Research Article

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Nonuniform dichotomy spectrum and reducibility for nonautonomous difference equations

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Abstract: For nonautonomous linear difference equations, we introduce the notion of the so-called nonuniform dichotomy spectrum and prove a spectral theorem. As an application of the spectral theorem, we prove a reducibility result.

Keywords: Dichotomy spectrum; nonuniform exponential dichotomy; reducibility

MSC: 37D25; 37B55

1 Introduction

Let $A_k \in \mathbb{R}^{N \times N}$, $k \in \mathbb{Z}$, be a sequence of invertible matrices. In this paper, we consider the following nonautonomous linear difference equations

$$x_{k+1} = A_k x_k, \quad (1.1)$$

where $x_k \in \mathbb{R}^N$, $k \in \mathbb{Z}$. Let $\Phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$, $(k, l) \mapsto \Phi(k, l)$, denote the evolution operator of (1.1), i.e.,

$$\Phi(k, l) = \begin{cases} A_{k-1} \cdots A_l, & \text{for } k > l, \\ \text{Id}, & \text{for } k = l, \\ A_k^{-1} \cdots A_{l-1}^{-1}, & \text{for } k < l. \end{cases}$$

Obviously, $\Phi(k, m)\Phi(m, l) = \Phi(k, l)$, $k, m, l \in \mathbb{Z}$, and $\Phi(\cdot, l)\xi$ solves the initial value problem (1.1), $x(l) = \xi$, for $l \in \mathbb{Z}$, $\xi \in \mathbb{R}^N$.

An invariant projector of (1.1) is defined to be a function $P : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$ of projections P_k , $k \in \mathbb{Z}$, such that for each P_k the following property holds

$$P_{k+1}A_k = A_kP_k, \quad k \in \mathbb{Z}.$$

We say that (1.1) admits an *exponential dichotomy* if there exist an invariant projector P and constants $0 < \alpha < 1$, $K \geq 1$ such that

$$\|\Phi(k, l)P_l\| \leq K\alpha^{k-l}, \quad k \geq l, \quad (1.2)$$

and

$$\|\Phi(k, l)Q_l\| \leq K\left(\frac{1}{\alpha}\right)^{k-l}, \quad k \leq l, \quad (1.3)$$

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where $Q_l = \text{Id} - P_l$ is the complementary projection.

The notion of exponential dichotomy was introduced by Perron in [25] and has attracted a lot of interest during the last few decades because it plays an important role in the study of hyperbolic dynamical behavior of differential equations and difference equations. For example, see [1, 19, 28] and the references therein. We also refer to the books [10, 16, 22] for details and further references related to exponential dichotomies. On the other hand, during the last decade, inspired both by the classical notion of exponential dichotomy and by the notion of nonuniformly hyperbolic trajectory introduced by Pesin (see [5]), Barreira and Valls have introduced the notion of nonuniform exponential dichotomies and have developed the corresponding theory in a systematic way (see [6] and the references therein). As explained by Barreira and Valls, in comparison to the notion of exponential dichotomies, nonuniform exponential dichotomy is a useful and weaker notion.

We say that (1.1) admits a *nonuniform exponential dichotomy* if there exist an invariant projector P and constants $0 < \alpha < 1$, $K \geq 1$, $\varepsilon \geq 1$ such that $\alpha\varepsilon^2 < 1$ and

$$\|\Phi(k, l)P_l\| \leq K\alpha^{k-l}\varepsilon^{|l|}, \quad k \geq l, \quad (1.4)$$

and

$$\|\Phi(k, l)Q_l\| \leq K\left(\frac{1}{\alpha}\right)^{k-l}\varepsilon^{|l|}, \quad k \leq l. \quad (1.5)$$

When $\varepsilon = 1$, (1.4)-(1.5) become (1.2)-(1.3), and therefore a nonuniform exponential dichotomy becomes an exponential dichotomy. Moreover, [5, Theorem 1.4.2] indicates that the condition $\alpha\varepsilon^2 < 1$ is reasonable, which means that the nonuniform part is small.

For example, given $\omega > 5a > 0$, then the linear equation

$$u_{k+1} = e^{-\omega+ak(-1)^k-a(k-1)(-1)^{(k-1)}} u_k, \quad v_{k+1} = e^{\omega-ak(-1)^k+a(k-1)(-1)^{(k-1)}} v_k \quad (1.6)$$

admits a nonuniform exponential dichotomy, but does not admit an exponential dichotomy. In fact, we have

$$\Phi(k, l)P_l = \begin{pmatrix} e^{-\omega(k-l-1)-a(k-l-1)(-1)^{k-1}-al(-1)^{(k-1)}+al(-1)^l} & 0 \\ 0 & 0 \end{pmatrix}$$

with $P_l = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore (1.4) holds with

$$K = e^{\omega-a} > 1, \quad \alpha = e^{(-\omega+a)} \in (0, 1), \quad \varepsilon = e^{2a} > 1.$$

Analogous arguments applied to the second equation yield the estimate (1.5). Moreover, when both k and l are even, we obtain the equality

$$\|\Phi(k, l)P_l\| = K\alpha^{k-l}\varepsilon^{|l|}, \quad k \geq l,$$

which means that the nonuniform part $\varepsilon^l = e^{2al}$ cannot be removed.

Among the different topics on classical exponential dichotomies, the dichotomy spectrum (also called dynamical spectrum, or Sacker-Sell spectrum) is very important and many results have been obtained. As far as we know, dynamical spectrum defined with exponential dichotomies was first introduced by Sacker and Sell in [29] in which they studied the linear skew product flows with compact base. For more results on dichotomy spectrum, we refer the reader to [2, 3, 13, 18, 23, 26, 27, 29, 31, 32] and the references therein. The definition and investigation for finite-time hyperbolicity have also been studied in [9, 17]. The dynamical spectral theorem has some important applications. For example, based on the dichotomy spectral theorem, normal forms for nonautonomous systems were established in [20, 21, 33]. However, all the results mentioned above were established in the setting of classical exponential dichotomies. In this paper, we establish the corresponding spectral theory for difference equations (1.1) with a nonuniform exponential dichotomy. To the best of our knowledge the nonuniform dichotomy spectral theory for linear differential equations was first studied in [14] and [35]. We refer the reader to [7, 8, 11, 12, 34] for further results on nonuniform dichotomy spectrum.

This paper is organized as follows. In Section 2 we propose a definition of spectrum based on nonuniform exponential dichotomies, which is called nonuniform dichotomy spectrum. Such a spectrum can be seen as a generalization of Sacker-Sell spectrum. We prove a nonuniform dichotomy spectral theorem. In Section 3, as an application of the spectral theorem, we prove a reducibility result for (1.1). Recall that system (1.1) is called reducible if it is kinematically similar to a block diagonal system with blocks of dimension less than N .

2 Nonuniform dichotomy spectrum

Consider the weighted system

$$x_{k+1} = \frac{1}{\gamma} A_k x_k, \quad (2.1)$$

where $\gamma \in \mathbb{R}^+ = (0, \infty)$. One can easily see that

$$\Phi_\gamma(k, l) := \left(\frac{1}{\gamma}\right)^{k-l} \Phi(k, l)$$

is its evolution operator. If for some $\gamma \in \mathbb{R}^+$, (2.1) admits a nonuniform exponential dichotomy with projector P_k and constants $K \geq 1$, $0 < \alpha < 1$ and $\varepsilon \geq 1$, then P_k is also invariant for (1.1), that is

$$P_{k+1} A_k = A_k P_k, \quad k \in \mathbb{Z},$$

and the dichotomy estimates of (2.1) are equivalent to

$$\|\Phi(k, l) P_l\| \leq K(\gamma \alpha)^{k-l} \varepsilon^{|l|}, \quad k \geq l, \quad (2.2)$$

and

$$\|\Phi(k, l) Q_l\| \leq K(\gamma \frac{1}{\alpha})^{k-l} \varepsilon^{|l|}, \quad k \leq l. \quad (2.3)$$

Definition 2.1. The nonuniform dichotomy spectrum of (1.1) is the set

$$\Sigma_{NED}(A) = \{\gamma \in \mathbb{R}^+ : (2.1) \text{ admits no nonuniform exponential dichotomy}\},$$

and the resolvent set $\rho_{NED}(A) = \mathbb{R}^+ \setminus \Sigma_{NED}(A)$ is its complement. The dichotomy spectrum of (1.1) is the set

$$\Sigma_{ED}(A) = \{\gamma \in \mathbb{R}^+ : (2.1) \text{ admits no exponential dichotomy}\},$$

and $\rho_{ED}(A) = \mathbb{R}^+ \setminus \Sigma_{ED}(A)$.

Proposition 1. $\Sigma_{NED}(A) \subset \Sigma_{ED}(A)$.

