Research Article

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Qualitative analysis for the nonlinear fractional Hartree type system with nonlocal interaction

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Abstract: In the present paper we study the existence of nontrivial solutions of a class of static coupled nonlinear fractional Hartree type system. First, we use the direct moving plane methods to establish the maximum principle (Decay at infinity and Narrow region principle) and prove the symmetry and nonexistence of positive solution of this nonlocal system. Second, we make complete classification of positive solutions of the system in the critical case when some parameters are equal. Finally, we prove the existence of multiple nontrivial solutions in the critical case according to the different parameters ranges by using variational methods. To accomplish our results we establish the maximum principle for the fractional nonlocal system.

Keywords: Fractional Laplacians, Nonlinear fractional Hartree system; Variational methods

MSC: 35J61, 35J20, 35Q55, 49J40

1 Introduction and main results

In the present paper we study the coupled nonlinear fractional Hartree system in the following form

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \mu_1 \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^{N-\gamma}} dy \right) |u|^{p-2} u + \beta \left(\int_{\mathbb{R}^N} \frac{|v(y)|^p}{|x-y|^{N-\gamma}} dy \right) |u|^{p-2} u, & x \in \mathbb{R}^N, \\ (-\Delta)^{\frac{\alpha}{2}} v = \mu_2 \left(\int_{\mathbb{R}^N} \frac{|v(y)|^p}{|x-y|^{N-\gamma}} dy \right) |v|^{p-2} v + \beta \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^{N-\gamma}} dy \right) |v|^{p-2} v, & x \in \mathbb{R}^N, \\ u, v > 0 \text{ and } u, v \in C_{loc}^{1,1}(\mathbb{R}^N) \cap \mathcal{F}_\alpha(\mathbb{R}^N), \end{cases} \quad (1.1)$$

where $N \geq 3$, $\gamma \in (0, N)$, $1 \leq p \leq \frac{N+\gamma}{N-\alpha}$ and

$$\mathcal{F}_\alpha = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \mid \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+\alpha}} < \infty \right\}. \quad (1.2)$$

Recall that the operator $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian in $\mathbb{R}^N$ which is defined as a nonlocal pseudo differential operator

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \mathcal{C}_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+\alpha}} dy := \mathcal{C}_{N,\alpha} \lim_{\varepsilon \rightarrow 0} \int_{|x-y| \geq \varepsilon} \frac{u(x) - u(y)}{|x-y|^{N+\alpha}} dy \quad (1.3)$$

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for each $u \in C_{loc}^{1,1}(\mathbb{R}^N) \cap \mathcal{F}_\alpha(\mathbb{R}^N)$, where the constant $\mathcal{C}_{N,\alpha} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(2\pi\omega_1)}{|\omega|^{N+\alpha}} d\omega \right)^{-1}$ and $\omega = (\omega_1, \dots, \omega_N)$ (see [3, 6, 7, 9, 49]). Note that both the fractional Laplacians $(-\Delta)^{\frac{\alpha}{2}}$ and the Hartree type nonlinearities are nonlocal in the system (1.1). Moreover, we can check that the system is closely related to the following integral equation

$$\begin{cases} u(x) = \mu_1 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}|u(y)|^{p-2}u(y)\phi_u(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}|u(y)|^{p-2}u(y)\phi_v(y)}{|x-y|^{N-\alpha}} dy, \\ v(x) = \mu_2 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}|v(y)|^{p-2}v(y)\phi_v(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}|v(y)|^{p-2}v(y)\phi_u(y)}{|x-y|^{N-\alpha}} dy, \end{cases} \quad (1.4)$$

where $\phi_w(y) = \int_{\mathbb{R}^N} \frac{|w(z)|^p}{|y-z|^{N-\gamma}} dz$ and $\mathcal{R}_{N,\alpha} = \frac{\Gamma(\frac{N-\alpha}{2})}{\pi^{N/2} 2^\alpha \Gamma(\frac{\alpha}{2})}$.

One of the motivation to study the system (1.1) is motivated by recent studies on the nonlinear fractional Schrödinger equation with nonlocal interaction

$$i \frac{\partial \psi}{\partial t} = (-\Delta)^{\frac{\alpha}{2}} \psi + V\psi - \chi (C(x) * |\psi|^p) |\psi|^{p-2} \psi, \quad x \in \mathbb{R}^N. \quad (1.5)$$

If $\alpha = 2$, $N = 3$ and $p = 2$, the system (1.5) is related to the condensate in the mean field regime. Indeed, concerning the movement of the identical and non-relativistic basic particles (such as bosons or electrons), under the influence of an external potential, the interaction between two particles is governed by the nonlinear nonlocal Hartree equation (see [17, 18, 23, 24]). Here the function ψ is a radially symmetric two-body potential function defined and $*$ denotes the convolution in $\mathbb{R}^3$. The most typical external potential is the Coulomb function $C(x) = |x|^{-1}$. On the other hand, the system (1.5) is also used in the description of the Bose-Einstein condensates, in which V is the trapping potential and the nonlocal interaction also describes the interaction between the bosons in the condensate [14, 44, 47]. When $V = 0$, (1.5) is also known as nonlinear Choquard equation [33, 37, 40], and the equation (1.5) with $V = 0$ also arises from the model of wave propagation in a media with a large response length [1, 26]. Many papers considered the general interaction case. That is,

$$C(x) = \frac{A_\gamma}{|x|^{N-\gamma}} \quad \text{and} \quad A_\gamma = \frac{\Gamma\left(\frac{N-\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right) \pi^{N/2} 2^\gamma}. \quad (1.6)$$

Then the stationary system of (1.5) reduces to

$$(-\Delta)^{\frac{\alpha}{2}} \psi + \lambda \psi = \mu (C(x) * |\psi|^p) |\psi|^{p-2} \psi, \quad x \in \mathbb{R}^N. \quad (1.7)$$

If $\alpha = 2$, the paper [41] proved the existence and some properties of solution of (1.7). Recently, the paper [20] prove the existence of nodal solution of (1.7). For more general information one can refer to the papers [19, 20, 30, 31, 40, 42, 46] and references therein. For the fractional case $0 < \alpha < 2$, the paper [12] proved the regularity and classification of the solution of (1.7). Recently, by using the direct moving plane methods, the paper [11] make the classification of the positive solution of (1.7) when $p = 2$ and $N - \gamma = 2\alpha$. The paper [28] make the classification of the positive solution of (1.7) for the general case.

The system (1.1) was studied in the several recent papers [51–54, 56] for the case $\alpha = 2$. This kind of systems are considered in the basic quantum chemistry model of small number of electrons interacting with static nuclei which can be approximated by Hartree or Hartree-Fock minimization problems (see [29, 35, 39]). In fact, the Euler-Lagrange equations corresponding to such Hartree problem are

$$-\Delta u_i + V(x)u_i + \sum_{j \neq i} \left(\int_{\mathbb{R}^3} \frac{u_j^2(y)}{|x-y|} dy \right) u_i + \varepsilon_i u_i = 0, \quad x \in \mathbb{R}^3, \quad 1 \leq i \leq k, \quad (1.8)$$

where $k \in \mathbb{N}$, $V(x)$ describes the attractive interaction between the electrons and the nuclei, the integral term shows the repulsive Coulomb interaction between the electrons, and $-\varepsilon_i$ are the Lagrange multipliers. Following the discussion in [39], we usually consider the case that some components are set to be equal in (1.8). For example, when $k = 2$ and $u_1 = u_2$, then (1.8) is reduced to a scalar equation

$$-\Delta u + V(x)u + \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u + \varepsilon u = 0, \quad x \in \mathbb{R}^3. \quad (1.9)$$

The solutions of (1.9) were considered in, for example, [19, 38, 39]. We notice that in (1.8), the interaction between electrons is repulsive while the one in (1.5) is attractive. On the other hand, if $k = 4$, $u_1 = u_2$ and $u_3 = u_4$, then we can also obtain (1.1) when $V = 0$, $p = q = 2$, $N = 3$ and $\gamma = \alpha = 2$. Recent years, the following nonlocal system

$$\begin{cases} -\varepsilon^2 \Delta u + \lambda_1(x)u = \mu_1 \phi_u |u|^{p-2} u + \beta \phi_v |u|^{p-2} u, & x \in \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + \lambda_2(x)v = \mu_2 \phi_v |v|^{p-2} v + \beta \phi_u |v|^{p-2} v, & x \in \mathbb{R}^N, \end{cases} \quad (1.10)$$

has been studied by many literatures, where $\phi_u(x) = \int_{\mathbb{R}^N} C(x-y)|u(y)|^p dy$ and $C(x)$ is given in (1.6). Note that the system (1.10)(or (1.1)) has two semitrivial solutions $(u, 0)$ and $(0, v)$, where u, v are solutions of (1.7). In order to make clear statement one gives the following definitions. We say (u, v) is a nontrivial solution of (1.10)(or (1.1)) if $u \neq 0$ and $v \neq 0$. If $u, v > 0$, we say (u, v) is a positive solution of (1.10)(or (1.1)). A solution is called a nontrivial ground state solution(or positive ground state solution) if its energy is minimal among all the nontrivial solutions(or all the positive solutions) of (1.10)(or (1.1)). The paper [56] considered the semi-classical case. Under some conditions for the potential function $\lambda_i(x)$, $i = 1, 2$, the existence of a ground state solution of (1.10) for $\varepsilon > 0$ small and $\beta > 0$ sufficiently large was proved. Later, the paper [53] studied the case $\lambda_i(x) = \lambda_i = \text{constant}$, $\varepsilon = 1$, $p = 2$, $N = 3$ and $\gamma = 2$. The authors proved the existence and nonexistence of positive ground state solutions of (1.10). Moreover, various qualitative properties of ground state solutions are also obtained. Recently, the papers [51, 54] studied the existence and properties of normolized solution of (1.10) with general interaction. Very recently, the paper [52] proved the existence of nontrivial solutions of the more general nonlocal interaction case of (1.10).

Motivated by the previous works, in the present paper we shall study the general case of the system (1.10) in fractional situation. Precisely, the main purpose of this paper is the following three parts. First, we use the direct moving plane methods to establish the maximum principle(Decay at infinity and Narrow region principle) and prove the symmetry and nonexistence of positive solution of this nonlocal system. Second, we make complete classification of positive solutions of the system for the critical case when $\mu_1 = \mu_2$. Finally, we prove the existence of multiple nontrivial solutions of the critical case for $\mu_1 \neq \mu_2$ by using variational methods. To accomplish the first two conclusions, we shall use a variant (for nonlocal nonlinearity) of the direct method of moving planes for fractional Laplacians due to the paper [6, 7] to obtain symmetry, monotonicity, nonexistence and classification of the positive solutions to (1.1). The methods of moving planes was initially invented by Alexanderoff in the early 1950s. Later, it was further developed by Serrin [48], Gidas, Ni and Nirenberg [21, 22], Caffarelli, Gidas and Spruck [2], Chen and Li [4], Li and Zhu [32] and many others. For more literatures on this direction, one can see the papers [4, 7, 8, 13, 16, 32, 36, 55] and the references therein. In this paper we establish the maximum principle(Decay at infinity and Narrow region principle, see Theorems 2.6-2.7 below) for the fractional nonlocal system. This is new and we believe that it will be useful to study the other nonlocal problems. To accomplish the third result, we should make careful study the uniqueness of the synchronous solutions of (1.1) and use the perturbation methods to obtain the nontrivial ground state solution of (1.1).

Then we first have the following main results for the subcritical and critical cases.

Theorem 1.1. Assume that $N \geq 2$, $0 < \alpha < 2$, $0 < \gamma < N$ and $\mu_1, \mu_2, \beta > 0$. Then the system (1.1) has no positive solution for $1 \leq p < \frac{N+\gamma}{N-\alpha}$. If $p = \frac{N+\gamma}{N-\alpha}$ and $\mu_1 = \mu_2$, then every positive solution (u, v) of (1.1) must has the following form $u = v = (\mu_1 + \beta)^{-\frac{N-\alpha}{2(\gamma+\alpha)}} \mu^{\frac{N-\alpha}{2}} U(\mu(x-y))$, where $U(x) = \tau_0 \left(\frac{1}{1+|x|^2} \right)^{\frac{N-\alpha}{2}}$ and $I(k) = \frac{\pi^{N/2} \Gamma(\frac{N-2k}{2})}{\Gamma(N-k)}$, where $\tau_0 = \left(\mathcal{R}_{N,\alpha} I\left(\frac{N-\alpha}{2}\right) I\left(\frac{N-\gamma}{2}\right) \right)^{-\frac{N-\alpha}{2(\gamma+\alpha)}}$.

From the results of Theorem 1.1, we have the following corollary.

Corollary 1.2. If $u = v$, $\mu_1 = \mu_2 = 1$ and $p = \frac{N+\gamma}{N-\alpha}$, then the following critical Hartree type equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{\frac{N+\gamma}{N-\alpha}}}{|x-y|^{N-\gamma}} dy \right) |u|^{\frac{2\alpha+\gamma-N}{N-\alpha}} u, \quad \text{in } \mathbb{R}^N \quad (1.11)$$

has a unique positive solution of the form

$$U_\mu(x) = C \left(\frac{\mu}{1 + \mu^2 |x - x_0|^2} \right)^{\frac{N-\alpha}{2}} \quad \text{for some } \mu > 0 \text{ and } x_0 \in \mathbb{R}^N. \quad (1.12)$$

Remark 1.3. The result of Corollary 1.2 has been obtained recently by [11, 28].

