

Research article

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The structure of \mathcal{A} -free measures revisited

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Abstract: We refine a recent result on the structure of measures satisfying a linear partial differential equation $\mathcal{A}\mu = \sigma$, μ, σ are Radon measures, considering the measure $\mu(\mathbf{x}) = g(\mathbf{x})d\mathbf{x} + \mu_{us}(\tilde{\mathbf{x}})(\mu_s(\tilde{\mathbf{x}}) + d\tilde{\mathbf{x}})$ where $\mathbf{x} = (\tilde{\mathbf{x}}, \tilde{\mathbf{x}}) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$, μ_{us} is a uniformly singular measure in $\tilde{\mathbf{x}}_0$ and the measure μ_s is a singular measure. We proved that for μ_{us} -a.e. $\tilde{\mathbf{x}}_0$ the range of the Radon-Nykodim derivative $\tilde{f}(\tilde{\mathbf{x}}_0) = \frac{d\mu_{us}}{d|\mu_{us}|}(\tilde{\mathbf{x}}_0)$ is in the set $\cap_{\tilde{\xi} \in \tilde{P}} \text{Ker} \mathbf{A}^{\tilde{P}}(\tilde{\xi})$ and, if μ_s is different to zero, for μ_s -a.e. $\tilde{\mathbf{x}}_0$ the range of the Radon-Nykodim derivative $\tilde{f}(\tilde{\mathbf{x}}_0) = \frac{d\mu_s}{d|\mu_s|}(\tilde{\mathbf{x}}_0)$ is in the set $\cup_{\tilde{\xi} \in \tilde{P}} \text{Ker} \mathbf{A}^{\tilde{P}}(\tilde{\xi})$ where $\tilde{P} \times \tilde{P} = P$ is a manifold determined by the main symbol $\mathbf{A}^P = \mathbf{A}^{\tilde{P}} \cdot \mathbf{A}^{\tilde{P}}$ of the operator \mathcal{A} .

Keywords: Structure of measures

MSC: 35D30

1 Introduction

In the paper, we consider a finite Radon measure $\mu = (\mu_1, \dots, \mu_m)$ defined on \mathbb{R}^d satisfying the system of partial differential equations

$$\mathcal{A}\mu = \sum_{\alpha \in I} \partial^\alpha (A_\alpha \mu) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d; \mathbb{R}^n) \tag{1.1}$$

where $I = I_1 \times I_2 \times \dots \times I_n \subset \{\alpha = (\alpha_1, \dots, \alpha_d) : \alpha_s \in \mathbb{N} \cup \{0\}, s = 1, \dots, d\}^n$ is a set of multi-indexes, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_d}^{\alpha_d}$, and $A_\alpha : \mathbb{R}^d \rightarrow M^{n \times m}$ are smooth mappings from \mathbb{R}^d into the space of real $n \times m$ matrices. Written coordinate-wise, we actually have the following system of equations

$$\mathcal{A}_j \mu = \sum_{\alpha \in I_j} \sum_{r=1}^m \partial^\alpha (a_{jr}^\alpha \mu_r) = 0, \quad j = 1, \dots, n, \tag{1.2}$$

where $I_j \subset \{\alpha = (\alpha_1, \dots, \alpha_d) : \alpha_s \in \mathbb{N} \cup \{0\}, s = 1, \dots, d\}$. Denote by $A_j, j = 1, \dots, n$, the principal symbol of the operator \mathcal{A}_j given by

$$A_j(\mathbf{x}, \xi) = \sum_{\alpha \in I'_j} \sum_{r=1}^m a_{jr}^\alpha(\mathbf{x})(2\pi i \xi)^\alpha, \quad I'_j \subset I_j. \tag{1.3}$$

The sum given above is taken over all terms from (1.2) whose order of derivative α is not dominated by any other multi-index from I_j . As usual, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$ for $\alpha = (\alpha_1, \dots, \alpha_d)$, and $|\alpha| = \alpha_1 + \dots + \alpha_d$.

For instance, for the (scalar) operator $\mathcal{A} = \partial_{x_1} + \partial_{x_2} + \partial_{x_2}^2$, we have $I = I_1 = \{(1, 0), (0, 1), (0, 2)\}$ and $I' = I'_1 = \{(1, 0), (0, 2)\}$.

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Let us emphasize the fact that the equation (1.1) includes the case

$$\mathcal{A}\mu = \sigma, \tag{1.4}$$

where $\sigma \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^n)$. Namely, regarding the equation (1.1) we may consider the measure $\tilde{\mu} = (\mu, \sigma) \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^{m+n})$ and the equation $\tilde{\mathcal{A}}\tilde{\mu} = 0$ (where 0th-order term was added to $\tilde{\mathcal{A}}$) which is equivalent to (1.4).

We are interested in the range of the Radon-Nikodym derivatives $\tilde{f}(\tilde{\mathbf{x}}) = \frac{d\mu_{us}}{d|\mu_{us}|}(\tilde{\mathbf{x}})$ and $\bar{f}(\tilde{\mathbf{x}}) = \frac{d\mu_s}{d|\mu_s|}(\tilde{\mathbf{x}})$ where the measures μ_{us} and μ_s are the parts of the measure

$$\mu = g(\mathbf{x})d\mathbf{x} + \mu_{us}(\tilde{\mathbf{x}})(\mu_s(\tilde{\mathbf{x}}) + d\tilde{\mathbf{x}}), \quad g \in L^1(\mathbb{R}^d), \tag{1.5}$$

satisfying (1.1). The measure μ_s is a singular measure while μ_{us} is uniformly singular measure. Roughly speaking, we require that μ_s is singular with respect to every of the variables. For instance, such a condition is not fulfilled by the measure $\mu = \delta(x_1 - x_2)dx_2$ since it is not singular with respect to x_2 . However, if we introduce the change of variables $z_1 = x_1 - x_2$ and $z_2 = x_2$, then μ becomes $\delta(z_1)dz_2$ and we have the combination of the uniformly singular measure and a regular measure.