Proof. For each $\gamma \in \rho_{ED}(A)$, the weighted system (2.1) admits an exponential dichotomy. Consequently, the weighted system (2.1) admits a nonuniform exponential dichotomy. Thus, $\gamma \in \rho_{NED}(A)$, which implies that $\rho_{ED}(A) \subset \rho_{NED}(A)$, and therefore $\Sigma_{NED}(A) \subset \Sigma_{ED}(A)$. $\square$

Let us define for $\gamma \in \rho_{NED}(A)$

$$\mathcal{S}_\gamma := \left\{ (l, \xi) \in \mathbb{Z} \times \mathbb{R}^N : \left(\sup_{k \geq l} \|\Phi(k, l) \xi\| \gamma^{-k} \right) \varepsilon^{-|l|} < \infty \right\},$$

and

$$\mathcal{U}_\gamma := \left\{ (l, \xi) \in \mathbb{Z} \times \mathbb{R}^N : \left(\sup_{k \leq l} \|\Phi(k, l) \xi\| \gamma^{-k} \right) \varepsilon^{-|l|} < \infty \right\},$$

where ε is the constant in (2.2)-(2.3). Note that $\mathcal{S}_\gamma$ and $\mathcal{U}_\gamma$ depend on $\varepsilon = \varepsilon(\gamma)$ for each $\gamma \in \rho_{NED}(A)$. If in addition $\gamma \in \rho_{ED}(A)$ then it is shown below that these sets do not depend on ε , more precisely, ε can be set to equal 1. One may readily verify that $\mathcal{S}_\gamma$ and $\mathcal{U}_\gamma$ are invariant vector bundles of (1.1), here we say that

a nonempty set $\mathcal{W} \subset \mathbb{Z} \times \mathbb{R}^N$ is an invariant vector bundle of (1.1) if (a) it is invariant, i.e., $(l, \xi) \in \mathcal{W} \Rightarrow (k, \Phi(k, l)\xi) \in \mathcal{W}$ for all $k \in \mathbb{Z}$; and (b) for every $l \in \mathbb{Z}$ the fiber $\mathcal{W}(l) = \{\xi \in \mathbb{R}^N : (l, \xi) \in \mathcal{W}\}$ is a linear subspace of $\mathbb{R}^N$.

The next lemma gives the relationship between $\mathcal{S}_\gamma$, $\mathcal{U}_\gamma$ and the projector P . In [16, Chapter 2] it is proved in the setting of exponential dichotomies that the invariant projector is unique. The proof for the invariant projectors for (1.1) and (2.1) in the setting of nonuniform exponential dichotomies is almost identical.

Lemma 2.2. *Assume that (2.1) admits a nonuniform exponential dichotomy with invariant projector P for $\gamma \in \mathbb{R}^+$. Then*

$$\mathcal{S}_\gamma = \operatorname{im} P, \quad \mathcal{U}_\gamma = \ker P \quad \text{and} \quad \mathcal{S}_\gamma \oplus \mathcal{U}_\gamma = \mathbb{Z} \times \mathbb{R}^N.$$

Proof. We show only $\mathcal{S}_\gamma = \operatorname{im} P$. The fact $\mathcal{U}_\gamma = \ker P$ is analog and the fact $\mathcal{S}_\gamma \oplus \mathcal{U}_\gamma = \mathbb{Z} \times \mathbb{R}^N$ is clear.

First we show $\mathcal{S}_\gamma \subset \operatorname{im} P$. Let $l \in \mathbb{Z}$ and $\xi \in \mathcal{S}_\gamma(l)$. Then there exists a positive constant C such that

$$\|\Phi(k, l)\xi\| \leq C\gamma^k \varepsilon^{|l|}, \quad k \geq l.$$

We write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \operatorname{im} P_l$ and $\xi_2 \in \ker P_l$. We show that $\xi_2 = 0$. By invariance of P we have for $k \in \mathbb{Z}$ the identity

$$\xi_2 = \Phi_\gamma(l, k)\Phi_\gamma(k, l)Q_l\xi = \Phi_\gamma(l, k)Q_k\Phi_\gamma(k, l)\xi.$$

Using the fact that (2.1) admits a nonuniform exponential dichotomy, it follows that

$$\|\Phi_\gamma(l, k)Q_k\| \leq K\left(\frac{1}{\alpha}\right)^{l-k} \varepsilon^{|k|}.$$

Thus

$$\begin{aligned} \|\xi_2\| &\leq K\left(\frac{1}{\alpha}\right)^{l-k} \varepsilon^{|k|} \|\Phi_\gamma(k, l)\xi\| \\ &= K(\alpha\varepsilon)^{k-l} \varepsilon^{l-k+|k|} \left(\frac{1}{\gamma}\right)^{k-l} \|\Phi(k, l)\xi\| \\ &\leq CK(\alpha\varepsilon)^{k-l} \varepsilon^{l+|l|-k+|k|} \left(\frac{1}{\gamma}\right)^{k-l} \gamma^k, \quad k \geq l, \end{aligned}$$

which implies that when $k \geq l$ and $k \rightarrow \infty$, we have

$$\|\xi_2\| \leq CK(\alpha\varepsilon)^{k-l} \varepsilon^{2|l|} \gamma^l,$$

and therefore $\xi_2 = 0$ by letting $k \rightarrow \infty$, since $\alpha\varepsilon < 1$.

Next we show $\operatorname{im} P \subset \mathcal{S}_\gamma$. Let $l \in \mathbb{Z}$ and $\xi \in \operatorname{im} P_l$, i.e., $P_l\xi = \xi$. Using the fact that $\alpha < 1$, the nonuniform exponential dichotomy estimate implies

$$\|\Phi_\gamma(k, l)\xi\| \leq K\alpha^{k-l} \varepsilon^{|l|} \|\xi\| \leq K\varepsilon^{|l|} \|\xi\|, \quad k \geq l,$$

which yields

$$\|\Phi(k, l)\xi\| \leq K\gamma^{k-l} \varepsilon^{|l|} \|\xi\|,$$

and thus $\xi \in \mathcal{S}_\gamma(l)$. $\square$

Lemma 2.3. *The resolvent set is open, i.e., for every $\gamma \in \rho_{\text{NED}}(A)$, there exists a $\beta = \beta(\gamma) \in (0, 1)$ with $(\beta\gamma, \frac{1}{\beta}\gamma) \subset \rho_{\text{NED}}(A)$. Furthermore,*

$$\mathcal{S}_\zeta = \mathcal{S}_\gamma \quad \text{and} \quad \mathcal{U}_\zeta = \mathcal{U}_\gamma \quad \text{for} \quad \zeta \in (\beta\gamma, \frac{1}{\beta}\gamma).$$

Proof. Let $\gamma \in \rho_{\text{NED}}(A)$. Then (2.1) admits a nonuniform exponential dichotomy, i.e., the estimates (2.2)-(2.3) hold with an invariant projector P and constants $K \geq 0$, $0 < \alpha < 1$ and $\varepsilon \geq 1$. For $\beta := \sqrt{\alpha} \in (0, 1)$ and $\zeta \in (\beta\gamma, \frac{1}{\beta}\gamma)$ we have

$$\Phi_\zeta(k, l) = \left(\frac{\gamma}{\zeta}\right)^{k-l} \Phi_\gamma(k, l).$$

Note that P is also an invariant projector for the equation

$$x_{k+1} = \frac{1}{\zeta} A_k x_k.$$

Moreover, we have the estimates

$$\|\Phi_\zeta(k, l)P_l\| \leq K\left(\frac{\gamma}{\zeta}\alpha\right)^{k-l}\varepsilon^{|l|} \leq K\beta^{k-l}\varepsilon^{|l|}, \quad k \geq l,$$

and

$$\|\Phi_\zeta(k, l)Q_l\| \leq K\left(\frac{\gamma}{\zeta}\frac{1}{\alpha}\right)^{k-l}\varepsilon^{|l|} \leq K\left(\frac{1}{\beta}\right)^{k-l}\varepsilon^{|l|}, \quad k \leq l.$$

Hence $\zeta \in \rho_{NED}(A)$. Therefore, $\rho_{NED}(A)$ is an open set. Using Lemma 2.2, we know that $\mathcal{S}_\zeta = \mathcal{S}_\gamma$ and $\mathcal{U}_\zeta = \mathcal{U}_\gamma$.
□

Corollary 2.4. $\Sigma_{NED}(A)$ is a closed set.

Using the facts proved above, we can obtain the following result, whose proof is similar as [4, Lemma 2.2], and therefore we omit the proof here.

