup_{u \in \Phi^{-1}(-\infty, v_2+v_3)} \Psi(u)}{v_3}, \\ \alpha(v_1, v_2, v_3) &= \max \{ \varphi(v_1), \varphi(v_2), \gamma(v_2, v_3) \}. \end{aligned}$$

We have the following variant of Theorem 3.3 of [43] (see also Corollary 3.1 and Remark 3.9 of [43]).

Lemma 2.3. *If condition (A) holds. Assume that:*

(F) Φ is convex and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(G) for each $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $I_\lambda := \Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has $\inf_{0 \leq t \leq 1} \Psi(tu_1 + (1-t)u_2) \geq 0$.

Further, assume that there are three positive constants ρ_1, ρ_2, ρ_3 with $\rho_1 < \rho_2$, such that

(a₅) $\varphi(\rho_1) < \beta(\rho_1, \rho_2)$;

(a₆) $\varphi(\rho_2) < \beta(\rho_1, \rho_2)$;

(a₇) $\gamma(\rho_2, \rho_3) < \beta(\rho_1, \rho_2)$.

Then, for all $\lambda \in \left(\frac{1}{\beta(\rho_1, \rho_2)}, \frac{1}{\alpha(\rho_1, \rho_2, \rho_3)} \right)$, the functional I_λ has three distinct critical points.

Let

$$\gamma = \liminf_{r \rightarrow +\infty} \varphi(r), \quad \delta = \liminf_{r \rightarrow \left(\inf_X \Phi \right)^+} \varphi(r).$$

Clearly, $\gamma \geq 0$ and $\delta \geq 0$. When $\gamma = 0$ (or $\delta = 0$), in the sequel, we agree to read $\frac{1}{\gamma}$ (or $\frac{1}{\delta}$) as $+\infty$.

We recall the following lemma (see Theorem 2.1 of [44]).

Lemma 2.4. *Assume that condition (A) holds. One has:*

(i) If $\gamma < +\infty$ then, for each $\lambda \in (0, \frac{1}{\gamma})$, the following alternative holds: either

(i₁) I_λ possesses a global minimum, or

(i₂) there is a sequence $\{u_t\}$ of critical points (local minima) of I_λ such that $\lim_{t \rightarrow +\infty} \Phi(u_t) = +\infty$.

(j) If $\delta < +\infty$ then, for each $\lambda \in (0, \frac{1}{\delta})$, the following alternative holds: either

(j₁) there is a global minimum of Φ which is a local minimum of I_λ , or

(j₂) there is a sequence $\{u_t\}$ of pairwise distinct critical points (local minima) of I_λ , with $\lim_{t \rightarrow +\infty} \Phi(u_t) = \inf_X \Phi$, which weakly converges to a global minimum of Φ .

We set the mn -dimensional Banach space

$$S = \left\{ u : \mathbb{Z}(0, m+1) \times \mathbb{Z}(0, n+1) \rightarrow \mathbb{R} \text{ such that } u(i, 0) = u(i, n+1) = 0, \right. \\ \left. i \in \mathbb{Z}(0, m+1) \text{ and } u(0, j) = u(m+1, j) = 0, j \in \mathbb{Z}(0, n+1) \right\},$$

endowed with the norm

$$\|u\| = \left(\sum_{j=1}^n \sum_{i=1}^{m+1} (\Delta_1 u(i-1, j))^2 + \sum_{i=1}^m \sum_{j=1}^{n+1} (\Delta_2 u(i, j-1))^2 \right)^{\frac{1}{2}}, \quad u \in S.$$

Put

$$\Phi(u) = \sum_{j=1}^n \sum_{i=1}^{m+1} \left(\sqrt{1 + (\Delta_1 u(i-1, j))^2} - 1 \right) + \sum_{i=1}^m \sum_{j=1}^{n+1} \left(\sqrt{1 + (\Delta_2 u(i, j-1))^2} - 1 \right),$$

$$\Psi(u) = \sum_{j=1}^n \sum_{i=1}^m F((i, j), u(i, j)),$$

for all $u \in S$, where $F((i, j), u) = \int_0^u f((i, j), \xi) d\xi$ for each $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$.