Next we consider the case $p = \frac{N+\gamma}{N-\alpha}$, $\mu_1 \neq \mu_2$ and $\beta \in \mathbb{R}$. Note that the system (1.1) has two semitrivial solutions $(u, 0)$ and $(0, v)$, where

$$u(x) = \mu_1^{-\frac{N-\alpha}{2(\gamma+\alpha)}} \mu^{\frac{N-\alpha}{2}} U(\mu(x-y)) \text{ and } v(x) = \mu_2^{-\frac{N-\alpha}{2(\gamma+\alpha)}} \mu^{\frac{N-\alpha}{2}} U(\mu(x-y)), \quad (1.13)$$

and U is given in Theorem 1.1. We define the space

$$D^{1,\alpha}(\mathbb{R}^N) := \{u \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N) : |(-\Delta)^{\frac{\alpha}{4}} u|_{L^2(\mathbb{R}^N)}^2 < \infty\} \quad (1.14)$$

with the norm $|u|_{D^{1,\alpha}}^2 = |(-\Delta)^{\frac{\alpha}{4}} u|_{L^2(\mathbb{R}^N)}^2$. As in the [15, Proposition 3.6], we know that

$$|u|_{D^{1,\alpha}}^2 = |(-\Delta)^{\frac{\alpha}{4}} u|_{L^2(\mathbb{R}^N)}^2 = \frac{C(N, \alpha)}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+\alpha}} dy dx, \quad (1.15)$$

where $C(N, \alpha) = \left(\int_{\mathbb{R}^N} \frac{1 - \cos \zeta_1}{|\zeta|^{N+\alpha}} d\zeta \right)^{-1}$. We infer from Hardy-Littlewood-Sobolev inequality (see Lemma 2.1 below) and the embedding $D^{1,\alpha}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ that there exists $S > 0$ such that

$$|u|_{D^{1,\alpha}}^2 = |(-\Delta)^{\frac{\alpha}{4}} u|_{L^2(\mathbb{R}^N)}^2 \geq S \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x - y|^{N-\gamma}} dx dy \right)^{\frac{1}{p}}, \quad (1.16)$$

where $p = \frac{N+\gamma}{N-\alpha}$. Since U_μ satisfies the equation (1.11), it follows that

$$|U_\mu|_{D^{1,\alpha}}^2 = |(-\Delta)^{\frac{\alpha}{4}} U_\mu|_{L^2(\mathbb{R}^N)}^2 = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|U_\mu(x)|^p |U_\mu(y)|^p}{|x - y|^{N-\gamma}} dx dy = S^{\frac{p}{p-1}} = S^{\frac{N+\gamma}{\gamma+\alpha}}. \quad (1.17)$$

Obviously, the system (1.1) has two semitrivial solutions $\left(\mu_1^{-\frac{N-\alpha}{2(\gamma+\alpha)}} U_\mu, 0 \right)$ and $\left(0, \mu_2^{-\frac{N-\alpha}{2(\gamma+\alpha)}} U_\mu \right)$. In order to find the nontrivial solutions of (1.1), we define $D^\alpha := D^{1,\alpha}(\mathbb{R}^N) \times D^{1,\alpha}(\mathbb{R}^N)$ and the functional

$$\mathcal{J}(u, v) = \frac{1}{2} \left(|u|_{D^{1,\alpha}}^2 + |v|_{D^{1,\alpha}}^2 \right) - \frac{1}{2p} \int_{\mathbb{R}^N} (\mu_1 \phi_u |u|^p + \mu_2 \phi_v |v|^p + 2\beta \phi_v |u|^p). \quad (1.18)$$

For $\beta < 0$, we consider the set

$$\mathcal{N} = \left\{ (u, v) \in D^\alpha : u \neq 0, v \neq 0, \mathcal{J}'(u, v)(u, 0) = \mathcal{J}'(u, v)(0, v) = 0 \right\}. \quad (1.19)$$

It is clear that any nontrivial solutions of (1.1) belong to $\mathcal{N}$ and $\mathcal{N} \neq \emptyset$ (see below). We define

$$\begin{aligned} K &= \inf_{(u,v) \in \mathcal{N}} \mathcal{J}(u, v) = \inf_{(u,v) \in \mathcal{N}} \left(\frac{1}{2} - \frac{1}{2p} \right) \left(|u|_{D^{1,\alpha}}^2 + |v|_{D^{1,\alpha}}^2 \right) \\ &= \inf_{(u,v) \in \mathcal{N}} \left(\frac{1}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} (\mu_1 \phi_u |u|^p + \mu_2 \phi_v |v|^p + 2\beta \phi_v |u|^p). \end{aligned} \quad (1.20)$$

Considering the following nonlinear problem

$$\begin{cases} \mu_1 k^{2p-2} + \beta l^p k^{p-2} = 1, \\ \mu_2 l^{2p-2} + \beta k^p l^{p-2} = 1, \quad k, l > 0. \end{cases} \quad (1.21)$$

Then we have the following existence results.

Theorem 1.4. Assume that $p = \frac{N+\gamma}{N-\alpha}$. Then the following conclusions hold true.

- (i) If $\beta < 0$, then the infimum K can not be attained.
- (ii) If $\beta > 0$, then (1.1) has a nontrivial least energy solution (U, V) with $\mathcal{J}(U, V) = K$.
- (iii) There exists $0 < \beta_0 < (p-1) \max\{\mu_1, \mu_2\}$ small such that for $0 < \beta < \beta_0$, there exists solution $(k(\beta), l(\beta))$ of (1.21). Moreover, $(k(\beta), l(\beta))$ satisfies

$$\mathcal{J}(U, V) < \mathcal{J}(k(\beta)U_1, l(\beta)U_1), \quad \forall \beta \in (0, \beta_0).$$

This implies that $(k(\beta)U_1, l(\beta)U_1)$ is a different positive solution of (1.1) from (U, V) .

- (iv) If $2\alpha + \gamma > N$, then there exists $\beta_1 > \beta_0$ such that $K = \mathcal{J}(k_0U_1, l_0U_1)$ for $\beta > \beta_1$. If $2\alpha + \gamma < N$ and $\beta \geq (p-1) \max\{\mu_1, \mu_2\}$, then $K = \mathcal{J}(k_1U_1, l_1U_1)$, where (k_0, l_0) and (k_1, l_1) satisfy (1.21). If $2\alpha + \gamma = N$ and $\beta \in (0, \min\{\mu_1, \mu_2\}) \cap (\max\{\mu_1, \mu_2\}, \infty)$, then $K = \mathcal{J}\left(\sqrt{\frac{\mu_2-\beta}{\mu_1\mu_2-\beta^2}}U_1, \sqrt{\frac{\mu_1-\beta}{\mu_1\mu_2-\beta^2}}U_1\right)$.

Remark 1.5. (1) The condition $2\alpha + \gamma \geq N$ (or $2\alpha + \gamma < N$) equals to $p \geq 2$ (or $p < 2$) in Theorem 1.4 (iv).

- (2) In Theorem 1.4 (iii), we obtain two different solutions to the system (1.1). That is, the nontrivial ground state solution and the positive solution of $(k(\beta)U_1, l(\beta)U_1)$. Due to the fractional nonlocal operator, we do not know whether the solution (U, V) is positive and $(U, V) = (k(\beta)U_1, l(\beta)U_1)$. We shall pursue this question in the future.
- (3) Combining Theorems 1.1 and 1.4, we know that if $\mu_1 = \mu_2$, then $k_0 = k_2 = l_0 = l_2 = (\mu_1 + \beta)^{-\frac{N-\alpha}{2(\gamma+\alpha)}}$, where k_0, k_2, l_0, l_2 are given in Theorem 1.4.

2 Main conclusions from the direct methods of moving planes

Throughout the paper, we use the following notations:

- $|\cdot|_p$ is the norm of $L^p(\mathbb{R}^N)$ defined by $|u|_p = \left(\int_{\mathbb{R}^N} |u|^p\right)^{1/p}$ for $0 < p < \infty$.
- Let $c > 0$ be an arbitrary constants.

In this section we shall establish the main maximum principle theorems (Decay at infinity and Narrow region principle) for the fractional nonlocal system by using the direct methods of moving planes. We first recall the following classical Hardy-Littlewood-Sobolev inequality (see [34, Theorem 4.3]) and Cauchy-Schwarz inequality (see [34, Theorem 5.9] or [20, Inequality (3.3)]) for nonlocal problem.

Lemma 2.1. (i) Assume that $f \in L^{p_1}(\mathbb{R}^N)$ and $g \in L^{q_1}(\mathbb{R}^N)$. Then one has

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^t} dx dy \leq c(p_1, q_1, t) |f|_{p_1} |g|_{q_1},$$

where $1 < p_1, q_1 < \infty$, $0 < t < N$ and $\frac{1}{p_1} + \frac{1}{q_1} + \frac{t}{N} = 2$.

- (ii) If $f \in L^{p_1}(\mathbb{R}^N)$, then

$$\left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\gamma}} dy \right)^{q_1} dx \right)^{\frac{1}{q_1}} \leq c |f|_{p_1},$$

where $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\gamma}{N}$.

- (iii)

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p |v(y)|^p}{|x-y|^{N-\gamma}} dx dy \leq \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\gamma}} dx dy \right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(x)|^p |v(y)|^p}{|x-y|^{N-\gamma}} dx dy \right)^{\frac{1}{2}},$$

where $\gamma \in (0, N)$ and $p \geq 1$.

For $0 < \alpha < 2$, we define

$$H^\alpha(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+\alpha}{2}}} \in L^2(\mathbb{R}^N) \right\}, \quad (2.1)$$

which is endowed with the natural norm

$$\|u\|_\alpha := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+\alpha}} dx dy + \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{2}}. \quad (2.2)$$

From [15, Theorem 6.5], we know that the embedding $H^\alpha(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ ($\forall q \in [2, 2_\alpha^*]$) and $2_\alpha^* = \frac{2N}{N-\alpha}$ ($N \geq 3$). From Hardy-Littlewood-Sobolev inequality Lemma 2.1 (i) and (iii), we know that if $u \in H^\alpha(\mathbb{R}^N)$, then the nonlinearity of the system (1.1) belongs to $L^p(\mathbb{R}^N)$ for each $1 \leq p \leq \frac{N+\gamma}{N-\alpha}$. Hence if $1 \leq p < \frac{N+\gamma}{N-\alpha}$, we call it a subcritical case in (1.1). If $p = \frac{N+\gamma}{N-\alpha}$, we say it a critical case in (1.1).

Next we shall study the properties of the positive solution (u, v) of (1.1). To describe the asymptotic behavior of u and v at infinity, we introduce the Kelvin transform $\tilde{u}, \tilde{v}$ of u, v center at 0 defined by

$$\tilde{u}(x) = \frac{1}{|x|^{N-\alpha}} u\left(\frac{x}{|x|^2}\right) \quad \text{and} \quad \tilde{v}(x) = \frac{1}{|x|^{N-\alpha}} v\left(\frac{x}{|x|^2}\right) \quad (2.3)$$

for each $x \in \mathbb{R}^N \setminus \{0\}$. It's obvious that the Kelvin transform $\tilde{u}, \tilde{v}$ may have singularity at 0. Moreover we have $\lim_{|x| \rightarrow \infty} |x|^{N-\alpha} \tilde{u}(x) = u(0) > 0$ and $\lim_{|x| \rightarrow \infty} |x|^{N-\alpha} \tilde{v}(x) = v(0) > 0$. We infer from the definition (2.3) and $u, v \in C_{loc}^{1,1}(\mathbb{R}^N) \cap \mathcal{F}_\alpha(\mathbb{R}^N)$ that $\tilde{u}, \tilde{v} \in C_{loc}^{1,1}(\mathbb{R}^N) \cap \mathcal{F}_\alpha(\mathbb{R}^N)$, and the integral property $\int_{\mathbb{R}^N} \frac{|u(x)|^{\frac{N+\gamma}{N-\alpha}}}{|x|^{N-\gamma}} dx < \infty$ equals to $\int_{\mathbb{R}^N} |\tilde{u}|^{\frac{N+\gamma}{N-\alpha}}(x) dx < \infty$. According to the system (1.1), we know that $\int_{\mathbb{R}^N} \frac{|u(x)|^{\frac{N+\gamma}{N-\alpha}}}{|x|^{N-\gamma}} dx < \infty$. This conclusion holds similarly for $\tilde{v}$. Now we are ready to calculate the system for $(\tilde{u}, \tilde{v})$. That is, one infers from (1.3) that

$$\begin{aligned} & (-\Delta)^{\frac{\alpha}{2}} \tilde{u}(x) \\ &= \mathcal{C}_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{u\left(\frac{x}{|x|^2}\right) \left(\frac{1}{|x|^{N-\alpha}} - \frac{1}{|y|^{N-\alpha}} \right) + \frac{1}{|y|^{N-\alpha}} \left(u\left(\frac{x}{|x|^2}\right) - u\left(\frac{y}{|y|^2}\right) \right)}{|x - y|^{N+\alpha}} dy \\ &= u\left(\frac{x}{|x|^2}\right) (-\Delta)^{\frac{\alpha}{2}} \left(\frac{1}{|x|^{N-\alpha}} \right) + \mathcal{C}_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{u\left(\frac{x}{|x|^2}\right) - u(y)}{|y|^{N-\alpha} |x - y| |y|^2 |y|^{N+\alpha}} dy \\ &= \frac{1}{|x|^{N+\alpha}} (-\Delta)^{\frac{\alpha}{2}} u\left(\frac{x}{|x|^2}\right) \\ &= \frac{\mu_1}{|x|^{N+\alpha}} \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x/|x|^2 - y|^{N-\gamma}} dy \right) |u\left(\frac{x}{|x|^2}\right)|^{p-2} u\left(\frac{x}{|x|^2}\right) \\ &\quad + \frac{\beta}{|x|^{N+\alpha}} \left(\int_{\mathbb{R}^N} \frac{|v(y)|^p}{|x/|x|^2 - y|^{N-\gamma}} dy \right) |u\left(\frac{x}{|x|^2}\right)|^{p-2} u\left(\frac{x}{|x|^2}\right) \\ &= \frac{\mu_1}{|x|^{\alpha+\gamma-(p-1)(N-\alpha)}} \left(\int_{\mathbb{R}^N} \frac{|y|^{p(N-\alpha)-(N+\gamma)} |\tilde{u}(y)|^p}{|x - y|^{N-\gamma}} dy \right) |\tilde{u}(x)|^{p-2} \tilde{u}(x) \\ &\quad + \frac{\beta}{|x|^{\alpha+\gamma-(p-1)(N-\alpha)}} \left(\int_{\mathbb{R}^N} \frac{|y|^{p(N-\alpha)-(N+\gamma)} |\tilde{v}(y)|^p}{|x - y|^{N-\gamma}} dy \right) |\tilde{u}(x)|^{p-2} \tilde{u}(x). \end{aligned} \quad (2.4)$$

