Research Article

Jing Yang* and Ting Zhou

Existence of single peak solutions for a nonlinear Schrödinger system with coupled quadratic nonlinearity

<https://doi.org/10.1515/anona-2021-0201>

Received April 22, 2021; accepted July 4, 2021.

Abstract: We are concerned with the following Schrödinger system with coupled quadratic nonlinearity

$$\begin{cases} -\varepsilon^2 \Delta v + P(x)v = \mu vw, & x \in \mathbb{R}^N, \\ -\varepsilon^2 \Delta w + Q(x)w = \frac{\mu}{2} v^2 + \gamma w^2, & x \in \mathbb{R}^N, \\ v > 0, \quad w > 0, & v, w \in H^1(\mathbb{R}^N), \end{cases}$$

which arises from second-harmonic generation in quadratic media. Here $\varepsilon > 0$ is a small parameter, $2 \leq N < 6$, $\mu > 0$ and $\mu > \gamma$, $P(x)$, $Q(x)$ are positive function potentials. By applying reduction method, we prove that if x_0 is a non-degenerate critical point of $\Delta(P + Q)$ on some closed $N - 1$ dimensional hypersurface, then the system above has a single peak solution $(v_\varepsilon, w_\varepsilon)$ concentrating at x_0 for ε small enough.

Keywords: second-harmonic generation; non-degenerate; single peak solutions; reduction method

MSC: 35J10, 35B99, 35J60

1 Introduction and main result

In this paper, we consider the following Schrödinger system with coupled quadratic nonlinearity

$$\begin{cases} -\varepsilon^2 \Delta v + P(x)v = \mu vw, & x \in \mathbb{R}^N, \\ -\varepsilon^2 \Delta w + Q(x)w = \frac{\mu}{2} v^2 + \gamma w^2, & x \in \mathbb{R}^N, \\ v > 0, \quad w > 0, & v, w \in H^1(\mathbb{R}^N), \end{cases} \quad (1.1)$$

where $\varepsilon > 0$ is a small parameter, $2 \leq N < 6$, $\mu > 0$ and $\mu > \gamma$, $P(x)$, $Q(x)$ are positive potentials.

System (1.1) arises from the cubic nonlinear Schrödinger equation

$$i \frac{\partial \phi}{\partial z} + r \nabla^2 \phi + \chi |\phi|^2 \phi = 0, \quad (1.2)$$

which appears in the nonlinear optic theory and can be used to describe the formation and propagation of optical solutions in Kerr-type materials [6, 19]. Here ϕ is a slowly varying envelope of electric field, the real-valued parameter r and χ represent the relative strength and sign of dispersion/diffraction and nonlinearity respectively, and z is the propagation distance coordinate. The Laplacian operator ∇^2 can either be $\frac{\partial^2}{\partial r^2}$ for

*Corresponding Author: **Jing Yang**, School of science, Jiangsu University of Science and Technology, Zhenjiang 212003, P. R. China, E-mail: yyangecho@163.com

Ting Zhou, School of Mathematics and Statistics and Hubei Key Laboratory Mathematical Sciences, Central China Normal University, Wuhan, 430079, P. R. China, E-mail: 1260602906@qq.com

temporal solitons with τ is the normalized retarded time, or $\nabla^2 = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, where $x = (x_1, \dots, x_N)$ is in the direction orthogonal to z . Solitary wave solutions to (1.2) and its generations have been studied in [4, 18].

Also, (1.1) appears in the study of standing waves for the following nonlinear system

$$\begin{cases} i \frac{\partial \phi_1}{\partial t} = -\varepsilon^2 \Delta \phi_1 + (P(x) + \mu) \phi_1 - \mu |\phi_1| \phi_2, & (x, t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ i \frac{\partial \phi_2}{\partial t} = -\varepsilon^2 \Delta \phi_2 + (Q(x) + \mu) \phi_2 - \frac{\mu}{2} |\phi_1|^2 - \gamma |\phi_2|^2, & (x, t) \in \mathbb{R}^N \times \mathbb{R}^+, \end{cases} \quad (1.3)$$

with the form $\phi_1(x, t) = v(x)e^{i\mu t}$, $\phi_2(x, t) = w(x)e^{i\mu t}$, where i is the imaginary unit and ε is the Planck constant. When $\varepsilon = 1$ and $\gamma = 0$, the existence of ground state solutions of (1.3) was proved in [27]. Besides, (1.1) is closely related to the general parabolic system with coupled nonlinearity and the nonlinear evolution equations. For this information, we can refer to [11, 16, 23, 25, 26] and references therein.

By contrast with the coupled Schrödinger system (1.3) with $\chi^{(2)}$ nonlinearities, the following χ^2 nonlinear Schrödinger system

$$\begin{cases} i \frac{\partial \phi_1}{\partial t} = -\varepsilon^2 \Delta \phi_1 + V_1(x) \phi_1 - \mu_1 |\phi_1|^2 \phi_1 - \beta |\phi_2|^2 \phi_1, & (x, t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ i \frac{\partial \phi_2}{\partial t} = -\varepsilon^2 \Delta \phi_2 + V_2(x) \phi_2 - \mu_2 |\phi_2|^2 \phi_2 - \beta |\phi_1|^2 \phi_2, & (x, t) \in \mathbb{R}^N \times \mathbb{R}^+ \end{cases} \quad (1.4)$$

has been extensively investigated. There are many interesting results about (1.4) under various assumptions of $V_1(x)$ and $V_2(x)$, one can refer to [1–4, 7, 8, 12–15, 17, 21, 24] and their references therein.

In recent decades, system (1.1) and its related problems have attracted a lot of attention. When $\varepsilon = 1$, (1.1) reduces to

$$\begin{cases} -\Delta v + P(x)v = \mu vw, & x \in \mathbb{R}^N, \\ -\Delta w + Q(x)w = \frac{\mu}{2} v^2 + \gamma w^2, & x \in \mathbb{R}^N, \\ v > 0, \quad w > 0, & v, w \in H^1(\mathbb{R}^N). \end{cases} \quad (1.5)$$

Applying the finite dimensional reduction method, Wang and Zhou [22] constructed the infinitely many non-radial positive solutions of (1.5) if the potential functions $P(x)$, $Q(x)$ are radial and satisfy some algebraic decay at infinity. Also, if ε is small, for any positive integer $k \leq N + 1$, Tang and Xie [20] proved that (1.1) has a k spikes solution concentrating at some strict local maximum of $P(x)$ and $Q(x)$ by using the finite dimensional reduction provided that $|P(x) - P(y)| \leq L_1 |x - y|^{\theta_1}$ and $|Q(x) - Q(y)| \leq L_2 |x - y|^{\theta_2}$ for some positive constants $L_1, L_2, \theta_1, \theta_2$.

Here we want to mention that, very recently, Luo, Peng and Yan [13] revisited the following Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = u^{p-1}, \quad u \in H^1(\mathbb{R}^N) \quad (1.6)$$

with $2 < p < 2^*$. Under the condition that $V(x)$ obtains its local minimum or local maximum x_0 at a closed $N - 1$ dimensional hypersurface, they obtained the existence of a positive single peak solution for (1.6) concentrating at x_0 if x_0 is non-degenerate critical point of ΔV and also verified the local uniqueness of single peak solutions by using local Pohazaev type identity.

Motivated by [13, 20, 22], we want to apply the finite-dimensional reduction to study the existence of positive single peak solutions for (1.1). Our purpose here is to prove that (1.1) has a single peak solution concentrating at some non-degenerate critical point of $\Delta(P + Q)$ on a closed $N - 1$ dimensional hypersurface.

To state our results, throughout this paper, we assume that $P(x)$, $Q(x)$ obtain their local minimum or local maximum at a closed $N - 1$ dimensional hypersurface Γ . Without loss of generality, we suppose that $P(x) = Q(x) = 1$ if $x \in \Gamma$ and more precisely, we assume that $P(x)$, $Q(x)$ satisfies the following conditions.

(H₁) There exist $\delta > 0$ and a closed smooth hypersurface Γ such that if $y \in \Gamma$, $P(y)$, $Q(y) = 1$ and $P(y)$, $Q(y) > 1$ (or $P(y)$, $Q(y) < 1$) for any $y \in W_\delta \setminus \Gamma$, where $W_\delta := \{x \in \mathbb{R}^N, \text{dist}(x, \Gamma) < \delta\}$.

(H₂) The level set $\Gamma_t = \{x : P(x), Q(x) = t\}$ is a closed smooth hypersurface for $t \in [1, 1 + \vartheta]$ (or $t \in [1 - \vartheta, 1]$) for some small $\vartheta > 0$. Also, for any $x_t \in \Gamma_t$ and $x_0 \in \Gamma$, there holds $|v_t - v| \leq C|x_t - x_0|$ and $|\zeta_{i,t} - \zeta_i| \leq C|x_t - x_0|$, $i = 1, 2, \dots, N - 1$. Here, throughout this paper, we denote by $v_t(v)$ the outward unit normal vector of $\Gamma_t(\Gamma)$ at $x_t(x_0)$, while we use $\zeta_{i,t}(\zeta_i)$ to denote the i -th principal tangential unit vector of $\Gamma_t(\Gamma)$ at $x_t(x_0)$.