Let

$$I_\lambda(u) = \Phi(u) - \lambda \Psi(u),$$

for each $u \in S$. Evidently, $I_\lambda \in C^1(S, \mathbb{R})$ with

$$\Psi'(u)(z) = \lim_{t \rightarrow 0} \frac{\Psi(u + tz) - \Psi(u)}{t} = \sum_{j=1}^n \sum_{i=1}^m f((i, j), u(i, j)) z(i, j)$$

and

$$\begin{aligned} \Phi'(u)(z) &= \lim_{t \rightarrow 0} \frac{\Phi(u + tz) - \Phi(u)}{t} \\ &= \sum_{j=1}^n \sum_{i=1}^{m+1} \phi_c(\Delta_1 u(i-1, j)) \Delta_1 z(i-1, j) + \sum_{i=1}^m \sum_{j=1}^{n+1} \phi_c(\Delta_2 u(i, j-1)) \Delta_2 z(i, j-1) \\ &= \sum_{j=1}^n \sum_{i=1}^m \phi_c(\Delta_1 u(i-1, j)) \Delta_1 z(i-1, j) - \sum_{j=1}^n \phi_c(\Delta_1 u(m, j)) z(m, j) \\ &\quad + \sum_{i=1}^m \sum_{j=1}^n \phi_c(\Delta_2 u(i, j-1)) \Delta_2 z(i, j-1) - \sum_{i=1}^m \phi_c(\Delta_2 u(i, n)) z(i, n) \\ &= - \sum_{j=1}^n \sum_{i=1}^m \Delta_1 \phi_c(\Delta_1 u(i-1, j)) z(i, j) - \sum_{i=1}^m \sum_{j=1}^n \Delta_2 \phi_c(\Delta_2 u(i, j-1)) z(i, j) \end{aligned}$$

for each $u, z \in S$. Taken together, we have

$$\begin{aligned} [\Phi'(u) - \lambda \Psi'(u)](z) &= - \sum_{j=1}^n \sum_{i=1}^m [\Delta_1 \phi_c(\Delta_1 u(i-1, j)) + \Delta_2 \phi_c(\Delta_2 u(i, j-1))] \\ &\quad + \lambda f((i, j), u(i, j)) z(i, j). \end{aligned}$$

Accordingly, the critical points of I_λ in S are the solutions of problem (H_λ^f) .

The following lemma comes from Proposition 1 of [42].

Lemma 2.5. *For each $u \in S$, one has the following relation*

$$\max_{\substack{i \in \mathbb{Z}(1, m) \\ j \in \mathbb{Z}(1, n)}} \{|u(i, j)|\} \leq \frac{\sqrt{m+n+2}}{4} \|u\|. \quad (2.1)$$

Remark 2.1. Clearly, when $m = 1$ and $n = 1$, then $\|u\| = 2|u(1, 1)|$ and the equality in (2.1) holds.

Proposition 2.1. *Suppose that there is $\bar{u} \in S$ such that either*

$$\bar{u}(i, j) > 0 \text{ or } \Delta_1(\phi_c(\Delta_1 \bar{u}(i-1, j))) + \Delta_2(\phi_c(\Delta_2 \bar{u}(i, j-1))) \leq 0, \quad (2.2)$$

for each $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$. Then, either $\bar{u}(i, j) > 0$ for all $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$ or $\bar{u} \equiv 0$.

Proof. Fix $\alpha \in \mathbb{Z}(1, m)$, $\beta \in \mathbb{Z}(1, n)$ such that

$$\bar{u}(\alpha, \beta) = \min \left\{ \bar{u}(i, j) : i \in \mathbb{Z}(1, m), j \in \mathbb{Z}(1, n) \right\}.$$

When $\bar{u}(\alpha, \beta) > 0$, we get $\bar{u}(i, j) > 0$ for every $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$. The proposition is proved. When $\bar{u}(\alpha, \beta) \leq 0$, we have $\bar{u}(\alpha, \beta) = \min \left\{ \bar{u}(i, j) : i \in \mathbb{Z}(0, m+1), j \in \mathbb{Z}(0, n+1) \right\}$. Clearly, $\Delta_1 \bar{u}(\alpha-1, \beta) = \bar{u}(\alpha, \beta) - \bar{u}(\alpha-1, \beta) \leq 0$, $\Delta_2 \bar{u}(\alpha, \beta-1) = \bar{u}(\alpha, \beta) - \bar{u}(\alpha, \beta-1) \leq 0$, $\Delta_1 \bar{u}(\alpha, \beta) = \bar{u}(\alpha+1, \beta) - \bar{u}(\alpha, \beta) \geq 0$, and $\Delta_2 \bar{u}(\alpha, \beta) = \bar{u}(\alpha, \beta+1) - \bar{u}(\alpha, \beta) \geq 0$. Since $\phi_c(\mu)$ is increasing in μ and $\phi_c(0) = 0$, we see that

$$\phi_c(\Delta_1 \bar{u}(\alpha, \beta)) \geq 0 \geq \phi_c(\Delta_1 \bar{u}(\alpha-1, \beta))$$

and

$$\phi_c(\Delta_2 \bar{u}(\alpha, \beta)) \geq 0 \geq \phi_c(\Delta_2 \bar{u}(\alpha, \beta-1)).$$

This further shows that

$$\Delta_1(\phi_c(\Delta_1 \bar{u}(\alpha-1, \beta))) + \Delta_2(\phi_c(\Delta_2 \bar{u}(\alpha, \beta-1))) \geq 0.$$