Thus, we know that $(\tilde{u}, \tilde{v})$ is the positive solution of

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} \tilde{u} = \frac{\mu_1}{|x|^m} \left(\int_{\mathbb{R}^N} \frac{|\tilde{u}(y)|^p}{|y|^m |x-y|^{N-\gamma}} dy \right) \tilde{u}(x)^{p-1} \\ \quad + \frac{\beta}{|x|^m} \left(\int_{\mathbb{R}^N} \frac{|\tilde{v}(y)|^p}{|y|^m |x-y|^{N-\gamma}} dy \right) \tilde{u}(x)^{p-1}, \quad x \in \mathbb{R}^N \setminus \{0\}, \\ (-\Delta)^{\frac{\alpha}{2}} \tilde{v} = \frac{\mu_2}{|x|^m} \left(\int_{\mathbb{R}^N} \frac{|\tilde{v}(y)|^p}{|y|^m |x-y|^{N-\gamma}} dy \right) \tilde{v}(x)^{p-1} \\ \quad + \frac{\beta}{|x|^m} \left(\int_{\mathbb{R}^N} \frac{|\tilde{u}(y)|^p}{|y|^m |x-y|^{N-\gamma}} dy \right) \tilde{v}(x)^{p-1}, \quad x \in \mathbb{R}^N \setminus \{0\}, \end{cases} \quad (2.5)$$

where $m := \alpha + \gamma - p(N - \alpha)$. Thus, we see that if $m > 0$, then $1 \leq p < \frac{N+\gamma}{N-\alpha}$ (subcritical case). If $m = 0$, then $p = \frac{N+\gamma}{N-\alpha}$ (critical case).

Before going further, we need the following maximum principle for $\frac{\alpha}{2}$ -superharmonic functions and Liouville theorem for $\frac{\alpha}{2}$ -harmonic functions. For the details of the proof one can refer to the papers [7, 49, 57].

Lemma 2.2. (i) (Maximum Principle) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume that $u \in C_{loc}^{1,1}(\mathbb{R}^N) \cap \mathcal{F}_\alpha(\mathbb{R}^N)$ and is lower semicontinuous on $\bar{\Omega}$. If

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) \geq 0, \quad x \in \Omega, \\ u(x) \geq 0, \quad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $u(x) \geq 0$ for all $x \in \mathbb{R}^N$. Furthermore, if $u = 0$ at some point in Ω , then $u(x) = 0$ almost everywhere in $\mathbb{R}^N$. In addition, these conclusions hold for unbounded region Ω if we further assume that

$$\lim_{|x| \rightarrow \infty} u(x) \geq 0.$$

(ii) (Liouville Theorem) Let $u \in C_{loc}^{1,1}(\mathbb{R}^N) \cap \mathcal{F}_\alpha(\mathbb{R}^N)$ be solution of

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) \geq 0, \quad x \in \mathbb{R}^N, \\ u(x) \geq 0, \quad x \in \mathbb{R}^N. \end{cases}$$

Then $u \equiv C \geq 0$.

Next we prove the equivalence between (1.1) and (1.4).

Lemma 2.3. Assume that $N \geq 3$, $0 < \alpha < 2$, $\gamma \in (0, N)$, $p \geq 1$ and $u \in C_{loc}^{1,1}(\mathbb{R}^N) \cap \mathcal{F}_\alpha(\mathbb{R}^N)$. If u is nonnegative solution of (1.1), then u satisfies the integral system (1.4), and vice versa.

Proof. Let G_R^α be the Green's function for $(-\Delta)^{\frac{\alpha}{2}}$ on $B_R(0)$ which is defined by

$$G_R^\alpha(x, y) = \frac{C_{N,\alpha}}{|x-y|^{N-\alpha}} \int_0^{\frac{t_R}{s_R}} \frac{l^{\frac{\alpha}{2}-1}}{(1+l)^{\frac{N}{2}}} dl \quad \text{for } x, y \in B_R(0)$$

$$\text{and } G_R^\alpha(x, y) = 0 \quad \text{if } x \text{ or } y \in \mathbb{R}^N \setminus B_R(0),$$

where $s_R = \frac{|x-y|^2}{R^2}$ and $t_R = \left(1 - \frac{|x|^2}{R^2}\right) \left(1 - \frac{|y|^2}{R^2}\right)$ (see [27]). For each $R > 0$, we define

$$\begin{aligned} w_R(x) &= \int_{B_R(0)} G_R^\alpha(x, y) \left(\mu_1 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} u(y)^{p-1} \phi_u(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} u(y)^{p-1} \phi_v(y)}{|x-y|^{N-\alpha}} dy \right) dy, \\ z_R(x) &= \int_{B_R(0)} G_R^\alpha(x, y) \left(\mu_2 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} v(y)^{p-1} \phi_v(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} v(y)^{p-1} \phi_u(y)}{|x-y|^{N-\alpha}} dy \right) dy. \end{aligned} \quad (2.6)$$

By using the properties of Green's function, we can derive that $(w_R(x), z_R(x))$ satisfies

$$\begin{cases} -\Delta w_R(x) = \mu_1 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} u(y)^{p-1} \phi_u(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} u(y)^{p-1} \phi_v(y)}{|x-y|^{N-\alpha}} dy, \quad x \in B_R(0) \\ -\Delta z_R(x) = \mu_2 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} v(y)^{p-1} \phi_v(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} v(y)^{p-1} \phi_u(y)}{|x-y|^{N-\alpha}} dy, \quad x \in B_R(0), \\ w_R(x) = z_R(x) = 0, \quad x \in \mathbb{R}^N \setminus B_R(0). \end{cases} \quad (2.7)$$

Let $\tilde{w}_R = u - w_R$ and $\tilde{z}_R = v - z_R$. We infer from (1.1) and (2.7) that

$$\begin{cases} -\Delta \tilde{w}_R(x) = -\Delta \tilde{z}_R(x) = 0, & x \in B_R(0), \\ w_R(x), z_R(x) \geq 0, & x \in \mathbb{R}^N \setminus B_R(0). \end{cases} \quad (2.8)$$

One deduces from the Maximum principle (Lemma 2.2) that for any $R > 0$

$$\tilde{w}(x) = u(x) - w_R(x) \geq 0 \quad \text{and} \quad \tilde{z}(x) = v(x) - z_R(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^N. \quad (2.9)$$

For each fixed $x \in \mathbb{R}^N$, we have that

$$\begin{aligned} u(x) &\geq w(x) := \mu_1 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} u(y)^{p-1} \phi_u(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} u(y)^{p-1} \phi_v(y)}{|x-y|^{N-\alpha}} dy, \\ v(x) &\geq z(x) := \mu_2 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} v(y)^{p-1} \phi_v(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} v(y)^{p-1} \phi_u(y)}{|x-y|^{N-\alpha}} dy, \end{aligned} \quad (2.10)$$

as $R \rightarrow \infty$. On the other hand, it is easy to see that (w, z) is a solution of

$$\begin{cases} -\Delta w(x) = \mu_1 u(y)^{p-1} \phi_u(y) + \beta u(y)^{p-1} \phi_v(y), & x \in \mathbb{R}^N, \\ -\Delta z(x) = \mu_2 v(y)^{p-1} \phi_v(y) + \beta v(y)^{p-1} \phi_u(y), & x \in \mathbb{R}^N. \end{cases} \quad (2.11)$$

Set $\tilde{w} = u - w$ and $\tilde{z} = v - z$. Hence we have

$$\begin{cases} -\Delta \tilde{w}(x) = -\Delta \tilde{z}(x) = 0, & x \in \mathbb{R}^N, \\ w(x), z(x) \geq 0, & x \in \mathbb{R}^N. \end{cases} \quad (2.12)$$

We infer from Lemma 2.2 (ii) that $\tilde{w} = u - w = C_0 \geq 0$ and $\tilde{z} = v - z = C_1 \geq 0$. This implies that

$$\begin{aligned} u(x) &= \mu_1 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} u(y)^{p-1} \phi_u(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} u(y)^{p-1} \phi_v(y)}{|x-y|^{N-\alpha}} dy + C_0, \\ v(x) &= \mu_2 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} v(y)^{p-1} \phi_v(y)}{|x-y|^{N-\alpha}} dy + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} v(y)^{p-1} \phi_u(y)}{|x-y|^{N-\alpha}} dy + C_1, \end{aligned} \quad (2.13)$$

Since $\mu_i, \beta \geq 0$ and $\mu_i + \beta > 0$ ($i = 1, 2$), we know that

$$\begin{aligned} \infty > u(0) &\geq w(0) = \mu_1 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} C_0^{2p-1}}{|y|^{N-\alpha} |y-z|^{N-\alpha}} dy dz + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} C_0^{p-1} C_1^p}{|y|^{N-\alpha} |y-z|^{N-\alpha}} dy dz, \\ \infty > v(0) &\geq z(0) = \mu_2 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} C_1^{2p-1}}{|y|^{N-\alpha} |y-z|^{N-\alpha}} dy dz + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha} C_1^{p-1} C_0^p}{|y|^{N-\alpha} |y-z|^{N-\alpha}} dy dz. \end{aligned} \quad (2.14)$$

This implies that $C_0 = C_1 = 0$. Thus, we arrive at (1.4). Conversely, we follow the idea of [5, 8, 57] to infer that if (u, v) is a nonnegative solution of (1.4), then (u, v) is also a solution of (1.1). $\square$

The next lemma states the basic properties for the Kelvin transform $\tilde{u}, \tilde{v} \in C_{loc}^{1,1}(\mathbb{R}^N) \cap \mathcal{F}_\alpha(\mathbb{R}^N)$.