(H₃) For any $x \in B_r(x_0)$, it holds $P(x) = Q(x)$, where $x_0 \in \Gamma$ and r is a small positive constant.

Remark 1.1. Let $F(x) = \sum_{i=1}^N \frac{x_i^2}{a_i^2} - 1$ with $a_i > 0$, $a_i \neq a_j (i \neq j)$ and $\Gamma = \{x \in \mathbb{R}^N : F(x) = 0\}$. Take

$$P(x) = Q(x) = F^2 + 1, \text{ in } W_0,$$

where $W_0 = \{x \in \mathbb{R}^N : |\sum_{i=1}^N \frac{x_i^2}{a_i^2} - 1| \leq \delta_0\}$ for some small fixed $\delta_0 > 0$. Then we can use the above conditions to the potentials $P(x)$, $Q(x)$.

Let us point out that if Γ is a local minimum (or local maximum) set of $P(x)$ and $Q(x)$, then for any $x \in \Gamma$,

$$P(x) = 1, \quad \nabla P(x) = 0$$

and

$$Q(x) = 1, \quad \nabla Q(x) = 0.$$

This implies that for any tangential vector ζ of Γ at x , one has

$$(D_\zeta \nabla)P(x) = 0, \quad (D_\zeta \nabla)Q(x) = 0, \quad \forall x \in \Gamma,$$

where D_ζ denotes the directional derivative at the direction ζ .

Let U be the unique positive radial solution of the following problem

$$\begin{cases} -\Delta u + u = u^2, & u > 0, \text{ in } \mathbb{R}^N, \\ u(0) = \max_{x \in \mathbb{R}^N} u(x), & u(x) \in H^1(\mathbb{R}^N). \end{cases} \quad (1.7)$$

It is well-known in [10] that $U(x)$ is strictly decreasing and its s order derivative satisfies

$$|D^s U(x)| e^{|x|} |x|^{\frac{1-N}{2}} \leq C$$

for $|s| \leq 1$ and some constant $C > 0$.

For x_ε close to x_0 , if we denote

$$\bar{V}_{\varepsilon, x_\varepsilon} = P(x_\varepsilon) U\left(\frac{\sqrt{P(x_\varepsilon)}(x - x_\varepsilon)}{\varepsilon}\right),$$

then $\bar{V}_{\varepsilon, x_\varepsilon}$ solves

$$\begin{cases} -\varepsilon^2 \Delta v + P(x_\varepsilon)v = v^2, & v > 0, \quad x \in \mathbb{R}^N, \\ v(x_\varepsilon) = \max_{x \in \mathbb{R}^N} v(x), & v \in H^1(\mathbb{R}^N). \end{cases} \quad (1.8)$$

Also, write

$$\bar{W}_{\varepsilon, x_\varepsilon} = Q(x_\varepsilon) U\left(\frac{\sqrt{Q(x_\varepsilon)}(x - x_\varepsilon)}{\varepsilon}\right),$$

which is the solution of

$$\begin{cases} -\varepsilon^2 \Delta w + Q(x_\varepsilon)w = w^2, & w > 0, \quad x \in \mathbb{R}^N, \\ w(x_\varepsilon) = \max_{x \in \mathbb{R}^N} w(x), & w \in H^1(\mathbb{R}^N). \end{cases} \quad (1.9)$$

Take $(V_{\varepsilon, x_\varepsilon}, W_{\varepsilon, x_\varepsilon}) = (\alpha \bar{V}_{\varepsilon, x_\varepsilon}, \beta \bar{W}_{\varepsilon, x_\varepsilon})$ with $\alpha = \frac{1}{\mu} \sqrt{\frac{2(\mu-\gamma)}{\mu}}$, $\beta = \frac{1}{\mu}$. Then $(V_{\varepsilon, x_\varepsilon}, W_{\varepsilon, x_\varepsilon})$ solves

$$\begin{cases} -\varepsilon^2 \Delta v + P(x_\varepsilon)v = \mu v w, & x \in \mathbb{R}^N, \\ -\varepsilon^2 \Delta w + Q(x_\varepsilon)w = \frac{\mu}{2} v^2 + \gamma w^2, & x \in \mathbb{R}^N, \\ v > 0, \quad w > 0, & v, w \in H^1(\mathbb{R}^N) \end{cases} \quad (1.10)$$

since (H_3) holds.

For $\varepsilon > 0$ small, we will use $(V_{\varepsilon, x_\varepsilon}, W_{\varepsilon, x_\varepsilon})$ to construct the single peak solutions concentrating at x_0 . First we give the following definitions.

Definition 1.2. We say that $(v_\varepsilon, w_\varepsilon)$ is a single peak solution of (1.1) concentrating at x_0 if there exist $x_\varepsilon \in \mathbb{R}^N$ with $|x_\varepsilon - x_0| = o(1)$ such that

$$\|(v_\varepsilon - V_{\varepsilon, x_\varepsilon}, w_\varepsilon - W_{\varepsilon, x_\varepsilon})\|_\varepsilon = o(\varepsilon^{\frac{N}{2}}),$$

where $\|(u, v)\|_\varepsilon = \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + P(x)u^2 + \varepsilon^2 |\nabla v|^2 + Q(x)v^2) dx$.

Definition 1.3. We say that a critical point $x_0 \in \Gamma$ of $\Delta(P + Q)$ on Γ is non-degenerate if it holds

$$\frac{\partial^2(P(x_0) + Q(x_0))}{\partial v^2} \neq 0, \quad \det \left(\left(\frac{\partial^2 \Delta P(x_0)}{\partial \zeta_i \partial \zeta_j} \right)_{1 \leq i, j \leq N-1} + \left(\frac{\partial^2 \Delta Q(x_0)}{\partial \zeta_i \partial \zeta_j} \right)_{1 \leq i, j \leq N-1} \right) \neq 0.$$

The main result of this paper is the following.

Theorem 1.4. Assume that $(H_1) - (H_3)$ hold. If $x_0 \in \Gamma$ is a non-degenerate critical point of $\Delta(P + Q)$, then there exists $\varepsilon_0 > 0$ such that (1.1) has a single peak solution $(v_\varepsilon, w_\varepsilon)$ concentrating at x_0 provided $\varepsilon \in (0, \varepsilon_0]$.

As in [13, 20, 22], we mainly apply the finite dimensional reduction method to prove our main result. Compared with [13], we have to overcome much difficulties in the reduction process which involves some technical and careful computations due to the $\chi^{(2)}$ nonlinearity appears. Moreover, to our best knowledge, our result exhibits a new phenomenon for the coupled Schrödinger system with $\chi^{(2)}$ nonlinearity.

Remark 1.5. Combining the ideas from [9, 13], where in [9] the coupled nonlinear Gross-Pitaevskii system was studied, we guess that the following conclusions may hold.

(1) On the basis of Theorem 1.4, further we can prove the local uniqueness of single peak solutions by using local Pohazaev type identity.

(2) Under the conditions of Theorem 1.4, if $\Delta(P + Q)$ has an isolated maximum or minimum point $x_0 \in \Gamma$, then for any integer $k > 0$, (1.1) has a k -peaks solution whose peaks cluster at x_0 .

The structure of this paper is organized as follows. In section 2, we do some preliminaries and then we carry out the finite dimensional reduction. We will prove our main result in section 3. In the sequel, for simplicity of notations we write $\int f$ to mean the Lebesgue integral of $f(x)$ in $\mathbb{R}^N$.

2 the finite dimensional reduction

In this section, we mainly give some preliminaries and do the finite dimensional reduction. Hereafter, for any function $K(x) > 0$, we define the Sobolev space

$$H_{\varepsilon, K}^1 = \{u \in H^1(\mathbb{R}^N) : \int (\varepsilon^2 |\nabla u|^2 + K(x)u^2) < \infty\},$$

endowed with the standard norm

$$\|u\|_{\varepsilon, K} = \left(\int (\varepsilon^2 |\nabla u|^2 + K(x)u^2) \right)^{\frac{1}{2}},$$

which is induced by the inner product

$$\langle u, v \rangle_{\varepsilon, K} = \int (\varepsilon^2 \nabla u \nabla v + K(x)uv).$$

Now we define H to be the product space $H_{\varepsilon, P}^1 \times H_{\varepsilon, Q}^1$ with the norm

$$\|(u, v)\|_\varepsilon^2 = \|u\|_{\varepsilon, P}^2 + \|v\|_{\varepsilon, Q}^2$$

and set

$$E_{\varepsilon, x_\varepsilon} = \left\{ (\varphi, \psi) \in H : \left\langle (\varphi, \psi), \left(\frac{\partial V_{\varepsilon, x_\varepsilon}}{\partial x_i}, \frac{\partial W_{\varepsilon, x_\varepsilon}}{\partial x_i} \right) \right\rangle_\varepsilon = 0 \right\}$$

for $i = 1, \dots, N$, where

$$\begin{aligned} & \left\langle (\varphi, \psi), \left(\frac{\partial V_{\varepsilon, x_\varepsilon}}{\partial x_i}, \frac{\partial W_{\varepsilon, x_\varepsilon}}{\partial x_i} \right) \right\rangle_\varepsilon \\ &= \int \left(\varepsilon^2 \nabla \varphi \nabla \frac{\partial V_{\varepsilon, x_\varepsilon}}{\partial x_i} + P(x) \frac{\partial V_{\varepsilon, x_\varepsilon}}{\partial x_i} \varphi + \varepsilon^2 \nabla \psi \nabla \frac{\partial W_{\varepsilon, x_\varepsilon}}{\partial x_i} + Q(x) \frac{\partial W_{\varepsilon, x_\varepsilon}}{\partial x_i} \psi \right). \end{aligned}$$

Note that the variational functional corresponding to (1.1) is

$$I_\varepsilon(v, w) = \frac{1}{2} \int (\varepsilon^2 |\nabla v|^2 + P(x)v^2 + \varepsilon^2 |\nabla w|^2 + Q(x)w^2) - \frac{\mu}{2} \int v^2 w - \frac{\gamma}{3} \int w^3. \quad (2.1)$$

Then $I \in C^2(H, \mathbb{R})$ and its critical points are solutions of (1.1).