Definition 1. We say that the measure μ_s is uniformly singular if for μ_s -almost every $\mathbf{x} \in \mathbb{R}^d$ there exist real positive functions $\alpha(\varepsilon), \beta(\varepsilon), \varepsilon \in \mathbb{R}$, satisfying $\frac{\alpha(\varepsilon)}{\varepsilon} \rightarrow 0, \frac{\varepsilon}{\beta(\varepsilon)} \rightarrow 0$, and a family of balls $B(\mathbf{x}, \alpha(\varepsilon))$ such that $\lim_{\varepsilon \rightarrow 0} \frac{\mu_s(B(\mathbf{x}, \beta(\varepsilon)) \setminus B(\mathbf{x}, \alpha(\varepsilon)))}{|\mu_s|(B(\mathbf{x}, \alpha(\varepsilon)))} = 0$.

For instance, it is clear that the measure $\delta(x_1)\delta(x_2)$ satisfies the latter condition with $E_\varepsilon = \{(0, 0)\}$ and arbitrary $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ satisfying $\frac{\alpha(\varepsilon)}{\varepsilon} \rightarrow 0$ and $\frac{\varepsilon}{\beta(\varepsilon)} \rightarrow 0$. In general, a measure supported on the set whose Hausdorff dimension is less than $n - 1$ is a candidate for the uniformly singular measure (see below for further explanations). It is not difficult to see that $\delta(x_1)dx_2$ is a singular measure on \mathbb{R}^2 , but not uniformly singular.

If we put $k = 0$ i.e. $m = d$ (implying that we do not have uniformly singular part: $\mu = g(\mathbf{x})d\mathbf{x} + \mu_s$) then the problem is solved elegantly in [3] confirming the conjecture from [1] that for the k -th order operator \mathcal{A} , the function $\bar{f}(\mathbf{x}) = \frac{d\mu_s}{d|\mu_s|}(\mathbf{x})$ must take values in the wave cone $\Lambda_{\mathcal{A}} = \cup_{|\xi|=1} \text{Ker} \mathbf{A}^k(\xi)$ where $\mathbf{A}^k(\xi)$ is the sum of all symbols of order k (see [3] for details and thorough information concerning history and applications of this issue; in particular in the calculus of variations and geometric measure theory).

In the case when μ_{us} is nontrivial, we are able to prove a stronger result as announced in the abstract and formally introduced in Theorem 2. However, in the latter case, we have the measure of very special form which actually separates variables. This shortens space of measures that fit into our considerations, but the space is far from trivial. For instance, consider the singular measure $\mu = \delta(x_1 - x_2)dx_2$. This measure is not uniformly singular and it satisfies the equation

$$\partial_{x_1}\mu + \partial_{x_2}\mu = 0$$

and after introducing the change of variables $x_1 - x_2 = y_1$ and $x_2 = y_2$ we reduce the measure μ on the form (1.5). Also, the $(n - 1)$ -dimensional Hausdorff measure that can be locally represented in the form $\delta(x_1 - g(x_2, \dots, x_d))dx_2 \dots dx_d$ and, after the change of variables $z_1 = x_1 - g(x_2, \dots, x_d), x_j = z_j, j = 2, \dots, d$, it gets the form (1.5). Moreover, if we augment (1.1) with initial conditions involving an uniformly singular measure, then, at least in the case of first order scalar equations, the solution will contain the uniformly singular measure as well (since the solution is given along characteristics).

To continue, we assume that all the principal symbols $A_j, j = 1, \dots, n$, can be represented in the form

$$A_j(\mathbf{x}, \xi) = \tilde{A}_j(\tilde{\mathbf{x}}, \tilde{\xi}) \cdot \bar{A}_j(\tilde{\mathbf{x}}, \tilde{\xi}). \tag{1.6}$$

For instance, the matrix A_j can be of the form $\xi_1 \xi_2$ where $\tilde{A}_j(\tilde{\xi}) = \xi_1$ and $\bar{A}_j(\tilde{\xi}) = \xi_2$. The operator determined by such a matrix A_1 is actually a hyperbolic operator.

The latter restriction essentially means that we can separate variables $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ while for the dual variables ξ , this means that we first move in the direction of $\tilde{\xi}$ determined by the matrix \bar{A}_j , and then in the direction $\tilde{\xi}$ determined by the matrix \tilde{A}_j .