Lemma 2.5. Let $\gamma_1, \gamma_2 \in \rho_{NED}(A)$ with $\gamma_1 < \gamma_2$. Then $\mathcal{F} = \mathcal{U}_{\gamma_1} \cap \mathcal{S}_{\gamma_2}$ is an invariant vector bundle which satisfies exactly one of the following two alternatives and the statements given in each alternative are equivalent:

Alternative I	Alternative II
(A) $\mathcal{F} = \mathbb{Z} \times \{0\}$.	(A') $\mathcal{F} \neq \mathbb{Z} \times \{0\}$.
(B) $[\gamma_1, \gamma_2] \subset \rho_{NED}(A)$.	(B') There is a $\zeta \in (\gamma_1, \gamma_2) \cap \Sigma_{NED}(A)$.
(C) $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ and $\mathcal{U}_{\gamma_1} = \mathcal{U}_{\gamma_2}$.	(C') $\dim \mathcal{S}_{\gamma_1} < \dim \mathcal{S}_{\gamma_2}$.
(D) $\mathcal{S}_\gamma = \mathcal{S}_{\gamma_2}$ and $\mathcal{U}_\gamma = \mathcal{U}_{\gamma_2}$ for $\gamma \in [\gamma_1, \gamma_2]$.	(D') $\dim \mathcal{U}_{\gamma_1} > \dim \mathcal{U}_{\gamma_2}$.

Now we are in a position to state and prove the nonuniform dichotomy spectral theorem which will be essential to prove the result on reducibility in Section 3. The proof follows the idea and technique of the classical dichotomy spectrum proposed in [30], we present the details for the reader's convenience.

Theorem 2.6. The nonuniform dichotomy spectrum $\Sigma_{NED}(A)$ of (1.1) is the disjoint union of n closed intervals (called spectral intervals) where $0 \leq n \leq N$, i.e., $\Sigma_{NED}(A) = \emptyset$ or $\Sigma_{NED}(A) = \mathbb{R}^+$ or one of the four cases

$$\Sigma_{NED}(A) = \left\{ \begin{array}{c} [a_1, b_1] \\ \text{or} \\ (0, b_1] \end{array} \right\} \cup [a_2, b_2] \cup \cdots \cup [a_{n-1}, b_{n-1}] \cup \left\{ \begin{array}{c} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{array} \right\},$$

where $0 < a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$. Choose a

$$\gamma_0 \in \rho_{NED}(A) \text{ with } (0, \gamma_0) \subset \rho_{NED}(A) \text{ if possible,} \quad (2.4)$$

otherwise define $\mathcal{U}_{\gamma_0} := \mathbb{Z} \times \mathbb{R}^N$, $\mathcal{S}_{\gamma_0} := \mathbb{Z} \times \{0\}$. Choose a

$$\gamma_n \in \rho_{NED}(A) \text{ with } (\gamma_n, +\infty) \subset \rho_{NED}(A) \text{ if possible,} \quad (2.5)$$

otherwise define $\mathcal{U}_{\gamma_n} := \mathbb{Z} \times \{0\}$, $\mathcal{S}_{\gamma_0} := \mathbb{Z} \times \mathbb{R}^N$. Then the sets

$$\mathcal{W}_0 = \mathcal{S}_{\gamma_0} \quad \text{and} \quad \mathcal{W}_{n+1} = \mathcal{S}_{\gamma_n}$$

are invariant vector bundles of (1.1). For $n \geq 2$, choose $\gamma_i \in \rho_{NED}(A)$ with

$$b_i < \gamma_i < a_{i+1} \quad \text{for } i = 1, \dots, n-1, \quad (2.6)$$

then for every $i = 1, \dots, n-1$ the intersection

$$\mathcal{W}_i = \mathcal{U}_{\gamma_{i-1}} \cap \mathcal{S}_{\gamma_i}$$

is an invariant vector bundle of (1.1) with $\dim \mathcal{W}_i \geq 1$. The invariant vector bundles $\mathcal{W}_i$, $i = 0, \dots, n+1$, are called spectral bundles and they are independent of the choice of $\gamma_0, \dots, \gamma_n$ in (2.4), (2.5) and (2.6). Moreover

$$\mathcal{W}_0 \oplus \dots \oplus \mathcal{W}_{n+1} = \mathbb{Z} \times \mathbb{R}^N$$

is a direct sum, i.e., $\mathcal{W}_i \cap \mathcal{W}_j = \mathbb{Z} \times \{0\}$ for $i \neq j$ and $\mathcal{W}_0 + \dots + \mathcal{W}_{n+1} = \mathbb{Z} \times \mathbb{R}^N$.

Proof. Recall that the resolvent set $\rho_{NED}(A)$ is open and therefore $\Sigma_{NED}(A)$ is the disjoint union of closed intervals. Next we will show that $\Sigma_{NED}(A)$ consists of at most N intervals. Indeed, if $\Sigma_{NED}(A)$ contains $N+1$ components, then one can choose a collection of points $\zeta_1, \dots, \zeta_N$ in $\rho_{NED}(A)$ such that $\zeta_1 < \dots < \zeta_N$ and each of the intervals $(0, \zeta_1), (\zeta_1, \zeta_2), \dots, (\zeta_{N-1}, \zeta_N), (\zeta_N, \infty)$ has nonempty intersection with the spectrum $\Sigma_{NED}(A)$. Now alternative II of Lemma 2.5 implies

$$0 \leq \dim \mathcal{S}_{\zeta_1} < \dots < \dim \mathcal{S}_{\zeta_N} \leq N$$

and therefore either $\dim \mathcal{S}_{\zeta_1} = 0$ or $\dim \mathcal{S}_{\zeta_N} = N$ or both. Without loss of generality, $\dim \mathcal{S}_{\zeta_N} = N$, i.e., $\mathcal{S}_{\zeta_N} = \mathbb{Z} \times \mathbb{R}^N$. Assume that

$$x_{k+1} = \frac{1}{\zeta_N} A_k x_k$$

admits a nonuniform exponential dichotomy with invariant projector $P \equiv \text{Id}$, then

$$x_{k+1} = \frac{1}{\zeta} A_k x_k$$

also admits for every $\zeta > \zeta_N$ a nonuniform exponential dichotomy with the same projector. We conclude $(\zeta_N, \infty) \subset \rho_{NED}(A)$, which is a contradiction. This proves the alternatives for $\Sigma_{NED}(A)$.

Due to Lemma 2.5, the sets $\mathcal{W}_0, \dots, \mathcal{W}_{n+1}$ are invariant vector bundles. To prove now that $\dim \mathcal{W}_1 \geq 1, \dots, \dim \mathcal{W}_n \geq 1$ for $n \geq 1$, let us assume that $\dim \mathcal{W}_1 = 0$, i.e., $\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1} = \mathbb{Z} \times \{0\}$. If $(0, b_1]$ is a spectral interval this implies that $\mathcal{S}_{\gamma_1} = \mathbb{Z} \times \{0\}$. Then the projector of the nonuniform exponential dichotomy of

$$x_{k+1} = \frac{1}{\gamma_1} A_k x_k$$

is 0 and then we get the contradiction $(0, \gamma_1) \subset \rho_{NED}(A)$. If $[a_1, b_1]$ is a spectral interval then $[\gamma_0, \gamma_1] \cap \Sigma_{NED}(A) \neq \emptyset$ and by alternative II of Lemma 2.5 we get a contradiction. Therefore $\dim \mathcal{W}_1 \geq 1$ and analogously $\dim \mathcal{W}_n \geq 1$. Furthermore for $n \geq 3$ and $i = 2, \dots, n-1$ one has $[\gamma_{i-1}, \gamma_i] \cap \Sigma_{NED}(A) \neq \emptyset$ and again alternative II of Lemma 2.5 yields $\dim \mathcal{W}_i \geq 1$.

For $i < j$ we have $\mathcal{W}_i \subset \mathcal{S}_{\gamma_i}$ and $\mathcal{W}_j \subset \mathcal{U}_{\gamma_{j-1}} \subset \mathcal{U}_{\gamma_i}$ and with Lemma 2.2 this gives $\mathcal{W}_i \cap \mathcal{W}_j \subset \mathcal{S}_{\gamma_i} \cap \mathcal{U}_{\gamma_i} = \mathbb{Z} \times \{0\}$, so $\mathcal{W}_i \cap \mathcal{W}_j = \mathbb{Z} \times \{0\}$ for $i \neq j$.