Set

$$J_\varepsilon(\varphi, \psi) = I_\varepsilon(V_{\varepsilon, x_\varepsilon} + \varphi, W_{\varepsilon, x_\varepsilon} + \psi), \quad (\varphi, \psi) \in E_{\varepsilon, x_\varepsilon}.$$

We can expand $J_\varepsilon(\varphi, \psi)$ as follows:

$$J_\varepsilon(\varphi, \psi) = J_\varepsilon(0, 0) + \ell_\varepsilon(\varphi, \psi) + \frac{1}{2} L_\varepsilon(\varphi, \psi) + R_\varepsilon(\varphi, \psi), \quad (2.2)$$

where

$$\begin{aligned} \ell_\varepsilon(\varphi, \psi) &= \int (\varepsilon^2 \nabla V_{\varepsilon, x_\varepsilon} \nabla \varphi + P(x)V_{\varepsilon, x_\varepsilon} \varphi + \varepsilon^2 \nabla W_{\varepsilon, x_\varepsilon} \nabla \psi + Q(x)W_{\varepsilon, x_\varepsilon} \psi) - \mu \int V_{\varepsilon, x_\varepsilon} W_{\varepsilon, x_\varepsilon} \varphi \\ &\quad - \frac{\mu}{2} \int V_{\varepsilon, x_\varepsilon}^2 \psi - \gamma \int W_{\varepsilon, x_\varepsilon}^2 \psi \\ &= \int (P(x) - P(x_\varepsilon))V_{\varepsilon, x_\varepsilon} \varphi + (Q(x) - Q(x_\varepsilon))W_{\varepsilon, x_\varepsilon} \psi, \end{aligned}$$

$$\begin{aligned} L_\varepsilon(\varphi, \psi) &= \int (\varepsilon^2 |\nabla \varphi|^2 + P(x)\varphi^2 + \varepsilon^2 |\nabla \psi|^2 + Q(x)\psi^2) - \int (\mu W_{\varepsilon, x_\varepsilon} \varphi^2 + 2\gamma W_{\varepsilon, x_\varepsilon} \psi^2 \\ &\quad + 2\mu V_{\varepsilon, x_\varepsilon} \varphi \psi) \end{aligned}$$

and

$$R_\varepsilon(\varphi, \psi) = -\frac{\mu}{2} \int \varphi^2 \psi - \frac{\gamma}{3} \int \psi^3.$$

It follows from [5] that $(V_{\varepsilon, x_\varepsilon} + \varphi, W_{\varepsilon, x_\varepsilon} + \psi)$ is a critical point of $I_\varepsilon(v, w)$ if and only if (φ, ψ) is a critical point of $J_\varepsilon(\varphi, \psi)$. In order to find a critical point for $J_\varepsilon(\varphi, \psi)$, we need to discuss each terms in the expansion (2.2).

Lemma 2.1. *There exists $C > 0$ independent of ε such that*

$$\|R_\varepsilon^i(\varphi, \psi)\| \leq C\varepsilon^{-\frac{N}{2}} \|(\varphi, \psi)\|_\varepsilon^{3-i}, \quad i = 0, 1, 2,$$

where $R_\varepsilon^i(\varphi, \psi)$ denotes the i -th derivative of $R_\varepsilon(\varphi, \psi)$.

Proof. Recall that

$$R_\varepsilon(\varphi, \psi) = -\frac{\mu}{2} \int \varphi^2 \psi - \frac{\gamma}{3} \int \psi^3.$$

By the direct computations, we get

$$\langle R'_\varepsilon(\varphi, \psi), (\xi, \eta) \rangle = -\frac{\mu}{2} \int (\varphi^2 \eta + 2\varphi \psi \xi) - \gamma \int \psi^2 \eta$$

and

$$\langle R_\varepsilon''(\varphi, \psi)(\xi, \eta), (g, h) \rangle = -\mu \int (\varphi g \eta + \varphi h \xi + \psi g \xi) - 2\gamma \int \psi h \eta.$$

Observe that by letting $u_\varepsilon(x) = u(\varepsilon x)$, it is easy to check

$$\|u\|_{L^p(\mathbb{R}^N)} \leq C\varepsilon^{(\frac{1}{p}-\frac{1}{2})N} \|u\|_\varepsilon \quad (2.3)$$

with $\|u\|_\varepsilon = \left(\int (\varepsilon^2 |\nabla u|^2 + u^2) \right)^{\frac{1}{2}}$ for $2 \leq p \leq 2^*$ and some $C > 0$. Then we have

$$\begin{aligned} |R_\varepsilon(\varphi, \psi)| &\leq C \left(\int |\varphi|^2 |\psi| + \int |\psi|^3 \right) \\ &\leq C \left(\int |\varphi|^3 \right)^{\frac{2}{3}} \left(\int |\psi|^3 \right)^{\frac{1}{3}} + C \int |\psi|^3 \\ &\leq C\varepsilon^{-\frac{N}{2}} \|\varphi\|_\varepsilon^2 \|\psi\|_\varepsilon + C\varepsilon^{-\frac{N}{2}} \|\varphi\|_\varepsilon^3 \\ &\leq C\varepsilon^{-\frac{N}{2}} \|(\varphi, \psi)\|_\varepsilon^3, \end{aligned}$$

$$\begin{aligned} |R'_\varepsilon(\varphi, \psi), (\xi, \eta)| &\leq C \int (|\varphi|^2 |\eta| + |\varphi| |\psi| |\xi| + |\psi|^2 |\eta|) \\ &\leq C \left(\int |\varphi|^3 \right)^{\frac{2}{3}} \left(\int |\eta|^3 \right)^{\frac{1}{3}} + C \left(\int |\varphi|^3 \right)^{\frac{1}{3}} \left(\int |\psi|^3 \right)^{\frac{1}{3}} \left(\int |\xi|^3 \right)^{\frac{1}{3}} + C \left(\int |\psi|^3 \right)^{\frac{2}{3}} \left(\int |\eta|^3 \right)^{\frac{1}{3}} \\ &\leq C\varepsilon^{-\frac{N}{2}} (\|\varphi\|_\varepsilon^2 \|\eta\|_\varepsilon + \|\varphi\|_\varepsilon \|\psi\|_\varepsilon \|\xi\|_\varepsilon + \|\psi\|_\varepsilon^2 \|\eta\|_\varepsilon) \\ &\leq C\varepsilon^{-\frac{N}{2}} \|(\varphi, \psi)\|_\varepsilon^2 \|(\xi, \eta)\|_\varepsilon \end{aligned}$$

and similarly,

$$\begin{aligned} |R''_\varepsilon(\varphi, \psi)(\xi, \eta), (g, h)| &\leq C \int (|\varphi| |g| |\eta| + |\varphi| |h| |\xi| + |\psi| |g| |\xi| + |\psi| |h| |\eta|) \\ &\leq C\varepsilon^{-\frac{N}{2}} \|(\varphi, \psi)\|_\varepsilon \|(\xi, \eta)\|_\varepsilon \|(g, h)\|_\varepsilon. \end{aligned}$$

This completes our proof. □

Lemma 2.2. *There exists $C > 0$ independent of ε such that*

$$\|\ell_\varepsilon\| \leq C \left((|\nabla P(x_\varepsilon)| + |\nabla Q(x_\varepsilon)|) \varepsilon^{\frac{N}{2}+1} + \varepsilon^{\frac{N}{2}+2} \right).$$