Next, we assume that there exist multi-indices $\beta^j = (\beta_1^j, \dots, \beta_k^j) \in \mathbb{N}_+^k$ (depending on $j = 1, \dots, n$) and $\beta = (\beta_{k+1}, \dots, \beta_d) \in \mathbb{N}_+^{d-k}$ (independent of $j = 1, \dots, n$), $(\beta_1^j, \dots, \beta_k^j, \beta_{k+1}, \dots, \beta_d) \in I_j^j$ such that for any positive $\lambda \in \mathbb{R}$ the following homogeneity assumption holds for every $j = 1, \dots, n$:

$$\begin{aligned} \tilde{A}_j(\tilde{\mathbf{x}}, \lambda^{\beta_1^j} \xi_1, \dots, \lambda^{\beta_k^j} \xi_k) &= \lambda \tilde{A}_j(\tilde{\mathbf{x}}, \tilde{\xi}), \\ \bar{A}_j(\tilde{\mathbf{x}}, \lambda^{\beta_{k+1}} \xi_{k+1}, \dots, \lambda^{\beta_d} \xi_d) &= \lambda \bar{A}_j(\tilde{\mathbf{x}}, \tilde{\xi}) \end{aligned} \tag{1.7}$$

implying that

$$\begin{aligned} \alpha_r^j \beta_r^j &= 1 \text{ for every } r \in 1, \dots, k \text{ and} \\ \alpha_r \beta_r &= 1 \text{ for every } r \in k+1, \dots, d. \end{aligned} \tag{1.8}$$

We then introduce the homogeneity manifolds:

$$\begin{aligned} \tilde{P}_j &= \{ \tilde{\xi} \in \mathbb{R}^k : |\xi_1|^{1/\beta_1^j} + \dots + |\xi_k|^{1/\beta_k^j} = 1 \} \\ \bar{P} &= \{ \tilde{\xi} \in \mathbb{R}^{d-k} : |\xi_{k+1}|^{1/\beta_{k+1}} + \dots + |\xi_d|^{1/\beta_d} = 1 \} \end{aligned}$$

and the corresponding projections

$$\begin{aligned} \pi_j(\tilde{\xi}) &= \left(\frac{\xi_1}{(|\xi_1|^{1/\beta_1^j} + \dots + |\xi_k|^{1/\beta_k^j})^{\beta_1^j}}, \dots, \frac{\xi_k}{(|\xi_1|^{1/\beta_1^j} + \dots + |\xi_k|^{1/\beta_k^j})^{\beta_k^j}} \right), \tilde{\xi} \in \mathbb{R}^k \\ \pi(\tilde{\xi}) &= \left(\frac{\xi_{k+1}}{(|\xi_{k+1}|^{1/\beta_{k+1}} + \dots + |\xi_d|^{1/\beta_d})^{\beta_{k+1}}}, \dots, \frac{\xi_d}{(|\xi_{k+1}|^{1/\beta_{k+1}} + \dots + |\xi_d|^{1/\beta_d})^{\beta_d}} \right), \tilde{\xi} \in \mathbb{R}^{d-k}. \end{aligned}$$

Finally, we can formulate the main theorem of the paper.

Theorem 2. *Let μ be a solution to (1.1) of the form (1.5). Then, for $|\mu_{us}|$ -almost every $\tilde{\mathbf{x}} \in \mathbb{R}^k$ and $|\mu^s|$ -almost every $\tilde{\mathbf{x}} \in \mathbb{R}^{d-k}$ there exists $\tilde{\xi} \in \bar{P}$ such that:*

$$\sum_{\alpha \in I_j^j} \sum_{l=1}^m a_{jl}^\alpha(\mathbf{x}) (2\pi i \xi)^\alpha \tilde{f}_l(\tilde{\mathbf{x}}) \bar{f}_l(\tilde{\mathbf{x}}) = 0, \quad \mathcal{H}^{k-1} - \text{a.e. } \tilde{\xi} \in \tilde{P}_j, \quad j = 1, \dots, n. \tag{1.9}$$

If for all $j = 1, \dots, n$, the manifolds \tilde{P}_j would be the same, say \tilde{P} , and we have the same set of dominating multi-indices $I' = I_j^j, j = 1, \dots, n$, then we could rewrite (1.9) in the form

$$\tilde{f}(\tilde{\mathbf{x}}) \bar{f}(\tilde{\mathbf{x}}) \in \cap_{\tilde{\xi} \in \tilde{P}} \cup_{\xi \in \tilde{P}} \text{Ker} \left(\sum_{\alpha \in I'} (2\pi i \xi)^\alpha A_\alpha(\mathbf{x}) \right) \text{ for } \mu_{us} \otimes \mu_s \text{ a.e. } \mathbf{x} \in \mathbb{R}^d \tag{1.10}$$

for appropriate matrices $A_\alpha, \alpha \in I'$. If we do not have the uniformly singular part and \tilde{P} is the sphere in \mathbb{R}^d , then (1.10) is actually the statement of the main result from [3]. In their elegant proof, the main tool was the concept of tangent measures in the sense of [7]. We will pursue this approach here as well but with slightly more refined arguments which take into account properties of the measure (splitting on singular and uniformly singular part) as well as properties of the operator \mathcal{A} itself (principal symbols do not have to be defined on a sphere).

We will dedicate the last section to the proof of the theorem. In the next section, we shall prove it in the case of first order constant coefficient operators and the scalar measure which captures all the elements of the general situation. The proof is based on the blow up method [8] (which naturally leads us to the tangent measures) and appropriate usage of Fourier multiplier operators (as in deriving appropriate defect functionals [2, 5]).

Let us recall that the Fourier multiplier operator T_ψ with the symbol ψ is defined via the Fourier and inverse Fourier transform

$$T_\psi u(\mathbf{x}) = \mathcal{F}^{-1}(\psi(\xi) \mathcal{F}(u))(\mathbf{x}), \quad u \in L^2(\mathbb{R}^d),$$

where the Fourier and the inverse Fourier transforms are given by

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = \int e^{-2\pi i \mathbf{x} \cdot \xi} u(\mathbf{x}) d\mathbf{x}, \quad \mathcal{F}^{-1}(u)(\mathbf{x}) = \check{u}(\mathbf{x}) = \int e^{2\pi i \mathbf{x} \cdot \xi} u(\xi) d\xi.$$

For properties of the Fourier multiplier operators one can consult [4].

