Research Article

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A variant of Clark's theorem and its applications for nonsmooth functionals without the global symmetric condition

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Abstract: We give a new non-smooth Clark's theorem without the global symmetric condition. The theorem can be applied to generalized quasi-linear elliptic equations with small continuous perturbations. Our results improve the abstract results about a semi-linear elliptic equation in Kajikiya [10] and Li-Liu [11].

Keywords: quasi-linear elliptic equations; non-smooth Clark's theorem; essential values

MSC: 35J20; 35J62; 35B45

1 Introduction

The Clark's theorem is an important result in critical point theory (see [4, 8]). Using this theorem for the even coercive functional, the existence of a sequence of negative critical values tending to 0 is obtained. Specifically, in [8], Heinz obtained a variant of the Clark theorem as follows:

Clark Theorem (see [21]). Let X be a Banach space and assume $J \in C^1(X)$ satisfies, $J(0) = 0$, the (P-S) conditions, is bounded from below and even. For any positive integer k , there exists a k -dimensional subspace X_k of X and $\beta_k > 0$ such that

$$\sup_{X_k \cap S_{\beta_k}} J < 0, \text{ here } S_{\beta} = \{u \in X : \|u\| = \beta\},$$

then there exists a sequence of negative critical values for J tending to 0.

This Clark's theorem was improved by Kajikiya in [9] and Liu-Wang in [18], under the same conditions as in the above Clark's theorem, they showed the critical points of J also tend to 0 in X . We remark that Liu-Wang also studied the existence of periodic solutions for sub-linear Hamiltonian systems and showed a new version of the Clark's theorem for non-smooth functionals. Very recently, in [3] Chen-Liu-Wang showed a version of the Clark's theorem without the Palais-Smale conditions ((P-S) conditions). And then they studied the existence of infinitely many solutions for a degenerate quasi-linear elliptic operator and a second-order Hamiltonian system via their abstract theory.

However, all those versions of Clark's theorem references to above rely on the symmetric condition about the Euler-Lagrange functional. In [10], Kajikiya established the existence of infinitely many critical points about C^1 functional without the global symmetric condition. As applications, they obtained the existence of infinitely many solutions of the sub-linear elliptic equation with a small perturbation. We note that since the

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perturbation term only satisfies continuity, the Euler-Lagrange functional corresponding to the sub-linear elliptic equation may not be even.

But for some quasi-linear elliptic problems with continuous perturbations, here the problems do not have a C^1 variational formulation and do not satisfy the global symmetric condition. For these reasons, both the classical Clark's Theorem in [8] and the abstract result in [10] cannot be applied directly. In this situation, we need develop a new non-smooth variational method based on the Clark's theorem.

In order to state the new variant of Clark's theorem, we firstly give the following assumption:

Condition (I). Let X be an infinite dimensional Banach space and E be dense subspace of X . For any $\varepsilon \in [0, 1]$, let I_ε be a continuous functional defined on X which is E -differentiable. I_ε satisfies $(I_1) - (I_5)$ below.

(I_1): For $u \in X$, $I_\varepsilon(u)$ is bounded from below;

(I_2): For $u \in X$, $|I_\varepsilon(u) - I_0(u)| \leq \psi(\varepsilon)$, where $\psi \in C([0, 1], \mathbb{R})$ and $\psi(0) = 0$;

(I_3): $I_\varepsilon(u)$ satisfies the (P-S) conditions uniformly on ε ;

(I_4): $I_0(u)$ is odd on u ;

(I_5): For any $u \in X \setminus \{0\}$ there exists a unique $t(u) > 0$ such that

$$I_0(tu) < 0 \text{ if } 0 < |t| < t(u) \text{ and } I_0(tu) \geq 0 \text{ if } |t| \geq t(u).$$

In order to explain some concepts in Condition (I), we recall some definitions as follows:

Definition 1.1. A continuous functional J is said to be E -differentiable if

(1) for all $u \in X$ and $\varphi \in E$ the derivative of J in the direction φ at u exists and will be denoted by $\langle D_E J(u), \varphi \rangle$:

$$\langle D_E J(u), \varphi \rangle = \lim_{t \rightarrow 0^+} \frac{1}{t} (J(u + t\varphi) - J(u)).$$

(2) The map $(u, \varphi) \mapsto \langle D_E J(u), \varphi \rangle$ satisfies:

(i) $\langle D_E J(u), \varphi \rangle$ is linear in $\varphi \in E$,

(ii) $\langle D_E J(u), \varphi \rangle$ is continuous in u , that is $\langle D_E J(u_n), \varphi \rangle \rightarrow \langle D_E J(u), \varphi \rangle$ as $u_n \rightarrow u$ in X .

Definition 1.2. The slope of an E -differentiable functional J at u denoted by $|D_E J(u)|$ is an extended number in $[0, \infty]$:

$$|D_E J(u)| = \sup \{ \langle D_E J(u), \varphi \rangle : \varphi \in E, \|\varphi\| = 1 \}.$$

A point $u \in X$ is said to be a critical point of J at the level c if $|D_E J(u)| = 0$ and $J(u) = c$.

Definition 1.3. $I_\varepsilon(u)$ is said to satisfy (P-S) conditions uniformly on ε if a sequence $(\varepsilon_n, u_n) \in [0, 1] \times X$ satisfies that

$$\sup_{n \in \mathbb{N}} |I_{\varepsilon_n}(u_n)| < \infty \text{ and } |D_E I_{\varepsilon_n}(u_n)| \text{ converges to zero,}$$

then (ε_n, u_n) has a convergent subsequence.

We now introduce the variant of Clark's theorem.

Theorem 1.1. Assume that Condition (I) holds. Denote

$$S^k := \{x \in \mathbb{R}^{k+1} : |x| = 1\},$$

$$\mathcal{A}_k := \{\alpha \in C(S^k, X) : \alpha \text{ is odd}\},$$

$$d_k := \inf_{\alpha \in \mathcal{A}_k} \max_{x \in S^k} I_0(\alpha(x)).$$

Let $k \in \mathbb{N} \setminus \{0\}$ satisfying $d_k < d_{k+1}$. Then there exist two constants $\varepsilon_{k+1}, c_{k+1}$ such that $0 < \varepsilon_{k+1} \leq 1$, $d_{k+1} \leq c_{k+1} < -\psi(\varepsilon)$ for $\varepsilon \in [0, \varepsilon_{k+1}]$ and for any $\varepsilon \in [0, \varepsilon_{k+1}]$, I_ε has a critical value in the interval $[d_{k+1} - \psi(\varepsilon), c_{k+1} + \psi(\varepsilon)]$.

From the above abstract theorem, we have the following corollary.

Corollary 1.1. Assume the condition (I) holds. Then, for any $k \in \mathbb{N}$, there exists an $\varepsilon_k > 0$ such that when $0 \leq \varepsilon \leq \varepsilon_k$, functional I_ε has at least k distinct critical points with negative critical values.

Secondly, we give another direct proof about Corollary 1.1 in Section 3. Our method is based on the approach developed by Degiovanni-Lancelotti [6]. By this approach, Li-Liu [11] considered a similar perturbation for semilinear elliptic equation in a bounded domain. Since our problems do not have a C^1 variational formulation, the method in [11] cannot be applied directly.

Finally, as an application of Theorem 1.1 and Corollary 1.1, we consider the following quasi-linear problem

$$\begin{cases} -\sum_{i,j=1}^N D_j(a_{ij}(x, u)D_i u) + \frac{1}{2} \sum_{i,j=1}^N D_s a_{ij}(x, u)D_i u D_j u + V(x)u \\ = f(x, u) + \varepsilon K(x)g(u), \quad x \in \mathbb{R}^N, \\ u \in W^{1,2}(\mathbb{R}^N), \end{cases} \quad (1.1)$$

where $N \geq 3$, $D_s a_{ij}(x, u) = \frac{\partial}{\partial u} a_{ij}(x, u)$, $D_i u = \frac{\partial}{\partial x_i} u(x)$ and $D_j u = \frac{\partial}{\partial x_j} u(x)$. Denoted $F(x, u)$ by $\int_0^u f(x, s)ds$. For given $a > 0$, let $f(x, u) \in C(\mathbb{R}^N \times [-a, a])$ and satisfy $(f_1) - (f_4)$ below

- (f_1) : $f(x, -u) = -f(x, u)$ for $x \in \mathbb{R}^N$ and $|u| \leq a$;
- (f_2) : $uf(x, u) - 2F(x, u) < 0$ when $0 < |u| < a$ and $x \in \mathbb{R}^N$;
- (f_3) : $|f(x, u)| \leq C|u|^{r-1}$ for $x \in \mathbb{R}^N$ and $|u| \leq a$, where $r \in [1, 2)$;
- (f_4) : $\lim_{u \rightarrow 0} (\min_{x \in \mathbb{R}^N} u^{-2} F(x, u)) = \infty$.