Lemma 2.4. *The Kelvin transform $(\tilde{u}, \tilde{v})$ satisfies the integral system*

$$\begin{cases} \tilde{u}(x) &= \mu_1 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{|x-y|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{|\tilde{u}(z)|^p}{|z|^m |y-z|^{N-\gamma}} dz \right) \frac{\tilde{u}(y)^{p-1}}{|y|^m} dy \\ &\quad + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{|x-y|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{|\tilde{v}(z)|^p}{|z|^m |y-z|^{N-\gamma}} dz \right) \frac{\tilde{u}(y)^{p-1}}{|y|^m} dy, & x \in \mathbb{R}^N \setminus \{0\}, \\ \tilde{v}(x) &= \mu_2 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{|x-y|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{|\tilde{v}(z)|^p}{|z|^m |y-z|^{N-\gamma}} dz \right) \frac{\tilde{v}(y)^{p-1}}{|y|^m} dy \\ &\quad + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{|x-y|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{|\tilde{u}(z)|^p}{|z|^m |y-z|^{N-\gamma}} dz \right) \frac{\tilde{v}(y)^{p-1}}{|y|^m} dy, & x \in \mathbb{R}^N \setminus \{0\}, \end{cases} \quad (2.15)$$

where $m = N + \gamma - p(N - \alpha)$. Furthermore, we have

$$\liminf_{|x| \rightarrow 0} \tilde{u}(x) > 0 \quad \text{and} \quad \liminf_{|x| \rightarrow 0} \tilde{v}(x) > 0. \quad (2.16)$$

Proof. A direct computation shows that

$$\begin{aligned}
 & \mu_1 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{|x-y|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{|\tilde{u}(z)|^p}{|z|^m |y-z|^{N-\gamma}} dz \right) \frac{\tilde{u}(y)^{p-1}}{|y|^m} dy \\
 & + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{|x-y|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{|\tilde{v}(z)|^p}{|z|^m |y-z|^{N-\gamma}} dz \right) \frac{\tilde{u}(y)^{p-1}}{|y|^m} dy \\
 & = \mu_1 \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{\left| x - \frac{y}{|y|^2} \right|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{|z|^m \left(|z|^{N-\gamma} u(z) \right)^p}{\left| \frac{y}{|y|^2} - \frac{z}{|z|^2} \right|^{N-\gamma}} \frac{dz}{|z|^{2N}} \right) |y|^m \left(|y|^{N-\gamma} u(y) \right)^{p-1} \frac{dy}{|y|^{2N}} \\
 & + \beta \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{\left| x - \frac{y}{|y|^2} \right|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{|z|^m \left(|z|^{N-\gamma} v(z) \right)^p}{\left| \frac{y}{|y|^2} - \frac{z}{|z|^2} \right|^{N-\gamma}} \frac{dz}{|z|^{2N}} \right) \left(|y|^{N-\gamma} u(y) \right)^{p-1} \frac{dy}{|y|^{2N}} \quad (2.17) \\
 & = \frac{\mu_1}{|x|^{N-\gamma}} \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{\left| \frac{x}{|x|^2} - y \right|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{u(z)^p}{|y-z|^{N-\gamma}} dz \right) u(y)^{p-1} dy \\
 & + \frac{\beta}{|x|^{N-\gamma}} \int_{\mathbb{R}^N} \frac{\mathcal{R}_{N,\alpha}}{\left| \frac{x}{|x|^2} - y \right|^{N-\alpha}} \left(\int_{\mathbb{R}^N} \frac{v(z)^p}{|y-z|^{N-\gamma}} dz \right) u(y)^{p-1} dy \\
 & = \frac{1}{|x|^{N-\gamma}} u \left(\frac{x}{|x|^2} \right) = \tilde{u}(x).
 \end{aligned}$$

Similarly, we can prove the second equality in (2.15). Next we shall prove (2.16). As in [11, Inequality (2.19)](or [28]), one can prove that $F_u, F_v \in L^\infty(B_2(0))$. Let $\chi \in C_0^\infty(\mathbb{R}^N)$ satisfy $0 \leq \chi \leq 1$, $\chi = 1$ in $B_1(0)$ and $\chi = 0$ in $\mathbb{R}^N \setminus B_2(0)$. We define

$$\psi(x) = \mu_1 \mathcal{R}_{N,\alpha} \int_{\mathbb{R}^N} \frac{\chi(y) u(y)^{p-1} \phi_u(y)}{|x-y|^{N-\alpha}} dy + \beta \mathcal{R}_{N,\alpha} \int_{\mathbb{R}^N} \frac{\chi(y) u(y)^{p-1} \phi_v(y)}{|x-y|^{N-\alpha}} dy. \quad (2.18)$$

Then we know that $\psi(x)$ satisfies (see [5, 8, 57])

$$(-\Delta)^{\frac{\alpha}{2}} \psi(x) = \mu_1 \chi(y) u(y)^{p-1} \phi_u(y) + \beta \chi(y) u(y)^{p-1} \phi_v(y), \quad x \in \mathbb{R}^N. \quad (2.19)$$

We infer from the properties of χ , $\mu_1 + \beta > 0$ and (2.18) that

$$\frac{\mathcal{R}_{N,\alpha}}{|x|^{N-\alpha}} \left(\mu_1 |\phi_u u^{p-1}|_{L^1(B_1(0))} + \beta |\phi_v u^{p-1}|_{L^1(B_1(0))} \right) \leq \psi(x) \leq \frac{C_{N,\alpha}^2}{|x|^{N-\alpha}}, \quad (2.20)$$

where $C_{N,\alpha}^2 > 0$ is a positive constant depending on N and α . This implies that

$$\frac{C_{N,\alpha}^1}{|x|^{N-\alpha}} \leq \psi(x) \leq \frac{C_{N,\alpha}^2}{|x|^{N-\alpha}}, \quad (2.21)$$

where $C_{N,\alpha}^1 > 0$ is a positive constant. For each $R > 0$, we let $h_R(x) = u(x) - \psi(x) + \frac{C_{N,\alpha}^2}{R^{N-\alpha}}$. Thus, it follows from (1.1) and (2.19) that

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} h_R(x) = (1 - \chi(x)) (\mu_1 u(y)^{p-1} \phi_u(y) + \beta u(y)^{p-1} \phi_v(y)) \geq 0, & x \in B_R(0), \\ h_R(x) \geq \frac{C_{N,\alpha}^2}{R^{N-\alpha}} - \psi(x), & x \in \mathbb{R}^N \setminus B_R(0). \end{cases} \quad (2.22)$$

We infer from Lemma 2.2 that $h_R \geq 0$ in $\mathbb{R}^N$ for each $R > 0$ large. Particularly, we have

$$u(x) - \psi(x) + \frac{C_{N,\alpha}^2}{R^{N-\alpha}} \geq 0 \quad \text{in } B_R(0). \quad (2.23)$$

For arbitrarily fixed x , letting $R \rightarrow \infty$ in (2.23), we get $u \geq \psi$ in $\mathbb{R}^N$. Hence we get that

$$u(x) \geq \frac{C_{N,\alpha}^1}{|x|^{N-\alpha}} \quad (2.24)$$

for $|x|$ large enough. Thus, we deduce from (2.24) that

$$\liminf_{|x| \rightarrow 0} \tilde{u}(x) = \liminf_{|x| \rightarrow 0} \frac{1}{|x|^{N-\alpha}} u \left(\frac{x}{|x|^2} \right) = \liminf_{|x| \rightarrow \infty} |x|^{N-\alpha} u(x) \geq C_{N,\alpha}^1 > 0. \quad (2.25)$$

Similarly, we can prove $\liminf_{|x| \rightarrow 0} \tilde{v}(x) > 0$. This finishes the proof. $\square$

Now we are ready to use the method of moving planes. For each $\lambda \in \mathbb{R}^N$, we define

$$T_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\} \quad \text{and} \quad \Sigma_\lambda = \{x \in \mathbb{R}^N : x_1 < \lambda\}. \quad (2.26)$$

Let

$$x^\lambda := (2\lambda - x_1, x') = (2\lambda - x_1, x_2, \dots, x_N) \quad (2.27)$$

be the reflection of x about the plane T_λ , and define $w_\lambda(x) = \tilde{u}(x^\lambda) - \tilde{u}(x)$ and $z_\lambda(x) = \tilde{v}(x^\lambda) - \tilde{v}(x)$. We first need to prove that for λ sufficiently negative

$$w_\lambda(x), z_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}. \quad (2.28)$$

Then we can start moving plane T_λ from near $x_1 = -\infty$ to the right as long as (2.28) holds, until its limiting position and finally derive the symmetry. To accomplish this we set

$$\Sigma_{\lambda,w}^- = \{x \in \Sigma_\lambda \setminus \{0\} : w_\lambda(x) < 0\} \quad \text{and} \quad \Sigma_{\lambda,z}^- = \{x \in \Sigma_\lambda \setminus \{0\} : z_\lambda(x) < 0\}. \quad (2.29)$$

It suffices to show that $\Sigma_{\lambda,w}^- = \Sigma_{\lambda,z}^- = \emptyset$. To this purpose we calculate the term $(-\Delta)^{\frac{\alpha}{2}} w_\lambda$ and $(-\Delta)^{\frac{\alpha}{2}} z_\lambda$ on $\Sigma_{\lambda,w}^-$ and $\Sigma_{\lambda,z}^-$ respectively. Let

$$\begin{aligned} P_{\tilde{u}}(x) &= \frac{1}{|x|^{N-\gamma}} * \frac{\tilde{u}^p}{|x|^m}, \quad P_{\tilde{v}}(x) = \frac{1}{|x|^{N-\gamma}} * \frac{\tilde{v}^p}{|x|^m}, \\ G_u(x) &= \int_{\Sigma_{\lambda,w}^-} \frac{\tilde{u}(y)^{p-1} w_\lambda(y)}{|x-y|^{N-\gamma} |y|^m} dy \quad \text{and} \quad G_v(x) = \int_{\Sigma_{\lambda,z}^-} \frac{\tilde{v}(y)^{p-1} z_\lambda(y)}{|x-y|^{N-\gamma} |y|^m} dy. \end{aligned} \quad (2.30)$$

Then we have the following conclusion.

Lemma 2.5. Assume that $\lambda \leq 0$. Then we have the following conclusions.

(1) If $w_\lambda(x) < 0$ and $z_\lambda(x) \geq 0$, then we have

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} w_\lambda(x) & \geq \max\{p-1, 1\} (\mu_1 P_{\tilde{u}}(x) + \beta P_{\tilde{v}}(x)) \frac{\tilde{u}(x)^{p-2}}{|x|^m} w_\lambda(x) \\ & \quad + \mu_1 p G_u(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m}, \\ (-\Delta)^{\frac{\alpha}{2}} z_\lambda(x) & \geq \beta p G_v(x) \frac{\tilde{v}(x)^{p-1}}{|x|^m}. \end{cases} \quad (2.31)$$

(2) If $w_\lambda(x) \geq 0$ and $z_\lambda(x) < 0$, then we have

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} w_\lambda(x) & \geq \beta p G_v(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m}, \\ (-\Delta)^{\frac{\alpha}{2}} z_\lambda(x) & \geq \max\{p-1, 1\} (\mu_2 P_{\tilde{v}}(x) + \beta P_{\tilde{u}}(x)) \frac{\tilde{v}(x)^{p-2}}{|x|^m} z_\lambda(x) \\ & \quad + \mu_2 p G_v(x) \frac{\tilde{v}(x)^{p-1}}{|x|^m}. \end{cases} \quad (2.32)$$

(3) If $w_\lambda(x) < 0$ and $z_\lambda(x) < 0$, then we have

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} w_\lambda(x) & \geq \max\{p-1, 1\} (\mu_1 P_{\tilde{u}}(x) + \beta P_{\tilde{v}}(x)) \frac{\tilde{u}(x)^{p-2}}{|x|^m} w_\lambda(x) \\ & \quad + \mu_1 p G_u(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m} + \beta p G_v(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m}, \\ (-\Delta)^{\frac{\alpha}{2}} z_\lambda(x) & \geq \max\{p-1, 1\} (\mu_2 P_{\tilde{v}}(x) + \beta P_{\tilde{u}}(x)) \frac{\tilde{v}(x)^{p-2}}{|x|^m} z_\lambda(x) \\ & \quad + \mu_2 p G_v(x) \frac{\tilde{v}(x)^{p-1}}{|x|^m} + \beta p G_u(x) \frac{\tilde{v}(x)^{p-1}}{|x|^m}. \end{cases} \quad (2.33)$$

Proof. We only give the proof of the case (3), other cases can be proved similarly. We infer from (2.5) that

$$\begin{aligned}
 (-\Delta)^{\frac{\alpha}{2}} w_{\lambda} &= (-\Delta)^{\frac{\alpha}{2}} \tilde{u}(x^{\lambda}) - (-\Delta)^{\frac{\alpha}{2}} \tilde{u}(x) \\
 &= \mu_1 P_{\tilde{u}}(x^{\lambda}) \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x^{\lambda}|^m} + \beta P_{\tilde{v}}(x^{\lambda}) \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x^{\lambda}|^m} - \mu_1 P_{\tilde{u}}(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m} - \beta P_{\tilde{v}}(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m} \\
 &\geq \mu_1 P_{\tilde{u}}(x^{\lambda}) \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x|^m} + \beta P_{\tilde{v}}(x^{\lambda}) \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x|^m} - \mu_1 P_{\tilde{u}}(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m} - \beta P_{\tilde{v}}(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m} \\
 &= \mu_1 P_{\tilde{u}}(x) \frac{\tilde{u}(x^{\lambda})^{p-1} - \tilde{u}(x)^{p-1}}{|x|^m} + \mu_1 (P_{\tilde{u}}(x^{\lambda}) - P_{\tilde{u}}(x)) \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x|^m} \\
 &\quad + \beta P_{\tilde{v}}(x) \frac{\tilde{u}(x^{\lambda})^{p-1} - \tilde{u}(x)^{p-1}}{|x|^m} + \beta (P_{\tilde{v}}(x^{\lambda}) - P_{\tilde{v}}(x)) \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x|^m} \\
 &= \mu_1 L_1 + \mu_1 L_2 + \beta L_3 + \beta L_4.
 \end{aligned} \tag{2.34}$$

We first give the estimate the L_1 . Since $\lambda \leq 0$ and $w_{\lambda}, z_{\lambda} \leq 0$, it follows that

$$L_1 \geq \mu_1 P_{\tilde{u}}(x) \frac{\max\{p-1, 1\} \tilde{u}(x)^{p-2} (\tilde{u}(x^{\lambda}) - \tilde{u}(x))}{|x|^m} = \mu_1 \max\{p-1, 1\} P_{\tilde{u}}(x) \frac{\tilde{u}(x)^{p-2}}{|x|^m}, \tag{2.35}$$

where we used the basic inequality

$$a^q - b^q \geq \max\{q, 1\} b^{q-1} (a - b) \quad \text{for } q \geq 0 \quad \text{and} \quad 0 < a \leq b. \tag{2.36}$$

Similarly we estimate the term L_2 as follows.