As for the tangent measure, we shall use the property of any locally finite measure [7, Lemma 2.4] and to recall the following theorem (see also [7, Theorem 2.5]) representing a special case of the tangent measures.

Theorem 3. *For any locally finite measure μ defined on \mathbb{R}^d and any multi-index $(\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and μ -almost every $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ there exists a locally finite measure ν such that along a subsequence as $r \rightarrow 0$ it holds for every $\varphi \in C_0(\mathbb{R}^d)$*

$$\frac{\int \varphi(\mathbf{y}) d\mu(x_1 + r^{\alpha_1} y_1, \dots, x_d + r^{\alpha_d} y_d)}{\mu((x_1 - r^{\alpha_1}, x_1 + r^{\alpha_1}) \times \dots \times (x_d - r^{\alpha_d}, x_d + r^{\alpha_d}))} \rightarrow \int \varphi(\mathbf{y}) d\nu(\mathbf{y}) \text{ as } r \rightarrow 0.$$

The paper is organized as follows. In the next section, we shall prove the theorem in the case $\mu = \mu_{us}(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}}$ which contains arguments concerning uniformly singular measures which are new with respect to the ones from [3]. In the last section, we prove the result in the full generality.

2 Proof of Theorem 2 in the case of the hyperbolic constant coefficients operator

Here, we shall prove Theorem 2 when the scalar finite Radon measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ of the form $\mu(\mathbf{x}) = \mu_{us}(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}}$ satisfies the equation

$$\sum_{|\alpha|=m} \partial_{\tilde{\mathbf{x}}}^\alpha (a_\alpha \mu) + a_0 \sigma = 0, \quad (2.11)$$

where a_α are constants and σ is also finite scalar Radon measure. The proof is essentially the same for the general operator of the form given in (1.1), but it is a bit less technical for (2.11). The proof in full generality is given in the next section.

Before we start, let $\tilde{f}(\mathbf{x}_0)$ be the Radon-Nykodim derivative of μ_{us} with respect to $|\mu_{us}|$:

$$\tilde{f}(\tilde{\mathbf{x}}_0) = \frac{d\mu_{us}}{d|\mu_{us}|}(\tilde{\mathbf{x}}_0).$$

We fix a convolution kernel $\rho : \mathbb{R} \rightarrow \mathbb{R}$ which is a smooth, compactly supported function of total mass one and convolve (2.11) by

$$\frac{1}{\varepsilon^k} \boldsymbol{\rho}\left(\frac{\tilde{\mathbf{x}}}{\varepsilon}\right) = \frac{1}{\varepsilon^k} \prod_{j=1}^k \rho\left(\frac{x_j}{\varepsilon}\right) = \frac{1}{\varepsilon^k} \prod_{j=1}^k \rho_{j,\varepsilon}(x_j) = \boldsymbol{\rho}_\varepsilon(x).$$

Then, we take an arbitrary $\varphi \in C_c^1(\mathbb{R}^k)$ and $\varphi_1 \in C_c^1(\mathbb{R}^{d-k})$ and test the convolved equation on the product of such functions. We get (below, we denote $\mu_\varepsilon^{us} = \mu_{us} \star \boldsymbol{\rho}_\varepsilon$ and $\langle \varphi(\tilde{\mathbf{y}}), \mu_{us}(\tilde{\mathbf{y}}) \rangle = \int_{\mathbb{R}^k} \varphi(\tilde{\mathbf{y}}) d\mu_{us}(\tilde{\mathbf{y}})$)

$$\sum_{|\alpha|=m} a_\alpha \int_{\mathbb{R}^d} \left\langle \frac{1}{\varepsilon^k} \prod_{j=1}^k \rho\left(\frac{x_j - y_j}{\varepsilon}\right), \mu_{us}(\tilde{\mathbf{y}}) \right\rangle \partial_{\tilde{\mathbf{x}}}^\alpha \varphi(\tilde{\mathbf{x}}) \varphi_1(\tilde{\mathbf{x}}) d\mathbf{x} + a_0 \int_{\mathbb{R}^d} \varphi(\tilde{\mathbf{x}}) \varphi_1(\tilde{\mathbf{x}}) d\sigma^\varepsilon(\mathbf{x}) = 0. \quad (2.12)$$

We now fix $\tilde{\mathbf{x}}_0 \in \mathbb{R}^k$ and take $\varphi\left(\frac{\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_0}{\varepsilon}\right)$ instead of φ in (2.12). We get (below, $\mathbf{w} = (\tilde{\mathbf{w}}, \bar{\mathbf{w}})$):

$$\varepsilon^{-m} \sum_{|\alpha|=m} a_\alpha \int_{\mathbb{R}^d} \left\langle \frac{1}{\varepsilon^k} \prod_{j=1}^k \rho\left(\frac{x_j - y_j}{\varepsilon}\right), \mu_{us}(\tilde{\mathbf{y}}) \right\rangle \partial_{\tilde{\mathbf{w}}}^\alpha \varphi(\tilde{\mathbf{w}}) \Big|_{\tilde{\mathbf{w}} = \frac{\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_0}{\varepsilon}} \varphi_1(\bar{\mathbf{w}}) d\mathbf{w} + a_0 \int_{\mathbb{R}^d} \varphi\left(\frac{\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_0}{\varepsilon}\right) \varphi_1(\tilde{\mathbf{x}}) d\sigma^\varepsilon(\mathbf{x}) = 0. \quad (2.13)$$