Besides, $a_{ij}(x, u)$ satisfies

(a): $a_{ij} \in C^1(\mathbb{R}^N \times [-a, a], \mathbb{R})$, a_{ij} is even, $a_{ij} = a_{ji}$, for all $x \in \mathbb{R}^N$ and $|s| \leq a$, $D_s a_{ij}(x, s) \geq 0$ and there exist C_0, C_1 such that $C_0 |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq C_1 |\xi|^2$.

And $V, K \in C(\mathbb{R}^N, \mathbb{R})$ satisfy:

(V): $V(x)^{-1} \in L^{r/(2-r)}(\mathbb{R}^N)$ and there exists a constant V_0 such that

$$0 < V_0 \leq V(x), \text{ for all } x \in \mathbb{R}^N;$$

(K): $K(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Quasi-linear elliptic equation of the form (1.1) contains the quasi-linear Schrödinger equation

$$\begin{cases} -\Delta u - \frac{1}{2} \Delta(u^2)u + V(x)u = f(x, u) + \varepsilon K(x)g(u), \quad x \in \mathbb{R}^N, \\ u \in W^{1,2}(\mathbb{R}^N), \end{cases} \quad (1.2)$$

which corresponds to the special case $a_{ij}(x, t) = (1 + 2t^2)\delta_{ij}$. The problem (1.2) arose in several models of physical phenomena, such as superfluid films in plasma physics (see e.g. [1, 2, 19]). And it has received considerable attention in mathematical analysis in the last twenty years (see [5, 12, 20, 22, 23]).

Since the variational functional of the quasi-linear problem (1.1) is merely continuous. Even though $\varepsilon = 0$, limited work has been done in the general form of the quasi-linear problem (1.1). In [13], a least energy sign-changing solution of (1.1) with $\varepsilon = 0$ is obtained via the Nehari manifold method. Multiple solutions for (1.1) with $\varepsilon = 0$ was first proved in [16], where a 4-Laplacian perturbation term is added to (1.1) so that the associated functionals are well-defined on $W_0^{1,4}(\Omega)$. Then in [15], they obtained multiplicity of sign-changing solutions for general form of the quasi-linear problem (1.1) with $\varepsilon = 0$. This idea is further developed in [17], where treated the critical exponent case giving new existence results. The previous mentioned four results yield only with the power range $f(x, u) = |u|^{s-2}u$ ($4 < s \leq 2 \cdot 2^*$). The case $1 < s < 2$ is investigated in [18], by using the variants of Clark's theorem, the quasi-linear problem (1.1) with $\varepsilon = 0$ has a sequence of solutions with L^∞ -norms tending to zero. For $2 < s < 4$, less results are known, by using the perturbation approach and the invariant sets approach, in [7] Jing-Liu-Wang showed the problem (1.1) with $\varepsilon = 0$ has at least six solutions.

When $\varepsilon \neq 0$, under our assumptions on g , the Euler-Lagrange functional corresponding to (1.1) may be not even with respect to u . In Section 4, for $|\varepsilon|$ small enough, implying Theorem 1.1, we prove that the generalized quasi-linear elliptic problem with small perturbations (1.1) has infinitely many solutions.

Next, we give our second main result.

Theorem 1.2. Let $g(u)$ be continuous on $[-a, a]$. Assume that conditions (a) , (V) , (K) and (f_1) – (f_4) hold. Then, for any $k \in \mathbb{N}$ and any $b > 0$, there exists an $\varepsilon_0(k, b) > 0$ such that when $|\varepsilon| \leq \varepsilon_0(k, b)$ the problem (1.1) possesses at least k distinct solutions whose L^∞ -norms are less than b .

Remark 1.1. We do not need any growth condition on the nonlinear term f at infinity. Thus there exist some supercritical growth examples satisfying our conditions (f_1) – (f_4) :

1. $f(x, u) = |u|^{r-1} \operatorname{sgn} u$ with $r \in (1, 2)$;

2. $f(x, u) = |u|^{r-1} \operatorname{sgn} u + |u|^{i-1} \operatorname{sgn} u$, where $1 < r < 2$, $i \geq 2^* := \frac{2N}{N-2}$.

Remark 1.2. Since our theorem does not need any growth or odd condition on the perturbation term g . The Euler-Lagrange functional $I_\varepsilon(u)$ may not satisfy symmetrical condition. By applying our abstract results, for $|\varepsilon|$ small enough, $I_\varepsilon(u)$ still possesses infinite many critical points.

This paper is organized as follows. In Section 2, we give the proofs of the abstract result Theorem 1.1 and Corollary 1.1. Then we give the another direct proof about Corollary 1.1 in Section 3. As an application of Theorem 1.1, we prove Theorem 1.2 in Sections 4.

In what follows, C denotes positive generic constants.

2 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. To this end, we use the following topological lemma, which is an analogue of a result of [10]. Throughout this section, X is an infinite dimensional Banach space and E be dense subspace of X . For any $\varepsilon \in [0, 1]$, functionals I_ε are an E -differentiable functional defined on X and satisfies (I_1) – (I_5) .

Definition 2.1. Let M be a compact subset of X such that $0 \notin M$. Denote

(1) $\mathbb{R}M := \{ru : r \in \mathbb{R}, u \in M\}$;

(2) $O := \{u \in X : I_0(u) < 0\}$.

Lemma 2.1. [Topological Lemma] (1) There exists $v \in X$ such that

$$\|v\| = 1 \text{ and } v \notin \mathbb{R}M.$$

(2) Let v be mentioned in Lemma 2.1-(1), there exists a $\delta_0 > 0$ such that

$$tu + (1-t)\delta v \in O \text{ for } u \in M, 0 \leq t \leq 1 \text{ and } 0 < \delta < \delta_0.$$

Proof. Notice that condition (I_5) means that

$$O := \{u \in X : I_0(u) < 0\} = \{tu : u \in X \setminus \{0\}, 0 < |t| < t(u)\},$$

which plays a key role in the proof of this lemma. The details in the proof see also [10]. □

Recall the definitions in Section 1, as follows:

$$S^k := \{x \in \mathbb{R}^{k+1} : |x| = 1\}, \mathcal{A}_k := \{\alpha \in C(S^k, X) : \alpha \text{ is odd}\},$$

and

$$d_k := \inf_{\alpha \in \mathcal{A}_k} \max_{x \in S^k} I_0(\alpha(x)). \quad (2.1)$$

Using these definitions and the condition (I_5) , we have the following lemma.

Lemma 2.2. For any $k \in \mathbb{N}$, there exists an $\alpha_k \in \mathcal{A}_k$ such that

$$\max_{x \in S^k} I_0(\alpha_k(x)) < 0.$$

Proof. Let k be a fixed positive integer. For any $\alpha \in A_k$, by the condition (I_5) , there exists $t(\alpha)$ such that

$$I_0\left(\frac{t(\alpha)}{2}\alpha\right) < 0.$$

Then we can take $\alpha_k = \frac{t(\alpha)}{2}\alpha$, which satisfies

$$\alpha_k(x) \in \mathcal{A}_k \text{ and } \max_{x \in S^k} I_0(\alpha_k(x)) < 0.$$

□

For any $k \in \mathbb{N}$, it follows from Lemma 2.2 and the definition of d_k that

$$d_k \leq \max_{x \in S^k} I_0(\alpha_k(x)) < 0 \text{ and } d_k \leq d_{k+1}.$$

Next we suppose that there exists a positive integer k such that $d_k < d_{k+1} < 0$. Define

$$S_+^{k+1} := \{(x_1, \dots, x_{k+2}) : \sum_{i=1}^{k+2} x_i^2 = 1, x_{k+2} \geq 0\},$$

$$S^k := \{(x_1, \dots, x_{k+2}) : \sum_{i=1}^{k+2} x_i^2 = 1, x_{k+2} = 0\},$$

$$\mathcal{H}_{k+1} := \{h \in C(S_+^{k+1}, X) : h \text{ satisfies } (H_1), (H_2)\}$$

(H_1) $h(-x) = -h(x)$ for $x \in S^k$.

(H_2) $I_0(h(x)) < d_k + r$ for $x \in S^k$, where the constant $r > 0$ small enough such that $d_k + r < d_{k+1}$.

Using the above definitions and Lemma 2.1, we show the next fundamental lemma holds. We use the same method as in Kajikiya [10]. For the completeness of the article, we give a detailed proof as follows.