To show that $\mathcal{W}_0 \oplus \dots \oplus \mathcal{W}_{n+1} = \mathbb{Z} \times \mathbb{R}^N$, recall the monotonicity relations $\mathcal{S}_{\gamma_0} \subset \dots \subset \mathcal{S}_{\gamma_n}, \mathcal{U}_{\gamma_0} \supset \dots \supset \mathcal{U}_{\gamma_n}$, and the identity $\mathcal{S}_{\gamma} \oplus \mathcal{U}_{\gamma} = \mathbb{Z} \times \mathbb{R}^N$ for $\gamma \in \mathbb{R}^+$. Therefore $\mathbb{Z} \times \mathbb{R}^N = \mathcal{W}_0 + \mathcal{U}_{\gamma_0}$. Now we have

$$\begin{aligned} \mathbb{Z} \times \mathbb{R}^N &= \mathcal{W}_0 + \mathcal{U}_{\gamma_0} \cap [\mathcal{S}_{\gamma_1} + \mathcal{U}_{\gamma_1}] \\ &= \mathcal{W}_0 + [\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1}] + \mathcal{U}_{\gamma_1} \\ &= \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{U}_{\gamma_1}. \end{aligned}$$

Doing the same for $\mathcal{U}_{\gamma_1}$, we get

$$\begin{aligned} \mathbb{Z} \times \mathbb{R}^N &= \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{U}_{\gamma_1} \cap [\mathcal{S}_{\gamma_2} + \mathcal{U}_{\gamma_2}] \\ &= \mathcal{W}_0 + \mathcal{W}_1 + [\mathcal{U}_{\gamma_1} \cap \mathcal{S}_{\gamma_2}] + \mathcal{U}_{\gamma_2} \\ &= \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{U}_{\gamma_2}, \end{aligned}$$

and mathematical induction yields $\mathbb{Z} \times \mathbb{R}^N = \mathcal{W}_0 + \dots + \mathcal{W}_{n+1}$. To finish the proof, let $\tilde{\gamma}_0, \dots, \tilde{\gamma}_n \in \rho_{NED}(A)$ be given with the properties (2.4), (2.5) and (2.6). Then alternative I of Lemma 2.5 implies

$$\mathcal{S}_{\gamma_i} = \mathcal{S}_{\tilde{\gamma}_i} \quad \text{and} \quad \mathcal{U}_{\gamma_i} = \mathcal{U}_{\tilde{\gamma}_i} \quad \text{for} \quad i = 0, \dots, n$$

and therefore the invariant vector bundles $\mathcal{W}_0, \dots, \mathcal{W}_{n+1}$ are independent of the choice of $\gamma_0, \dots, \gamma_n$ in (2.4), (2.5) and (2.6). $\square$

Definition 2.7. We say that the evolution operator of (1.1) is nonuniformly exponentially bounded if there exist $K > 0$, $\varepsilon \geq 1$ and $a \geq 1$ with

$$\|\Phi(k, l)\| \leq K\alpha^{|k-l|}\varepsilon^{|l|}, \quad k, l \in \mathbb{Z}. \quad (2.7)$$

Lemma 2.8. Assume the evolution operator of (1.1) is nonuniformly exponentially bounded. Then $\Sigma_{NED}(A)$ is a bounded closed set and $\Sigma_{NED}(A) \subset [\frac{1}{a}, a]$.

Proof. Assume that (2.7) holds. Let $\gamma > a$ and $0 < \alpha := \frac{a}{\gamma} < 1$, then estimate (2.7) implies

$$\|\Phi_\gamma(k, l)\| \leq K\alpha^{k-l}\varepsilon^{|l|}, \quad k \geq l.$$

Therefore (1.1) admits a nonuniform exponential dichotomy with invariant projector $P = \text{Id}$. It follows that $\gamma \in \rho_{NED}(A)$ and also similarly for $0 < \gamma < \frac{1}{a}$, hence $\Sigma_{NED}(A) \subset [\frac{1}{a}, a]$. $\square$

Lemma 2.9. The evolution operator of (1.1) is nonuniformly exponentially bounded if and only if the nonuniform dichotomy spectrum $\Sigma_{NED}(A)$ of (1.1) is the disjoint union of n closed intervals where $0 \leq n \leq N$, i.e.,

$$\Sigma_{NED}(A) = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_{n-1}, b_{n-1}] \cup [a_n, b_n],$$

where $0 < a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n < \infty$. In addition, $\mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n = \mathbb{Z} \times \mathbb{R}^N$, $\mathcal{W}_0 = \mathcal{W}_{n+1} = \mathbb{Z} \times \{0\}$, where the sets $\mathcal{W}_0, \dots, \mathcal{W}_{n+1}$ are invariant vector bundles defined in Theorem 2.6.

Proof. Necessity. It is easy to know that $\Sigma_{NED}(A)$ is bounded from Lemma 2.8. Now we prove that $\Sigma_{NED}(A) \neq \emptyset$. It is easy to verify that $\mathcal{S}_\gamma = \text{im } P = \mathbb{Z} \times \mathbb{R}^N$ and $\mathcal{U}_\gamma = \ker P = \mathbb{Z} \times \{0\}$ for $\gamma > a$. Set

$$\gamma^* = \inf\{\gamma \in \rho_{NED}(A) : \mathcal{S}_\gamma = \mathbb{Z} \times \mathbb{R}^N\}.$$

Clearly, $\gamma^* \in [\frac{1}{a}, a]$. In addition, we have $\gamma^* \in \Sigma_{NED}(A)$. Otherwise, by using Lemma 2.3, there exists a neighborhood $\mathcal{J}$ of γ^* such that $\mathcal{J} \subset \rho_{NED}(A)$ and for any $\gamma < \gamma^*$ of $\mathcal{J}$ we have $\mathcal{S}_\gamma = \mathcal{S}_{\gamma^*} = \mathbb{Z} \times \mathbb{R}^N$, which is a contradiction to the definition of γ^* . So $\Sigma_{NED}(A) \neq \emptyset$, which means that

$$\Sigma_{NED}(A) = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_{n-1}, b_{n-1}] \cup [a_n, b_n].$$

Let $\gamma_0 \in (0, a_1)$, $\gamma_i \in (b_i, a_{i+1})$ for $i = 1, \dots, n-1$ and $\gamma_n \in (b_n, \infty)$. Clearly, from the proof of Lemma 2.8, we have $\mathcal{U}_{\gamma_n} = \mathbb{Z} \times \{0\}$ for $\gamma_n > b_n$ and $\mathcal{S}_{\gamma_0} = \mathbb{Z} \times \{0\}$ for $0 < \gamma_0 < a_1$. Therefore, $\mathcal{W}_0 = \mathcal{W}_{n+1} = \mathbb{Z} \times \{0\}$, and then it follows from Theorem 2.6 that $\mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n = \mathbb{Z} \times \mathbb{R}^N$.

Sufficiency. Let $\gamma_0 \in (0, a_1)$, $\gamma_i \in (b_i, a_{i+1})$ for $i = 1, \dots, n-1$ and $\gamma_n \in (b_n, \infty)$. Clearly, from the proof of Lemma 2.8, we have $\mathcal{S}_{\gamma_n} = \mathbb{Z} \times \mathbb{R}^N$ for $\gamma_n > b_n$ and $\mathcal{U}_{\gamma_0} = \mathbb{Z} \times \mathbb{R}^N$ for $0 < \gamma_0 < a_1$. This means that the invariant projectors associated to γ_0 and γ_n are $P_l = 0$ and $P_l = \text{Id}$ respectively.

Hence, there exist $K_0 > 1$, $0 < \alpha_0 < 1$ and $\varepsilon_0 > 1$ such that

$$\|\Phi_{\gamma_0}(k, l)\| \leq K_0\left(\frac{1}{\alpha_0}\right)^{k-l}\varepsilon_0^{|l|}, \quad k \leq l,$$

and there exist $K_n > 1$, $0 < \alpha_n < 1$ and $\varepsilon_n > 1$ such that

$$\|\Phi_{\gamma_n}(k, l)\| \leq K_n\alpha_n^{k-l}\varepsilon_n^{|l|}, \quad k \geq l.$$

Now taking $K = \max\{K_0, K_n\}$, $\varepsilon = \max\{\varepsilon_0, \varepsilon_n\}$ and $a = \max\{\frac{\alpha_0}{\gamma_0}, \alpha_n\gamma_n\}$, then we have

$$\|\Phi(k, l)\| \leq K\alpha^{|k-l|}\varepsilon^{|l|}, \quad \text{for } k, l \in \mathbb{Z},$$

which shows that the evolution operator of (1.1) is nonuniformly exponentially bounded. $\square$

From Proposition 1, we know $\Sigma_{NED}(A) \subset \Sigma_{ED}(A)$. Finally in this section, we present an example to illustrate that $\Sigma_{NED}(A) \neq \Sigma_{ED}(A)$ can occur.

Example 2.10. Given $\omega > 5a > 0$. Consider the scalar equation

$$u_{k+1} = A_k u_k \quad (2.8)$$

with

$$A_k = e^{-\omega + ak(-1)^k - a(k-1)(-1)^{(k-1)}}.$$

Then $\Sigma_{NED}(A) = [e^{-\omega-5a}, e^{-\omega+5a}]$ and $\Sigma_{ED}(A) = \mathbb{R}^+$.