We now introduce in the first integral above the change of variables $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_0 + \varepsilon\tilde{\mathbf{w}}$ and multiply the entire expression by ε^m . We get (we denote $\tilde{\mathbf{x}}_0 = (x_1^0, \dots, x_k^0)$ and $\tilde{\mathbf{w}} = (w_1, \dots, w_k)$):

$$\sum_{|\alpha|=m} a_\alpha \int_{\mathbb{R}^d} \langle \Pi_{j=1}^k \rho\left(\frac{x_j^0 - y_j}{\varepsilon} + w_j\right), \mu_{us}(\tilde{\mathbf{y}}) \rangle \partial_{\tilde{\mathbf{w}}}^\alpha \varphi(\tilde{\mathbf{w}}) \varphi_1(\tilde{\mathbf{w}}) d\tilde{\mathbf{w}} + \varepsilon^m a_0 \int_{\mathbb{R}^d} \varphi\left(\frac{\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_0}{\varepsilon}\right) \varphi_1(\tilde{\mathbf{x}}) d\sigma^\varepsilon(\tilde{\mathbf{x}}) = 0. \quad (2.14)$$

We consider separately terms involving the measure μ_{us} :

$$M_{us,\varepsilon} := \int_{\mathbb{R}^k} \langle \Pi_{j=1}^k \rho\left(\frac{x_j^0 - y_j}{\varepsilon} + w_j\right), \mu_{us}(\tilde{\mathbf{y}}) \rangle a_\alpha \partial_{\tilde{\mathbf{w}}}^\alpha \varphi(\tilde{\mathbf{w}}) d\tilde{\mathbf{w}}, \quad (2.15)$$

We rewrite (2.15) in the form (for α and β given in the uniform singularity definition)

$$M_{us,\varepsilon} = \int_{\mathbb{R}^k} \left(\int_{B(\tilde{\mathbf{x}}_0, \alpha(\varepsilon))} + \int_{B(\tilde{\mathbf{x}}_0, \beta(\varepsilon)) \setminus B(\tilde{\mathbf{x}}_0, \alpha(\varepsilon))} + \int_{\mathbb{R}^k \setminus B(\tilde{\mathbf{x}}_0, \beta(\varepsilon))} \right) \Pi_{j=1}^k \rho\left(\frac{x_j^0 - y_j}{\varepsilon} + w_j\right) d\mu_{us}(\tilde{\mathbf{y}}) \times a_\alpha \partial_{\tilde{\mathbf{w}}}^\alpha \varphi(\tilde{\mathbf{w}}) d\tilde{\mathbf{w}}. \quad (2.16)$$

Now, according to the assumptions for the uniformly singular measures (see Definition 1) and the fact that ρ is compactly supported:

$$\begin{aligned} \frac{\mu_{us}(B(\tilde{\mathbf{x}}_0, \beta(\varepsilon)) \setminus B(\tilde{\mathbf{x}}_0, \alpha(\varepsilon)))}{|\mu_{us}|(B(\tilde{\mathbf{x}}_0, \alpha(\varepsilon)))} &\rightarrow 0 \text{ as } \varepsilon \rightarrow 0; \\ \rho\left(\frac{\tilde{\mathbf{x}}_0 - \mathbf{y}}{\varepsilon} + \tilde{\mathbf{w}}\right) &\rightarrow 0, \mathbf{y} \notin B(\tilde{\mathbf{x}}_0, \beta(\varepsilon)) \text{ and, obviously} \\ \rho\left(\frac{\tilde{\mathbf{x}}_0 - \mathbf{y}}{\varepsilon} + \tilde{\mathbf{w}}\right) &\rightarrow \rho(\tilde{\mathbf{w}}), \mathbf{y} \in B(\tilde{\mathbf{x}}_0, \alpha(\varepsilon)) \end{aligned}$$

we get after dividing (2.16) by $|\mu_{us}|(B(\tilde{\mathbf{x}}_0, \alpha(\varepsilon)))$ and letting $\varepsilon \rightarrow 0$ in (2.16) (for $|\mu_{us}|$ -a.e. $\tilde{\mathbf{x}}_0 \in \mathbb{R}^d$):

$$\lim_{\varepsilon \rightarrow 0} \frac{M_{us,\varepsilon}}{|\mu_{us}|(B(\tilde{\mathbf{x}}_0, \alpha(\varepsilon)))} = \tilde{f}(\tilde{\mathbf{x}}_0) \int_{\mathbb{R}^k} \rho(\tilde{\mathbf{w}}) a_\alpha \partial_{\tilde{\mathbf{w}}}^\alpha \varphi(\tilde{\mathbf{w}}) d\tilde{\mathbf{w}}. \quad (2.17)$$

If we take here

$$\varphi(\tilde{\mathbf{w}}) = T_{\frac{\psi(\tilde{\xi})}{|\tilde{\xi}|^m}}(\rho(\tilde{\mathbf{w}})), \quad (2.18)$$

where $T_{\frac{\psi(\tilde{\xi})}{|\tilde{\xi}|^m}}$ is the Fourier multiplier operator with the symbol $\frac{\psi(\tilde{\xi})}{|\tilde{\xi}|^m}$, we find after taking the Plancherel theorem into account:

$$\lim_{\varepsilon \rightarrow 0} \frac{M_{us,\varepsilon}}{|\mu_{us}|(B(\tilde{\mathbf{x}}_0, \alpha(\varepsilon)))} = \tilde{f}(\tilde{\mathbf{x}}_0) \int_{\mathbb{R}^k} |\hat{\rho}|^2(\tilde{\xi}) a_\alpha \frac{(i\tilde{\xi})^\alpha}{|\tilde{\xi}|^m} \psi(\tilde{\xi}) d\tilde{\xi}. \quad (2.19)$$