Lemma 2.3. *There exists an $f_{k+1} \in \mathcal{A}_{k+1} \cap \mathcal{H}_{k+1}$ such that*

$$\max_{x \in S^{k+1}} I_0(f_{k+1}(x)) < 0.$$

Proof. Let d_k and $r > 0$ be defined in (H_2) . From the definition of d_k , we can choose an $\alpha \in \mathcal{A}_k$ satisfying

$$I_0(\alpha(x)) < d_k + r < 0 \text{ for } x \in S^k. \quad (2.2)$$

Then take $M = \alpha(S^k)$. It follows from (2.2) that M is compact and $0 \notin M$. From Lemma 2.1-(2), there exist $v \in X$ and $\delta_0 > 0$ such that

$$t\alpha(x) + (1-t)\delta_0 v \in O \text{ for } x \in S^k, 0 \leq t \leq 1 \text{ and } 0 < \delta < \delta_0. \quad (2.3)$$

This means

$$I_0(t\alpha(x) + (1-t)\frac{\delta_0}{2}v) < 0 \text{ for } x \in S^k, 0 \leq t \leq 1.$$

Next, we denote

$$x = (x_1, \dots, x_{k+1}, x_{k+2}) = (x', x_{k+2}),$$

$$x' = (x_1, \dots, x_{k+1}), \text{ and } |x'| = \left(\sum_{i=1}^{k+1} x_i^2 \right)^{1/2}.$$

Then for $x \in S_+^{k+1}$, we take

$$f_{k+1}(x) = \begin{cases} |x'| \alpha(x'/|x'|) + \frac{\delta_0}{2}(1-|x'|)v, & \text{if } x' \neq 0, \\ \frac{\delta_0}{2}v, & \text{if } x' = 0. \end{cases}$$

We only need extend the continuous function $f_{k+1}(x)$ onto S^{k+1} as an odd mapping $\tilde{f}_{k+1}(x)$. Then $\tilde{f}_{k+1} \in \mathcal{A}_{k+1} \cap \mathcal{H}_{k+1}$ and (2.3) imply that

$$I_0(\tilde{f}_{k+1}(x)) = I_0(f_{k+1}(x)) < 0, \text{ for } x \in S_+^{k+1}.$$

□

Lemma 2.4. Each d_k is a critical value of $I_0(u)$ and

$$d_k \leq d_{k+1} < 0 \text{ for } k \in \mathbb{N}, \lim_{k \rightarrow \infty} d_k = 0.$$

Before completing the proof of Lemma 2.4, we need the following non-smooth deformation lemma and a notion of genus (see [14, 21]):

Lemma 2.5 (The First Deformation Lemma). [14] Assume J is an E -differentiable functional defined on X and satisfies the (P-S) conditions. For some $c \in \mathbb{R}$, let N be a neighborhood of $K_c = \{u \in X : J(u) = c, |D_E J(u)| = 0\}$. Then there exists a deformation map $\sigma : [0, 1] \times X \rightarrow X$ and $\bar{\delta} > \delta > 0$ such that

- (i) If $J(u) \leq c - \bar{\delta}$, then $\sigma(t, u) = u$.
- (ii) If $J(u) \leq c + \delta$ and $u \notin N$, then $J(\sigma(1, u)) \leq c - \delta$.
- (iii) If $J(\cdot)$ is even functional, then $\sigma(t, \cdot)$ is odd mapping.
- (iv) If c is a regular value of J with $J(u) \leq c + \delta$, then $J(\sigma(1, u)) \leq c - \delta$.

Definition 2.2. Denote $\Gamma = \{A \subset X \setminus \{0\} : A \text{ is closed, } -A = A\}$. Let $A \in \Gamma$, define a genus $\gamma(A)$ of A by

$$\gamma(A) = \min\{n \in \mathbb{N} : \text{there exists an odd, continuous } \phi : A \rightarrow \mathbb{R}^n \setminus \{0\}\},$$

If such a minimum does not exist, then we define $\gamma(A) = +\infty$. Moreover, set $\gamma(\emptyset) = 0$. For all $k \in \mathbb{N}$, let $\Gamma_k = \{A \in \Gamma : \gamma(A) \geq k\}$. We define

$$e_k = \inf_{A \in \Gamma_k} \sup_{u \in A} I_0(u).$$

Remark 2.1. Due to the Borsuk-Ulam theorem, we have $\gamma(S^k) = k + 1$. Then from the property of genus, for any $\alpha \in \mathcal{A}_k$ we have $\gamma(\alpha(S^k)) \geq k + 1$. Besides, since $d_k < 0$, we can suppose $0 \notin \alpha(S^k)$. Thus, we deduce $\mathcal{A}_k \subset \Gamma_k$ and then $e_k \leq d_k < 0$.

With the help of lemma 2.5, and using the standard method as in Rabinowitz's argument [21] (see Proposition 9.33), we show the following lemma holds.

Lemma 2.6. For all $k \in \mathbb{N}$, e_k is a critical value of $I_0(u)$ and $e_k \rightarrow 0$ as $k \rightarrow \infty$.

Proof. Under the condition (I_1) , we know that

$$e_k > -\infty.$$

Remark 2.1 means $e_k \leq d_k < 0$. Since $\Gamma_{k+1} \subset \Gamma_k$, it follows that $e_k \leq e_{k+1}$. Then from the above facts, we have

$$\lim_{k \rightarrow \infty} e_k := e_\infty \leq 0.$$

In contradiction to Lemma 2.6, we suppose that

$$e_\infty < 0. \tag{2.4}$$

Next we use the following notation:

$$K := \{u \in X : D_E I_0(u) = 0, I_0(u) \leq e_\infty\}.$$

By (P-S) conditions on I_0 , we have K is compact. It is clear that K is symmetric. Due to (2.4), we have $0 \notin K$. Then by the properties of genus, there exists a $\delta' > 0$ small enough such that

$$\gamma(\overline{N_{\delta'}(K)}) = \gamma(K) := i < +\infty,$$

where $N_{\delta'}(K) = \{u \in X : \text{dist}(u, K) < \delta'\}$.

By Lemma 2.5 with $c = e_\infty$, there exists $\delta > 0$ such that

$$I_0(\sigma(1, u)) \leq e_\infty - \delta, \text{ if } I_0(u) \leq e_\infty + \delta \text{ and } u \notin N_{\delta'}(K). \quad (2.5)$$

Now we fix an integer $j \in \mathbb{N}$ such that

$$e_\infty - \delta < e_j. \quad (2.6)$$

By the definition of e_{i+j} , there exists $P \in \Gamma_{i+j}$ such that

$$\sup_{u \in P} I_0(u) < e_{i+j} + \delta. \quad (2.7)$$

Let $Q = \overline{P \setminus N_{\delta'}(K)}$, then from (2.5) and (2.7), we have

$$I_0(\sigma(1, u)) \leq e_\infty - \delta \text{ for } u \in Q. \quad (2.8)$$

Since $\gamma(\sigma(1, Q)) \geq \gamma(Q) \geq \gamma(P) - \gamma(\overline{N_{\delta'}(K)}) \geq j$, we have

$$\sigma(1, Q) \in \Gamma_j.$$

From (2.6), (2.8) and the above fact, we get

$$e_\infty - \delta < e_j \leq \sup_{u \in Q} I_0(\sigma(1, u)) \leq e_\infty - \delta.$$

This is a contradiction. $\square$

The proof of Lemma 2.4. From the definition of d_k , we know that for any $\epsilon > 0$, there exists an $\alpha_k \in \mathcal{A}_k$ such that

$$\sup_{x \in S^k} I_0(\alpha_k(x)) < d_k + \epsilon.$$

Here we fix $\epsilon = \delta$ which is mentioned in Lemma 2.5. Assume to the contrary that the conclusions are false. d_k is a regular value. Using Lemma 2.5 with $c = d_k$, there exists $\sigma : [0, 1] \times X \rightarrow X$ satisfying

$$\sup_{x \in S^k} I_0(\sigma(1, \alpha_k(x))) < d_k - \delta. \quad (2.9)$$

On the other hand, it is straightforward to show that $\sigma(1, \alpha_k) \in \mathcal{A}_k$. Then by the definition of d_k , it implies

$$\sup_{x \in S^k} I_0(\sigma(1, \alpha_k(x))) \geq d_k,$$

which contradicts (2.9). Hence d_k is a critical value of I_0 .

Next we shall prove that $d_k \rightarrow 0$ as $k \rightarrow \infty$. Due to $e_k \leq d_k < 0$, it is enough to show the convergence of e_k to zero. This fact follows from Lemma 2.6. The proof is complete. $\square$

Now, we are ready to prove the variant of Clark's theorem.