Proof. The evolution operator of (2.8) is given by

$$\Phi(k, l) = e^{-\omega(k-l-1) - a(k-l-1)(-1)^{k-1} - al(-1)^{(k-1)} + al(-1)^l}.$$

For each $\gamma \in \mathbb{R}^+$ the evolution operator of

$$u_{k+1} = \frac{1}{\gamma} A_k u_k \quad (2.9)$$

is given by

$$\Phi_\gamma(k, l) = \left(\frac{1}{\gamma}\right)^{(k-l)} e^{-\omega(k-l-1) - a(k-l-1)(-1)^{k-1} - al(-1)^{(k-1)} + al(-1)^l}. \quad (2.10)$$

For any $\gamma \in (e^{(-\omega+5a)}, +\infty)$, it follows from (2.10) that

$$|\Phi_\gamma(k, l)| \leq e^{\omega-a} \left(\frac{e^{-\omega+a}}{\gamma}\right)^{k-l} e^{2a|l|}, \quad k \geq l, \quad (2.11)$$

which implies that the equation (2.9) admits a nonuniform exponential dichotomy with invariant projector $P = \text{Id}$, by setting

$$K = e^{\omega-a}, \quad \alpha = \frac{e^{-\omega+a}}{\gamma} < 1, \quad \varepsilon = e^{2a} > 0.$$

Thus,

$$(e^{-\omega+5a}, +\infty) \subset \rho_{NED}(A). \quad (2.12)$$

For any $\tilde{\gamma} \in (0, e^{-\omega-5a})$, it follows from (2.10) that

$$|\Phi_\gamma(k, l)| \leq e^{\omega+a} \left(\frac{e^{-\omega-a}}{\gamma}\right)^{k-l} e^{2a|l|}, \quad k \leq l, \quad (2.13)$$

which implies that (2.9) admits a nonuniform exponential dichotomy with $P = 0$, by taking

$$K = e^{\omega+a}, \quad \alpha = \frac{\gamma}{e^{-\omega-a}} < 1, \quad \varepsilon = e^{2a} > 0.$$

Thus,

$$(0, e^{-\omega-5a}) \subset \rho_{NED}(A). \quad (2.14)$$

It follows from (2.12) and (2.14) that

$$(0, e^{-\omega-5a}) \cup (e^{-\omega+5a}, +\infty) \subset \rho_{NED}(A),$$

which implies that

$$\Sigma_{NED}(A) \subset [e^{-\omega-5a}, e^{-\omega+5a}].$$

Next we show that

$$[e^{-\omega-5a}, e^{-\omega+5a}] \subset \Sigma_{NED}(A).$$

To do this, we first prove that $\gamma_1 = e^{-\omega+5a} \in \Sigma_{NED}(A)$. The evolution operator of the system

$$u_{k+1} = \frac{1}{\gamma_1} A_k u_k$$

is given as

$$\Phi_{\gamma_1}(k, l) = e^{\omega-a} e^{-a(k-l-1)(1+(-1)^{k-1}) - al(-1)^{(k-1)} + al(-1)^l}.$$

It is easy to see that there do not exist $K, \alpha > 0$ and $\varepsilon > 0$ such that

$$\|\Phi_{\gamma_1}(k, l)\| \leq K\alpha^{k-l}\varepsilon^{|l|}, \quad \text{for } k \geq l,$$

or

$$\|\Phi_{\gamma_1}(k, l)\| \leq K\left(\frac{1}{\alpha}\right)^{k-l}\varepsilon^{|l|}, \quad \text{for } k \leq l.$$

Therefore $\gamma_1 = e^{-\omega+5a} \in \Sigma_{NED}(A)$. In a similar manner, we can prove $\gamma_2 = e^{-\omega-5a} \in \Sigma_{NED}(A)$. We can see from Theorem 2.6 that (2.8) has at most one nonuniform dichotomy spectral interval, which means that $[e^{-\omega-5a}, e^{-\omega+5a}] \subset \Sigma_{NED}(A)$ and therefore $[e^{-\omega-5a}, e^{-\omega+5a}] = \Sigma_{NED}(A)$.

On the other hand, using a similar argument as in equations (1.6), we know that the nonuniform part $\varepsilon^{|l|}$ cannot be removed in the estimates (2.11) and (2.13). Therefore, (2.8) does not admit an exponential dichotomy, which means that $\Sigma_{ED}(A) = \mathbb{R}^+$. $\square$

3 Reducibility

In this section we utilize Theorem 2.6 to show a reducibility result. For the reducibility results in the setting of an exponential dichotomy, we refer the reader to [15, 24, 32] and the references therein. First we recall the definition of kinematic similarity given in Coppel [16] and Aulbach et al. [2].

Definition 3.1. Equation (1.1) is said to be kinematically similar to another equation

$$y_{k+1} = B_k y_k \tag{3.1}$$

with $k \in \mathbb{Z}$, if there exists an invertible matrix S_k with $\|S_k\| \leq M$ and $\|S_k^{-1}\| \leq M$ ($M > 0$), which satisfies the difference equation

$$S_{k+1}B_k = A_kS_k.$$

The change of variables $x_k = S_k y_k$ then transforms (1.1) into (3.1).

The next lemma is important to establish the reducibility results and its proof follows along the lines of the proof of Siegmund [32]. See also Coppel [16] and Aulbach et al. [2].

Lemma 3.2. [16, Chapter 5] Let P be an orthogonal projection ($P^T = P$) and let X be an invertible matrix function. Then there exists an invertible matrix function $S : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$ such that

$$S_k P S_k^{-1} = X_k P X_k^{-1}, \quad S_k Q S_k^{-1} = X_k Q X_k^{-1},$$

and

$$\|S_k\| \leq \sqrt{2},$$

$$\|S_k^{-1}\| \leq \left[\|X_k P X_k^{-1}\|^2 + \|X_k (I - P) X_k^{-1}\|^2 \right]^{\frac{1}{2}},$$

where $k \in \mathbb{Z}$ and $Q = \text{Id} - P$. Define

$$\tilde{R} : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}, \quad k \mapsto P X_k^T X_k P + [\text{Id} - P] X_k^T X_k [\text{Id} - P].$$

Then the mapping is a positive definite, symmetric matrix for every $k \in \mathbb{Z}$. Moreover, there is a unique function

$$R : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$$

of positive definite symmetric matrices $R_k, k \in \mathbb{Z}$, with

$$R_k^2 = \tilde{R}_k, \quad P R_k = R_k P.$$

We remark that in the setting of an exponential dichotomy the function $k \mapsto S_k^{-1}$ in Lemma 3.2 is bounded. However, in the setting of a nonuniform exponential dichotomy, S_k^{-1} can be unbounded, because $\|\Phi(k, k)P_k\| \leq K\varepsilon^k$ for $k \geq 0$. In fact, we can see that

$$\|S_k^{-1}\| \leq \sqrt{2}K\varepsilon^{|k|}. \quad (3.2)$$

To overcome the difficulty, we introduce a new version of non-degeneracy, so-called weak non-degeneracy and define the concept of weak kinematical similarity, which is a very natural notion if we note the fact (3.2).

Definition 3.3. $S : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$ is called weakly non-degenerate if there exists an $M = M(\varepsilon) > 0$ such that

$$\|S_k\| \leq M\varepsilon^{|k|} \quad \text{and} \quad \|S_k^{-1}\| \leq M\varepsilon^{|k|}, \quad \text{for all } k \in \mathbb{Z},$$

where ε is the same constant in (1.4)-(1.5).

Definition 3.4. If there exists a weakly non-degenerate matrix S_k such that

$$S_{k+1}B_k = A_kS_k,$$

then equation (1.1) is weakly kinematically similar to equation (3.1). For short, we denote (1.1) $\overset{w}{\sim}$ (3.1) or $A_k \overset{w}{\sim} B_k$.

We analogously denote kinematical similarity by (1.1) $\sim$ (3.1) or $A_k \sim B_k$.

Definition 3.5. Equation (1.1) is called reducible, if it is weakly kinematically similar to equation (3.1) whose coefficient matrix B_k has the block diagonal form

$$\begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^2 \end{pmatrix}, \quad (3.3)$$

where B_k^1 and B_k^2 are matrices of smaller size than B_k .

Before stating the main results of this section, we prove several preliminary lemmas.

Lemma 3.6. Assume that (1.1) admits a nonuniform exponential dichotomy. Then the projector of equation (1.1) can be chosen as $\tilde{P} = \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0_{N_2} \end{pmatrix}$ with $N_1 = \dim \operatorname{im} \tilde{P}$ and $N_2 = \dim \ker \tilde{P}$, and the fundamental matrix X_k can be chosen suitably such that the estimates (1.4)-(1.5) can be rewritten as

$$\|X_k \tilde{P} X_l^{-1}\| \leq K\alpha^{k-l}\varepsilon^{|l|}, \quad k \geq l, \quad (3.4)$$

and

$$\|X_k \tilde{Q} X_l^{-1}\| \leq K\left(\frac{1}{\alpha}\right)^{k-l}\varepsilon^{|l|}, \quad k \leq l, \quad (3.5)$$

where $\tilde{Q} = \operatorname{Id} - \tilde{P}$.