Proof. For any $(\varphi, \psi) \in E_{\varepsilon, x_\varepsilon}$, taking into account the decay property of U , we find for any fixed $d > 0$,

$$\begin{aligned} \ell_\varepsilon(\varphi, \psi) &= \int (P(x) - P(x_\varepsilon)) V_{\varepsilon, x_\varepsilon} \varphi + (Q(x) - Q(x_\varepsilon)) W_{\varepsilon, x_\varepsilon} \psi \\ &= \varepsilon^N \left(\int (P(\varepsilon y + x_\varepsilon) - P(x_\varepsilon)) \alpha P(x_\varepsilon) U(\sqrt{P(x_\varepsilon)} y) \varphi(\varepsilon y + x_\varepsilon) \right. \\ &\quad \left. + \int (Q(\varepsilon y + x_\varepsilon) - Q(x_\varepsilon)) \beta Q(x_\varepsilon) U(\sqrt{Q(x_\varepsilon)} y) \psi(\varepsilon y + x_\varepsilon) \right) \\ &= \varepsilon^N \left\{ \int_{B_{\frac{d}{\varepsilon}}(0)} (P(\varepsilon y + x_\varepsilon) - P(x_\varepsilon)) \alpha P(x_\varepsilon) U(\sqrt{P(x_\varepsilon)} y) \varphi(\varepsilon y + x_\varepsilon) \right. \\ &\quad \left. + \int_{B_{\frac{d}{\varepsilon}}(0)} (Q(\varepsilon y + x_\varepsilon) - Q(x_\varepsilon)) \beta Q(x_\varepsilon) U(\sqrt{Q(x_\varepsilon)} y) \psi(\varepsilon y + x_\varepsilon) \right. \\ &\quad \left. + \int_{\mathbb{R}^N \setminus B_{\frac{d}{\varepsilon}}(0)} (P(\varepsilon y + x_\varepsilon) - P(x_\varepsilon)) \alpha P(x_\varepsilon) U(\sqrt{P(x_\varepsilon)} y) \varphi(\varepsilon y + x_\varepsilon) \right. \\ &\quad \left. + \int_{\mathbb{R}^N \setminus B_{\frac{d}{\varepsilon}}(0)} (Q(\varepsilon y + x_\varepsilon) - Q(x_\varepsilon)) \beta Q(x_\varepsilon) U(\sqrt{Q(x_\varepsilon)} y) \psi(\varepsilon y + x_\varepsilon) \right\} \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^N \setminus B_{\frac{d}{\varepsilon}}(0)} (Q(\varepsilon y + x_\varepsilon) - Q(x_\varepsilon)) \beta Q(x_\varepsilon) U(\sqrt{Q(x_\varepsilon)} y) \psi(\varepsilon y + x_\varepsilon) \Big\} \\
& \leq C \varepsilon^{\frac{N}{2}} \left[\left(\int_{B_{\frac{d}{\varepsilon}}(0)} \alpha^2 (P(\varepsilon y + x_\varepsilon) - P(x_\varepsilon))^2 P^2(x_\varepsilon) U^2(\sqrt{P(x_\varepsilon)} y) \right)^{\frac{1}{2}} \right. \\
& \quad \left. + \left(\int_{B_{\frac{d}{\varepsilon}}(0)} \beta^2 (Q(\varepsilon y + x_\varepsilon) - Q(x_\varepsilon))^2 Q^2(x_\varepsilon) U^2(\sqrt{Q(x_\varepsilon)} y) \right)^{\frac{1}{2}} \right] \|(\varphi, \psi)\|_\varepsilon \\
& \quad + C \varepsilon^{\frac{N}{2}} \left[\left(\int_{\mathbb{R}^N \setminus B_{\frac{d}{\varepsilon}}(0)} U^2(\sqrt{P(x_\varepsilon)} y) \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N \setminus B_{\frac{d}{\varepsilon}}(0)} U^2(\sqrt{Q(x_\varepsilon)} y) \right)^{\frac{1}{2}} \right] \|(\varphi, \psi)\|_\varepsilon \\
& = O(\varepsilon^{\frac{N}{2}} e^{-\frac{\tau}{\varepsilon}}) \|(\varphi, \psi)\|_\varepsilon + O\left[\varepsilon^{\frac{N}{2}} \left(\int_{B_{\frac{d}{\varepsilon}}(0)} (|\nabla P(x_\varepsilon)| |\varepsilon y| + |\varepsilon y|^2)^2 U^2(\sqrt{P(x_\varepsilon)} y) \right)^{\frac{1}{2}} \right. \\
& \quad \left. + \varepsilon^{\frac{N}{2}} \left(\int_{B_{\frac{d}{\varepsilon}}(0)} (|\nabla Q(x_\varepsilon)| |\varepsilon y| + |\varepsilon y|^2)^2 U^2(\sqrt{Q(x_\varepsilon)} y) \right)^{\frac{1}{2}} \right] \|(\varphi, \psi)\|_\varepsilon \\
& = O\left((|\nabla P(x_\varepsilon)| + |\nabla Q(x_\varepsilon)|) \varepsilon^{\frac{N}{2}+1} + \varepsilon^{\frac{N}{2}+2}\right) \|(\varphi, \psi)\|_\varepsilon
\end{aligned}$$

for some $\tau > 0$.

□

Now it is easy to check that

$$\begin{aligned}
& \int (\varepsilon^2 (\nabla v \nabla \varphi + \nabla w \nabla \psi) + P(x) v \varphi + Q(x) w \psi) - \int (\mu W_{\varepsilon, x_\varepsilon} v \varphi + \mu V_{\varepsilon, x_\varepsilon} v \psi + \mu V_{\varepsilon, x_\varepsilon} w \varphi \\
& \quad + 2\gamma W_{\varepsilon, x_\varepsilon} w \psi)
\end{aligned}$$

is a bounded bi-linear functional in $E_{\varepsilon, x_\varepsilon}$. Hence we can define L to be a bounded linear map from $E_{\varepsilon, x_\varepsilon}$ to $E_{\varepsilon, x_\varepsilon}$ such that for any $(v, w), (\varphi, \psi) \in E_{\varepsilon, x_\varepsilon}$,

$$\begin{aligned}
\langle L(v, w), (\varphi, \psi) \rangle &= \int (\varepsilon^2 (\nabla v \nabla \varphi + \nabla w \nabla \psi) + P(x) v \varphi + Q(x) w \psi) \\
&\quad - \int (\mu W_{\varepsilon, x_\varepsilon} v \varphi + \mu V_{\varepsilon, x_\varepsilon} v \psi + \mu V_{\varepsilon, x_\varepsilon} w \varphi + 2\gamma W_{\varepsilon, x_\varepsilon} w \psi).
\end{aligned}$$

Next we want to discuss the invertibility of L in $E_{\varepsilon, x_\varepsilon}$. To this end, we denote $(V, W) = (\alpha U, \beta U)$. Then (V, W) solves

$$\begin{cases} -\Delta v + v = \mu v w, & x \in \mathbb{R}^N, \\ -\Delta w + w = \frac{\mu}{2} v^2 + \gamma w^2, & x \in \mathbb{R}^N. \end{cases} \quad (2.4)$$

It follows from Proposition 2.2 in [22] that

Proposition 2.3. For any $\mu > 0$ and $\mu > \gamma$, (V, W) is non-degenerate for the system (2.4) in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ in the sense that the kernel is given by $\text{span} \left\{ \lambda(\mu, \gamma) \frac{\partial V}{\partial x_i}, \frac{\partial W}{\partial x_i} \mid i = 1, 2, \dots, N \right\}$, where $\lambda(\mu, \gamma) \neq 0$.

Using the above result, we come to discuss the invertibility of L in $E_{\varepsilon, x_\varepsilon}$.

Lemma 2.4. There exist constants $\rho > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$,

$$\|L(v, w)\| \geq \rho \|(v, w)\|_\varepsilon, \quad \forall (v, w) \in E_{\varepsilon, x_\varepsilon}.$$

Proof. We argue by contradiction. Suppose that there exist $\varepsilon_n \rightarrow 0$, $x_{\varepsilon_n} \rightarrow x_0$ and $(v_n, w_n) \in E_{\varepsilon_n, x_{\varepsilon_n}}$ such that for any $(\varphi_n, \psi_n) \in E_{\varepsilon_n, x_{\varepsilon_n}}$,

$$\langle L(v_n, w_n), (\varphi_n, \psi_n) \rangle = o_n(1) \| (v_n, w_n) \|_{\varepsilon_n} \| (\varphi_n, \psi_n) \|_{\varepsilon_n}. \quad (2.5)$$

Without loss of generality, we can assume that $\| (v_n, w_n) \|_{\varepsilon_n}^2 = \varepsilon_n^N$ and let

$$\tilde{v}_n(x) = \frac{1}{\sqrt{P(x_{\varepsilon_n})}} v_n \left(\frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right)$$

and

$$\tilde{w}_n(x) = \frac{1}{\sqrt{Q(x_{\varepsilon_n})}} w_n \left(\frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right).$$

Then, in view of $\| (v_n, w_n) \|_{\varepsilon_n}^2 = \varepsilon_n^N$, we get

$$\| (\tilde{v}_n, \tilde{w}_n) \|_{H^1 \times H^1} = \int (|\nabla \tilde{v}_n|^2 + \tilde{v}_n^2 + |\nabla \tilde{w}_n|^2 + \tilde{w}_n^2) \leq C,$$

which implies that up to a subsequence, there exists $v, w \in H^1(\mathbb{R}^N)$ such that as $n \rightarrow +\infty$,

$$\begin{cases} \tilde{v}_n \rightharpoonup v, & \tilde{w}_n \rightharpoonup w & \text{in } H^1(\mathbb{R}^N), \\ \tilde{v}_n \rightarrow v, & \tilde{w}_n \rightarrow w & \text{in } L_{loc}^2(\mathbb{R}^N). \end{cases}$$