The proof of Theorem 1.1. Fixed a positive integer k , such that

$$d_k < d_{k+1} < 0 \text{ and } d_k + r < d_{k+1}, \text{ for some } r > 0$$

From Lemma 2.3 and $I_0(u)$ is even on u , we have

$$c_{k+1} := \max_{S_+^{k+1}} I_0(f_{k+1}(x)) = \max_{S^{k+1}} I_0(f_{k+1}(x)) < 0.$$

Choose $\varepsilon_{k+1} \in (0, 1]$ so small that

$$d_k + r + 2\psi(\varepsilon) < d_{k+1} \text{ and } c_{k+1} + \psi(\varepsilon) < 0 \text{ for } \varepsilon \in [0, \varepsilon_{k+1}].$$

For all $\varepsilon \in [0, \varepsilon_{k+1}]$, define

$$b_{k+1}(\varepsilon) := \inf_{h \in \mathcal{H}_{k+1}} \max_{S_+^{k+1}} I_\varepsilon(h(x)).$$

On one hand, from condition (I_1) , it implies

$$b_{k+1}(\varepsilon) \leq \max_{S_+^{k+1}} I_0(f_{k+1}(x)) + \psi(\varepsilon) = c_{k+1} + \psi(\varepsilon) < 0. \quad (2.10)$$

On the other hand, for any $h \in \mathcal{H}_{k+1}$ fixed, denote the odd extension of h on S^{k+1} by $\bar{h}$, then $\bar{h} \in \mathcal{A}_{k+1}$. Since $I_0(u)$ is even, it holds that

$$\max_{S_+^{k+1}} I_0(h(x)) = \max_{S^{k+1}} I_0(\bar{h}(x)).$$

Then

$$\begin{aligned} \max_{S_+^{k+1}} I_\varepsilon(h(x)) &\geq \max_{S_+^{k+1}} I_0(h(x)) - \psi(\varepsilon) \\ &= \max_{S^{k+1}} I_0(\bar{h}(x)) - \psi(\varepsilon) \\ &\geq d_{k+1} - \psi(\varepsilon). \end{aligned}$$

Taking the infimum on $h \in \mathcal{H}_{k+1}$ in the above inequality, we have

$$b_{k+1}(\varepsilon) \geq d_{k+1} - \psi(\varepsilon) > d_k + r + \psi(\varepsilon). \quad (2.11)$$

Then, from (2.10) and (2.11), it implies

$$d_{k+1} - \psi(\varepsilon) \leq b_{k+1}(\varepsilon) \leq c_{k+1} + \psi(\varepsilon).$$

Next we shall prove $b_{k+1}(\varepsilon)$ is critical value of I_ε . Assume to the contrary that the conclusions are false. $b_{k+1}(\varepsilon)$ is a regular value. Then by Lemma 2.5 with $c = b_{k+1}(\varepsilon)$ and $c - \bar{\delta} = d_k + r + \psi(\varepsilon)$, we have an $\delta \in (0, \bar{\delta})$ and $\sigma : [0, 1] \times X \rightarrow X$ satisfying the conditions below:

(i) If $I_\varepsilon(u) \leq b_{k+1}(\varepsilon) + \delta$, then $I_\varepsilon(\sigma(1, u)) \leq b_{k+1}(\varepsilon) - \delta$.

(ii) If $I_\varepsilon(u) \leq d_k + r + \psi(\varepsilon)$, then $\sigma(1, u) = u$.

By the definition of $b_{k+1}(\varepsilon)$, there exists an $h_0 \in \mathcal{H}_{k+1}$ such that

$$\max_{S_+^{k+1}} I_\varepsilon(h_0(x)) < b_{k+1}(\varepsilon) + \delta.$$

By the deformation property (i), we have

$$\max_{S_+^{k+1}} I_\varepsilon(\sigma(1, h_0(x))) < b_{k+1}(\varepsilon) - \delta. \quad (2.12)$$

Since $h_0 \in \mathcal{H}_{k+1}$, we get

$$I_\varepsilon(h_0(x)) \leq I_0(h_0(x)) + \psi(\varepsilon) < d_k + r + \psi(\varepsilon) \text{ for } x \in S^k.$$

From this, we have

$$\sigma(1, h_0(x)) = h_0(x) \text{ for } x \in S^k.$$

Thus $\sigma(1, h_0(x))$ satisfies (H_1) and (H_2) and then

$$\sigma(1, h_0(x)) \in \mathcal{H}_{k+1}.$$

Then, by the definition of $b_{k+1}(\varepsilon)$, we obtain

$$\max_{S_+^{k+1}} I_\varepsilon(\sigma(1, h_0(x))) \geq b_{k+1}(\varepsilon),$$

which contradicts (2.12). □

The proof of Corollary 1.1. From Theorem 1.1, there exist sequences $\{\varepsilon_{k+1}\}$, $\{d_{k+1}\}$ and $\{b_{k+1}(\varepsilon)\}$ such that $b_{k+1}(\varepsilon)$ is a critical value of I_ε for $\varepsilon \in [0, \varepsilon_{k+1}]$ and

$$d_{k+1} - \psi(\varepsilon) \leq b_{k+1}(\varepsilon) \leq c_{k+1} + \psi(\varepsilon) < 0. \quad (2.13)$$

For any $\delta > 0$ and $k \in \mathbb{N}$ fixed. Let $n(i)$ ($i \in \{1, 2, \dots, k\}$) be a increasing positive integers sequence, such that

$$-\delta < d_{n(1)} \text{ and } d_{n(i)} \leq c_{n(i)} < d_{n(i+1)} \text{ for } i \in \{1, 2, \dots, k\}.$$

Then for all $i \in \{1, 2, \dots, k\}$, there exists $\varepsilon_k > 0$ small enough such that

$$-\delta < d_{n(1)} - \psi(\varepsilon) \text{ and } c_{n(i)} + \psi(\varepsilon) < d_{n(i+1)} - \psi(\varepsilon) \text{ on } \varepsilon \in [0, \varepsilon_0]. \quad (2.14)$$

Combining (2.13) and (2.14), for all $\varepsilon \in [0, \varepsilon_k]$, we have

$$-\delta < b_{n(1)}(\varepsilon) < b_{n(2)}(\varepsilon) < \dots < b_{n(k)}(\varepsilon) < 0,$$

which means I_ε has at least k distinct critical values. $\square$

3 The another proof of Corollary 1.1

In this section, we shall give the another proof of Corollary 1.1. To this end, let us recall some notions and facts from Degiovanni-Lancelotti (see [6]). And set $I^b = \{u \in X : I(u) \leq b\}$.

Definition 3.1. A real number c is said to be an essential value of I , if for every $\varepsilon > 0$ there exist $a, b \in (c - \varepsilon, c + \varepsilon)$ with $a < b$ such that the pair (I^b, I^a) is not trivial.

Definition 3.2. Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $a \leq b$. The pair (I^b, I^a) is said to be trivial, if for any neighborhood $[\alpha', \alpha'']$ of a and $[\beta', \beta'']$ of b , there exist two closed subsets A and B such that $I^{\alpha'} \subset A \subset I^{\alpha''}$, $I^{\beta'} \subset B \subset I^{\beta''}$ and such that A is a strong deformation retract of B .

Let I be a merely continuous functional in X . The next lemma shows the main property of essential values.

Lemma 3.1. [see Theorem 2.6 in [6]] Let c be an essential value of I . Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $J \in C(X, \mathbb{R})$ with $\sup_{u \in X} |J(u) - I(u)| < \delta$ admits an essential value in $(c - \varepsilon, c + \varepsilon)$.

In what follows, let I be a E -differentiable functional defined on X , then by the following lemma, we show the relationship between essential values and critical values of I .

Lemma 3.2. Let functional I be E -differentiable functional defined on X and c be an essential value of I . If (P-S) conditions hold for I , then c is a critical value of I .

Proof. By contradiction, let us assume that c is not a critical value of I . By (P-S) conditions of I , there exist positive constants ε and d such that

$$|D_E I(u)| \geq d > 0, \text{ for all } u \in \{u : c - \varepsilon < I(u) < c + \varepsilon\}.$$

Then let $a, b \in (c - \varepsilon, c + \varepsilon)$ with $a < b$. By Lemma 2.5, there exists a deformation map $\sigma : [0, 1] \times X \rightarrow X$ such that

$$\sigma(0, u) = u, \quad I(\sigma(t, u)) \leq I(u),$$

$$\text{if } u \in I^b, \text{ then } \sigma(1, u) \in I^a, \text{ and if } u \in I^a, \text{ then } \sigma(t, u) = u.$$

This means I^a is a strong deformation retract of I^b , so that the pair (I^b, I^a) is trivial, which is a contradiction since c is an essential value of I . $\square$

The idea of the proof of Corollary 1.1 is taken from [11]. But our abstract result extends Theorem 1.2 in [11] by relaxing the C^1 assumption of I . Moreover, our abstract result is powerful in application such as quasi-linear elliptic problems (see Section 4).

The proof of Corollary 1.1. Set $S^\infty = \{u \in X : \|u\| = 1\}$ and $O = \{tu : 0 < t < t(u), u \in S^\infty\}$, where for $u \in S^\infty$, $t(u)$ is mentioned in condition (I_5) . From the above definitions, we first show O is contractible, which will be used in the sequel. Define $G = \{t(u)u/2 : u \in S^\infty\}$. Consider $g : S^\infty \rightarrow G$ as $g(u) = t(u)u/2$. Then the inverse of g is given by $g^{-1} : G \rightarrow S^\infty$, $g^{-1}(u) = u/\|u\|$ and both g and g^{-1} are continuous. It implies G is homeomorphic to S^∞ . On the other hand, G is a strong deformation retract of O . Thus O is contractible since S^∞ is contractible.