Proof. Let $n \in \mathbb{Z}$ be arbitrary but fixed. Note that the rank of the projector P_n is independent of $n \in \mathbb{Z}$ (see [9, Page 1100]), then there exists a nondegenerate matrix $T \in \mathbb{R}^{N \times N}$ such that

$$\tilde{P} := \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0_{N_2} \end{pmatrix} = TP_nT^{-1}$$

with $N_1 = \dim \operatorname{im} \tilde{P}$ and $N_2 = \dim \ker \tilde{P}$. Define

$$X_k := \Phi(k, n)T^{-1} \quad \text{for } k \in \mathbb{Z} \quad \text{and} \quad \tilde{P} := \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0_{N_2} \end{pmatrix} = TP_nT^{-1}.$$

Then

$$\|X_k \tilde{P} X_l^{-1}\| = \|\Phi(k, n) T^{-1} \tilde{P} T \Phi^{-1}(l, n)\| = \|\Phi(k, n) P_n \Phi^{-1}(l, n)\|. \quad (3.6)$$

On the other hand, we have

$$\begin{aligned} \|\Phi(k, l) P_l\| &= \|\Phi(k, n) \Phi(n, l) P_l\| \\ &= \|\Phi(k, n) P_n \Phi(n, l)\| \\ &= \|\Phi(k, n) P_n \Phi^{-1}(l, n)\|. \end{aligned} \quad (3.7)$$

It follows from (3.6) and (3.7) that (1.4)-(1.5) can be rewritten in the form (3.4)-(3.5). $\square$

Lemma 3.7. Assume that (1.1) admits a nonuniform exponential dichotomy with the form of estimates (3.4)-(3.5) and $\text{rank}(\tilde{P}) = N_1$, ($0 < N_1 < N$), and there exists a weakly non-degenerate transformation S_k such that $A_k \stackrel{w}{\sim} B_k$. Then system (3.1) also admits a nonuniform exponential dichotomy, and the projector has the same rank.

Proof. Suppose that S_k is weakly non-degenerate, which means that there exists $M = M(\varepsilon) > 0$ such that $\|S_k\| \leq M\varepsilon^{|k|}$ and $\|S_k^{-1}\| \leq M\varepsilon^{|k|}$ and such that $A_k \stackrel{w}{\sim} B_k$. Let $X_k = S_k Y_k$. It is easy to see that Y_k is the fundamental matrix of system (3.1). To prove that system (3.1) admits a nonuniform exponential dichotomy, we first consider the case $k \geq l$ and obtain

$$\begin{aligned} \|Y_k \tilde{P} Y_l^{-1}\| &= \|S_k^{-1} X_k \tilde{P} X_l^{-1} S_l\| \\ &\leq \|S_k^{-1}\| \cdot \|X_k \tilde{P} X_l^{-1}\| \cdot \|S_l\| \\ &\leq KM^2 \varepsilon^{|k|} \alpha^{k-l} \varepsilon^{2|l|} \\ &= KM^2 (\alpha \varepsilon^2)^{k-l} \varepsilon^{2l-2k+|k|+2|l|} \\ &\leq KM^2 (\alpha \varepsilon^2)^{k-l} \varepsilon^{l-k+3|l|} \\ &\leq KM^2 (\alpha \varepsilon^2)^{k-l} \varepsilon^{3|l|}, \quad k \geq l. \end{aligned} \quad (3.8)$$

A similar argument shows that

$$\|Y_k \tilde{Q} Y_l^{-1}\| \leq KM^2 \left(\frac{1}{\alpha \varepsilon^2}\right)^{k-l} \varepsilon^{3|l|}, \quad k \leq l. \quad (3.9)$$

Form (3.8) and (3.9), it is easy to see that system (3.1) admits a nonuniform exponential dichotomy. Clearly, the rank of the projector is N_1 . $\square$

Lemma 3.8. Assume that the systems (1.1) and (3.1) are weakly kinematically similar via S_k . If for a constant $\gamma \in \mathbb{R}^+$ the system (2.1) admits a nonuniform exponential dichotomy with constants $K > 0$, $0 < \alpha < 1$, $\varepsilon \geq 1$ and invariant projector P , then the system

$$y_{k+1} = \frac{1}{\gamma} B_k y_k \quad (3.10)$$

also admits a nonuniform exponential dichotomy.

Proof. Obviously, P is also an invariant projector for (1.1). The dichotomy estimates are equivalent to

$$\|X_k P X_l^{-1}\| \leq K \alpha^{k-l} \varepsilon^{|l|}, \quad k \geq l,$$

and

$$\|X_k P X_l^{-1}\| \leq K \left(\frac{1}{\alpha}\right)^{k-l} \varepsilon^{|l|}, \quad k \leq l.$$

Using Lemma 3.7, it is easy to see that

$$\|Y_k P Y_l^{-1}\| \leq K'_\gamma (\varepsilon \alpha)^{k-l} \varepsilon^{|l|}, \quad k \geq l,$$

and

$$\|Y_k P Y_l^{-1}\| \leq K'_\gamma \left(\frac{1}{\varepsilon \alpha}\right)^{k-l} \varepsilon^{|l|}, \quad k \leq l,$$

for some constant $K'_\gamma \geq 1$. Therefore, (3.10) admits a nonuniform exponential dichotomy. $\square$

The following result follows directly from Lemma 3.8.

Corollary 3.9. Assume that there exists a weakly non-degenerate transformation S_k such that $A_k \stackrel{w}{\sim} B_k$. Then $\Sigma_{NED}(A) = \Sigma_{NED}(B)$, i.e.,

$$\Sigma_{NED}(A) = \left\{ \begin{array}{c} [a_1, b_1] \\ \text{or} \\ (0, b_1] \end{array} \right\} \cup [a_2, b_2] \cup \cdots \cup [a_{n-1}, b_{n-1}] \cup \left\{ \begin{array}{c} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{array} \right\} = \Sigma_{NED}(B).$$

The following theorem states that if (1.1) admits a nonuniform exponential dichotomy, then there exists a weakly non-degenerate transformation such that $A_k \stackrel{w}{\sim} B_k$ and B_k has the block form (3.3), i.e., system (1.1) is reducible.

Theorem 3.10. Assume that (1.1) admits a nonuniform exponential dichotomy (not necessary) of the form (3.4)-(3.5) with the invariant projector satisfying $P_k \neq 0, \text{Id}$. Then (1.1) is reducible. More precisely, it is weakly kinematically similar to a decoupled system

$$x_{k+1} = \begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^2 \end{pmatrix} x_k \quad (3.11)$$

for some matrix functions

$$B^1 : \mathbb{Z} \rightarrow \mathbb{R}^{N_1 \times N_1} \quad \text{and} \quad B^2 : \mathbb{Z} \rightarrow \mathbb{R}^{N_2 \times N_2}$$

where $N_1 := \dim \text{im } \tilde{P}$ and $N_2 := \dim \ker \tilde{P}$.

Proof. Since equation (1.1) admits a nonuniform exponential dichotomy of the form (1.4)-(1.5) with the invariant projector satisfying $P_k \neq 0, \text{Id}$, by Lemma 3.6, one can choose a suitable fundamental matrix X_k and the projector $\tilde{P} = \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0 \end{pmatrix}$, ($0 < N_1 < N$) such that the estimates (3.4)-(3.5) hold. By Lemma 3.2 and the estimates (3.4)-(3.5), there exists an $M = M(\varepsilon) > 0$ large enough such that

$$\|S_k\| \leq \sqrt{2} \leq M\varepsilon^{|k|},$$

$$\|S_k^{-1}\| \leq [\|X_k \tilde{P} X_k^{-1}\|^2 + \|X_k(\text{Id} - \tilde{P})X_k^{-1}\|^2]^{\frac{1}{2}} \leq \sqrt{2}K\varepsilon^{|k|}.$$

Thus, S is weakly non-degenerate. Setting

$$B_k = R_{k+1} R_k^{-1},$$

where R_k is defined in Lemma 3.2 and $X_k = S_k R_k$. Obviously, R_k is the fundamental matrix of the linear system

$$y_{k+1} = B_k y_k.$$

Now we need to show that $A_k \stackrel{w}{\sim} B_k$ and B_k has the block diagonal form

$$B_k = \begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^2 \end{pmatrix}, \quad \text{for } k \in \mathbb{Z}.$$

First, we show that $A_k \stackrel{w}{\sim} B_k$. In fact,

$$\begin{aligned} S_{k+1} B_k &= X_{k+1} R_{k+1}^{-1} B_k \\ &= A_k X_k R_k^{-1} B_k \\ &= A_k S_k, \end{aligned}$$

which implies that $A_k \stackrel{w}{\sim} B_k$.