We obtain

$$\phi_c(\Delta_1 \bar{u}(\alpha, \beta)) = \phi_c(\Delta_1 \bar{u}(\alpha-1, \beta)) = 0.$$

Consequently, $\bar{u}(\alpha+1, \beta) = \bar{u}(\alpha, \beta) = \bar{u}(\alpha-1, \beta)$. We claim that $\bar{u}(\alpha, \beta) = 0$ when $\alpha+1 = m+1$. If not, then $(\alpha+1) \in \mathbb{Z}(1, m)$. By taking the place of α with $\alpha+1$, we have $\bar{u}(\alpha+2, \beta) = \bar{u}(\alpha+1, \beta)$. Continuing this process $(m+1-\alpha)$ times, we get $\bar{u}(\alpha, \beta) = \bar{u}(\alpha+1, \beta) = \bar{u}(\alpha+2, \beta) = \cdots = \bar{u}(m+1, \beta) = 0$. Similarly, we obtain $\bar{u}(\alpha, \beta) = \bar{u}(\alpha-1, \beta) = \bar{u}(\alpha-2, \beta) = \cdots = \bar{u}(0, \beta) = 0$. Then, it holds that $\bar{u}(i, \beta) = 0$ for $i \in \mathbb{Z}(1, m)$. Analogously, we get $\bar{u} \equiv 0$.

3 Main results and the proofs

Throughout the rest of this paper, for each $h > 0$, we let

$$\rho(h) := \frac{\sqrt{\frac{16h^2}{m+n+2} + 1} - 1}{m+n}, \quad \Gamma(h) := \frac{\sum_{j=1}^n \sum_{i=1}^m F((i, j), h)}{\sqrt{\frac{16h^2}{n+m+2} + 1} - 1} \quad \text{and} \quad \Lambda(h) := \frac{(n+m)(\sqrt{1+h^2} - 1)}{\sum_{j=1}^n \sum_{i=1}^m F((i, j), h)}.$$

Theorem 3.1. Suppose that $f((i, j), \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a non-negative continuous function for each $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$ and there are positive constants b_1, b_2 and l with

$$\left(\frac{\rho(b_1)}{2} + 1 \right)^2 < 1 + l^2 < \left(\frac{\rho(b_2)}{4} + 1 \right)^2$$

such that

$$(g_1) \quad \max \{ \Gamma(b_1), 2\Gamma(b_2) \} < \frac{1}{4\Lambda(l)}.$$

Then, for each

$$\lambda \in \left(4\Lambda(l), \min \left\{ \frac{1}{\Gamma(b_1)}, \frac{1}{2\Gamma(b_2)} \right\} \right),$$

problem (H_λ^f) admits at least two positive solutions.

Proof. Obviously, Φ and Ψ satisfy conditions (A) and (B) of Lemma 2.1 by setting Φ, Ψ, I_λ as defined in Section 2 for each $u \in S$. Furthermore, let u_1 and u_2 be two local minima of I_λ , which implies that u_1 and u_2 are solutions of (H_λ^f) . We have $u_1(i, j) \geq 0$ and $u_2(i, j) \geq 0$ for every $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$. Then, $tu_1(i, j) + (1-t)u_2(i, j) \geq 0$ for all $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$ and $0 \leq t \leq 1$. It is clear that $\Psi(tu_1 + (1-t)u_2) \geq 0$ for $0 \leq t \leq 1$ and condition (C) is satisfied.

Now, we set

$$\rho_1 = (m+n)\rho(b_1) \quad \text{and} \quad \rho_2 = (m+n)\rho(b_2).$$

For $u \in S$, let

$$\xi_1(i, j) = \sqrt{1 + (\Delta_1 u(i-1, j))^2} - 1, \quad (i, j) \in \mathbb{Z}(1, m+1) \times \mathbb{Z}(1, n),$$

$$\xi_2(i, j) = \sqrt{1 + (\Delta_2 u(i, j - 1))^2} - 1, \quad (i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n + 1).$$

One has

$$\Phi(u) = \sum_{j=1}^n \sum_{i=1}^{m+1} \xi_1(i, j) + \sum_{i=1}^m \sum_{j=1}^{n+1} \xi_2(i, j).$$