From here, we conclude after letting $\varepsilon \rightarrow 0$ in (2.14) with φ given by (2.18)

$$\tilde{f}(\tilde{\mathbf{x}}_0) \sum_{|\alpha|=m} \int_{\mathbb{R}^d} |\hat{\rho}|^2(\tilde{\xi}) a_\alpha \frac{(i\tilde{\xi})^\alpha}{|\tilde{\xi}|^m} \psi(\tilde{\xi}) \varphi_1(\tilde{\mathbf{x}}) d\tilde{\xi} d\tilde{\mathbf{x}} = 0. \quad (2.20)$$

From here, due to arbitrariness of ψ , we conclude that (1.10) holds (without the \cup sign since we do not have the singular part, but only the uniformly singular part of the measure).

3 Proof of Theorem 2; general case

In this section, we consider equations (1.2) under the assumption $\mu(\mathbf{x}) = \mu_{us}(\tilde{\mathbf{x}})\mu_s(\bar{\mathbf{x}})$. We have omitted the terms $g(\mathbf{x})d\mathbf{x}$ and $\mu_{us}(\tilde{\mathbf{x}})d\tilde{\mathbf{x}}$ appearing in (1.5) since for the purely Lebesgue part there is nothing to prove and the term $\mu_{us}(\tilde{\mathbf{x}})d\tilde{\mathbf{x}}$ is handled as the $\mu(\mathbf{x}) = \mu_{us}(\tilde{\mathbf{x}})\mu_s(\bar{\mathbf{x}})$ by replacing $\mu_s(\bar{\mathbf{x}})$ by $d\bar{\mathbf{x}}$.

We shall rewrite in the form:

$$\sum_{\alpha \in \tilde{I}_j, \beta \in \bar{I}} \sum_{r=1}^m \partial_{\tilde{\mathbf{x}}}^{\alpha} \partial_{\bar{\mathbf{x}}}^{\beta} (a_{jr}^{\alpha\beta}(\mathbf{x}) \mu_{us}^r(\tilde{\mathbf{x}}) \mu_s^r(\bar{\mathbf{x}})) + \mathcal{A}_{lower}^j \mu = 0 \quad (3.21)$$

where \tilde{I}_j and \bar{I} are set of indexes corresponding to the principal symbol of the operator \mathcal{A} .

We shall prove Theorem 2 by following the steps from the previous section together with the approach taken in [3] and we refer the reader there for clarifications.

We start by fixing j in (1.2) and the convolution kernel $\rho : \mathbb{R} \rightarrow \mathbb{R}$ which is smooth, compactly supported with total mass one. We then denote

$$\rho_{j,\varepsilon}(\tilde{\mathbf{x}}) = \frac{1}{\varepsilon^{\beta_1 + \dots + \beta_k}} \prod_{s=1}^k \rho\left(\frac{x_s}{\varepsilon^{\beta_s}}\right) \quad \text{and} \quad \rho(\tilde{\mathbf{w}}) = \prod_{s=1}^k \rho(w_s),$$

and convolve (3.21) by $\rho_{j,\varepsilon}$. We have for $(a_{jr}^{\alpha\beta} \mu_{us})^{\varepsilon} = (a_{jr}^{\alpha\beta} \mu_{us}) * \rho_{j,\varepsilon}(\tilde{\mathbf{x}})$:

$$\sum_{\alpha \in \tilde{I}_j, \beta \in \bar{I}} \sum_{r=1}^m \partial^{\alpha\beta} ((a_{jr}^{\alpha\beta} \mu_{us}^r)^{\varepsilon} \mu_s^r + (\mathcal{A}_{lower}^j \mu) * \Pi_{r=1}^k \rho_{j,\varepsilon}(\tilde{\mathbf{x}})) = 0. \quad (3.22)$$

We then apply a test function $\varphi \in C_c^{\infty}(\mathbb{R}^k)$ on (3.22) to get

$$\sum_{\alpha \in \tilde{I}_j, \beta \in \bar{I}} (-1)^{|\alpha|} \sum_{r=1}^m \partial^{\beta} \int_{\mathbb{R}^k} \langle \rho_{j,\varepsilon}(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}), a_{jr}^{\alpha\beta}(\tilde{\mathbf{y}}, \bar{\mathbf{x}}) \mu_{us}^r(\tilde{\mathbf{y}}) \rangle \partial^{\alpha} \varphi(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}} d\mu_s^r(\bar{\mathbf{x}}) + \left\langle (\mathcal{A}_{lower}^j \mu) * \rho_{j,\varepsilon}(\tilde{\mathbf{x}}), \varphi(\tilde{\mathbf{x}}) \right\rangle = 0. \quad (3.23)$$