$$\begin{aligned}
 L_2 &= \left(\int_{\mathbb{R}^N} \frac{\tilde{u}(y)^p}{|x^{\lambda} - y|^{N-\gamma} |y|^m} dy - \int_{\mathbb{R}^N} \frac{\tilde{u}(y)^p}{|x - y|^{N-\gamma} |y|^m} dy \right) \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x|^m} \\
 &= \int_{\Sigma_{\lambda}} \left(\frac{1}{|x - y|^{N-\gamma}} - \frac{1}{|x^{\lambda} - y|^{N-\gamma}} \right) \left(\frac{\tilde{u}(y^{\lambda})^p}{|y^{\lambda}|^m} - \frac{\tilde{u}(y)^p}{|y|^m} \right) dy \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x|^m} \\
 &\geq \int_{\Sigma_{\lambda}} \left(\frac{1}{|x - y|^{N-\gamma}} - \frac{1}{|x^{\lambda} - y|^{N-\gamma}} \right) \frac{\tilde{u}(y^{\lambda})^p - \tilde{u}(y)^p}{|y|^m} dy \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x|^m} \\
 &\geq \int_{\Sigma_{\lambda}} \frac{1}{|x - y|^{N-\gamma}} \frac{\tilde{u}(y^{\lambda})^p - \tilde{u}(y)^p}{|y|^m} dy \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x|^m} \\
 &\geq \int_{\Sigma_{\lambda}} \frac{1}{|x - y|^{N-\gamma}} \frac{p \tilde{u}(y)^{p-1} (\tilde{u}(y^{\lambda}) - \tilde{u}(y))}{|y|^m} dy \frac{\tilde{u}(x^{\lambda})^{p-1}}{|x|^m} = p G_u(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m},
 \end{aligned} \tag{2.37}$$

where we used the following basic inequality

$$a^q - b^q \geq q b^{q-1} (a - b) \quad \text{for } q \geq 1 \quad \text{and} \quad 0 < a < b. \tag{2.38}$$

We can similarly get the estimates of L_3, L_4 . Thus, we get the first inequality of (2.32). By using similar arguments one can obtain the second inequality of (2.32). $\square$

Then we have the following decay at infinity maximum principle for the nonlocal problem (1.1).

Theorem 2.6. Assume $\lambda < 0$ such that $w_{\lambda} \in \mathcal{F}_{\alpha} \cap C_{loc}^{1,1}(\Sigma_{\lambda,w}^{-})$ or $z_{\lambda} \in \mathcal{F}_{\alpha} \cap C_{loc}^{1,1}(\Sigma_{\lambda,z}^{-})$ and the negative minimum of w_{λ} or z_{λ} is attained in the interior of $\Sigma_{\lambda} \setminus \{0\}$. Then, there exists some $R_0 > 0$ (depending on u, v , but is independent of λ) such that, if $x_0 \in \Sigma_{\lambda,w}^{-}$ or $x_0 \in \Sigma_{\lambda,z}^{-}$ satisfying $w_{\lambda}(x_0) = \min_{\Sigma_{\lambda,w}^{-}} w_{\lambda}(x) < 0$ or $z_{\lambda}(x_0) = \min_{\Sigma_{\lambda,z}^{-}} z_{\lambda}(x) < 0$, then $|x_0| \leq R_0$.

Proof. We divide into the following three cases to prove the conclusions.

Case 1. $w_\lambda(x_0) < 0$ and $z_\lambda(x_0) \geq 0$ for $x_0 \in \Sigma_{\lambda,w}^-$. Then we know that (w_λ, z_λ) satisfies (2.31). We infer from the definition of (1.3) and $w_\lambda(y^\lambda) = -w_\lambda(y)$ that

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0) &= \mathbb{C}_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{w_\lambda(x_0) - w_\lambda(y)}{|x_0 - y|^{N+\alpha}} dy \\ &= \mathbb{C}_{N,\alpha} \left(P.V. \int_{\Sigma_\lambda} \frac{w_\lambda(x_0) - w_\lambda(y)}{|x_0 - y|^{N+\alpha}} dy + \int_{\mathbb{R}^N \setminus \Sigma_\lambda} \frac{w_\lambda(x_0) - w_\lambda(y)}{|x_0 - y|^{N+\alpha}} dy \right) \\ &\leq \mathbb{C}_{N,\alpha} \left(\int_{\Sigma_\lambda} \frac{w_\lambda(x_0) - w_\lambda(y)}{|x_0 - y^\lambda|^{N+\alpha}} dy + \int_{\Sigma_\lambda} \frac{w_\lambda(x_0) - w_\lambda(y^\lambda)}{|x_0 - y^\lambda|^{N+\alpha}} dy \right) \\ &= 2\mathbb{C}_{N,\alpha} w_\lambda(x_0) \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy. \end{aligned} \quad (2.39)$$

For each fixed λ , we know that $B_{|x_0|}(\tilde{x}) \subset \mathbb{R}^N \setminus \Sigma_\lambda$ for $\tilde{x} = (x_0^1 + 4|x_0|, (x_0)')'$, where $x_0 = (x_0^1, (x_0)')'$. Hence it follows that

$$\begin{aligned} \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy &= \int_{\mathbb{R}^N \setminus \Sigma_\lambda} \frac{1}{|x_0 - y|^{N+\alpha}} dy \\ &\geq \int_{B_{|x_0|}(\tilde{x})} \frac{1}{|x_0 - y|^{N+\alpha}} dy \\ &\geq \int_{B_{|x_0|}(\tilde{x})} \frac{1}{5^{N+\alpha} |x_0|^{N+\alpha}} dy = \frac{\omega_N}{5^{N+\alpha} |x_0|^\alpha}, \end{aligned} \quad (2.40)$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. Combining (2.39) and (2.40), we know that

$$(-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0) \leq \frac{\omega_N}{5^{N+\alpha} |x_0|^\alpha} w_\lambda(x_0). \quad (2.41)$$

On the other hand, it follows from $w_\lambda(x_0) = \min_{\Sigma_{\lambda,w}^-} w_\lambda(x)$ that

$$(-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0) \geq A(x_0) w_\lambda(x_0), \quad (2.42)$$

where

$$A(x) = \left(\max\{p-1, 1\} (\mu_1 P_{\tilde{u}}(x) + \beta P_{\tilde{v}}(x)) \frac{\tilde{u}(x)^{p-2}}{|x|^m} + \mu_1 p H_u(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m} \right) \quad (2.43)$$

and

$$H_u(x) = \int_{\mathbb{R}^N} \frac{\tilde{u}(y)^{p-1}}{|x - y|^{N-\gamma} |y|^m} dy. \quad (2.44)$$

We claim that for $x \in \Sigma_\lambda$

$$A(x) \leq \frac{C}{|x|^{\alpha+\sigma}}, \quad (2.45)$$

where $\sigma = \min \left\{ \alpha, N \left(1 - \frac{1}{p} \right) + \frac{m}{p} \right\}$ and $C > 0$ is independent of x and λ . Indeed, it is clear that

$$\tilde{u}(x), \tilde{v}(x) \leq \frac{C}{|x|^{N-\alpha}} \quad \text{for } x \in \mathbb{R}^N \setminus B_R(0) \text{ and } R > 0. \quad (2.46)$$

Moreover, we infer from [28, Proposition 9] that

$$P_{\tilde{u}}(x), P_{\tilde{v}}(x) \leq \frac{C}{|x|^{N-\gamma}} \quad \text{and} \quad H_u(x) \leq \frac{C}{|x|^{N-\gamma-\frac{N-m}{p}}}. \quad (2.47)$$

This implies that

$$\begin{aligned} A(x) &\leq \frac{C}{|x|^{N-\gamma+m+(p-2)(N-\alpha)}} + \frac{C}{|x|^{N-\gamma-\frac{N-m}{p}+m+(p-1)(N-\alpha)}} \\ &\leq \frac{C}{|x|^{2\alpha}} + \frac{C}{|x|^{\alpha+N\left(1-\frac{1}{p}\right)+\frac{m}{p}}}. \end{aligned} \quad (2.48)$$

Hence we get the claim (2.45). Combining (2.41)-(2.42) and (2.45), we know that

$$0 \leq \left(\frac{\omega_N}{5^{N+\alpha}|x_0|^\alpha} - \frac{C}{|x_0|^{\alpha+\sigma}} \right) w_\lambda(x_0). \quad (2.49)$$

If $|x_0| \rightarrow \infty$, we know that the right hand of (2.49) is strictly less than zero. This is a contradiction. Thus there exists $R_0 > 0$ such that $|x_0| \leq R_0$.

Case 2. $w_\lambda(x_0) \geq 0$ and $z_\lambda(x_0) < 0$ for $x_0 \in \Sigma_{\lambda,z}^-$. We infer from (2.32) that

$$(-\Delta)^{\frac{\alpha}{2}} z_\lambda(x_0) \geq A_1(x_0) z_\lambda(x_0), \quad (2.50)$$

where

$$A_1(x) = \left(\max\{p-1, 1\} (\mu_2 P_{\tilde{v}}(x) + \beta P_{\tilde{u}}(x)) \frac{\tilde{v}(x)^{p-2}}{|x|^m} + \mu_2 p H_v(x) \frac{\tilde{v}(x)^{p-1}}{|x|^m} \right) \quad (2.51)$$

and

$$H_u(x) = \int_{\mathbb{R}^N} \frac{\tilde{v}(y)^{p-1}}{|x-y|^{N-\gamma}|y|^m} dy. \quad (2.52)$$

Similar to **Case 1**, we know that

$$0 \leq \left(\frac{\omega_N}{5^{N+\alpha}|x_0|^\alpha} - \frac{C}{|x_0|^{\alpha+\sigma}} \right) z_\lambda(x_0). \quad (2.53)$$

Hence we get $|x_0| \leq R_0$ for some $R_0 > 0$.

Case 3. $w_\lambda(x_0) < 0$ and $z_\lambda(x_0) < 0$ for $x_0 \in \Sigma_{\lambda,w}^-$ or $x_0 \in \Sigma_{\lambda,z}^-$. One deduces from (2.33) that

$$(-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0) \geq A_3(x_0) w_\lambda(x_0) \quad \text{and} \quad (-\Delta)^{\frac{\alpha}{2}} z_\lambda(x_0) \geq A_4(x_0) z_\lambda(x_0), \quad (2.54)$$

where

$$\begin{aligned} A_3(x) &= \left(\max\{p-1, 1\} (\mu_1 P_{\tilde{u}}(x) + \beta P_{\tilde{v}}(x)) \frac{\tilde{u}(x)^{p-2}}{|x|^m} \right. \\ &\quad \left. + \mu_1 p H_u(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m} + \beta p H_v(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m} \right) \\ A_4(x) &= \left(\max\{p-1, 1\} (\mu_1 P_{\tilde{v}}(x) + \beta P_{\tilde{u}}(x)) \frac{\tilde{v}(x)^{p-2}}{|x|^m} \right. \\ &\quad \left. + \mu_2 p H_v(x) \frac{\tilde{v}(x)^{p-1}}{|x|^m} + \beta p H_u(x) \frac{\tilde{v}(x)^{p-1}}{|x|^m} \right). \end{aligned} \quad (2.55)$$

One deduces from (2.41) and (2.54) that

$$\left(\frac{\omega_N}{5^{N+\alpha}|x_0|^\alpha} - A_3(x) \right) w_\lambda(x_0) + \left(\frac{\omega_N}{5^{N+\alpha}|x_0|^\alpha} - A_4(x) \right) z_\lambda(x_0) \geq 0. \quad (2.56)$$

Since $w_\lambda(x_0), z_\lambda(x_0) < 0$, we infer that at least one of

$$\frac{\omega_N}{5^{N+\alpha}|x_0|^\alpha} - A_3(x) \leq 0 \quad \text{or} \quad \frac{\omega_N}{5^{N+\alpha}|x_0|^\alpha} - A_4(x) \leq 0 \quad (2.57)$$

holds. However, similar to the estimates in (2.45), we know that

$$A_3(x), A_4(x) \leq \frac{C}{|x|^{\alpha+\sigma}}. \quad (2.58)$$

Thus we infer that

$$\frac{\omega_N}{5^{N+\alpha}|x_0|^\alpha} - A_3(x) > 0 \quad \text{and} \quad \frac{\omega_N}{5^{N+\alpha}|x_0|^\alpha} - A_4(x) > 0, \quad (2.59)$$

as $|x_0| \rightarrow \infty$. This is a contradiction. Hence we get $|x_0| \leq R_0$ for some $R_0 > 0$. $\square$

Next we prove the narrow region principle for the system (1.1).

Theorem 2.7. Let $\Omega \subset \{x \in \mathbb{R}^N : \lambda - l < x_1 < l\}$ be a narrow region in $\Sigma_\lambda \setminus \{0^\lambda\}$, where $\lambda < 0$ and $l > 0$ small. Assume that $(w_\lambda, z_\lambda) \in \mathcal{F}_\alpha \cap C_{loc}^{1,1}$ satisfies the following conditions:

- (1) Lemma 2.5 holds with domain in $\Omega \cap \Sigma_\lambda^-$;
- (2) the negative minimum of w_λ or z_λ is attained in the interior of $\Sigma_\lambda \setminus \{0^\lambda\}$ if $\Sigma_\lambda^- \neq \emptyset$;
- (3) the negative minimum of w_λ or z_λ can not be attained in $(\Sigma_\lambda \setminus \{0^\lambda\}) \setminus \Omega$.