Since

$$\begin{aligned} \sum_{j=1}^n \sum_{i=1}^{m+1} (\Delta_1 u(i - 1, j))^2 &= \sum_{j=1}^n \sum_{i=1}^{m+1} [(\xi_1(i, j))^2 + 2(\xi_1(i, j))] \\ &\leq \left(\sum_{j=1}^n \sum_{i=1}^{m+1} \xi_1(i, j) \right)^2 + 2 \sum_{j=1}^n \sum_{i=1}^{m+1} \xi_1(i, j) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^{n+1} (\Delta_2 u(i, j - 1))^2 &= \sum_{i=1}^m \sum_{j=1}^{n+1} [(\xi_2(i, j))^2 + 2(\xi_2(i, j))] \\ &\leq \left(\sum_{i=1}^m \sum_{j=1}^{n+1} \xi_2(i, j) \right)^2 + 2 \sum_{i=1}^m \sum_{j=1}^{n+1} \xi_2(i, j), \end{aligned}$$

we get

$$\begin{aligned} &\sum_{j=1}^n \sum_{i=1}^{m+1} (\Delta_1 u(i - 1, j))^2 + \sum_{i=1}^m \sum_{j=1}^{n+1} (\Delta_2 u(i, j - 1))^2 \\ &\leq \left(\sum_{j=1}^n \sum_{i=1}^{m+1} \xi_1(i, j) \right)^2 + 2 \sum_{j=1}^n \sum_{i=1}^{m+1} \xi_1(i, j) + \left(\sum_{i=1}^m \sum_{j=1}^{n+1} \xi_2(i, j) \right)^2 + 2 \sum_{i=1}^m \sum_{j=1}^{n+1} \xi_2(i, j) \\ &\leq \left(\sum_{j=1}^n \sum_{i=1}^{m+1} \xi_1(i, j) + \sum_{i=1}^m \sum_{j=1}^{n+1} \xi_2(i, j) \right)^2 + 2 \sum_{j=1}^n \sum_{i=1}^{m+1} \xi_1(i, j) + 2 \sum_{i=1}^m \sum_{j=1}^{n+1} \xi_2(i, j). \end{aligned}$$

Then,

$$\sum_{j=1}^n \sum_{i=1}^{m+1} (\Delta_1 u(i - 1, j))^2 + \sum_{i=1}^m \sum_{j=1}^{n+1} (\Delta_2 u(i, j - 1))^2 + 1 \leq \left[\sum_{j=1}^n \sum_{i=1}^{m+1} \xi_1(i, j) + \sum_{i=1}^m \sum_{j=1}^{n+1} \xi_2(i, j) + 1 \right]^2.$$

For each $u \in S$, we have

$$\sqrt{\|u\|^2 + 1} - 1 \leq \Phi(u),$$

and

$$\max_{\substack{i \in \mathbb{Z}(1, m) \\ j \in \mathbb{Z}(1, n)}} \{|u(i, j)|\} \leq \frac{\sqrt{m+n+2}}{4} \|u\|.$$

This implies that

$$\max_{\substack{i \in \mathbb{Z}(1, m) \\ j \in \mathbb{Z}(1, n)}} \{|u(i, j)|\} \leq \frac{\sqrt{m+n+2}}{4} \cdot \sqrt{(\rho_1 + 1)^2 - 1} = b_1.$$

Similarly, for $u \in S$, one has

$$\max_{\substack{i \in \mathbb{Z}(1, m) \\ j \in \mathbb{Z}(1, n)}} \{|u(i, j)|\} \leq \frac{\sqrt{m+n+2}}{4} \cdot \sqrt{(\rho_2 + 1)^2 - 1} = b_2.$$

It follows that

$$\frac{\sup_{u \in \Phi^{-1}(-\infty, \rho_1)} \Psi(u)}{\rho_1} = \frac{\sup_{u \in \Phi^{-1}(-\infty, \rho_1)} \sum_{j=1}^n \sum_{i=1}^m F((i, j), u(i, j))}{\rho_1} \leq \Gamma(b_1),$$

and

$$\frac{\sup_{u \in \Phi^{-1}(-\infty, \rho_2)} \Psi(u)}{\rho_2} = \frac{\sup_{u \in \Phi^{-1}(-\infty, \rho_2)} \sum_{j=1}^n \sum_{i=1}^m F((i, j), u(i, j))}{\rho_2} \leq \Gamma(b_2).$$

For $y \in S$ defined by

$$y(i, j) = \begin{cases} l, & \text{if } (i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n), \\ 0, & \text{if } i = 0, j \in \mathbb{Z}(0, n+1) \text{ or } i = m+1, j \in \mathbb{Z}(0, n+1), \\ 0, & \text{if } j = 0, i \in \mathbb{Z}(0, m+1) \text{ or } j = n+1, i \in \mathbb{Z}(0, m+1), \end{cases}$$

it holds that $\Phi(y) = (2m + 2n)(\sqrt{1 + l^2} - 1)$. It is clear that $\rho_1 < \Phi(y) < \frac{\rho_2}{2}$. Then,

$$\frac{\Psi(y)}{\Phi(y)} = \frac{1}{2\Lambda(l)}.$$

In terms of (g_1) , we can verify assumptions (a_1) and (a_2) of Lemma 2.1. Thus, for each

$$\lambda \in \left(4\Lambda(l), \min \left\{ \frac{1}{\Gamma(b_1)}, \frac{1}{2\Gamma(b_2)} \right\} \right),$$

problem (H_λ^f) admits at least two positive solutions. The proof of the theorem is completed.