Now, we fix $\mathbf{z} = (\tilde{\mathbf{z}}, \bar{\mathbf{z}}) \in \mathbb{R}^d$ and take

$$\varphi_{\varepsilon}(\tilde{\mathbf{x}} - \tilde{\mathbf{z}}) = \varphi\left(\frac{x_1 - z_1}{\varepsilon^{\beta_1}}, \dots, \frac{x_k - z_k}{\varepsilon^{\beta_k}}\right)$$

in (3.23) instead of φ . At the same time, for the variable $\bar{\mathbf{x}}$, we fix $\mathbf{z} = (\tilde{\mathbf{z}}, \bar{\mathbf{z}}) \in \mathbb{R}^d$ we introduce the change $\bar{\mathbf{x}} = (z_{k+1} + \varepsilon^{\beta_{k+1}} w_{k+1}, \dots, z_d + \varepsilon^{\beta_d} w_d) = \bar{\mathbf{z}} + \varepsilon^{\bar{\beta}} \bar{\mathbf{w}}$ in (3.23) and apply the test function $\varphi_1(\bar{\mathbf{w}})$. Multiplying the obtained expression by ε and taking into account (1.8), we conclude

$$\sum_{\alpha \in \tilde{I}_j, \beta \in \bar{I}} (-1)^{|\alpha| + |\beta|} \sum_{r=1}^m \int_{\mathbb{R}^d} \langle \rho_{j,\varepsilon}(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}), a_{jr}^{\alpha\beta}(\tilde{\mathbf{y}}, \bar{\mathbf{x}}) \mu_{us}^r(\tilde{\mathbf{y}}) \rangle \partial^{\alpha} \varphi_{\varepsilon}(\tilde{\mathbf{x}} - \tilde{\mathbf{z}}) \partial^{\beta} \varphi_1(\bar{\mathbf{w}}) d\mu_s^r(\bar{\mathbf{z}} + \varepsilon^{\bar{\beta}} \bar{\mathbf{w}}) d\tilde{\mathbf{x}} + \varepsilon \langle (\mathcal{A}_{lower}^j \mu) * \rho_{j,\varepsilon}(\tilde{\mathbf{x}}), \varphi \varphi_1 \rangle = 0. \quad (3.24)$$

Next, we introduce the change of variables $\tilde{\mathbf{x}} = (z_1 + \varepsilon^{\beta_1} w_1, \dots, z_k + \varepsilon^{\beta_k} w_k)$ in the first term on the left-hand side of (3.24). We get

$$\sum_{\alpha \in \tilde{I}_j, \beta \in \bar{I}} (-1)^{|\alpha| + |\beta|} \sum_{r=1}^m \int_{\mathbb{R}^d} \left\langle \prod_{s=1}^k \rho\left(\frac{z_s - y_s}{\varepsilon^{\beta_s}} + w_s\right), (a_{jr}^{\alpha\beta}(\tilde{\mathbf{y}}, \bar{\mathbf{x}}) \mu_{us}^r(\tilde{\mathbf{y}})) \right\rangle \partial^{\alpha} \varphi(\tilde{\mathbf{w}}) \partial^{\beta} \varphi_1(\bar{\mathbf{w}}) d\mu_s^r(\bar{\mathbf{z}} + \varepsilon^{\bar{\beta}} \bar{\mathbf{w}}) d\mathbf{w} + R_{\varepsilon} = 0, \quad (3.25)$$

where

$$R_{\varepsilon} = \varepsilon \langle (\mathcal{A}_{lower}^j \mu) * \Pi_{r=1}^k \rho_{j,\varepsilon}(\tilde{\mathbf{x}}), \varphi \varphi_1 \rangle \rightarrow 0$$

by definition of the principal symbol. Now, we divide (3.25) by

$$|\mu_s|(B(0, \alpha(\varepsilon))) |\mu_s|((z_{k+1} - r^{\alpha_1}, x_{k+1} + r^{\alpha_1}) \times \cdots \times (z_d - r^{\alpha_d}, x_d + r^{\alpha_d}))$$

where $\alpha(\varepsilon)$ is given in the definition of the uniform singularity and let $\varepsilon \rightarrow 0$. Taking into account the uniform singularity assumptions as in (2.16) and Theorem 3, we get

$$\sum_{\alpha \in \bar{I}_j, \beta \in \bar{I}} (-1)^{|\alpha|+|\beta|} \sum_{r=1}^m a_{j_r}^{\alpha\beta}(\mathbf{z}) \int_{\mathbb{R}^d} \rho(\tilde{\mathbf{w}}) \partial_{\tilde{\mathbf{w}}}^\alpha \varphi(\tilde{\mathbf{w}}) d\tilde{\mathbf{w}} \partial_{\tilde{\mathbf{w}}}^\beta \varphi_1(\tilde{\mathbf{w}}) \tilde{f}_r(\tilde{\mathbf{z}}) \bar{f}_r(\tilde{\mathbf{z}}) d\nu(\tilde{\mathbf{w}}) = 0, \quad (3.26)$$

where (along a subsequence; see Theorem 3)

$$\nu^\varepsilon = \frac{d|\mu_s|(\tilde{\mathbf{z}} + \varepsilon^\beta \tilde{\mathbf{w}})}{|\mu_s|((z_{k+1} - r^{\alpha_1}, x_{k+1} + r^{\alpha_1}) \times \cdots \times (z_d - r^{\alpha_d}, x_d + r^{\alpha_d}))} \rightarrow \nu \text{ as } \varepsilon \rightarrow 0. \quad (3.27)$$