Now we claim that $v = w = 0$. Considering (2.5), we find

$$\begin{aligned} & \int \left[P^2(x_{\varepsilon_n}) \left(\nabla \tilde{v}_n \nabla \tilde{\varphi}_n + \nabla \tilde{w}_n \nabla \tilde{\psi}_n - \mu(\beta U) \tilde{v}_n \tilde{\varphi}_n - \mu(\alpha U) \tilde{w}_n \tilde{\varphi}_n - \mu(\alpha U) \tilde{v}_n \tilde{\psi}_n \right. \right. \\ & \quad \left. \left. - 2\gamma(\beta U) \tilde{w}_n \tilde{\psi}_n \right) + P(x_{\varepsilon_n}) P \left(\frac{\varepsilon_n y}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \tilde{v}_n \tilde{\varphi}_n \right. \\ & \quad \left. + Q(x_{\varepsilon_n}) Q \left(\frac{\varepsilon_n y}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \tilde{w}_n \tilde{\psi}_n \right] \\ & = (P(x_{\varepsilon_n}))^{\frac{N}{2}} o_n(1) \left[\int P^2(x_{\varepsilon_n}) |\nabla \tilde{\varphi}_n|^2 + Q^2(x_{\varepsilon_n}) |\nabla \tilde{\psi}_n|^2 + P(x_{\varepsilon_n}) P \left(\frac{\varepsilon_n y}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \tilde{\varphi}_n^2 \right. \\ & \quad \left. + Q(x_{\varepsilon_n}) Q \left(\frac{\varepsilon_n y}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \tilde{\psi}_n^2 \right]^{\frac{1}{2}}, \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} \tilde{\varphi}_n(x) &= \frac{1}{\sqrt{P(x_{\varepsilon_n})}} \varphi_n \left(\frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right), \\ \tilde{\psi}_n(x) &= \frac{1}{\sqrt{Q(x_{\varepsilon_n})}} \psi_n \left(\frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \end{aligned}$$

and $(\tilde{\varphi}_n, \tilde{\psi}_n) \in \tilde{E}_{\varepsilon_n, x_{\varepsilon_n}}$ with

$$\tilde{E}_{\varepsilon_n, x_{\varepsilon_n}} = \left\{ (\tilde{\varphi}, \tilde{\psi}) : \left(\tilde{\varphi} \left(\frac{\sqrt{P(x_{\varepsilon_n})}(x - x_{\varepsilon_n})}{\varepsilon_n} \right), \tilde{\psi} \left(\frac{\sqrt{Q(x_{\varepsilon_n})}(x - x_{\varepsilon_n})}{\varepsilon_n} \right) \right) \in E_{\varepsilon_n, x_{\varepsilon_n}} \right\}.$$

On the other hand, being $\| (v_n, w_n) \|_{\varepsilon_n}^2 = \varepsilon_n^N$, we have

$$\begin{aligned} & \int \left[P(x_{\varepsilon_n}) (|\nabla \tilde{v}_n|^2 + |\nabla \tilde{w}_n|^2) + P \left(\frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \tilde{v}_n^2 \right. \\ & \quad \left. + Q \left(\frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \tilde{w}_n^2 \right] = (P(x_{\varepsilon_n}))^{\frac{N}{2}-1}. \end{aligned} \quad (2.7)$$

So, taking $\varphi_n = v_n$, $\psi_n = w_n$, (2.6) gives

$$\begin{aligned} & \int \left[P(x_{\varepsilon_n}) (|\nabla \tilde{v}_n|^2 + |\nabla \tilde{w}_n|^2 - \mu(\beta U) \tilde{v}_n^2 - 2\mu(\alpha U) \tilde{v}_n \tilde{w}_n - 2\gamma(\beta U) |\tilde{w}_n|^2) \right. \\ & \quad \left. + P \left(\frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \tilde{v}_n^2 + Q \left(\frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right) \tilde{w}_n^2 \right] \\ & = o_n(1) (P(x_{\varepsilon_n}))^{\frac{N}{2}-1}. \end{aligned} \quad (2.8)$$

Also, since $(\tilde{v}_n, \tilde{w}_n) \in \tilde{E}_{\varepsilon_n, x_{\varepsilon_n}}$, we obtain

$$\begin{aligned} & \int \left[P(x_{\varepsilon_n}) \nabla \tilde{v}_n \nabla \frac{\partial(\alpha U)}{\partial y_i} + P\left(\frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n}\right) \tilde{v}_n \frac{\partial(\alpha U)}{\partial y_i} \right. \\ & \quad \left. + Q(x_{\varepsilon_n}) \nabla \tilde{w}_n \nabla \frac{\partial(\beta U)}{\partial y_i} + Q\left(\frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n}\right) \tilde{w}_n \frac{\partial(\beta U)}{\partial y_i} \right] \\ & = 0, \end{aligned}$$

which gives that by letting $n \rightarrow +\infty$,

$$\int \left[\nabla v \nabla \frac{\partial(\alpha U)}{\partial x_i} + v(x) \frac{\partial(\alpha U)}{\partial x_i} + \nabla w \nabla \frac{\partial(\beta U)}{\partial x_i} + w(x) \frac{\partial(\beta U)}{\partial x_i} \right] = 0. \quad (2.9)$$

Now we take $(\tilde{\varphi}, \tilde{\psi}) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ satisfying

$$\int \left[\nabla \tilde{\varphi} \nabla \frac{\partial(\alpha U)}{\partial x_i} + \tilde{\varphi} \frac{\partial(\alpha U)}{\partial x_i} + \nabla \tilde{\psi} \nabla \frac{\partial(\beta U)}{\partial x_i} + \tilde{\psi} \frac{\partial(\beta U)}{\partial x_i} \right] = 0. \quad (2.10)$$

Meanwhile, we can decompose $(\tilde{\varphi}, \tilde{\psi}) \in \tilde{E}_{\varepsilon_n, x_{\varepsilon_n}}$ as follows

$$(\tilde{\varphi}, \tilde{\psi}) = (\tilde{\varphi}_n, \tilde{\psi}_n) - \sum_{i=1}^N a_{i,n} \left(\frac{\partial V_{\varepsilon_n, x_{\varepsilon_n}} \left(\frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right)}{\partial x_i}, \frac{\partial W_{\varepsilon_n, x_{\varepsilon_n}} \left(\frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n} \right)}{\partial x_i} \right).$$

Then from $(\tilde{\varphi}_n, \tilde{\psi}_n) \in \tilde{E}_{\varepsilon_n, x_{\varepsilon_n}}$ and the definition of $\tilde{E}_{\varepsilon_n, x_{\varepsilon_n}}$, we get

$$\begin{aligned} & \int \left[P(x_{\varepsilon_n}) \nabla \tilde{\varphi}_n \nabla \frac{\partial(\alpha U)}{\partial x_i} + P\left(\frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n}\right) \tilde{\varphi}_n \frac{\partial(\alpha U)}{\partial x_i} + Q(x_{\varepsilon_n}) \nabla \tilde{\psi}_n \nabla \frac{\partial(\beta U)}{\partial x_i} \right. \\ & \quad \left. + Q\left(\frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n}\right) \tilde{\psi}_n \frac{\partial(\beta U)}{\partial x_i} \right] + \int \left[\sum_{i=1}^N a_{i,n} \nabla \frac{\partial(\alpha U)}{\partial x_i} \nabla \frac{\partial(\alpha U)}{\partial x_i} \right. \\ & \quad \left. + P\left(\frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n}\right) \sum_{i=1}^N a_{i,n} \frac{\partial(\alpha U)}{\partial x_i} \frac{\partial(\alpha U)}{\partial x_i} + \sum_{i=1}^N a_{i,n} \nabla \frac{\partial(\beta U)}{\partial x_i} \nabla \frac{\partial(\beta U)}{\partial x_i} \right. \\ & \quad \left. + Q\left(\frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n}\right) \sum_{i=1}^N a_{i,n} \frac{\partial(\beta U)}{\partial x_i} \frac{\partial(\beta U)}{\partial x_i} \right] = 0. \end{aligned} \quad (2.11)$$

But by decay property of U , there exists $C > 0$ such that

$$\int \left[\left| \nabla \frac{\partial(\alpha U)}{\partial x_i} \right|^2 + \left| \nabla \frac{\partial(\beta U)}{\partial x_i} \right|^2 + \left| \frac{\partial(\alpha U)}{\partial x_i} \right|^2 + \left| \frac{\partial(\beta U)}{\partial x_i} \right|^2 \right] \geq C > 0. \quad (2.12)$$

Hence, combining (2.10)-(2.12), we can easily check that $a_{i,n} \rightarrow 0$ as $n \rightarrow +\infty$ and then it follows from (2.6) that

$$\begin{aligned} & \int \left[P^2(x_0) (\nabla v \nabla \tilde{\varphi} + \nabla w \nabla \tilde{\psi} + v \tilde{\varphi} + w \tilde{\psi} - \mu(\beta U) v \tilde{\varphi} - \mu(\alpha U) w \tilde{\varphi} \right. \\ & \quad \left. - \mu(\alpha U) v \tilde{\psi} - 2\gamma(\beta U) w \tilde{\psi}) \right] = 0. \end{aligned} \quad (2.13)$$