Next, for $k \in \mathbb{N}$, let S^{k-1} be the unit sphere in $\mathbb{R}^k$ and define

$$H_k = \{h \in C(S^{k-1}, O) : h \text{ is odd}\}$$

and

$$p_k = \inf_{h \in H_k} \max_{x \in S^{k-1}} I_0(h(x)).$$

It is easy to see $p_1 \leq p_2 \leq \dots \leq p_k$. For any $k \in \mathbb{N}$, by Lemma 3.1 and 3.2, there exists $\varepsilon_k > 0$ such that if $0 \leq \varepsilon \leq \varepsilon_k$ then

$$p_k \leq \max_{x \in S^{k-1}} I_0(g_k(x)) < 0.$$

Recall the definition of e_k in Definition 2.2. By $\{h(S^{k-1}) : h \in H_k\} \subset \gamma_k$, it implies that $p_k \geq e_k$. From Lemma 2.6, we see that $e_k \rightarrow 0$ as $k \rightarrow \infty$. So $p_k \rightarrow 0$ as $k \rightarrow \infty$.

Finally, we set $\mathcal{E} = \{c < 0 : c \text{ is an essential value of } I_0\}$. It suffices to prove $\mathcal{E} \neq \emptyset$ and $\sup \mathcal{E} = 0$. By contradiction, there exists $k \in \mathbb{N}$ such that $p_k < p_{k+1}$ and $[p_k, 0) \cap \mathcal{E} = \emptyset$. Then there exist constants α', α'' such that

$$p_k < \alpha' < \alpha < \alpha'' < p_{k+1}.$$

By the definition of p_k , we can choose $h \in H_k$ satisfying

$$\max_{x \in S^{k-1}} I_0(h(x)) < \alpha'.$$

Let $S_+^k = \{x : x = (x', x_{k+1}), x' \in \mathbb{R}^k, x_{k+1} \geq 0, |x| = 1\}$. From O is contractible, it implies that h can extend to $h' \in C(S_+^k, O)$. Take $\beta = \max\{I_0(h'(x)) : x \in S_+^k\}$. Since $h' \in C(S_+^k, O)$, we know $\beta < 0$. Then there exist constants b and β' such that

$$\beta < b < \beta' < 0.$$

Due to $[p_k, 0) \cap \mathcal{E} = \emptyset$, the pair (I_0^b, I_0^a) is trivial. This means that we can choose two closed subsets A and B of X satisfying

$$I_0^{\alpha'} \subset A \subset I_0^{\alpha''}, I_0^{\beta'} \subset B \subset I_0^{\beta''}$$

and there exists a deformation map $\sigma : [0, 1] \times B \rightarrow B$ such that

$$\sigma(0, \cdot) = id, \sigma(1, B) \subset A.$$

Then we consider $h''(x) = \sigma(1, h'(x))$ which satisfies

$$I_0(h''(S_+^k)) \subset I_0^{\alpha''} \text{ and } h''|_{S^{k-1}} \text{ is odd.}$$

Denote the odd extension of h'' on S^k by h^* , which satisfies $I_0(h^*(S^k)) \subset I_0^{\alpha''}$. However this contradicts the definition of p_{k+1} , i.e.

$$p_{k+1} \leq \max_{x \in S^k} I_0(h^*(x)) \leq \alpha'' < p_{k+1}.$$

Consequently, there exists $\{c_k\} \subset \mathcal{E}$ such that

$$c_1 < c_2 < c_3 < \dots < c_k < 0, \text{ and } c_k \rightarrow 0 \text{ as } k \rightarrow \infty.$$

For any $k \in \mathbb{N}$, by Lemma 3.1 and 3.2, there exists $\varepsilon_k > 0$ such that if $|\varepsilon| \leq \varepsilon_k$ then I_ε has at least k distinct critical points with negative critical values. $\square$

4 Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. For this purpose, we firstly make some modifies. Motivated by the similar modifies in [10, 18], we make use of the following approach: From condition (f_4) , decreasing a if necessary, we may assume

$$F(x, u) > 0, \text{ for } 0 < |u| < a, x \in \mathbb{R}^N.$$

For fixed $a > 0$. Let $\eta \in C_0^\infty(\mathbb{R}, [0, 1])$ be a cut-off even function such that

$$\eta(t) = 1 \text{ for } |t| \leq a/2; \quad \eta(t) > 0 \text{ for } |t| < a;$$

$$\eta(t) = 0 \text{ for } |t| \geq a; \quad \eta'(t) \in [-\frac{4}{a}, 0] \text{ for } \frac{a}{2} \leq t \leq a.$$

Using this cut-off function η , we consider the following modified functions:

$$\tilde{f}(x, u) := \frac{\partial}{\partial u}(\eta(u)F(x, u)),$$

$$\tilde{g}(u) := \eta(u)g(u),$$

$$\tilde{F}(x, u) := \int_0^u \tilde{f}(x, s) ds = \eta(u)F(x, u),$$

$$\tilde{a}_{ij}(x, u) := \eta(u)a_{ij}(x, u) + (1 - \eta(u))C_1\delta_{ij},$$

where $C_1 > 0$ is a positive constant mentioned in condition (a).

Next, we give some properties for $\tilde{f}$, $\tilde{g}$ and $\tilde{a}_{ij}$.

Lemma 4.1. $\tilde{f}(x, u)$ and $\tilde{g}(u)$ are continuous functions defined on $\mathbb{R}^N \times \mathbb{R}$ and satisfy the conditions below

$$(f'_1): \tilde{f}(x, -u) = -\tilde{f}(x, u) \text{ for } (x, u) \in \mathbb{R}^N \times \mathbb{R};$$

$$(f'_2): u\tilde{f}(x, u) - 2\tilde{F}(x, u) < 0 \text{ when } 0 < |u| < a \text{ and } x \in \mathbb{R}^N;$$

$$(f'_3): \tilde{f}(x, u) = \tilde{g}(u) = \tilde{F}(x, u) = 0 \text{ when } |u| \geq a \text{ and } x \in \mathbb{R}^N;$$

$$(f'_4): |\tilde{f}(x, u)| \leq C|u|^{r-1} \text{ for } (x, u) \in \mathbb{R}^N \times \mathbb{R};$$

$$(f'_5): \lim_{u \rightarrow 0} (\min_{x \in \mathbb{R}^N} u^{-2}\tilde{F}(x, u)) = \infty.$$

Proof. From the definition of $\tilde{f}(x, u)$ and $\tilde{g}(u)$, it is easy to show that (f'_1) , (f'_3) , (f'_4) and (f'_5) hold. To obtain (f'_2) a little manipulation is needed. For $0 < u < a$ and $x \in \mathbb{R}^N$, from the definition of η , we consider

$$\begin{aligned} \frac{\partial}{\partial u}(u^{-2}\tilde{F}(x, u)) &= \eta'(u)u^{-2}F(x, u) + \eta(u)\frac{\partial}{\partial u}(u^{-2}F(x, u)) \\ &= \eta'(u)u^{-2}F(x, u) + \eta(u)u^{-3}(uF(x, u) - 2F(x, u)) \\ &< 0. \end{aligned}$$

On the other hand, $\frac{\partial}{\partial u}(u^{-2}\tilde{F}(x, u))$ can also be denoted by

$$\frac{\partial}{\partial u}(u^{-2}\tilde{F}(x, u)) = u^{-3}(u\tilde{f}(x, u) - 2\tilde{F}(x, u)).$$

Combining the above two equations and the fact that $u\tilde{f}(x, u) - 2\tilde{F}(x, u)$ is even on u , we see that $\tilde{f}(x, u)$ satisfies condition (f'_2) . $\square$

Lemma 4.2. $\tilde{a}_{ij}(x, u)$ satisfies

$(a'_0): \tilde{a}_{ij} \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $\tilde{a}_{ij}$ is even, $\tilde{a}_{ij} = \tilde{a}_{ji}$ and there exist two constants C_0 and C_1 such that

$$C_0|\xi|^2 \leq \tilde{a}_{ij}\xi_i\xi_j \leq C_1|\xi|^2;$$

(a'_1) : there exists some $C > 0$, such that

$$|\tilde{a}_{ij}(x, u)| + |D_s(\tilde{a}_{ij}(x, u))| \leq C \text{ for } u \in \mathbb{R}, x \in \mathbb{R}^N;$$

(a'_2) : $\sum_{i,j=1}^N D_s(\tilde{a}_{ij}(x, u))u\xi_i\xi_j \geq 0$ for $u \in \mathbb{R}, x \in \mathbb{R}^N$.