With a similar proof of Theorem 3.1, we have the following

Theorem 3.2. Suppose that $f((i, j), \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a non-negative continuous function for each $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$ and there are positive constants b_1, b_2 and l with

$$(\rho(b_1) + 1)^2 < 1 + l^2 < \left(\frac{\rho(b_2)}{4} + 1 \right)^2$$

such that

$$(g_1) \quad \max \{ \Gamma(b_1), 2\Gamma(b_2) \} < \frac{1}{3\Lambda(l)}.$$

Then, for each

$$\lambda \in \left(3\Lambda(l), \min \left\{ \frac{1}{\Gamma(b_1)}, \frac{1}{2\Gamma(b_2)} \right\} \right),$$

problem (H_λ^f) admits at least two positive solutions.

Given $h_1 > 0$ and $h_2 > 0$, we further introduce another notation $\Omega(h_1, h_2)$ as follows

$$\Omega(h_1, h_2) := \frac{\sum_{j=1}^n \sum_{i=1}^m F((i, j), h_2)}{\sqrt{\frac{16h_2^2}{n+m+2} + 1} - \sqrt{\frac{16h_1^2}{n+m+2} + 1}}.$$

Theorem 3.3. Assume that $f((i, j), \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a non-negative continuous function for every $(i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n)$ and there are positive constants d, e_1, e_2 and e_3 with $e_3 > e_2$ such that

$$\left(\frac{\rho(e_1)}{2} + 1 \right)^2 < 1 + d^2 < \left(\frac{\rho(e_2)}{2} + 1 \right)^2$$

and

$$(g_4) \quad \max \{ \Gamma(e_1), \Gamma(e_2), \Omega(e_2, e_3) \} < \frac{1}{4\Lambda(d)}.$$

Then, for each

$$\lambda \in \left(\frac{2\Lambda(d)}{1 - 2\Lambda(d)\Gamma(e_1)}, \frac{1}{\max \{ \Gamma(e_1), \Gamma(e_2), \Omega(e_2, e_3) \}} \right),$$

problem (H_λ^f) admits at least two positive solutions.

Proof. Let

$$v_1 = \sqrt{\frac{16e_1^2}{m+n+2} + 1} - 1,$$

$$v_2 = \sqrt{\frac{16e_2^2}{m+n+2} + 1} - 1,$$

$$v_3 = \sqrt{\frac{16e_3^2}{m+n+2} + 1} - \sqrt{\frac{16e_2^2}{m+n+2} + 1}.$$

When $u \in S$ and $\Phi(u) \leq v_1$, we have

$$\max_{\substack{i \in \mathbb{Z}(1,m) \\ j \in \mathbb{Z}(1,n)}} \{|u(i,j)|\} \leq \frac{\sqrt{m+n+2}}{4} \cdot \sqrt{(v_1+1)^2 - 1} = e_1.$$

When $u \in S$ and $\Phi(u) \leq v_2$, we get

$$\max_{\substack{i \in \mathbb{Z}(1,m) \\ j \in \mathbb{Z}(1,n)}} \{|u(i,j)|\} \leq \frac{\sqrt{m+n+2}}{4} \cdot \sqrt{(v_2+1)^2 - 1} = e_2.$$

When $u \in S$ and $\Phi(u) \leq (v_2 + v_3)$, we obtain

$$\max_{\substack{i \in \mathbb{Z}(1,m) \\ j \in \mathbb{Z}(1,n)}} \{|u(i,j)|\} \leq \frac{\sqrt{m+n+2}}{4} \cdot \sqrt{(v_2 + v_3 + 1)^2 - 1} = e_3.$$

Define $y \in S$ by

$$y(i,j) = \begin{cases} d, & \text{if } (i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n), \\ 0, & \text{if } i = 0, j \in \mathbb{Z}(0, n+1) \text{ or } i = m+1, j \in \mathbb{Z}(0, n+1), \\ 0, & \text{if } j = 0, i \in \mathbb{Z}(0, m+1) \text{ or } j = n+1, i \in \mathbb{Z}(0, m+1). \end{cases}$$