This implies that for any $(\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N) \cap \tilde{E}_{\varepsilon_n, x_{\varepsilon_n}}$,

$$\begin{aligned} & \int \left[\nabla v \nabla \varphi + \nabla w \nabla \psi + v \varphi + w \psi - \mu(\beta U) v \varphi - \mu(\alpha U) w \varphi \right. \\ & \quad \left. - \mu(\alpha U) v \psi - 2\gamma(\beta U) w \psi \right] = 0. \end{aligned} \quad (2.14)$$

On the other hand, from the fact that $(\alpha U, \beta U)$ solves (2.4), we see

$$\begin{aligned} & \int \left[\nabla v \nabla \frac{\partial(\alpha U)}{\partial x_i} + \nabla w \nabla \frac{\partial(\beta U)}{\partial x_i} + v \frac{\partial(\alpha U)}{\partial x_i} + w \frac{\partial(\beta U)}{\partial x_i} - \mu(\beta U) v \frac{\partial(\alpha U)}{\partial x_i} - \mu(\alpha U) w \frac{\partial(\alpha U)}{\partial x_i} \right. \\ & \quad \left. - \mu(\alpha U) v \frac{\partial(\beta U)}{\partial x_i} - 2\gamma(\beta U) w \frac{\partial(\beta U)}{\partial x_i} \right] = 0, \end{aligned} \quad (2.15)$$

which, together with (2.14), yields that for any $(\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$

$$\int \left[\nabla v \nabla \varphi + \nabla w \nabla \psi + v \varphi + w \psi - \mu(\beta U) v \varphi - \mu(\alpha U) w \varphi - \mu(\alpha U) v \psi - 2\gamma(\beta U) w \psi \right] = 0.$$

So (v, w) is a solution of

$$\begin{cases} -\Delta v + v = \mu(\beta U)v + \mu(\alpha U)w, & x \in \mathbb{R}^N, \\ -\Delta w + w = \mu(\alpha U)v + 2\gamma(\beta U)w, & x \in \mathbb{R}^N. \end{cases}$$

Using Proposition 2.3, there exist $b_i \in \mathbb{R}$, $i = 1, 2, \dots, N$ such that

$$(v, w) = \sum_{i=1}^N b_i \left(\frac{\partial(\alpha U)}{\partial x_i}, \frac{\partial(\beta U)}{\partial x_i} \right).$$

But (2.9) gives that $b_i = 0$, $i = 1, 2, \dots, N$. That is $(v, w) = (0, 0)$, which is exactly our claim. Finally, taking into account that $\tilde{v}_n \rightarrow 0$ in $L_{loc}^2(\mathbb{R}^N)$ and the exponential decay of U , we have

$$\int U |\tilde{v}_n|^2 = \int_{B_R(0)} U |\tilde{v}_n|^2 + \int_{\mathbb{R}^N \setminus B_R(0)} U |\tilde{v}_n|^2 = o_R(1) + O(e^{-R}),$$

where $o_R(1) \rightarrow 0$ as $R \rightarrow +\infty$.

As a result, from (2.8) and (2.7), we deduce that

$$\begin{aligned} o_n(1)(P(x_{\varepsilon_n}))^{\frac{N}{2}-1} &= \int \left[P(x_{\varepsilon_n})(|\nabla \tilde{v}_n|^2 + |\nabla \tilde{w}_n|^2 - \mu(\beta U)\tilde{v}_n^2 - 2\mu(\alpha U)\tilde{v}_n\tilde{w}_n - 2\gamma(\beta U)|\tilde{w}_n|^2) \right. \\ &\quad \left. + P\left(\frac{\varepsilon_n x}{\sqrt{P(x_{\varepsilon_n})}} + x_{\varepsilon_n}\right)\tilde{v}_n^2 + Q\left(\frac{\varepsilon_n x}{\sqrt{Q(x_{\varepsilon_n})}} + x_{\varepsilon_n}\right)\tilde{w}_n^2 \right] \\ &= (P(x_{\varepsilon_n}))^{\frac{N}{2}-1} + o_R(1) + O(e^{-R}), \end{aligned}$$

which is impossible for large n and R . So we complete this proof. $\square$

Proposition 2.5. For $\varepsilon > 0$ sufficiently small, there is $(\varphi_\varepsilon, \psi_\varepsilon) \in E_{\varepsilon, x_\varepsilon}$ such that

$$\langle J'_\varepsilon(\varphi_\varepsilon, \psi_\varepsilon), (g, h) \rangle = 0, \quad \forall (g, h) \in E_{\varepsilon, x_\varepsilon}.$$

Moreover,

$$\|(\varphi_\varepsilon, \psi_\varepsilon)\|_\varepsilon \leq C \left[(|\nabla P(x_\varepsilon)| + |\nabla Q(x_\varepsilon)|) \varepsilon^{\frac{N}{2}+1} + \varepsilon^{\frac{N}{2}+2} \right]$$

for some constant $C > 0$ independent of ε .

Proof. We will use the contraction mapping theorem to prove the wanted result. By Lemma 2.2, $\ell_\varepsilon(\varphi, \psi)$ is a bounded linear function in $E_{\varepsilon, x_\varepsilon}$. So using Riesz representation theorem, we obtain that there is an $\bar{\ell}_\varepsilon \in E_{\varepsilon, x_\varepsilon}$, such that

$$\ell_\varepsilon(\varphi, \psi) = \langle \bar{\ell}_\varepsilon, (\varphi, \psi) \rangle.$$

Hence finding a critical point for $J_\varepsilon(\varphi, \psi)$ is equivalent to solving

$$\ell_\varepsilon + L(\varphi, \psi) + R'_\varepsilon(\varphi, \psi) = 0. \quad (2.16)$$

It follows from Lemma 2.4 that (2.16) can be rewritten as

$$(\varphi, \psi) = A(\varphi, \psi) := -L^{-1}(\bar{\ell}_\varepsilon + R'_\varepsilon(\varphi, \psi)). \quad (2.17)$$

Now we set

$$S_\varepsilon = \{(\varphi, \psi) \in E_{\varepsilon, x_\varepsilon}, \|(\varphi, \psi)\|_\varepsilon \leq [(|\nabla P(x_\varepsilon)| + |\nabla Q(x_\varepsilon)|) \varepsilon^{\frac{N}{2}+1-\theta} + \varepsilon^{\frac{N}{2}+2-\theta}] \}$$

with $\theta \in (0, 1)$. Then for any $(\varphi, \psi) \in S_\varepsilon$,

$$\begin{aligned} \|A(\varphi, \psi)\| &\leq C(\|\bar{\ell}_\varepsilon\|_\varepsilon + \|R'_\varepsilon(\varphi, \psi)\|) \\ &\leq C\|\bar{\ell}_\varepsilon\|_\varepsilon + C\varepsilon^{-\frac{N}{2}}\|(\varphi, \psi)\|_\varepsilon^2 \\ &\leq [(|\nabla P(x_\varepsilon)| + |\nabla Q(x_\varepsilon)|)\varepsilon^{\frac{N}{2}+1-\theta} + \varepsilon^{\frac{N}{2}+2-\theta}]. \end{aligned}$$

Then A maps S_ε to S_ε . On the other hand, for any $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in S_\varepsilon$, from Lemma 2.1,

$$\begin{aligned} \|A(\varphi_1, \psi_1) - A(\varphi_2, \psi_2)\| &= \|L^{-1}(R'_\varepsilon(\varphi_1, \psi_1) - R'_\varepsilon(\varphi_2, \psi_2))\| \\ &\leq C\|R''_\varepsilon(t(\varphi_1, \psi_1) + (1-t)(\varphi_2, \psi_2))\| \|(\varphi_1, \psi_1) - (\varphi_2, \psi_2)\|_\varepsilon \\ &\leq \frac{1}{2}\|(\varphi_1, \psi_1) - (\varphi_2, \psi_2)\|_\varepsilon, \end{aligned}$$

where $t \in (0, 1)$. This gives that A is a contraction map from S_ε to S_ε . Applying the contraction mapping theorem, we can find a unique $(\varphi_\varepsilon, \psi_\varepsilon) \in S_\varepsilon$ satisfying (2.17) and

$$\|(\varphi_\varepsilon, \psi_\varepsilon)\|_\varepsilon \leq [(|\nabla P(x_\varepsilon)| + |\nabla Q(x_\varepsilon)|)\varepsilon^{\frac{N}{2}+1-\theta} + \varepsilon^{\frac{N}{2}+2-\theta}].$$

Furthermore, in view of (2.17), we get

$$\begin{aligned} \|(\varphi_\varepsilon, \psi_\varepsilon)\|_\varepsilon &= \|L^{-1}(\bar{\ell}_\varepsilon) + L^{-1}(R'_\varepsilon(\varphi_\varepsilon, \psi_\varepsilon))\|_\varepsilon \\ &\leq C\|\bar{\ell}_\varepsilon\|_\varepsilon + C\varepsilon^{-\frac{N}{2}}\|(\varphi_\varepsilon, \psi_\varepsilon)\|_\varepsilon^2 \\ &\leq C\|\bar{\ell}_\varepsilon\|_\varepsilon + C[(|\nabla P(x_\varepsilon)| + |\nabla Q(x_\varepsilon)|)\varepsilon^{1-\theta} + \varepsilon^{2-\theta}]\|(\varphi_\varepsilon, \psi_\varepsilon)\|_\varepsilon, \end{aligned}$$

which implies that

$$\|(\varphi_\varepsilon, \psi_\varepsilon)\|_\varepsilon \leq C\|\bar{\ell}_\varepsilon\|_\varepsilon \leq C[(|\nabla P(x_\varepsilon)| + |\nabla Q(x_\varepsilon)|)\varepsilon^{\frac{N}{2}+1} + \varepsilon^{\frac{N}{2}+2}].$$

□

3 Proof of our main result

In this section, we assume that $x_0 \in \Gamma$ is a non-degenerate critical point of $\Delta(P + Q)$ and we will construct a single peak solution $(v_\varepsilon, w_\varepsilon)$ of (1.1) concentrating at x_0 .