Then there exists $l_0 > 0$ (depending on λ continuously) sufficiently small such that for all $0 < l \leq l_0$,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad z_\lambda(x) \geq 0, \quad \forall x \in \Omega. \quad (2.60)$$

Proof. We use the contradiction argument. If (2.60) does not hold, then there exists an $x_0 \in \bar{\Omega}$ such that

$$w_\lambda(x_0) = \min_{x \in \Sigma_\lambda \setminus \{0\}} w_\lambda(x) < 0 \quad \text{or} \quad z_\lambda(x_0) = \min_{x \in \Sigma_\lambda \setminus \{0\}} z_\lambda(x) < 0. \quad (2.61)$$

We divide into the following three cases to prove our conclusion.

Case 1. $w_\lambda(x_0) = \min_{x \in \Sigma_\lambda \setminus \{0\}} w_\lambda(x) < 0$ and $z_\lambda(x_0) \geq 0$. Similar to (2.39), we know that $x_0 \in (\Omega \cap \Sigma_\lambda^-) \subset \{x \in \mathbb{R}^N : \lambda - l < x_1 < l\}$ such that

$$(-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0) \leq 2\mathcal{C}_{N,\alpha} w_\lambda(x_0) \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy. \quad (2.62)$$

Let $A_l = \{y \in \mathbb{R}^N : l < y_1 - x_0^1 < 1, |y' - (x_0')'| < 1\} \subset \Sigma_\lambda$. By using the similar arguments as in [7, Inequality (22)], one sees that

$$\int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy \geq \int_{A_l} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy \geq c \int_l^1 \frac{1}{s^{1+\alpha}} ds \geq \frac{c}{l^\alpha} \rightarrow \infty, \quad (2.63)$$

as $l \rightarrow 0$. On the other hand, as in (2.42)-(2.45) we know that

$$(-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0) \geq A(x_0) w_\lambda(x_0), \quad (2.64)$$

where

$$A(x) = \left(\max\{p-1, 1\} (\mu_1 P_{\tilde{u}}(x) + \beta P_{\tilde{v}}(x)) \frac{\tilde{u}(x)^{p-2}}{|x|^m} + \mu_1 p H_u(x) \frac{\tilde{u}(x)^{p-1}}{|x|^m} \right)$$

and

$$H_u(x) = \int_{\mathbb{R}^N} \frac{\tilde{u}(y)^{p-1}}{|x - y|^{N-\gamma} |y|^m} dy.$$

Moreover, we have

$$A(x) \leq \frac{C}{|x|^{\alpha+\sigma}}, \quad \text{where } \sigma = \min \left\{ \alpha, N \left(1 - \frac{1}{p} \right) + \frac{m}{p} \right\}. \quad (2.65)$$

Since $\Omega \subset \{x \in \mathbb{R}^N : |x| \geq \lambda\}$, it follows from (2.65) that for each $x \in \Omega$

$$A(x) \leq \frac{C}{|\lambda|^{\alpha+\sigma}} := D_\lambda. \quad (2.66)$$

We infer from (2.62)-(2.63) that there exists $l_0 > 0$ such that for $0 < l \leq l_0$

$$2\mathcal{C}_{N,\alpha} \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy \geq 2D_\lambda. \quad (2.67)$$

Combining (2.64)-(2.66), we obtain that

$$0 \leq w_\lambda(x_0) \left(2\mathcal{C}_{N,\alpha} \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy - A(x_0) \right) \leq w_\lambda(x_0) D_\lambda < 0. \quad (2.68)$$

This is a contradiction.

Case 2. $z_\lambda(x_0) = \min_{x \in \Sigma_\lambda \setminus \{0\}} z_\lambda(x) < 0$ and $w_\lambda(x_0) \geq 0$. As in (2.50)-(2.52) we infer that

$$(-\Delta)^{\frac{\alpha}{2}} z_\lambda(x) \geq A_1(x_0) z_\lambda(x_0), \quad (2.69)$$

where A_1 and H_u are given in (2.51)-(2.52). Moreover, we have

$$A_1(x) \leq \frac{C}{|x|^{\alpha+\sigma}}, \quad \text{where } \sigma = \min \left\{ \alpha, N \left(1 - \frac{1}{p} \right) + \frac{m}{p} \right\}. \quad (2.70)$$

By using the same arguments as in Case 1, one can find the contradiction.

Case 3. $w_\lambda(x_0) = \min_{x \in \Sigma_\lambda \setminus \{0\}} w_\lambda(x) < 0$ and $z_\lambda(x_0) < 0$ or $z_\lambda(x_0) = \min_{x \in \Sigma_\lambda \setminus \{0\}} z_\lambda(x) < 0$ and $w_\lambda(x_0) < 0$. As in (2.54) we know that

$$(-\Delta)^{\frac{\alpha}{2}} w_\lambda(x_0) \geq A_3(x_0) w_\lambda(x_0) \quad \text{and} \quad (-\Delta)^{\frac{\alpha}{2}} z_\lambda(x_0) \geq A_4(x_0) z_\lambda(x_0), \quad (2.71)$$

where A_3, A_4 are given in (2.55). Furthermore, one has

$$A_3(x), A_4(x) \leq \frac{C}{|x|^{\alpha+\sigma}}, \quad \text{where } \sigma = \min \left\{ \alpha, N \left(1 - \frac{1}{p} \right) + \frac{m}{p} \right\}. \quad (2.72)$$

Thus, we get

$$\begin{aligned} & \left(2\mathcal{C}_{N,\alpha} \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy - A_3(x_0) \right) w_\lambda(x_0) \\ & + \left(2\mathcal{C}_{N,\alpha} \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy - A_4(x_0) \right) z_\lambda(x_0) \geq 0. \end{aligned} \quad (2.73)$$

Since $w_\lambda(x_0), z_\lambda(x_0) < 0$, we know that at least one of

$$\left(2\mathcal{C}_{N,\alpha} \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy - A_3(x_0) \right) \leq 0 \quad \text{or} \quad \left(2\mathcal{C}_{N,\alpha} \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy - A_4(x_0) \right) \leq 0 \quad (2.74)$$

holds. However, as in the cases 1-2, one can infer that

$$\left(2\mathcal{C}_{N,\alpha} \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy - A_3(x_0) \right) > 0 \quad \text{and} \quad \left(2\mathcal{C}_{N,\alpha} \int_{\Sigma_\lambda} \frac{1}{|x_0 - y^\lambda|^{N+\alpha}} dy - A_4(x_0) \right) > 0. \quad (2.75)$$

This is a contradiction. $\square$

3 Proof of Theorem 1.1

In this section we shall use the method of moving plane to prove the main results of Theorem 1.1. Precisely, we shall show the symmetry of u and v about T_{λ_0} ($\lambda_0 = 0$) by moving plane T_λ along x_1 direction from $-\infty$ to the right. We first divide into the following three steps to prove the subcritical case $1 \leq p < \frac{N+\gamma}{N-\alpha}$.

Step 1. We claim that for $\lambda < 0$ sufficiently negative,

$$w_\lambda(x) \geq 0 \quad \text{and} \quad z_\lambda(x) \geq 0 \quad \text{for } x \in \Sigma_\lambda \setminus \{0\}. \quad (3.1)$$

We use the contradiction argument. Assume that there exists $x_0 \in \Sigma_\lambda \setminus \{0\}$ such that $w_\lambda(x_0) < 0$ or $z_\lambda(x_0) < 0$. From (2.26), we deduce that there exist $\sigma_0 > 0$ and $\varepsilon > 0$ small such that $\tilde{u}(x) \geq \sigma_0$ and $\tilde{v}(x) \geq \sigma_0$ for all $x \in B_\varepsilon(0) \setminus \{0\}$. This implies that $\tilde{u}(x) \geq \sigma_0$ and $\tilde{v}(x) \geq \sigma_0$ for all $x \in B_\varepsilon(0^\lambda) \setminus \{0^\lambda\}$. On the other hand, we infer from the definition of $\tilde{u}, \tilde{v}$ that there exists $\Lambda_0 > 0$ large such that $\tilde{u}(x), \tilde{v}(x) < \frac{\sigma_0}{2}$ for $\lambda < -\Lambda_0$. Hence we get

$$w_\lambda(x) = \tilde{u}(x^\lambda) - \tilde{u}(x) \geq \frac{\sigma_0}{2} \quad \text{and} \quad z_\lambda(x) = \tilde{v}(x^\lambda) - \tilde{v}(x) \geq \frac{\sigma_0}{2} \quad \forall x \in B_\varepsilon(0^\lambda) \setminus \{0^\lambda\}. \quad (3.2)$$

Since $\lim_{|x| \rightarrow \infty} w_\lambda(x) = \lim_{|x| \rightarrow \infty} z_\lambda(x) = 0$, we infer from Theorem 2.6 that if $x_0 \in \Sigma_\lambda^-$ such that $w_\lambda(x_0) = \min_{x \in \Sigma_\lambda} w_\lambda(x) < 0$ or $z_\lambda(x_0) = \min_{x \in \Sigma_\lambda} z_\lambda(x) < 0$, then $|x_0| \leq R_0$. Thus, for $\lambda < -\max\{R_0, \Lambda_0\}$ we get the contradiction. This implies (3.1).

Step 2. Step 1 provides a starting point, from which we can move the plane T_λ to the right as long as (3.1) holds to its limiting position. Let

$$\lambda_0 = \sup\{\lambda \leq 0 : w_\mu(x) \geq 0 \quad \text{and} \quad z_\mu(x) \geq 0 \text{ for all } x \in \Sigma_\mu, \mu \leq \lambda\}. \quad (3.3)$$

In the following we shall prove that

$$\lambda_0 = 0 \quad \text{and} \quad w_{\lambda_0}(x) = z_{\lambda_0}(x) \equiv 0 \quad \text{for all } x \in \Sigma_{\lambda_0}. \quad (3.4)$$

We use the contradiction argument. Suppose that $\lambda_0 < 0$. We show that the plane T_λ can be moved further right. To be more rigorous, there exists some $\varepsilon > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, we have

$$w_\lambda(x) \geq 0 \quad \text{and} \quad z_\lambda(x) \geq 0 \quad \text{for all } x \in \Sigma_\lambda. \quad (3.5)$$

This is a contradiction with the definition of λ_0 . Hence we must have $\lambda_0 = 0$. We shall prove (3.5) by using the narrow region principle. To accomplish this we first claim that

$$w_{\lambda_0}(x) > 0 \quad \text{and} \quad z_{\lambda_0}(x) > 0 \quad \text{for all } x \in \Sigma_{\lambda_0} \setminus \{0^\lambda\}. \quad (3.6)$$

Since the proof is similar, we only prove $w_{\lambda_0}(x) > 0$ for $x \in \Sigma_{\lambda_0} \setminus \{0^\lambda\}$. We infer from Lemma 2.4 that there exists a constant $\sigma_0 > 0$ such that

$$w_{\lambda_0}(x) \geq \sigma_0 > 0 \quad \forall x \in B_{|\lambda_0|}(0^{\lambda_0}). \quad (3.7)$$

We infer from the integral equations (2.15) that

$$\begin{aligned} w_{\lambda_0}(x) &= \mu_1 \mathcal{R}_{N,\alpha} \left(\int_{\mathbb{R}^N} \frac{P_{\tilde{u}}(y) \tilde{u}(y)^{p-1}}{|y|^m |x^{\lambda_0} - y|^{N-\alpha}} dy - \int_{\mathbb{R}^N} \frac{P_{\tilde{u}}(y) \tilde{u}(y)^{p-1}}{|y|^m |x - y|^{N-\alpha}} dy \right) \\ &\quad + \beta \mathcal{R}_{N,\alpha} \left(\int_{\mathbb{R}^N} \frac{P_{\tilde{v}}(y) \tilde{u}(y)^{p-1}}{|y|^m |x^{\lambda_0} - y|^{N-\alpha}} dy - \int_{\mathbb{R}^N} \frac{P_{\tilde{v}}(y) \tilde{u}(y)^{p-1}}{|y|^m |x - y|^{N-\alpha}} dy \right) \\ &= \mu_1 \mathcal{R}_{N,\alpha} \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x - y|^{N-\alpha}} - \frac{1}{|x^{\lambda_0} - y|^{N-\alpha}} \right) \times \\ &\quad \left(\frac{P_{\tilde{u}}(y^{\lambda_0}) \tilde{u}(y^{\lambda_0})^{p-1}}{|y^{\lambda_0}|^m} - \frac{P_{\tilde{u}}(y) \tilde{u}(y)^{p-1}}{|y|^m} \right) dy \\ &\quad + \beta \mathcal{R}_{N,\alpha} \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x - y|^{N-\alpha}} - \frac{1}{|x^{\lambda_0} - y|^{N-\alpha}} \right) \times \\ &\quad \left(\frac{P_{\tilde{v}}(y^{\lambda_0}) \tilde{u}(y^{\lambda_0})^{p-1}}{|y^{\lambda_0}|^m} - \frac{P_{\tilde{v}}(y) \tilde{u}(y)^{p-1}}{|y|^m} \right) dy \\ &:= \mu_1 \mathcal{R}_{N,\alpha} L_1 + \beta \mathcal{R}_{N,\alpha} L_2 \end{aligned} \quad (3.8)$$