We now take

$$\varphi(\tilde{\mathbf{w}}) = T \frac{\psi(\pi_j(\tilde{\xi}))}{|\xi_1|^{\beta_1} + \cdots + |\xi_k|^{\beta_k}} \rho(\tilde{\mathbf{w}}),$$

where T_m is the Fourier multiplier operator with the symbol m . After inserting this in (3.26) and applying the Plancherel theorem with respect to $\tilde{\mathbf{w}}$, we obtain:

$$\sum_{r=1}^m \sum_{\alpha \in \bar{I}_j, \beta \in \bar{I}} (-1)^{|\alpha|+|\beta|} a_{j_r}^{\alpha\beta}(\mathbf{z}) \int_{\mathbb{R}^d} \frac{(2\pi i \tilde{\xi})^\alpha}{|\xi_1|^{\beta_1} + \cdots + |\xi_k|^{\beta_k}} \psi(\pi_j(\tilde{\xi})) |\hat{\rho}(\tilde{\xi})|^2 d\tilde{\xi} \partial_{\tilde{\mathbf{w}}}^\beta \varphi_1(\tilde{\mathbf{w}}) \tilde{f}_r(\tilde{\mathbf{z}}) \bar{f}_r(\tilde{\mathbf{z}}) d\nu(\tilde{\mathbf{w}}) = 0. \quad (3.28)$$

If we apply here the Plancherel theorem with respect to $\tilde{\mathbf{w}}$, we get

$$\sum_{r=1}^m \sum_{\alpha \in \bar{I}_j, \beta \in \bar{I}} (-1)^{|\alpha|+|\beta|} a_{j_r}^{\alpha\beta}(\mathbf{z}) \int_{\mathbb{R}^d} \frac{(2\pi i \tilde{\xi})^\alpha}{|\xi_1|^{\beta_1} + \cdots + |\xi_k|^{\beta_k}} \psi(\pi_j(\tilde{\xi})) |\hat{\rho}(\tilde{\xi})|^2 d\tilde{\xi} (i\tilde{\xi})^\beta \hat{\varphi}_1(\tilde{\xi}) \tilde{f}_r(\tilde{\mathbf{z}}) \bar{f}_r(\tilde{\mathbf{z}}) \hat{\nu}(\tilde{\xi}) d\tilde{\xi} = 0. \quad (3.29)$$

From here, we conclude that if $\tilde{f}(\tilde{\mathbf{z}}) \bar{f}(\tilde{\mathbf{z}}) = (\tilde{f}_1(\tilde{\mathbf{z}}) \bar{f}_1(\tilde{\mathbf{z}})), \dots, \tilde{f}_m(\tilde{\mathbf{z}}) \bar{f}_m(\tilde{\mathbf{z}})$ does not satisfy conditions of Theorem 2, we conclude that $\text{supp } \hat{\nu}_r = \{0\}$ which in turn implies that ν_r is the Lebesgue measure. By repeating the procedure from [3] where we replace the sphere by the manifold \tilde{P} , we conclude that the convergence from Theorem 3 used to get (3.26) is not only weak but also strong which contradicts the fact that μ_s is singular.

Let us briefly recall the arguments from [3]. First, since ψ and φ_1 in (3.29) are arbitrary, we know that it holds for every $(\tilde{\xi}, \tilde{\xi}) \in \tilde{P}_j \times \mathbb{R}^{d-k}, j = 1, \dots, n$:

$$\sum_{r=1}^m \sum_{\alpha \in \bar{I}_j, \beta \in \bar{I}} (-1)^{|\alpha|+|\beta|} a_{j_r}^{\alpha\beta}(\mathbf{z}) (2\pi i \tilde{\xi})^\alpha (i\tilde{\xi})^\beta \tilde{f}_r(\tilde{\mathbf{z}}) \bar{f}_r(\tilde{\mathbf{z}}) \hat{\nu}(\tilde{\xi}) = 0. \quad (3.30)$$

Due to linearity, with the expense of the small right-hand side, the same holds when we replace ν by $\nu - \nu^\varepsilon$ for ν^ε given by (3.27). In the matrix notation and after collecting the $\tilde{\xi}$ -terms with the coefficients $a_{\alpha\beta}$ this means

$$A(\tilde{\xi}) \bar{f}(\tilde{\mathbf{z}}) (\hat{\nu}(\tilde{\xi}) - \hat{\nu}^\varepsilon(\tilde{\xi})) = \text{low order terms.}$$

We multiply the latter equation by $[A(\tilde{\xi}) \bar{f}(\tilde{\mathbf{z}})]^*$ to get

$$|A(\tilde{\xi}) \bar{f}(\tilde{\mathbf{z}})|^2 (\hat{\nu}(\tilde{\xi}) - \hat{\nu}^\varepsilon(\tilde{\xi})) = \text{low order terms.} \quad (3.31)$$

If we assume that for every $\tilde{\xi} \in \mathbb{R}^{d-k}$, we have

$$|A(\tilde{\xi}) \bar{f}(\tilde{\mathbf{z}})|^2 \neq 0$$

and from here and (3.30) it follows that $\hat{\nu} \equiv 0$ i.e. that ν is the Lebesgue measure. Moreover, we can divide (3.31) by $|A(\tilde{\xi}) \bar{f}(\tilde{\mathbf{z}})|^2$ from where, after invoking the Marcinkiewicz multiplier theorem and letting $\varepsilon \rightarrow 0$, we reach to

$$\|\hat{\nu}(\tilde{\xi}) - \hat{\nu}^\varepsilon(\tilde{\xi})\|_{L^1(K)}, \quad K \subset \subset \mathbb{R}^d.$$

This implies that ν^ε strongly converges toward ν which is impossible since ν is the Lebesgue measure and ν^ε are not. Thus, we conclude that for some ξ it holds $A(\xi)(\bar{z}) = 0$ which we wanted to prove.

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