From Proposition 2.5, we can get the following result.

Proposition 3.1. *There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and y close to x_0 , there is $(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y}) \in E_{\varepsilon,y}$ such that for any $(g, h) \in E_{\varepsilon,y}$,*

$$\begin{aligned} &\int [\varepsilon^2 \nabla(V_{\varepsilon,y} + \varphi_{\varepsilon,y}) \nabla g + P(x)(V_{\varepsilon,y} + \varphi_{\varepsilon,y})g + \varepsilon^2 \nabla(W_{\varepsilon,y} + \psi_{\varepsilon,y}) \nabla h \\ &\quad + Q(x)(W_{\varepsilon,y} + \psi_{\varepsilon,y})h] - \int [\mu(V_{\varepsilon,y} + \varphi_{\varepsilon,y})g(W_{\varepsilon,y} + \psi_{\varepsilon,y}) + \frac{\mu}{2}(V_{\varepsilon,y} + \varphi_{\varepsilon,y})^2 h \\ &\quad + \gamma(W_{\varepsilon,y} + \psi_{\varepsilon,y})^2 h] = 0. \end{aligned}$$

Moreover,

$$\|(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y})\|_\varepsilon = O((|\nabla P(y)| + |\nabla Q(y)|)\varepsilon^{\frac{N}{2}+1} + \varepsilon^{\frac{N}{2}+2}).$$

To get a true solution of (1.1), we need to choose y such that

$$\begin{aligned} &\int [\varepsilon^2 \nabla v_\varepsilon \nabla \frac{\partial v_\varepsilon}{\partial x_i} + P(x)v_\varepsilon \frac{\partial v_\varepsilon}{\partial x_i} + \varepsilon^2 \nabla w_\varepsilon \nabla \frac{\partial w_\varepsilon}{\partial x_i} + Q(x)w_\varepsilon \frac{\partial w_\varepsilon}{\partial x_i}] - \int [\mu v_\varepsilon \frac{\partial v_\varepsilon}{\partial x_i} w_\varepsilon \\ &\quad + \frac{\mu}{2} v_\varepsilon^2 \frac{\partial w_\varepsilon}{\partial x_i} + \gamma w_\varepsilon^2 \frac{\partial w_\varepsilon}{\partial x_i}] = 0, \end{aligned} \quad (3.1)$$

where $v_\varepsilon = V_{\varepsilon,y} + \varphi_{\varepsilon,y}$, $w_\varepsilon = W_{\varepsilon,y} + \psi_{\varepsilon,y}$ and $i = 1, \dots, N$. It is easy to check that (3.1) is equivalent to

$$\int \left[\frac{\partial P(x)}{\partial x_i} v_\varepsilon^2 + \frac{\partial Q(x)}{\partial x_i} w_\varepsilon^2 \right] = 0, \quad i = 1, \dots, N, \quad (3.2)$$

which is exact the Pohozaev type identity.

For y close to x_0 , $y \in \Gamma_t$ for some t close to 1. In the following, we denote by ν the unit normal vector of Γ_t at y and we use ζ_i , $i = 1, \dots, N-1$, to denote the principal direction of Γ_t at y . Then, at y , one has

$$D_{\zeta_i} P(y) = 0, \quad |\nabla P(y)| = |D_\nu P(y)|$$

and

$$D_{\zeta_i} Q(y) = 0, \quad |\nabla Q(y)| = |D_\nu Q(y)|.$$

First, we prove the following results.

Lemma 3.2. *If $(H_1) - (H_3)$ hold, then*

$$\int (D_\nu P(x) v_\varepsilon^2 + D_\nu Q(x) w_\varepsilon^2) = 0$$

is equivalent to

$$D_\nu P(y) + D_\nu Q(y) = O(\varepsilon^2).$$

Proof. By the direct computations, we have

$$\begin{aligned} & \int (D_\nu P(x) V_{\varepsilon,y}^2 + D_\nu Q(x) W_{\varepsilon,y}^2) \\ &= - \int [2D_\nu P(x) V_{\varepsilon,y} \varphi_{\varepsilon,y} + D_\nu P(x) \varphi_{\varepsilon,y}^2 + 2D_\nu Q(x) W_{\varepsilon,y} \psi_{\varepsilon,y} + D_\nu Q(x) \psi_{\varepsilon,y}^2] \\ &= O((|D_\nu P(y)| + |D_\nu Q(y)|) \varepsilon^{\frac{N}{2}} \|(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y})\|_\varepsilon + \varepsilon^{\frac{N}{2}+1} \|(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y})\|_\varepsilon + \|(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y})\|_\varepsilon^2) \\ &= \varepsilon^N O((|D_\nu P(y)| + |D_\nu Q(y)|) \varepsilon + \varepsilon^2). \end{aligned} \quad (3.3)$$

On the other hand, using Taylor's expansion, we get

$$\begin{aligned} & \int (D_\nu P(x) V_{\varepsilon,y}^2 + D_\nu Q(x) W_{\varepsilon,y}^2) \\ &= \varepsilon^N \left[\int_{B_{\frac{\delta}{\varepsilon}}(0)} \left(D_\nu P(y) + \frac{\varepsilon^2 |x|^2}{2!} (\Delta D_\nu P)(y) \right) \alpha^2 P^2(y) U^2(\sqrt{P(y)} x) \right. \\ & \quad \left. + \int_{B_{\frac{\delta}{\varepsilon}}(0)} \left(D_\nu Q(y) + \frac{\varepsilon^2 |x|^2}{2!} (\Delta D_\nu Q)(y) \right) \beta^2 Q^2(y) U^2(\sqrt{Q(y)} x) \right] + O(\varepsilon^{N+4}) \\ &= O((|D_\nu P(y)| + |D_\nu Q(y)|) \varepsilon^N + ((\Delta D_\nu P)(y) + (\Delta D_\nu Q)(y)) \varepsilon^{N+2} + \varepsilon^{N+4}). \end{aligned} \quad (3.4)$$

Combining (3.3) and (3.4), we find

$$D_\nu P(y) + D_\nu Q(y) = O(\varepsilon^2).$$

□

Lemma 3.3. *Under the conditions $(H_1) - (H_3)$,*

$$\int (D_\zeta P(x) v_\varepsilon^2 + D_\zeta Q(x) w_\varepsilon^2) = 0$$

is equivalent to

$$\begin{aligned} & (D_\zeta \Delta P)(y) + (D_\zeta \Delta Q)(y) + B_1 K(y) \varepsilon^2 \\ &= O((|\nabla G_1(y)| + |\nabla G_2(y)|)(|\nabla P(y)| + |\nabla Q(y)|) + (|\nabla G_1(y)| + |\nabla G_2(y)|) \varepsilon), \end{aligned}$$

where B_1 is some constant, $K(y)$ is a smooth function, and $G_1(x) = \langle \nabla P(x), \zeta \rangle$, $G_2(x) = \langle \nabla Q(x), \zeta \rangle$.

Proof. Since for any fixed $d > 0$ and $j = 1, 2$

$$G_j(x) = \sum_{i=1}^N \frac{\partial G_j(y)}{\partial y_i} (x_i - y_i) + \frac{1}{2} \sum_{i=1}^N \sum_{\ell=1}^N \frac{\partial^2 G_j(y)}{\partial y_i \partial y_\ell} (x_i - y_i)(x_\ell - y_\ell) + o(|x - y|^2), \text{ in } B_d(y),$$

we have

$$\begin{aligned} & \int (G_1(x)V_{\varepsilon,y}^2 + G_2(x)W_{\varepsilon,y}^2) \\ &= -2 \int G_1(x)V_{\varepsilon,y}\varphi_{\varepsilon,y} - \int G_1(x)\varphi_{\varepsilon,y}^2 - 2 \int G_2(x)W_{\varepsilon,y}\psi_{\varepsilon,y} - \int G_2(x)\psi_{\varepsilon,y}^2 \\ &= O\left(\varepsilon^{\frac{N}{2}}(\varepsilon|\nabla G_1(y)| + \varepsilon|\nabla G_2(y)| + \varepsilon^2)\|(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y})\|_\varepsilon + \varepsilon(|\nabla G_1(y)| \right. \\ &\quad \left. + |\nabla G_2(y)|)\|(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y})\|_\varepsilon^2\right) \\ &= O\left((|\nabla G_1(y)| + |\nabla G_2(y)|)(|\nabla P(y)| + |\nabla Q(y)|)\varepsilon^{N+2} + (|\nabla G_1(y)| + |\nabla G_2(y)|)\varepsilon^{N+3} \right. \\ &\quad \left. + \varepsilon^{N+4}\right). \end{aligned}$$