Proof. It is clear that $\tilde{a}_{ij}$ satisfies (a'_0) and (a'_1) . We now verify (a'_2) . Observe the relation,

$$\begin{aligned} \sum_{i,j=1}^N D_s(\tilde{a}_{ij}(x, u))u\xi_i\xi_j &= \sum_{i,j=1}^N \left(\eta'(u)u a_{ij}(x, u)\xi_i\xi_j + \eta(u)D_s(a_{ij}(x, u))u\xi_i\xi_j - C_1\eta'(u)u\delta_{ij}\xi_i\xi_j \right) \\ &\geq \eta'(u)u \sum_{i,j=1}^N (a_{ij}(x, u)\xi_i\xi_j - C_1\delta_{ij}\xi_i\xi_j). \end{aligned}$$

Using condition (a) and the definition of η , we get

$$\eta'(u)u \sum_{i,j=1}^N (a_{ij}(x, u)\xi_i\xi_j - C_1\delta_{ij}\xi_i\xi_j) \geq 0.$$

□

We set $X := \{W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\}$ in which the norm is given by

$$\|u\|_X := \left(\int_{\mathbb{R}^N} (|Du|^2 + V(x)u^2) dx \right)^{1/2},$$

and $E := X \cap L^\infty(\mathbb{R}^N)$. For $u \in X$, we consider the following modified functional corresponding to (1.1),

$$I_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^N} \left(\sum_{i,j=1}^N \tilde{a}_{ij}(x, u)D_i u D_j u dx + V(x)u^2 \right) dx - \int_{\mathbb{R}^N} \tilde{F}(x, u) dx - \varepsilon \int_{\mathbb{R}^N} K(x)\tilde{G}(u) dx.$$

Now by Lemma 4.1, 4.2, I_ε is well defined in X . And it is easy to say functionals I_ε are E -differentiable in the directions $\varphi \in E$.

Next, we verify the (P-S) conditions for functional I_ε .

Lemma 4.3. For $\varepsilon \in [0, 1]$, I_ε satisfies (P-S) conditions uniformly on ε .

Proof. For any fixed $u \in X$ and $\varepsilon \in [0, 1]$, we have

$$\int_{\mathbb{R}^N} |\tilde{F}(x, u)| dx \leq C \int_{\mathbb{R}^N} |u|^r dx \leq C \left(\int_{\mathbb{R}^N} (V(x)^{-1})^{r/(2-r)} dx \right)^{(2-r)/2} \left(\int_{\mathbb{R}^N} V(x)u^2 dx \right)^{r/2} \leq C \|u\|_X^r,$$

and

$$\begin{aligned} \left| \varepsilon \int_{\mathbb{R}^N} K(x)\tilde{G}(u) dx \right| &= |\varepsilon| \int_{|u| \leq a} K(x)|\tilde{G}(u)| dx + |\varepsilon| \int_{|u| > a} K(x)|\tilde{G}(u)| dx \\ &= |\varepsilon| \int_{|u| \leq a} K(x)|\tilde{G}(u)| dx \\ &\leq |\varepsilon| C \int_{\mathbb{R}^N} K(x) dx \\ &\leq |\varepsilon| C. \end{aligned}$$

Thus,

$$I_\varepsilon(u) \geq \frac{C_0}{2} \|u\|_X^2 - C \|u\|_X^r - |\varepsilon|C, \quad u \in X, \quad (4.1)$$

and then $I_\varepsilon(u)$ is coercive and bounded below. Let $(\varepsilon_n, u_n) \in [0, 1] \times X$ be any sequence such that

$$I_{\varepsilon_n}(u_n) \rightarrow c \text{ and } |D_E I_{\varepsilon_n}(u_n)| \rightarrow 0.$$

Then $\{u_n\}$ and $\{\varepsilon_n\}$ are bounded. Therefore, there exists a subsequence of $\{\varepsilon_n\}$ converges to ε and a subsequence of $\{u_n\}$ converges to u weakly in X and a.e. on $\mathbb{R}^N$. Next, we shall show this convergence becomes a strong one.

Step 1. u is critical point of I_ε .

Take $T > a$, and define

$$u^T = \begin{cases} -T, & \text{if } u \leq -T, \\ u, & \text{if } -T < u < T, \\ T, & \text{if } u \geq T. \end{cases}$$

Choose $\varphi \in E$, $\varphi \geq 0$ and $\psi_n = \varphi \exp(-Hu_n^T)$ where $H > 0$ large enough such that $-H\tilde{a}_{ij} + \frac{1}{2}D_s\tilde{a}_{ij}$ is negatively definite.

Obviously, ψ_n can be seen as a test function in $\langle D_E I_\varepsilon(u_n), \psi_n \rangle \rightarrow 0$, that is

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_n) D_i u_n D_j \psi_n dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s \tilde{a}_{ij}(x, u_n) D_i u_n D_j u_n \psi_n dx \\ &\quad + \int_{\mathbb{R}^N} V(x) u_n \psi_n dx - \int_{\mathbb{R}^N} \tilde{f}(x, u_n) \psi_n dx - \varepsilon \int_{\mathbb{R}^N} K(x) \tilde{g}(u_n) \psi_n dx \\ &= \int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_n) D_i u_n D_j \varphi \exp(-Hu_n^T) dx \\ &\quad + \int_{\mathbb{R}^N} \sum_{i,j=1}^N \left(-H\tilde{a}_{ij}(x, u_n) + \frac{1}{2} D_s \tilde{a}_{ij}(x, u_n) \right) D_i u_n D_j u_n \varphi \exp(-Hu_n^T) dx \\ &\quad + \int_{\mathbb{R}^N} V(x) u_n \varphi \exp(-Hu_n^T) dx - \int_{\mathbb{R}^N} \tilde{f}(x, u_n) \varphi \exp(-Hu_n^T) dx - \varepsilon \int_{\mathbb{R}^N} K(x) \tilde{g}(u_n) \varphi \exp(-Hu_n^T) dx \\ &\leq \int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_i u D_j \varphi \exp(-Hu^T) dx + \int_{\mathbb{R}^N} \sum_{i,j=1}^N \left(-H\tilde{a}_{ij}(x, u) + \frac{1}{2} D_s \tilde{a}_{ij}(x, u) \right) D_i u D_j u \varphi \exp(-Hu^T) dx \\ &\quad + \int_{\mathbb{R}^N} V(x) u \varphi \exp(-Hu^T) dx - \int_{\mathbb{R}^N} \tilde{f}(x, u) \varphi \exp(-Hu^T) dx - \varepsilon \int_{\mathbb{R}^N} K(x) \tilde{g}(u) \varphi \exp(-Hu^T) dx + o(1) \\ &= \int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_i u D_j (\varphi \exp(-Hu^T)) dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s \tilde{a}_{ij}(x, u) D_i u D_j u \varphi \exp(-Hu^T) dx + \int_{\mathbb{R}^N} V(x) u \varphi \exp(-Hu^T) dx \\ &\quad - \int_{\mathbb{R}^N} \tilde{f}(x, u) \varphi \exp(-Hu^T) dx - \varepsilon \int_{\mathbb{R}^N} K(x) \tilde{g}(u) \varphi \exp(-Hu^T) dx + o(1), \end{aligned}$$

where we used the Fatou's Lemma and the lower semi-continuity. Thus, for all $\varphi \in E$, $\varphi \geq 0$, we have

$$\begin{aligned} 0 \leq & \int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_i u D_j (\varphi \exp(-Hu^T)) dx \\ & + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s \tilde{a}_{ij}(x, u) D_i u D_j u \varphi \exp(-Hu^T) dx + \int_{\mathbb{R}^N} V(x) u \varphi \exp(-Hu^T) dx \\ & - \int_{\mathbb{R}^N} \tilde{f}(x, u) \varphi \exp(-Hu^T) dx - \varepsilon \int_{\mathbb{R}^N} K(x) \tilde{g}(u) \varphi \exp(-Hu^T) dx. \end{aligned} \quad (4.2)$$

We can choose $\varphi = \phi \exp(Hu^T)$ in (4.2) for $\phi \in E$ and $\phi \geq 0$ we obtain

$$\begin{aligned} 0 \leq & \int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_i u D_j \phi dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s \tilde{a}_{ij}(x, u) D_i u D_j u \phi dx \\ & + \int_{\mathbb{R}^N} V(x) u \phi dx - \int_{\mathbb{R}^N} \tilde{f}(x, u) \phi dx \\ & - \varepsilon \int_{\mathbb{R}^N} K(x) \tilde{g}(u) \phi dx. \end{aligned}$$

Similarly, by choosing $\psi = \varphi \exp(Hu_n^T)$, we can get an opposite inequality. Hence, u is a critical point of I_ε .