Noticing that

$$\varphi(v_1) = \inf_{u \in \Phi^{-1}(-\infty, v_1)} \frac{\left(\sup_{u \in \Phi^{-1}(-\infty, v_1)} \Psi(u) \right) - \Psi(u)}{v_1 - \Phi(u)} \leq \Gamma(e_1),$$

$$\varphi(v_2) = \inf_{u \in \Phi^{-1}(-\infty, v_2)} \frac{\left(\sup_{u \in \Phi^{-1}(-\infty, v_2)} \Psi(u) \right) - \Psi(u)}{v_2 - \Phi(u)} \leq \Gamma(e_2),$$

and

$$\gamma(v_2, v_3) = \frac{\sup_{u \in \Phi^{-1}(-\infty, v_2+v_3)} \Psi(u)}{v_3} \leq \Omega(e_2, e_3),$$

we obtain

$$\begin{aligned} \beta(v_1, v_2) &= \inf_{u \in \Phi^{-1}(-\infty, v_1)} \sup_{\omega \in \Phi^{-1}[v_1, v_2]} \frac{\Psi(\omega) - \Psi(u)}{\Phi(\omega) - \Phi(u)} \\ &\geq \inf_{u \in \Phi^{-1}(-\infty, v_1)} \frac{\Psi(y) - \Psi(u)}{\Phi(y) - \Phi(u)} \\ &\geq \inf_{u \in \Phi^{-1}(-\infty, v_1)} \frac{\Psi(y) - \Psi(u)}{\Phi(y)} \\ &\geq \frac{\Psi(y)}{\Phi(y)} - \frac{\sup_{u \in \Phi^{-1}(-\infty, v_1)} \Psi(u)}{v_1} \\ &\geq \frac{1}{2\lambda(\bar{d})} - \Gamma(e_1). \end{aligned}$$

Therefore, we can see from (g_4) that $\varphi(v_1) < \beta(v_1, v_2)$, $\varphi(v_2) < \beta(v_1, v_2)$ and $\gamma(v_2, v_3) < \beta(v_1, v_2)$. This together with Lemma 2.3 finishes the proof.

Set

$$E^\infty = \limsup_{u \rightarrow +\infty} \frac{\sum_{j=1}^n \sum_{i=1}^m F((i,j), u)}{u}.$$

When $E^\infty = +\infty$, we agree to read $\frac{1}{E^\infty} = 0$.

Theorem 3.4. Suppose that there are two nonnegative sequences $\{\delta_t\}$ and $\{\omega_t\}$, with $\lim_{t \rightarrow +\infty} \omega_t = +\infty$, such that

$$1 + \delta_t^2 < \left(\frac{\rho(\omega_t)}{2} + 1 \right)^2, \quad t \in \mathbb{Z}(1),$$

and

$$D_{\infty} = \liminf_{t \rightarrow +\infty} \frac{\sum_{j=1}^n \sum_{i=1}^m \max_{|u| \leq \omega_t} F((i, j), u) - \sum_{j=1}^n \sum_{i=1}^m F((i, j), \delta_t)}{\sqrt{\frac{16\omega_t^2}{m+n+2} + 1 - (2n+2m)(\sqrt{1+(\delta_t)^2} - 1)}} < \frac{E^{\infty}}{2n+2m}.$$

Then, for all $\lambda \in \left(\frac{2n+2m}{E^{\infty}}, \frac{1}{D_{\infty}}\right)$, problem (H_{λ}^f) admits an unbounded sequence of solutions.

Proof. Let

$$\rho_t = (m+n)\rho(\omega_t), \quad t \in \mathbb{Z}(1).$$

We have

$$\max_{\substack{i \in \mathbb{Z}(1, m) \\ j \in \mathbb{Z}(1, n)}} \{|u(i, j)|\} \leq \frac{\sqrt{m+n+2}}{4} \cdot \sqrt{(\rho_t + 1)^2 - 1} = \omega_t,$$

for all $u \in S$, $t \in \mathbb{Z}(1)$. Obviously,

$$\varphi(\rho_t) \leq \inf_{u \in \Phi^{-1}(-\infty, \rho_t)} \frac{\sum_{j=1}^n \sum_{i=1}^m \max_{|u| \leq \omega_t} F((i, j), u) - \sum_{j=1}^n \sum_{i=1}^m F((i, j), u)}{\sqrt{\frac{16\omega_t^2}{m+n+2} + 1 - \Phi(u)}}.$$

Denote $\theta_t \in S$ by

$$\theta_t(i, j) = \begin{cases} \delta_t, & \text{if } (i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n), \\ 0, & \text{if } i = 0, j \in \mathbb{Z}(0, n+1) \text{ or } i = m+1, j \in \mathbb{Z}(0, n+1), \\ 0, & \text{if } j = 0, i \in \mathbb{Z}(0, m+1) \text{ or } j = n+1, i \in \mathbb{Z}(0, m+1). \end{cases}$$

Then we get $\Phi(\theta_t) = (2n+2m)(\sqrt{1+(\delta_t)^2} - 1)$. We know that

$$\varphi(\rho_t) \leq \frac{\sum_{j=1}^n \sum_{i=1}^m \max_{|u| \leq \omega_t} F((i, j), u) - \sum_{j=1}^n \sum_{i=1}^m F((i, j), \delta_t)}{\sqrt{\frac{16\omega_t^2}{m+n+2} + 1 - (2n+2m)(\sqrt{1+(\delta_t)^2} - 1)}}$$

and

$$\gamma \leq \liminf_{t \rightarrow +\infty} \varphi(\rho_t) \leq D_{\infty} < +\infty.$$

In the following, we distinguish two cases of $E^{\infty} = +\infty$ and $E^{\infty} < +\infty$ to finish the proof.