Next we show that $L_1, L_2 > 0$. We infer from Lemma 2.4 that for $y \in B_{|\lambda_0|/4}(0^{\lambda_0})$

$$\begin{aligned} P_{\tilde{u}}(y) &= \int_{\mathbb{R}^N} \frac{\tilde{u}^p(z)}{|y-z|^{N-\gamma}|z|^m} dz \geq \left(\min_{B_{|\lambda_0|}(0^{\lambda_0})} \tilde{u} \right)^p \int_{B_{|\lambda_0|/4}(y)} \frac{1}{|y-z|^{N-\gamma}|z|^m} dz \\ &\geq \left(\min_{B_{|\lambda_0|}(0^{\lambda_0})} \tilde{u} \right)^p \frac{1}{|\lambda_0|^m} \int_{B_{|\lambda_0|/4}(0)} \frac{1}{|z|^{N-\gamma}} dz = c > 0. \end{aligned} \quad (3.9)$$

Similar we obtain

$$P_{\tilde{v}}(y) \geq c > 0. \quad (3.10)$$

Now we are ready to deduce the estimates for L_1 . We infer from (3.9) that

$$\begin{aligned} L_1 &= \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{N-\alpha}} \right) \left(\frac{\tilde{u}(y^{\lambda_0})^{p-1}}{|y^{\lambda_0}|^m} - \frac{\tilde{u}(y)^{p-1}}{|y|^m} \right) P_{\tilde{u}}(y) dy \\ &\quad + \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{N-\alpha}} \right) \frac{\tilde{u}(y^{\lambda_0})^{p-1}}{|y^{\lambda_0}|^m} (P_{\tilde{u}}(y^{\lambda_0}) - P_{\tilde{u}}(y)) dy \\ &\geq \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{N-\alpha}} \right) \left(\frac{1}{|y^{\lambda_0}|^m} - \frac{1}{|y|^m} \right) \tilde{u}(y)^{p-1} P_{\tilde{u}}(y) dy \\ &\quad + \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{N-\alpha}} \right) \frac{\tilde{u}(y^{\lambda_0})^{p-1}}{|y^{\lambda_0}|^m} \times \\ &\quad \left[\int_{\Sigma_{\lambda_0}} \left(\frac{1}{|y-z|^{N-\gamma}} - \frac{1}{|y^{\lambda_0}-z|^{N-\gamma}} \right) \left(\frac{\tilde{u}(z^{\lambda_0})^{p-1}}{|z^{\lambda_0}|^m} - \frac{\tilde{u}(z)^{p-1}}{|z|^m} \right) dz \right] dy \\ &\geq \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{N-\alpha}} \right) \left(\frac{1}{|y^{\lambda_0}|^m} - \frac{1}{|y|^m} \right) \tilde{u}(y)^{p-1} P_{\tilde{u}}(y) dy \\ &\geq c \int_{B_{|\lambda_0|/2}(0^{\lambda_0})} \left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^{\lambda_0}-y|^{N-\alpha}} \right) \left(\frac{1}{|y^{\lambda_0}|^m} - \frac{1}{|y|^m} \right) dy > 0. \end{aligned} \quad (3.11)$$

Similarly, one can prove $L_2 > 0$. Hence we prove the claim (3.6). Furthermore, as in the proof of (3.11) we know that

$$w_{\lambda_0}(x) \geq c > 0 \quad \text{and} \quad z_{\lambda_0}(x) \geq c > 0 \quad \text{for } B_{\varrho}(0^{\lambda}) \setminus \{0^{\lambda_0}\}, \quad (3.12)$$

where $\varrho \in (0, |\lambda_0|/2)$. Thus, we get

$$\tilde{u}(x) - \tilde{u}_{\lambda_0}(x) \geq c > 0 \quad \text{and} \quad \tilde{v}(x) - \tilde{v}_{\lambda_0}(x) \geq c > 0 \quad \text{for } B_{\varrho}(0) \setminus \{0\}. \quad (3.13)$$

Since $\tilde{u}, \tilde{v}$ are uniformly continuous on $B_{\varrho+|\lambda_0|/2}(0^{\lambda_0})$, one infers from (3.13) that there exists $\varepsilon_0 \in (0, |\lambda_0|/4)$ sufficiently small such that for each $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_0]$

$$\tilde{u}(x) - \tilde{u}_{\lambda_0}(x) \geq \frac{c}{2} > 0 \quad \text{and} \quad \tilde{v}(x) - \tilde{v}_{\lambda_0}(x) \geq \frac{c}{2} > 0 \quad \text{for } B_{\varrho}(0) \setminus \{0\}. \quad (3.14)$$

This implies that for each $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_0]$

$$w_{\lambda}(x) \geq \frac{c}{2} > 0 \quad \text{and} \quad z_{\lambda}(x) \geq \frac{c}{2} > 0 \quad \text{for } B_{\varrho}(0^{\lambda}) \setminus \{0^{\lambda_0}\}. \quad (3.15)$$

Now we are ready to use the contradiction argument to give the proof of (3.5). Let $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1]$ where $\varepsilon_1 > 0$ is sufficiently small parameter. Assume that there exists $x_0 \in \Sigma_{\lambda} \setminus \{0\}$ such that

$$w_{\lambda}(x_0) = \min_{\Sigma_{\lambda} \setminus \{0\}} w_{\lambda} < 0 \quad \text{or} \quad z_{\lambda}(x_0) = \min_{\Sigma_{\lambda} \setminus \{0\}} z_{\lambda} < 0. \quad (3.16)$$

From Theorem 2.6 we infer that

$$x_0 \in B_{R_0}(0). \quad (3.17)$$

On the other hand, we infer from (3.12) that there exists $\varsigma \in (0, c/2)$ such that

$$w_{\lambda_0}(x) \geq \varsigma > 0 \quad \text{and} \quad z_{\lambda_0}(x) \geq \varsigma > 0 \quad \text{for } \overline{B_{R_0}(0) \cap \Sigma_{\lambda_0-l_0/2}} \setminus B_{\varrho/2}(0^{\lambda_0}), \quad (3.18)$$

By continuity, we may find $\varepsilon_1 \in (0, \min\{\varepsilon_0, l_0/2, |\lambda_0|/2, \varrho/4\})$ such that for each $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1)$

$$w_\lambda(x) \geq \frac{\varsigma}{2} > 0 \quad \text{and} \quad z_\lambda(x) \geq \frac{\varsigma}{2} > 0 \quad \text{for } \overline{B_{R_0}(0) \cap \Sigma_{\lambda_0-l_0/2}} \setminus B_{\varrho/2}(0^{\lambda_0}). \quad (3.19)$$

Since $0^\lambda \in B_{\varrho/2}(0^{\lambda_0})$ and $B_{\varrho/2}(0^{\lambda_0}) \subset B_\varrho(0^\lambda)$, we infer from (3.15) and (3.19) that

$$w_\lambda(x) \geq \frac{\varsigma}{2} > 0 \quad \text{and} \quad z_\lambda(x) \geq \frac{\varsigma}{2} > 0 \quad \text{for } \overline{B_{R_0}(0) \cap \Sigma_{\lambda_0-l_0/2}} \setminus \{0^\lambda\}. \quad (3.20)$$

We infer from (3.17) and (3.20) that the negative minimum of w_λ or z_λ can not be attained in $\overline{\Sigma_{\lambda_0-l_0/2}} \setminus \{0^\lambda\}$. We take $\Omega = \Sigma_\lambda \setminus \overline{\Sigma_{\lambda_0-l_0/2}}$ in Theorem 2.7. Then it follows that

$$w_\lambda(x) \geq 0 \quad \text{and} \quad z_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda \setminus \overline{\Sigma_{\lambda_0-l_0/2}} \setminus \{0^\lambda\}. \quad (3.21)$$

Thus, we infer from (3.21) and the definition of λ_0 that for $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1)$

$$w_\lambda(x) \geq 0 \quad \text{and} \quad z_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}. \quad (3.22)$$

This is contradicts with the definition of λ_0 . Hence we obtain

$$\lambda_0 = 0, \quad w_{\lambda_0}(x) \geq 0 \quad \text{and} \quad z_{\lambda_0}(x) \geq 0, \quad \forall x \in \Sigma_{\lambda_0}. \quad (3.23)$$

Similarly, we can move the plane T_λ from $+\infty$ to the left and show that

$$\lambda_0 = 0 \quad \text{and} \quad w_{\lambda_0}(x) = z_{\lambda_0}(x) \equiv 0 \quad \text{for all } x \in \Sigma_{\lambda_0}. \quad (3.24)$$

Step 3. We complete the proof. For any two points $X^i \in \mathbb{R}^N (i = 1, 2)$, one can choose X^0 to be the midpoint, i.e., $X^0 = \frac{X^1 + X^2}{2}$. Similar to Steps 1-2 (replacing 0 by X^0), one can prove that $(\tilde{u}, \tilde{v})$ is radially symmetric about X^0 , and so (u, v) . That is, we obtain $(u(X^1), v(X^1)) = (u(X^2), v(X^2))$. This implies that (u, v) is constant. Since a positive constant does not satisfy (1.1). This proves the nonexistence of positive solutions for (1.1) when $1 \leq p < \frac{N+\gamma}{N-\alpha}$. This finishes the proof for the subcritical case.

Finally, we make the classification of positive solutions in the critical case $p = \frac{N+\gamma}{N-\alpha}$ and $\mu_1 = \mu_2$. The next lemma finds the exact formula solution of (1.1).

Lemma 3.1. For each $\mu > 0$ and $y \in \mathbb{R}^N$, we know that $u = v = (\mu_1 + \beta)^{-\frac{N-\alpha}{2(\gamma+\alpha)}} \mu^{\frac{N-\alpha}{2}} U(\mu(x-y))$ is a solution of (1.1), where $U(x) = \tau_0 \left(\frac{1}{1+|x|^2} \right)^{\frac{N-\alpha}{2}}$ and $I(k) = \frac{\pi^{N/2} \Gamma(\frac{N-2k}{2})}{\Gamma(N-k)}$, where $\tau_0 = \left(\mathcal{R}_{N,\alpha} I \left(\frac{N-\alpha}{2} \right) I \left(\frac{N-\gamma}{2} \right) \right)^{-\frac{N-\alpha}{2(\gamma+\alpha)}}$.

Proof. From the invariance of the system (1.1), we may assume $\mu = 1$ and $y = 0$. Moreover, by using the Fourier transforms of the kernels of Riesz and Bessel potentials, we infer from Lemm [12, Lemma 4.2] that

$$\int_{\mathbb{R}^N} \frac{1}{|x-y|^{2s}} \left(\frac{1}{1+|y|^2} \right)^{N-s} dy = I(s) \left(\frac{1}{1+|y|^2} \right)^s. \quad (3.25)$$

A direct computation shows that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{U^p(y)}{|x-y|^{N-\gamma}} dy &= \int_{\mathbb{R}^N} \frac{\tau_0^p}{|x-y|^{N-\gamma}} \left(\frac{1}{1+|y|^2} \right)^{\frac{N+\gamma}{2}} dy \\ &= \tau_0^{\frac{N+\gamma}{N-\alpha}} I \left(\frac{N-\gamma}{2} \right) \left(\frac{1}{1+|x|^2} \right)^{\frac{N-\gamma}{2}}. \end{aligned} \quad (3.26)$$

Hence it follows that

$$\begin{aligned}
 & \frac{\mathcal{R}_{N,\alpha}}{(\mu_1 + \beta)^{\frac{2p-1}{2(p-1)}}} \int_{\mathbb{R}^N} \frac{U^{p-1}(y)}{|x-y|^{N-\alpha}} \int_{\mathbb{R}^N} \frac{U^p(z)}{|z-y|^{N-\gamma}} dz dy \\
 &= \frac{\mathcal{R}_{N,\alpha} \tau_0^{\frac{N+\alpha+2\gamma}{N-\alpha}}}{(\mu_1 + \beta)^{\frac{2p-1}{2(p-1)}}} I\left(\frac{N-\gamma}{2}\right) \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\alpha}} \left(\frac{1}{1+|y|^2}\right)^{\frac{N-\alpha}{2}} dy \\
 &= \frac{\mathcal{R}_{N,\alpha} \tau_0^{\frac{N+\alpha+2\gamma}{N-\alpha}}}{(\mu_1 + \beta)^{\frac{2p-1}{2(p-1)}}} I\left(\frac{N-\gamma}{2}\right) I\left(\frac{N-\alpha}{2}\right) \left(\frac{1}{1+|x|^2}\right)^{\frac{N-\alpha}{2}} \\
 &= \frac{\tau_0}{(\mu_1 + \beta)^{\frac{2p-1}{2(p-1)}}} \left(\frac{1}{1+|x|^2}\right)^{\frac{N-\alpha}{2}} = \frac{1}{(\mu_1 + \beta)^{\frac{2p-1}{2(p-1)}}} U(x).
 \end{aligned} \tag{3.27}$$

Thus, we know that $u = v = (\mu_1 + \beta)^{-\frac{1}{2(p-1)}} \mu^{\frac{N-\alpha}{2}} U(\mu(x-y))$ is a solution of (1.1). $\square$

Lemma 3.2. Assume that $N \geq 2$, $0 < \alpha < 2$, $0 < \gamma < N$, $\mu_1 = \mu_2$, $\beta > 0$ and $p = \frac{N+\gamma}{N-\alpha}$. Then every positive solution of (1.1) must assume the form

$$u = v = (\mu_1 + \beta)^{-\frac{N-\alpha}{2(\gamma+\alpha)}} \mu^{\frac{N-\alpha}{2}} U(\mu(x-y)) \quad \text{for some } \mu > 0 \text{ and } y \in \mathbb{R}^N, \tag{3.28}$$

where U is given in Lemma 3.1.