Step 2. We shall show the facts that

$$\int_{\mathbb{R}^N} |\tilde{f}(x, u_n) - \tilde{f}(x, u)| |u_n - u| dx = o(1) \text{ and } \int_{\mathbb{R}^N} K(x) |\tilde{g}(u_n) - \tilde{g}(u)| |u_n - u| dx = o(1).$$

From $\{u_n\}$ converges to u weakly in X and a.e. on $\mathbb{R}^N$, we know

$$u_n \rightarrow u \text{ strongly } L^r_{loc}(\mathbb{R}^N), \text{ with } 1 \leq r < \frac{2N}{N-2}.$$

For any $R > 0$, by the Young inequality and the Hölder inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}^N} |\tilde{f}(x, u_n) - \tilde{f}(x, u)| |u_n - u| dx \\ & \leq C \int_{\mathbb{R}^N \setminus B_R(0)} (|u_n|^{r-1} + |u|^{r-1}) (|u_n| + |u|) dx \\ & \quad + C \int_{B_R(0)} (|u_n|^{r-1} + |u|^{r-1}) |u_n - u| dx \\ & \leq C \int_{\mathbb{R}^N \setminus B_R(0)} (|u_n|^r + |u|^r) dx + C \int_{B_R(0)} (|u_n|^{r-1} + |u|^{r-1}) |u_n - u| dx \\ & \leq C \|V(x)^{-1}\|_{L^{r/(2-r)}(\mathbb{R}^N \setminus B_R(0))}^{r/2} \left(\|V(x) u_n^2\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{r/2} + \|V(x) u^2\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{r/2} \right) \\ & \quad + C \left(\|u_n\|_{L^r(B_R(0))}^{r-1} + \|u\|_{L^r(B_R(0))}^{r-1} \right) \|u_n - u\|_{L^r(B_R(0))} \\ & \leq C \|V(x)^{-1}\|_{L^{r/(2-r)}(\mathbb{R}^N \setminus B_R(0))}^{r/2} + C \|u_n - u\|_{L^r(B_R(0))}, \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\tilde{f}(x, u_n) - \tilde{f}(x, u)| |u_n - u| dx = 0.$$

Recall that for $|u| \geq a$ it implies $\tilde{g}(u) = 0$. From this and Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |\tilde{g}(u_n) - \tilde{g}(u)| |u_n - u| dx = 0.$$

Step 3. $u_n \rightarrow u$ strongly in X .

Due to $\|u_n^T\|_X \leq C$ with C independent of T and n , we get

$$|\langle D_E I_\varepsilon(u_n), u_n^T \rangle| \leq C |D_E I_\varepsilon(u_n)|.$$

Taking $T \rightarrow +\infty$, we obtain that

$$|\langle D_E I_\varepsilon(u_n), u_n \rangle| \leq C |D_E I_\varepsilon(u_n)| = o(1).$$

Similarly, we have $|\langle D_E I_\varepsilon(u), u \rangle| = 0$. From the definition of $\tilde{a}_{ij}$ and condition (a'_2) , we obtain

$$\left(\tilde{a}_{ij}(x, u) + \frac{1}{2} D_s(\tilde{a}_{ij}(x, u))u \right) \xi_i \xi_j \geq C |\xi|^2.$$

Then

$$\begin{aligned} o(1) &= \langle D_E I_\varepsilon(u_n), u_n \rangle - \langle D_E I_\varepsilon(u), u \rangle \\ &\geq \int_{\mathbb{R}^N} \sum_{i,j=1}^N \left(\tilde{a}_{ij}(x, u_n) + \frac{1}{2} D_s(\tilde{a}_{ij}(x, u_n))u_n \right) D_i(u_n - u) D_j(u_n - u) dx \\ &\quad + \int_{\mathbb{R}^N} V(x)(u_n - u)^2 dx - \int_{\mathbb{R}^N} |\tilde{f}(x, u_n) - \tilde{f}(x, u)| |u_n - u| dx \\ &\quad - \varepsilon \int_{\mathbb{R}^N} K(x) |\tilde{g}(u_n) - \tilde{g}(u)| |u_n - u| dx \\ &\geq C \|u_n - u\|_X^2 + o(1), \end{aligned}$$

which implies that $\|u_n - u\|_X^2 \rightarrow 0$ as $n \rightarrow \infty$. □

Lemma 4.4. For any $u \in X \setminus \{0\}$, $I_0(u)$ satisfies condition (I_5) .

Proof. For any $u \in X \setminus \{0\}$ fixed. For $t > 0$, we consider

$$P(t) := \frac{C_0}{2} \int_{\mathbb{R}^N} |Du|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - t^{-2} \int_{\mathbb{R}^N} \tilde{F}(x, tu) dx,$$

and

$$Q(t) = \frac{1}{2} \left(\int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, tu) D_i u D_j u dx - C_0 \int_{\mathbb{R}^N} |Du|^2 dx \right).$$

Then observe the relation,

$$I_0(tu) = t^2(P(t) + Q(t)).$$

Define $J(t) := P(t) + Q(t)$. By condition (f'_2) and (a'_2) , we have

$$P'(t) = -t^{-3} \int_{\mathbb{R}^N} (tuf(x, tu) - 2\tilde{F}(x, tu)) dx > 0 \text{ for } t > 0,$$

and

$$Q'(t) = \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s \tilde{a}_{ij}(x, tu) u D_i u D_j u dx > 0 \text{ for } t > 0.$$

Thus, $J'(t) > 0$ for $t > 0$.

Take $\epsilon > 0$ small enough such that

$$\mu(D_\epsilon) > 0 \text{ and } D_\epsilon := \{x \in \mathbb{R}^N : \epsilon < |u(x)| < 1/\epsilon\},$$

where μ denotes the Lebesgue measure of $\mathbb{R}^N$.

Due to $\tilde{F}(x, u) \geq 0$, we can estimate the function $J(t)$ as follows:

$$\begin{aligned} J(t) &\leq \frac{1}{2} \left(\int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, tu) D_i u D_j u dx + \int_{\mathbb{R}^N} V(x) u^2 dx \right) \\ &\quad - t^{-2} \int_{D_\epsilon} \tilde{F}(x, tu) dx \\ &\leq \frac{1}{2} \max\{1, C_1\} \|u\|_X^2 - \epsilon^2 \mu(D_\epsilon) \inf_{x \in D_\epsilon} ((tu(x))^{-2} \tilde{F}(x, tu)). \end{aligned}$$

By condition (f'_4) in Lemma 4.1, it implies that $\lim_{t \rightarrow 0^+} J(t) = -\infty$. Hence, $J(t) < 0$ for $t > 0$ small enough.

On the other hand, for $t > 0$ large, by $1 \leq r < 2$ we have

$$\begin{aligned} J(t) &\geq \frac{1}{2} \left(\int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, tu) D_i u D_j u dx + \int_{\mathbb{R}^N} V(x) u^2 dx \right) \\ &\quad - t^{-2} C \int_{\mathbb{R}^N} |tu|^r dx \\ &\geq \frac{1}{2} \min\{1, C_0\} \|u\|_X^2 - t^{r-2} \|u\|_X^r > 0. \end{aligned}$$

Accordingly, for fixed $u \in X \setminus \{0\}$, $J(t)$ has a unique zero $t(u)$ such that

$$J(t) < 0, \text{ for } 0 < t < t(u), \quad J(t) \geq 0, \quad t(u) \leq t.$$

Thus the above result holds for $I_0(tu)$. □

Before proving the multiplicity of the critical points for I_ϵ , by the Moser's iteration (see Lemma 3.7 in [7]) and condition (f'_2) , we give some priori estimates.