Now we show that system (1.1) is weakly kinematically similar to (3.11). By Lemma 3.2, R_{k+1} and R_k^{-1} commute with the matrix $\tilde{P}$ for every $k \in \mathbb{Z}$. It follows that

$$\tilde{P} B_k = B_k \tilde{P} \quad (3.12)$$

for all $k \in \mathbb{Z}$. Now we decompose $B_k : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$ into four functions

$$\begin{aligned} B_k^1 : \mathbb{Z} &\rightarrow \mathbb{R}^{N_1 \times N_1}, & B_k^2 : \mathbb{Z} &\rightarrow \mathbb{R}^{N_2 \times N_2}, \\ B_k^3 : \mathbb{Z} &\rightarrow \mathbb{R}^{N_1 \times N_2}, & B_k^4 : \mathbb{Z} &\rightarrow \mathbb{R}^{N_2 \times N_1}, \end{aligned}$$

with

$$B_k = \begin{pmatrix} B_k^1 & B_k^3 \\ B_k^4 & B_k^2 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Identity (3.12) implies that

$$\begin{pmatrix} B_k^1 & B_k^3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_k^1 & 0 \\ B_k^4 & 0 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Therefore $B_k^3 \equiv 0$ and $B_k^4 \equiv 0$. Thus B_k has the block form

$$B_k = \begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^2 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Now the proof is finished. $\square$

From Theorem 3.10, we know that if (1.1) admits a nonuniform exponential dichotomy, then there exists a weakly non-degenerate transformation S_k such that $A_k \stackrel{w}{\sim} B_k$ and B_k has two blocks of the form (3.3). Now we are in a position to prove the reducibility result.

Theorem 3.11. Assume that (1.1) admits a nonuniform exponential dichotomy. Due to Theorem 2.6, the dichotomy spectrum is either empty or the disjoint union of n closed spectral intervals $\mathcal{J}_1, \dots, \mathcal{J}_n$ with $1 \leq n \leq N$, i.e.,

$$\Sigma_{NED}(A) = \emptyset \quad (n = 0) \quad \text{or} \quad \Sigma_{NED}(A) = \mathcal{J}_1 \cup \dots \cup \mathcal{J}_n.$$

Then there exists a weakly kinematic similarity action $S : \mathbb{Z} \rightarrow \mathbb{R}^{N \times N}$ between (1.1) and a block diagonal system

$$x_{k+1} = \begin{pmatrix} B_k^0 & & \\ & \ddots & \\ & & B_k^{n+1} \end{pmatrix} x_k$$

with $B^i : \mathbb{Z} \rightarrow \mathbb{R}^{N_i \times N_i}$, $N_i = \dim \mathcal{W}_i$, and

$$\Sigma_{NED}(B^0) = \emptyset, \Sigma_{NED}(B^1) = \mathcal{J}_1, \dots, \Sigma_{NED}(B^n) = \mathcal{J}_n, \Sigma_{NED}(B^{n+1}) = \emptyset.$$

Proof. If for any $\gamma \in \mathbb{R}^+$, system (2.1) admits a nonuniform exponential dichotomy, then $\Sigma_{NED}(A) = \emptyset$. Conversely, for any $\gamma \in \mathbb{R}^+$, system (2.1) does not admit a nonuniform exponential dichotomy, then $\Sigma_{NED}(A) = \mathbb{R}^+$. Now, we prove the theorem for the nontrivial case ($\Sigma_{NED}(A) \neq \emptyset$ and $\Sigma_{NED}(A) \neq \mathbb{R}^+$).

Recall that the resolvent set $\rho_{NED}(A)$ is open and therefore the dichotomy spectrum $\Sigma_{NED}(A)$ is the disjoint union of closed intervals. Using Theorem 2.6, we can assume

$$\mathcal{J}_1 = \left\{ \begin{array}{c} [a_1, b_1] \\ \text{or} \\ (0, b_1] \end{array} \right\}, \mathcal{J}_2 = [a_2, b_2], \dots, \mathcal{J}_{n-1} = [a_{n-1}, b_{n-1}], \mathcal{J}_n = \left\{ \begin{array}{c} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{array} \right\}$$

with $0 < a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_n \leq b_n$.

If $\mathcal{J}_1 = [a_1, b_1]$ is a spectral interval, then $(0, \gamma_0) \subset \rho_{NED}(A)$ and $\mathcal{W}_0 = \mathcal{S}_{\gamma_0}$ for some $\gamma_0 < a_1$ due to Theorem 2.6, which implies that

$$x_{k+1} = \frac{1}{\gamma_0} A_k x_k$$

admits a nonuniform exponential dichotomy, let its invariant projector be denoted by $\tilde{P}_0$. By Theorem 3.10 and Corollary 3.9, there exists a weakly non-degenerate transformation $x_k = S_k^0 x_k^{(0)}$ with $\|S_k^0\| \leq M_0 \varepsilon^{|k|}$ and $\|(S_k^0)^{-1}\| \leq M_0 \varepsilon^{|k|}$ for some positive constant $M_0 = M_0(\varepsilon)$ and such that $A_k \stackrel{w}{\sim} A_k^0$ and A_k^0 has two blocks of

the form $A_k^0 = \begin{pmatrix} B_k^0 & 0 \\ 0 & B_k^{0,*} \end{pmatrix}$ with $\dim B_k^0 = \dim \operatorname{im} \tilde{P}_0 = \dim \mathcal{S}_{\gamma_0} = \dim \mathcal{W}_0 =: N_0$ due to Theorem 3.10, Lemma 2.2 and Theorem 2.6. If $\mathcal{I}_1 = (0, b_1]$ is a spectral interval, a block B_k^0 is omitted.

Now we consider the following system

$$x_{k+1}^{(0)} = A_k^0 x_k^{(0)} = \begin{pmatrix} B_k^0 & 0 \\ 0 & B_k^{0,*} \end{pmatrix} x_k^{(0)}.$$

By using Lemma 2.5, we take $\gamma_1 \in (b_1, a_2)$. In view of $(b_1, a_2) \subset \rho_{NED}(B_k^{0,*})$, $\gamma_1 \in \rho_{NED}(B_k^{0,*})$, which implies that

$$x_{k+1}^{(0)} = \frac{1}{\gamma_1} \begin{pmatrix} B_k^0 & 0 \\ 0 & B_k^{0,*} \end{pmatrix} x_k^{(0)}$$

admits a nonuniform exponential dichotomy. Its invariant projector $\tilde{P}_1$ satisfies $\tilde{P}_1 \neq 0, I$. Similarly, by Theorem 3.10 and Corollary 3.9, there exists a weakly non-degenerate transformation

$$x_k^{(0)} = S_k^1 x_k^{(1)} = \begin{pmatrix} I_{N_0} & 0 \\ 0 & \tilde{S}_k^1 \end{pmatrix} x_k^{(1)}$$

with $\|\tilde{S}_k^1\| \leq M_1 \varepsilon^{|k|}$ and $\|(\tilde{S}_k^1)^{-1}\| \leq M_1 \varepsilon^{|k|}$ for some positive constant $M_1 = M_1(\varepsilon)$ and such that $B_k^{0,*} \stackrel{w}{\sim} \tilde{B}_k^{0,*}$ and $\tilde{B}_k^{0,*}$ has two blocks of the form $\tilde{B}_k^{0,*} = \begin{pmatrix} B_k^1 & 0 \\ 0 & B_k^{1,*} \end{pmatrix}$ with $\dim B_k^1 = \dim \operatorname{im} \tilde{P}_1 = \dim \mathcal{S}_{\gamma_1} \geq \dim(\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1}) = \dim \mathcal{W}_1 =: N_1$ due to Theorem 3.10, Lemma 2.2 and Theorem 2.6. In addition, using Theorem 3.10 and Corollary 3.9, we have

$$\Sigma_{NED}(B_k^1) = \begin{cases} [a_1, b_1] \\ \text{or} \\ (0, b_1] \end{cases}, \quad \Sigma_{NED}(B_k^{1,*}) = [a_2, b_2] \cup \dots \cup [a_{n-1}, b_{n-1}] \cup \begin{cases} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{cases}.$$