On the other hand, from $G_j(y) = 0$, $j = 1, 2$, we get

$$\begin{aligned} & \int (G_1(x)V_{\varepsilon,y}^2 + G_2(x)W_{\varepsilon,y}^2) \\ &= O\left(\varepsilon^{N+2}(\Delta G_1(y) + \Delta G_2(y)) + B_1 K(y)\varepsilon^{N+4} + \varepsilon^{N+6}\right). \end{aligned}$$

As a result, the result follows. $\square$

Proof of Theorem 1.4:. Now, by Lemmas 3.2 and 3.3, (3.2) is equivalent to

$$D_v P(y) + D_v Q(y) = O(\varepsilon^2), \quad (D_\zeta \Delta P)(y) + (D_\zeta \Delta Q)(y) = O(|\nabla P(y)| + |\nabla Q(y)| + \varepsilon^2), \quad (3.5)$$

which is also equivalent to

$$D_v P(y) + D_v Q(y) = O(\varepsilon^2), \quad (D_\zeta \Delta P)(y) + (D_\zeta \Delta Q)(y) = O(\varepsilon^2). \quad (3.6)$$

Let $\bar{y} \in \Gamma$ be the point such that $y - \bar{y} = \kappa v$ for some $\kappa \in \mathbb{R}$. We have $D_v P(\bar{y}) = 0$ and $D_v Q(\bar{y}) = 0$. As a result,

$$\begin{aligned} D_v P(y) + D_v Q(y) &= D_v P(y) - D_v P(\bar{y}) + D_v Q(y) - D_v Q(\bar{y}) \\ &= (D_{vv}^2 P(\bar{y}) + D_{vv}^2 Q(\bar{y}))\langle y - \bar{y}, v \rangle + O(|y - \bar{y}|^2), \end{aligned}$$

which, together with the non-degenerate assumption, yields that $D_v P(y) + D_v Q(y) = O(\varepsilon^2)$ can be written as

$$\langle y - \bar{y}, v \rangle = O(\varepsilon^2 + |y - \bar{y}|^2). \quad (3.7)$$

Let $\bar{\zeta}_i$ be the i -th tangential unit vector of Γ at $\bar{y}$. It follows from the assumption (H_2) that

$$\begin{aligned} (D_{\bar{\zeta}_i} \Delta P)(y) + (D_{\bar{\zeta}_i} \Delta Q)(y) &= (D_{\bar{\zeta}_i} \Delta P)(\bar{y}) + (D_{\bar{\zeta}_i} \Delta Q)(\bar{y}) + O(|y - \bar{y}|) \\ &= (D_{\bar{\zeta}_i} \Delta P)(\bar{y}) + (D_{\bar{\zeta}_i} \Delta Q)(\bar{y}) + O(\varepsilon^2) \end{aligned}$$

and

$$\begin{aligned} & (D_{\bar{\zeta}_i} \Delta P)(\bar{y}) + (D_{\bar{\zeta}_i} \Delta Q)(\bar{y}) \\ &= (D_{\bar{\zeta}_i} \Delta P)(\bar{y}) - (D_{\bar{\zeta}_{i,0}} \Delta P)(x_0) + (D_{\bar{\zeta}_i} \Delta Q)(\bar{y}) - (D_{\bar{\zeta}_{i,0}} \Delta Q)(x_0) \\ &= \langle (\nabla_T D_{\bar{\zeta}_{i,0}} \Delta P)(x_0), \bar{y} - x_0 \rangle + \langle (\nabla_T D_{\bar{\zeta}_{i,0}} \Delta Q)(x_0), \bar{y} - x_0 \rangle + O(|\bar{y} - x_0|^2), \end{aligned}$$

where ∇_T is the tangential gradient on Γ at x_0 and $\bar{\zeta}_{i,0}$ is the i -th tangential unit vector of Γ at x_0 . Hence $(D_\zeta \Delta P)(y) + (D_\zeta \Delta Q)(y) = O(\varepsilon^2)$ can be rewritten as

$$\langle (\nabla_T D_{\bar{\zeta}_{i,0}} \Delta P)(x_0) + (\nabla_T D_{\bar{\zeta}_{i,0}} \Delta Q)(x_0), \bar{y} - x_0 \rangle = O(\varepsilon^2 + |\bar{y} - x_0|^2). \quad (3.8)$$

So we can solve (3.7) and (3.8) to get $y = x_\varepsilon$ with $x_\varepsilon \rightarrow x_0$ as $\varepsilon \rightarrow 0$. $\square$

Acknowledgements: The authors sincerely thank Dr. Chunhua Wang for her helpful discussions and suggestions. This paper was partially supported by NSFC (No.11601194, 11961043, 11901249).

Conflicts of Interest: Authors state no conflict of interest.

Data Availability: Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

References

- [1] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, *C. R. Math. Acad. Sci. Paris*, 342 (2006), no. 7, 453-458.
- [2] A. Ambrosetti and E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, *J. Lond. Math. Soc.* 75 (2007), no. 1, 67-82.
- [3] T. Bartsch and Z. Wang, Note on ground states of nonlinear Schrödinger systems, *J. Partial Differential Equations* 19 (2006), no. 3, 200-207.
- [4] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.* 82 (1983), no. 4, 313-345.
- [5] D. Cao, E.S. Noussair, S. Yan, Solutions with multiple peaks for nonlinear elliptic equations, *Proc. R. Soc. Edinburgh Sect. A*, 129 (1999), no. 2, 235-264.
- [6] T. Cazenave, Semilinear Schrödinger equations, *Courant Lecture Notes in Mathematics*, vol.10 of Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 2003.
- [7] M. Conti, S. Terracini and G. Verzini, Nehari's problem and competing species systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire*. 19 (2002), no. 6, 871-888.
- [8] E. Dancer, J. Wei and T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system, *Ann. Inst. H. Poincaré Anal. Non Linéaire*. 27 (2010), no. 3, 953-969.
- [9] Y. Guo, S. Li, J. Wei and X. Zeng, Ground states of two-component attractive Bose-Einstein condensates I: Existence and uniqueness, *J. Funct. Anal.* 276 (2019), no. 1, 183-230.
- [10] M. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$, *Arch. Rational Mech. Anal.* 105 (1989), no. 3, 243-266.
- [11] W. Lian, J. Wang, and R. Xu, Global existence and blow up of solutions for pseudo-parabolic equation with singular potential, *J. Differential Equations* 269 (2020), no. 6, 4914-4959.
- [12] Z. Liu and Z. Wang, Ground states and bound states of a nonlinear Schrödinger system, *Adv. Nonlinear Stud.* 10 (2010), no. 1, 175-193.
- [13] P. Luo, S. Peng and S. Yan, On the nonlinear Schrödinger equations and application to Bose-Einstein condensates, preprint.
- [14] E. Montefusco, B. Pellacci and M. Squassina, Semiclassical states for weakly coupled nonlinear Schrödinger systems, *J. Eur. Math. Soc.* 10 (2008), no. 1, 41-71.
- [15] B. Noris, H. Tavares, S. Terracini and G. Verzini, Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition, *Comm. Pure Appl. Math.* 63 (2010), no. 3, 267-302.
- [16] Nikolaos S. Papageorgiou, Vicentiu D. Rădulescu and Dušan D. Repovš, *Nonlinear Analysis—Theory and Methods*, Springer Monographs in Mathematics, Springer, Cham, 2019.
- [17] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equation in $\mathbb{R}^N$, *Comm. Math. Phys.* 271 (2007), no. 1, 199-221.
- [18] W.A. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* 55 (1977), no. 2, 149-162.
- [19] C. Sulem and P.L. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*, vol.39 of Applied Mathematical Sciences. Springer-Verlag, New York, 1999.
- [20] Z. Tang and H. Xie, Multi-spikes solutions for a system of coupled elliptic equations with quadratic nonlinearity, *Commun. Pure Appl. Anal.* 19 (2020), no. 1, 311-328.
- [21] R. Tian and Z. Wang, Multiple solitary wave solutions of nonlinear Schrödinger system, *Topol. Methods Nonlinear Anal.* 37 (2011), no. 2, 203-223.
- [22] C. Wang and J. Zhou, Infinitely many solitary waves due to the second-harmonic generation in quadratic media, *Acta Math. Sci. Ser. B (Engl. Ed.)* 40 (2020), no. 1, 16-34.
- [23] X. Wang and R. Xu, Global existence and finite time blow-up for a nonlocal semilinear pseudo-parabolic equation, *Adv. Nonlinear Anal.* 10 (2021), 261-288.
- [24] J. Wei and T. Weth, Radial solutions and phase separation in a system of two coupled Schrödinger equations, *Arch. Ration. Mech. Anal.* 190 (2008), no. 1, 83-106.

- [25] R. Xu and J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.* 264 (2013), no. 10, 2732-2763.
- [26] R. Xu, W. Lian and Y. Niu, Global well-posedness of coupled parabolic systems, *Sci. China Math.* 63 (2020), no. 2, 321-356.
- [27] L. Zhao, F. Zhao and J. Shi, Higher dimensional Solitary waves generated by second-harmonic generation in quadratic media, *Calc. Var. Partial Differential Equations* 54 (2015), no. 3, 2657-2691.