When $E^{\infty} = +\infty$, let $\{c_t\}$ be a sequence of positive numbers with $\lim_{t \rightarrow +\infty} c_t = +\infty$ such that

$\sum_{j=1}^n \sum_{i=1}^m F((i, j), c_t) \geq \frac{(3n+3m)c_t}{\lambda}$ for $t \in \mathbb{Z}(1)$. Setting a sequence $\{e_t\}$ in S with

$$e_t(i, j) = \begin{cases} c_t, & \text{if } (i, j) \in \mathbb{Z}(1, m) \times \mathbb{Z}(1, n), \\ 0, & \text{if } i = 0, j \in \mathbb{Z}(0, n+1) \text{ or } i = m+1, j \in \mathbb{Z}(0, n+1), \\ 0, & \text{if } j = 0, i \in \mathbb{Z}(0, m+1) \text{ or } j = n+1, i \in \mathbb{Z}(0, m+1), \end{cases}$$

we get

$$\begin{aligned} I_{\lambda}(e_t) &= (2n+2m)(\sqrt{1+(c_t)^2} - 1) - \lambda \sum_{j=1}^n \sum_{i=1}^m F((i, j), c_t) \\ &\leq (2n+2m)c_t - (3n+3m)c_t \\ &= -(n+m)c_t. \end{aligned}$$

Clearly, $\lim_{t \rightarrow +\infty} I_{\lambda}(e_t) = -\infty$.

When $E^{\infty} < +\infty$, fix $\varepsilon > 0$ such that $2n+2m - \lambda E^{\infty} + \lambda \varepsilon < 0$. Then there is a sequence of positive numbers $\{c_t\}$ such that $\lim_{t \rightarrow +\infty} c_t = +\infty$ and

$$(E^{\infty} - \varepsilon)c_t \leq \sum_{j=1}^n \sum_{i=1}^m F((i, j), c_t) \leq (E^{\infty} + \varepsilon)c_t, \quad t \in \mathbb{Z}(1).$$

We obtain

$$\begin{aligned} I_\lambda(e_t) &= (2n+2m)(\sqrt{1+(c_t)^2}-1) - \lambda \sum_{j=1}^n \sum_{i=1}^m F((i,j), c_t) \\ &\leq (2n+2m)c_t - \lambda(E^\infty - \varepsilon)c_t \\ &= (2n+2m - \lambda E^\infty + \lambda \varepsilon)c_t. \end{aligned}$$

Owing to $2n+2m - \lambda E^\infty + \lambda \varepsilon < 0$, we have $\lim_{t \rightarrow +\infty} I_\lambda(e_t) = -\infty$. Therefore, I_λ is unbounded from below. The proof is completed.

Remark 3.1. When $E^\infty = +\infty$, we have from Theorem 3.4 that for each $\lambda \in (0, \frac{1}{\hat{D}_\infty})$, problem (H_λ^f) admits an unbounded sequence of solutions.

Let

$$\hat{D}_\infty = \liminf_{s \rightarrow +\infty} \frac{\sqrt{m+n+2} \sum_{j=1}^n \sum_{i=1}^m \max_{|\xi| \leq s} F((i,j), \xi)}{4s}.$$

When $\hat{D}_\infty = 0$, we agree to read $\frac{1}{\hat{D}_\infty} = +\infty$. There is a sequence $\{\omega_t\}$ of positive numbers with $\lim_{t \rightarrow +\infty} \omega_t = +\infty$, such that

$$\lim_{t \rightarrow +\infty} \frac{\sum_{j=1}^n \sum_{i=1}^m \max_{|u| \leq \omega_t} F((i,j), u)}{\sqrt{\frac{16\omega_t^2}{m+n+2} + 1} - 1} = \hat{D}_\infty.$$

From Theorem 3.4, we get the following conclusion.

Corollary 3.2. When

$$\hat{D}_\infty < \frac{E^\infty}{2n+2m},$$

for each $\lambda \in (\frac{2n+2m}{E^\infty}, \frac{1}{\hat{D}_\infty})$, problem (H_λ^f) admits an unbounded sequence of solutions.