Lemma 4.5. *There exist positive constants μ and C^* such that if $|D_E I_\epsilon(u)| = 0$ with $|\epsilon| \leq 1$, then $\|u\|_{L^\infty(\mathbb{R}^N)} \leq C^* \|u\|_X^\mu$.*

Proof. Notice that from conditions (VW) , (K) and (f'_3) , for $u \in X$, we have

$$|\tilde{f}(x, u)| \leq C|u|^{r-1} \text{ and } K(x)|\tilde{g}(u)| \leq C \text{ for some } C > 0. \quad (4.3)$$

Let $u \in X$ be a critical point of I_ϵ . Using the equation, for any $\psi \in X \cap L^\infty(\mathbb{R}^N)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_i u D_j u \psi dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s \tilde{a}_{ij}(x, u) D_i u D_j u \psi dx \\ + \int_{\mathbb{R}^N} V(x) u \psi dx = \int_{\mathbb{R}^N} \tilde{f}(x, u) \psi dx + \epsilon \int_{\mathbb{R}^N} K(x) \tilde{g}(u) \psi dx \end{aligned} \quad (4.4)$$

Choosing $\psi = |u^T|^\eta u^T$ in (4.4), where $\eta > 0$ (will be fixed by some constants) and u^T is defined in Lemma 4.3, we obtain

$$\begin{aligned} \int_{|u| \leq T} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u) D_i u D_j u (\eta + 1) |u^T|^\eta dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s \tilde{a}_{ij}(x, u) D_i u D_j u |u^T|^\eta u^T dx \\ + \int_{\mathbb{R}^N} V(x) u |u^T|^\eta u^T dx = \int_{\mathbb{R}^N} \tilde{f}(x, u) |u^T|^\eta u^T dx + \epsilon \int_{\mathbb{R}^N} K(x) \tilde{g}(u) |u^T|^\eta u^T dx \end{aligned} \quad (4.5)$$

Combining the second, third term in the left side of the above equation is nonnegative and (4.3), we obtain

$$\frac{C_0}{(\eta+2)^2} \int_{\mathbb{R}^N} |D|u^T|^{\frac{\eta}{2}+1}|^2 \leq C \int_{\mathbb{R}^N} |u|^{\eta+r} + C \int_{\mathbb{R}^N} |u|^{\eta+1}.$$

Without loss of generality, for fixed $u \in X$, we have the following estimate

$$\frac{1}{(\eta+2)^2} \int_{\mathbb{R}^N} |D|u^T|^{\frac{\eta}{2}+1}|^2 \leq C \int_{\mathbb{R}^N} |u|^{\eta+r}. \quad (4.6)$$

Since we can assume $\int_{\mathbb{R}^N} |u|^{\eta+r} \geq \int_{\mathbb{R}^N} |u|^{\eta+1}$, the case $\int_{\mathbb{R}^N} |u|^{\eta+r} < \int_{\mathbb{R}^N} |u|^{\eta+1}$ is similarly treated by the following arguments.

On the other hand, using the Sobolev inequality, we deduce

$$\frac{S}{(\eta+2)^2} \|u^T\|_{(\eta+2)N/(N-2)}^{\eta+2} \leq \frac{1}{(\eta+2)^2} \int_{\mathbb{R}^N} |D|u^T|^{\frac{\eta}{2}+1}|^2 \leq C \int_{\mathbb{R}^N} |u|^{\eta+r}, \quad (4.7)$$

where we used that $S = \inf\{\int_{\mathbb{R}^N} |Dv|^2 dx : \int_{\mathbb{R}^N} |v|^{2^*} dx = 1\}$. From the Fatou's lemma, sending $T \rightarrow \infty$ in (4.7), it implies that

$$\|u\|_{(\eta+2)N/(N-2)} \leq (\tilde{C}(\eta+2))^{\frac{2}{\eta+2}} \|u\|_{\eta+r}^{(\eta+r)/(\eta+2)}. \quad (4.8)$$

Let us define $\eta_k = \frac{(\eta_{k-1}+2)N}{N-2} - r$, where $k = 1, 2, \dots$ and $\eta_0 = 2^* - r$. It is easy to see that $\eta_k \rightarrow +\infty$ as $k \rightarrow +\infty$. We may assume $\tilde{C} > 1$, then for $i < j$, it follows that

$$(\tilde{C}(\eta_i+2))^{(\eta_j+r)/(\eta_j+2)} \leq \tilde{C}(\eta_i+2).$$

By Moser's iteration method we have

$$\|u\|_{\eta_{k+1}+r} \leq \exp\left(\sum_{i=0}^k \frac{2\ln(\tilde{C}(\eta_i+2))}{\eta_i+2}\right) \|u\|_{2^*}^{\mu_k},$$

where $\mu_k = \prod_{i=0}^k \frac{\eta_i+r}{\eta_i+2}$. Letting $k \rightarrow \infty$, we obtain that

$$\|u\|_{\infty} \leq \exp\left(\sum_{i=0}^{\infty} \frac{2\ln(\tilde{C}(\eta_i+2))}{\eta_i+2}\right) \|u\|_{2^*}^{\mu},$$

where $\mu = \prod_{i=0}^{\infty} \frac{\eta_i+r}{\eta_i+2}$ with $0 < \mu < 1$ and $\exp\left(\sum_{i=0}^{\infty} \frac{2\ln(\tilde{C}(\eta_i+2))}{\eta_i+2}\right)$ is a positive constant. This ends the proof. $\square$

Lemma 4.6. For any $b > 0$, there exists a $\delta(b) > 0$ such that if $|\varepsilon| \leq \delta(b)$, $|D_E I_{\varepsilon}(u)| = 0$ and $|I_{\varepsilon}(u)| \leq \delta(b)$, then $\|u\|_X \leq b$.

Proof. Suppose on the contrary that there exist two sequences $\{u_n\} \subset X$ and $\{\varepsilon_n\}$ such that $\varepsilon_n \rightarrow 0$, $I_{\varepsilon_n}(u_n) \rightarrow 0$ as $n \rightarrow \infty$, $|D_E I_{\varepsilon_n}(u_n)| = 0$, and $\|u_n\|_X \geq b_0 > 0$, where b_0 is independent of n . Obviously, we can see $\{u_n\}$ as the (P-S) sequence of I_0 . Then from Lemma 4.3, we obtain that a subsequence of $\{u_n\}$ which converges to u_0 in X , which satisfies

$$\begin{aligned} \langle D_E I_0(u_0), u_0 \rangle &= \int_{\mathbb{R}^N} \sum_{i,j=1}^N \tilde{a}_{ij}(x, u_0) D_i u_0 D_j u_0 dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s \tilde{a}_{ij}(x, u_0) u_0 D_i u_0 D_j u_0 dx \\ &\quad + \int_{\mathbb{R}^N} V(x)(u_0)^2 dx - \int_{\mathbb{R}^N} \tilde{f}(x, u_0) u_0 dx = 0 \end{aligned}$$

and

$$I_0(u_0) = \frac{1}{2} \int_{\mathbb{R}^N} \left(\sum_{i,j=1}^N \tilde{a}_{ij}(x, u_0) D_i u_0 D_j u_0 + V(x) u_0^2 \right) dx \\ - \int_{\mathbb{R}^N} \tilde{F}(x, u_0) dx = 0.$$

From the above two equations, it follows that

$$I_0(u_0) - \frac{1}{2} \langle D_E I_0(u_0), u_0 \rangle = -\frac{1}{4} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s \tilde{a}_{ij}(x, u_0) u_0 D_i u_0 D_j u_0 dx \\ + \int_{\mathbb{R}^N} \left(\frac{1}{2} \tilde{f}(x, u_0) u_0 - \tilde{F}(x, u_0) \right) dx \\ = 0.$$

Since $u_0 \in X$, conditions (a'_2) and (f'_2) imply that $u_0 \equiv 0$.

On the other hand, from $\|u_n\|_X \geq b_0 > 0$ and $u_n \rightarrow u_0$ in X , we have $\|u_0\|_X \geq b_0 > 0$, which contradicts the fact that $u_0 \equiv 0$. The proof is complete. $\square$

From the above two lemmas, it is straightforward to show the following corollary.

Corollary 4.1. *If u is critical point of $I_\varepsilon(u)$ with $|\varepsilon| \leq \delta(\frac{a}{2C^*})$ and $|I_\varepsilon(u)| \leq \delta(\frac{a}{2C^*})$, then $\|u\|_{L^\infty(\mathbb{R}^N)} \leq \frac{a}{2}$, this means u is a critical of the original problem (1.1).*

Now, we are ready to prove the second main result.

The proof of Theorem 1.2. Without loss of generality, we assume $\varepsilon > 0$. Because the case $\varepsilon < 0$ is same studied by replacing $\tilde{g}(u)$ by $-\tilde{g}(u)$.

Next we are ready to verify that $I_\varepsilon(u)$ satisfies conditions $(I_1) - (I_5)$ in Theorem 1.1. To verify condition (I_1) , by (4.1), we have

$$\inf_{\varepsilon \in [0,1], u \in X} I_\varepsilon(u) > -\infty.$$

Condition (I_2) follows from

$$|I_\varepsilon(u) - I_0(u)| \leq |\varepsilon| \int_{\mathbb{R}^N} |K(x)| |\tilde{G}(u)| dx \leq |\varepsilon| C := \psi(\varepsilon),$$

where C is a constant independent of u and ε . Conditions (I_3) and (I_5) follow from Lemma 4.3 and 4.4 respectively. Thus $I_\varepsilon(u)$ satisfies all the conditions in Theorem 1.1. Then by the proof of Corollary 1.1, for any $\delta > 0$, we have k distinct critical values of I_ε satisfying

$$-\delta < b_{n(1)}(\varepsilon) < b_{n(2)}(\varepsilon) < \dots < b_{n(k)}(\varepsilon) < 0.$$

Finally, due to the arbitrariness of δ , take $0 < \delta < \delta(\frac{a}{2C^*})$, by Corollary 4.1, the original problem (1.1) has at least k solutions whose L^∞ -norms are less than $\frac{a}{2}$.

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