Now we can construct a weakly non-degenerate transformation $x_k = \tilde{S}_k x_k^{(1)}$ with $\tilde{S}_k = S_k^0 S_k^1 = S_k^0 \begin{pmatrix} I_{N_0} & 0 \\ 0 & \tilde{S}_k^1 \end{pmatrix}$, where $\|\tilde{S}_k\| \leq M_0 M_1 \varepsilon^{2|k|}$ and $\|\tilde{S}_k^{-1}\| \leq M_0 M_1 \varepsilon^{2|k|}$. Then $A_k \stackrel{w}{\sim} A_k^1$ and A_k^1 has three blocks of the form

$$A_k^1 = \begin{pmatrix} B_k^0 & & \\ & B_k^1 & \\ & & B_k^{1,*} \end{pmatrix}.$$

Applying similar procedures to $\gamma_2 \in (b_2, a_3)$, $\gamma_3 \in (b_3, a_4), \dots$, we can construct a weakly non-degenerate transformation $x_k = S_k x_k^{(n+1)}$ with

$$S_k = S_k^0 \begin{pmatrix} I_{N_0} & 0 \\ 0 & \tilde{S}_k^1 \end{pmatrix} \begin{pmatrix} I_{N_0+N_1} & 0 \\ 0 & \tilde{S}_k^2 \end{pmatrix} \dots \begin{pmatrix} I_{N_0+\dots+N_{n-1}} & 0 \\ 0 & \tilde{S}_k^n \end{pmatrix}$$

such that $\|S_k\| \leq M_\varepsilon \varepsilon^{n|k|}$ and $\|S_k^{-1}\| \leq M_\varepsilon \varepsilon^{n|k|}$ with $M_\varepsilon = M_0 \times \dots \times M_n$. Now we can prove

$$A_k \stackrel{w}{\sim} A_k^n := B_k = \begin{pmatrix} B_k^0 & & \\ & \ddots & \\ & & B_k^{n+1} \end{pmatrix}$$

with locally integrable functions $B^i : \mathbb{Z} \rightarrow \mathbb{R}^{N_i \times N_i}$ and

$$\Sigma_{NED}(B^0) = \emptyset, \Sigma_{NED}(B^1) = \mathcal{I}_1, \dots, \Sigma_{NED}(B^n) = \mathcal{I}_n, \Sigma_{NED}(B^{n+1}) = \emptyset.$$

Finally, we show that $N_i = \dim \mathcal{W}_i$. From the claim above, we note that $\dim B_k^0 = \dim \mathcal{W}_0$, $\dim B_k^1 \geq \dim \mathcal{W}_1, \dots, \dim B_k^n \geq \dim \mathcal{W}_n$, $\dim B_k^{n+1} = \dim \mathcal{W}_{n+1}$ and using Theorem 2.6 this yields

$\dim \mathcal{W}_0 + \cdots + \dim \mathcal{W}_{n+1} = N$, so $\dim B_k^i = \dim \mathcal{W}_i$ for $i = 0, \dots, n+1$. Now the proof is finished. $\square$

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References

- [1] B. Aulbach, The fundamental existence theorem on invariant fiber bundles, *J. Difference Equ. Appl.* 3 (1998), 267-312.
- [2] B. Aulbach, N. V. Minh, P. P. Zabrejko, The concept of spectral dichotomy for linear difference equations, *J. Math. Anal. Appl.* 185 (1994), 275-287.
- [3] B. Aulbach, N. V. Minh, The concept of spectral dichotomy for linear difference equations II, *J. Difference Equ. Appl.* 2 (1996), 251-262.
- [4] B. Aulbach, S. Siegmund, A spectral theory for nonautonomous difference equations, *New trends in difference equations* (Temuco, 2000), 45-55, Taylor & Francis, London, 2002.
- [5] L. Barreira, Ya. Pesin, *Lyapunov Exponents and Smooth Ergodic Theory*, University Lecture Series 23, Amer. Math. Soc. 2002.
- [6] L. Barreira, C. Valls, *Stability of Nonautonomous Differential Equations*, Lecture Notes in Mathematics, vol. 1926, Springer, 2008.
- [7] L. Barreira, D. Dragičević, C. Valls, Nonuniform spectrum on Banach spaces, *Adv. Math.* 321 (2017), 547-591.
- [8] L. Barreira, C. Valls, Spectral theory for invertible cocycles under nonuniform hyperbolicity, *São Paulo J. Math. Sci.* 12 (2018), 6-17.
- [9] A. Berger, T.S. Doan, S. Siegmund, A definition of spectrum for differential equations on finite time, *J. Differential Equations*, 246 (2009), 1098-1118.
- [10] A. Carvalho, J. Langa, J. Robinson, *Attractors for infinite-dimensional non-autonomous dynamical systems*, Springer, 2013.
- [11] A. Castañeda, I. Huerta, Nonuniform almost reducibility of nonautonomous linear differential equations, *J. Math. Anal. Appl.* 485 (2020), 123822, 22 pp.
- [12] A. Castañeda, G. Robledo, Dichotomy spectrum and almost topological conjugacy on nonautonomous unbounded difference systems, *Discrete Contin. Dyn. Syst.* 38 (2018), 2287-2304.
- [13] S. N. Chow, H. Leiva, Dynamical spectrum for time dependent linear systems in Banach spaces, *Japan J. Indust. Appl. Math.* 11 (1994), 379-415.
- [14] J. Chu, F. Liao, S. Siegmund, Y. Xia, W. Zhang, Nonuniform dichotomy spectrum and reducibility for nonautonomous equations, *Bull. Sci. Math.* 139 (2015), 538-557.
- [15] W. A. Coppel, Dichotomies and reducibility, *J. Differential Equations*, 3 (1967), 500-521.
- [16] W. A. Coppel, *Dichotomy in stability theory*, Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, New York/Berlin, 1978.
- [17] T. S. Doan, D. Karrasch, T. Y. Nguyen, S. Siegmund, A unified approach to finite-time hyperbolicity which extends finite-time Lyapunov exponents, *J. Differential Equations*, 252 (2012), 5535-5554.
- [18] T. S. Doan, K. J. Palmer, M. Rasmussen, The Bohl spectrum for linear nonautonomous differential equations, *J. Dynam. Differential Equations* 29 (2017), 1459-1485.
- [19] I. Gohberg, M. A. Kaashoek, J. Kos, Classification of linear time-varying difference equations under kinematic similarity, *Integral Equations Operator Theory*, 25 (1996), 445-480.
- [20] W. Li, J. Llibre, H. Wu, Polynomial and linearized normal forms for almost periodic difference systems, *J. Difference Equ. Appl.* 15 (2009), 927-948.
- [21] W. Li, J. Llibre, H. Wu, Normal forms for almost periodic differential systems, *Ergodic Theory Dynam. Systems* 29 (2009), 637-656.
- [22] J. Massera, J. Schäffer, Linear differential equations and function spaces, in: *Pure and Applied Mathematics*, vol. 21, Academic Press, 1966.
- [23] D. C. Nguyen, S. Siegmund, Dichotomy spectrum of nonautonomous linear stochastic differential equations, *Stoch. Dyn.* 2 (2002), 175-201.
- [24] K. J. Palmer, On the reducibility of almost periodic systems of linear differential equations, *J. Differential Equations*, 36 (1980), 374-390.

- [25] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, *Math. Z.* 32 (1930), 703-728.
- [26] C. Pötzsche, A note on the dichotomy spectrum, *J. Difference Equ. Appl.* 15 (2009), 1021-1025.
- [27] C. Pötzsche, Fine structure of the dichotomy spectrum, *Integral Equations Operator Theory*, 73 (2012), 107-151.
- [28] R. Sacker, G. Sell, Existence of dichotomies and invariant splitting for linear differential systems I [II, III], *J. Differential Equations*, 15 (1974), 429-458 [22 (1976), 478-496, 497-522].
- [29] R. J. Sacker, G. R. Sell, A spectral theory for linear differential systems, *J. Differential Equations*, 27 (1978), 320-358.
- [30] S. Siegmund, *Spektral-Theorie, glatte Faserungen und Normalformen für Differentialgleichungen vom Carathéodory-Typ*, Dissertation, Augsburger Mathematisch-Naturwissenschaftliche Schriften 30, Wißner Verlag, Augsburg, 1999.
- [31] S. Siegmund, Dichotomy spectrum for nonautonomous differential equations, *J. Dynam. Differential Equations*, 14 (2002), 243-258.
- [32] S. Siegmund, Reducibility of nonautonomous linear differential equations, *J. London Math. Soc.* 65 (2002), 397-410.
- [33] H. Wu, W. Li, Poincaré type theorems for non-autonomous systems, *J. Differential Equations*, 245 (2008), 2958-2978.
- [34] Y. Xia, Y. Bai, D. O'Regan, A new method to prove the nonuniform dichotomy spectrum theorem in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* 147 (2019), 3905-3917.
- [35] X. Zhang, Nonuniform dichotomy spectrum and normal forms for nonautonomous differential systems, *J. Funct. Anal.* 267 (2014), 1889-1916.