Remark 3.2. When $E^\infty = +\infty$, we see from Corollary 3.2 that for every $\lambda \in (0, \frac{1}{\hat{D}_\infty})$, problem (H_λ^f) admits an unbounded sequence of solutions. When $\hat{D}_\infty = 0$, according to Corollary 3.2, we have that for every $\lambda \in (\frac{2n+2m}{E^\infty}, +\infty)$, problem (H_λ^f) admits an unbounded sequence of solutions. Taken together, when $E^\infty = +\infty$ and $\hat{D}_\infty = 0$, we have that for every $\lambda > 0$, problem (H_λ^f) admits an unbounded sequence of solutions.

Corollary 3.3. Suppose that $f((i,j), 0) \geq 0$ for each $(i,j) \in \mathbb{Z}(1,m) \times \mathbb{Z}(1,n)$ and

$$\check{D}_\infty = \liminf_{s \rightarrow +\infty} \frac{\sqrt{m+n+2} \sum_{j=1}^n \sum_{i=1}^m \max_{0 \leq \xi \leq s} \int_0^\xi f((i,j), \tau) d\tau}{4s} < \frac{E^\infty}{2n+2m}.$$

Then for every $\lambda \in (\frac{2n+2m}{E^\infty}, \frac{1}{\check{D}_\infty})$, problem (H_λ^f) admits an unbounded sequence of positive solutions.

Proof. Set

$$\check{f}((i,j), s) = \begin{cases} f((i,j), s), & \text{if } s > 0, \\ f((i,j), 0), & \text{if } s \leq 0. \end{cases}$$

Clearly, we have $f((i,j), 0) \geq 0$, then

$$\max_{0 \leq |\xi| \leq s} \int_0^\xi \check{f}((i,j), \tau) d\tau = \max_{0 \leq \xi \leq s} \int_0^\xi f((i,j), \tau) d\tau,$$

for each $s \geq 0$. From Corollary 3.2 and Proposition 2.1, we have the desired conclusion.

4 An example

In this section, we give a feasible example to explain our result.

Example 4.1. Set $m = 2$, $n = 2$ and let

$$f((i, j), u) = f(u) = \frac{5}{4} + \cos\left(\frac{1}{4} \ln(|u| + 1)\right) - \frac{1}{4} \sin\left(\frac{1}{4} \ln(|u| + 1)\right),$$

for all $(i, j) \in \mathbb{Z}(1, 2) \times \mathbb{Z}(1, 2)$. This implies

$$F((i, j), u) = F(u) = \int_0^u f(\xi) d\xi = (1 + u) \left(\frac{5}{4} + \cos\left(\frac{1}{4} \ln(u + 1)\right) \right) - \frac{9}{4}, \quad u \geq 0.$$

We have $f(u) \geq 0$ for $u \geq 0$ and $F(u)$ is increasing on $u \in [0, +\infty)$. Put

$$a_t = e^{8t\pi} - 1 \quad \text{and} \quad b_t = e^{8t\pi+4\pi} - 1, \quad t \in \mathbb{Z}(1).$$

We get

$$\begin{aligned} F(a_t) &= e^{8t\pi} \cdot \left(\frac{5}{4} + \cos\left(\frac{1}{4} \ln(e^{8t\pi})\right) \right) - \frac{9}{4} \\ &= e^{8t\pi} \cdot \left(\frac{5}{4} + \cos(2t\pi) \right) - \frac{9}{4} \\ &= \frac{9}{4}(e^{8t\pi} - 1) \end{aligned}$$

and

$$\begin{aligned} F(b_t) &= e^{8t\pi+4\pi} \cdot \left(\frac{5}{4} + \cos\left(\frac{1}{4} \ln(e^{8t\pi+4\pi})\right) \right) - \frac{9}{4} \\ &= e^{8t\pi+4\pi} \cdot \left(\frac{5}{4} + \cos(2t\pi + \pi) \right) - \frac{9}{4} \\ &= \frac{1}{4}e^{8t\pi+4\pi} - \frac{9}{4}. \end{aligned}$$

Then, it holds that

$$\check{D}_\infty \leq \lim_{t \rightarrow +\infty} \frac{\sqrt{2+2+2} \times 2 \times 2 \times F(b_t)}{4b_t} = \frac{\sqrt{6}}{4}$$

and

$$E^\infty \geq \lim_{t \rightarrow +\infty} \frac{2 \times 2 \times F(a_t)}{a_t} = 9.$$

Thus, we have

$$\check{D}_\infty \leq \frac{\sqrt{6}}{4} < \frac{9}{2 \times 2 + 2 \times 2} \leq \frac{E^\infty}{2n + 2m}.$$

By Corollary 3.3, for each $\lambda \in (\frac{8}{9}, \frac{4}{\sqrt{6}})$, problem (H_λ^f) admits an unbounded sequence of positive solutions.

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