Proof. As in the subcritical case, we know that every positive solution of (1.1) is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^N$. We claim that (u, v) has the desired asymptotic behavior at ∞ . That is, it satisfies

$$\lim_{|x| \rightarrow \infty} |x|^{N-\alpha} u(x) = u_\infty \quad \text{and} \quad \lim_{|x| \rightarrow \infty} |x|^{N-\alpha} v(x) = v_\infty, \tag{3.29}$$

where u_∞ and v_∞ are some positive constants. We use the contradiction argument. Assume that (3.29) does not hold. Suppose that x_1 and x_2 be any two different points in $\mathbb{R}^N$ and let x_0 be the midpoint of the line segment $\overline{x_1 x_2}$. Consider the Kelvin type transform centered at x_0

$$z(x) = \frac{1}{|x-x_0|^{N-\alpha}} u\left(\frac{x-x_0}{|x-x_0|^2} + x_0\right) \quad \text{and} \quad w(x) = \frac{1}{|x-x_0|^{N-\alpha}} v\left(\frac{x-x_0}{|x-x_0|^2} + x_0\right). \tag{3.30}$$

Then $z(x)$ and $w(x)$ must have a singularity at x_0 . By using the same arguments as in the proof of the subcritical case, we can deduce that z, w must be radially symmetric and monotone decreasing about its singular point x_0 and hence $u(x_1) = u(x_2)$. Since x_1 and x_2 are arbitrarily chosen in $\mathbb{R}^N$, u must be constant, thus $u \equiv 0$, which contradicts with $u > 0$. Hence the claim (3.29) holds.

Now we borrow an idea of [5, 8, 43] to give the proof of this lemma. Assume that (u, v) is a nonnegative solution of (1.1). Then if $x_0 = 0$, we get that

$$u(s_1 x + a) = \frac{1}{|x|^{N-\alpha}} u\left(\frac{s_1 x}{|x|^2} + a\right) \quad \text{and} \quad v(s_2 x + a) = \frac{1}{|x|^{N-\alpha}} v\left(\frac{s_2 x}{|x|^2} + a\right) \tag{3.31}$$

for $x \in \mathbb{R}^N \setminus \{0\}$ and $a \in \mathbb{R}$, where $s_1 = (u(x_0)/u_\infty)^{\frac{1}{N-\alpha}}$ and $s_2 = (v(x_0)/v_\infty)^{\frac{1}{N-\alpha}}$. Without loss of generality we assume that $a = 0$ and $\mu_1 + \beta = 1$. Since the proof is similar, we only give the proof for the first conclusion of (3.31). Let $x^0 \in \mathbb{R}^N \setminus \{0\}$ be any fixed point and $e = \frac{x^0}{|x^0|}$. We define the function

$$w(x) = \frac{1}{|x|^{N-\alpha}} u\left(s\left(\frac{x}{|x|^2} - e\right)\right). \tag{3.32}$$

Then we know that $w(0) = s^{\alpha-N} u_\infty = u(0) = w(e)$ and $u = v = s^{\frac{N-\alpha}{2}} w$ is also a positive solution of (1.1). Thus w is radially symmetric with respect to some point $\tilde{x}$ that lies on the hyperplane $e^\perp + \frac{1}{2}e$ through $\frac{1}{2}e$ which is

perpendicular to e . Moreover, since u, v are radially symmetric about 0, it follows that for any $\frac{1}{2} < r < 1$ and $x^1, x^2 \in \partial B_r(0) \cap \partial B_r(e)$, we can deduce from (3.32) that

$$w(x^1) = \frac{1}{|x^1|^{N-\alpha}} u \left(s \left(\frac{x^1}{|x^1|^2} - e \right) \right) = \frac{1}{|x^2|^{N-\alpha}} u \left(s \left(\frac{x^2}{|x^2|^2} - e \right) \right) = w(x^2). \quad (3.33)$$

Hence we obtain that $w(x) = w(|x - \frac{1}{2}e|)$ on $e^\perp + \frac{1}{2}e$, and $\tilde{x} = \frac{e}{2}$ and w is actually radially symmetric about $\frac{e}{2}$. It is clear that there exists $\sigma \in (-\frac{1}{2}, \frac{1}{2})$ such that $|x^0| = \frac{\frac{1}{2} + \sigma}{\frac{1}{2} - \sigma}$. We infer from (3.32) that

$$\frac{1}{|\frac{1}{2} - \sigma|^{N-\alpha}} u \left(s \frac{\frac{1}{2} + \sigma}{\frac{1}{2} - \sigma} e \right) = w \left(\left(\frac{1}{2} - \sigma \right) e \right) = w \left(\left(\frac{1}{2} + \sigma \right) e \right) = \frac{1}{|\frac{1}{2} + \sigma|^{N-\alpha}} u \left(s \frac{\frac{1}{2} - \sigma}{\frac{1}{2} + \sigma} e \right). \quad (3.34)$$

This implies that

$$u(sx^0) = \frac{1}{|x^0|^{N-\alpha}} u \left(\frac{sx^0}{|x^0|^2} \right). \quad (3.35)$$

For the case $a \neq 0$, we can consider $u(\cdot + a)$ instead of u itself. As in [8, Lemma 3.2], one can prove that

$$u(m)u_\infty = v(m)v_\infty = \left(\mathcal{R}_{N,\alpha} I \left(\frac{N-\alpha}{2} \right) I \left(\frac{N-\gamma}{2} \right) \right)^{-\frac{N-\alpha}{\gamma+\alpha}}, \quad (3.36)$$

where (u, v) is a positive solution to (1.1) with symmetric center m . Finally, we define

$$\hat{u}(x) = \chi_1^{-\frac{N-\alpha}{2}} u(\chi_1^{-1}x + x_0) \quad \text{and} \quad \hat{v}(x) = \chi_2^{-\frac{N-\alpha}{2}} v(\chi_2^{-1}x + x_0), \quad (3.37)$$

where $\chi_1 = (u(x_0)/u_\infty)^{\frac{1}{N-\alpha}}$ and $\chi_2 = (v(x_0)/v_\infty)^{\frac{1}{N-\alpha}}$. By using the similar arguments as in [8, Section 3.1], we know that

$$\hat{u}(x) = U(x) = \hat{v}(x). \quad (3.38)$$

This finishes the proof. $\square$

4 Proof of Theorem 1.4

In this section we focus on the existence of solution of (1.1) when $\beta \in \mathbb{R}$ and $\mu_1 \neq \mu_2$. In the following we use (1.11)-(1.12) to construct the synchronous solutions of (1.1). That is, the solution of the form

$$(u, v) = (kU_1, lU_1). \quad (4.1)$$

Then (4.1) solves (1.1) if and only if (k, l) satisfies (1.21). In the following we first find the solution of (1.21). To this purpose we borrow an idea of [10] to define

$$\begin{aligned} g_1(k, l) &= \mu_1 k^{2p-2} + \beta l^p k^{p-2} - 1, \quad k > 0, l \geq 0, \\ g_2(k, l) &= \mu_2 l^{2p-2} + \beta k^p l^{p-2} - 1, \quad l > 0, k \geq 0, \\ h_1(k) &= \beta^{-\frac{1}{p}} \left(k^{2-p} - \mu_1 k^p \right)^{\frac{1}{p}}, \quad 0 < k \leq \mu_1^{-\frac{1}{2p-2}} \\ \text{and } h_2(l) &= \beta^{-\frac{1}{p}} \left(l^{2-p} - \mu_2 l^p \right)^{\frac{1}{p}}, \quad 0 < l \leq \mu_2^{-\frac{1}{2p-2}}, \end{aligned} \quad (4.2)$$

where $p = \frac{N+\gamma}{N-\alpha}$. By this definition we know that $g_1(k, h_1(k)) = g_2(h_2(l), l) = 0$. In the next lemma we find the solution (k, l) for (1.21).

Lemma 4.1. Assume that $\beta \in \mathbb{R}$. Then we have the following results.

- (1) If $2\alpha + \gamma = N$, then we have $\left(\sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}}, \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}} \right)$ is a solution of (1.21) for $\beta \in (-\sqrt{\mu_1 \mu_2}, \min\{\mu_1, \mu_2\}) \cap (\max\{\mu_1, \mu_2\}, \infty)$.

- (2) If $2\alpha + \gamma > N$ and $\beta > 0$, then (1.21) has a positive solution (k_0, l_0) , which satisfies $g_2(k, h_1(k)) > 0$ for $0 < k < k_0$. Moreover, (1.21) has another positive solution (k_1, l_1) , which satisfies $g_1(h_2(l), l) > 0$ for $0 < l < l_0$.
- (3) If $2\alpha + \gamma < N$ and $\beta > 0$, then (1.21) has a positive solution (k_2, l_2) , which satisfies $g_2(k, h_2(k)) < 0$ for $0 < k < k_2$. Moreover, (1.21) has another positive solution (k_3, l_3) , which satisfies $g_1(h_2(l), l) < 0$ for $0 < l < l_3$.

Proof. We first prove the conclusion (1). If $2\alpha + \gamma = N$, then we know that (1.21) equals

$$\begin{cases} \mu_1 k^2 + \beta l^2 = 1, \\ \mu_2 l^2 + \beta k^2 = 1, \quad k, l > 0. \end{cases} \quad (4.3)$$

Thus, if $\beta \in (-\sqrt{\mu_1 \mu_2}, \min\{\mu_1, \mu_2\}) \cap (\max\{\mu_1, \mu_2\}, \infty)$, then $(\sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}}, \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}})$ is a solution of (4.3).

Next we prove the conclusion (2) and (3). It is clear that $g_1(k, l) = 0, k, l > 0$ equals

$$l = h_1(k) = \beta^{-\frac{1}{p}} (k^{2-p} - \mu_1 k^p)^{\frac{1}{p}}, \quad 0 < k \leq \mu_1^{-\frac{1}{2p-2}}. \quad (4.4)$$

Substituting this into $g_2(k, l) = 0$, then we obtain that

$$\mu_2 \frac{1 - \mu_1 k^{2p-2}}{\beta} + \beta k^{2p-2} = \left(\frac{1 - \mu_1 k^{2p-2}}{\beta k^{2p-2}} \right)^{\frac{2-p}{p}} \quad (4.5)$$

In order to finish the proof we define

$$F(k) = \frac{\mu_2}{\beta} + \frac{\beta^2 - \mu_1 \mu_2}{\beta} k^{2p-2} - \left(\frac{1}{\beta k^{2p-2}} - \frac{\mu_1}{\beta} \right)^{\frac{2-p}{p}}, \quad 0 < k < \mu_1^{-\frac{1}{2p-2}}. \quad (4.6)$$

We first consider the case $2\alpha + \gamma > N$. This implies that $p > 2$ in this case. A direct computation shows that

$$\lim_{k \rightarrow 0^+} F(k) = \frac{\mu_2}{\beta} \quad \text{and} \quad \lim_{k \rightarrow \left(\mu_1^{-\frac{1}{2p-2}} \right)^-} F(k) = -\infty \quad (4.7)$$

This implies that there exists $k_0 \in (0, \mu_1^{-\frac{1}{2p-2}})$ such that $F(k_0) = 0$ and $F(k) > 0$ for each $k \in (0, k_0)$. Hence $g_2(k, h_1(k)) > 0$ for $0 < k < k_0$. Set $l_0 = h_1(k_0)$. Then (k_0, l_0) is a solution of $g_1(k, l) = 0$ and $g_2(k, l) = 0$. Similarly, one can find the (k_1, l_1) satisfies the conclusion (2). Finally, if $2\alpha + \gamma < N$ and $\beta > 0$, by using the similar argument as in the case (2), one can find (k_2, l_2) and (k_3, l_3) , which satisfy the conclusion (3). $\square$

Next we study the uniqueness of (k_0, l_0) and (k_2, l_2) .

Lemma 4.2. Assume that $2\alpha + \gamma > N$. Then (k_0, l_0) (see Lemma 4.1 (2)) is the unique solution of (1.21) for $0 < \beta \leq \sqrt{\mu_1 \mu_2}$. Furthermore, if (k, l) satisfies

$$\begin{cases} k^2 + l^2 \leq k_0^2 + l_0^2, \quad g_1(k, l) \geq 0, \quad g_2(k, l) \geq 0, \\ k, l \geq 0, \quad (k^2, l^2) \neq (0, 0), \end{cases} \quad (4.8)$$

then (4.8) has unique solution $(k, l) = (k_0, l_0)$. On the other hand, if $2\alpha + \gamma < N$ and $\beta \geq (p-1) \max\{\mu_1, \mu_2\}$ then the equation (4.8) (with (k_0, l_0) is replaced by (k_2, l_2)) has a unique solution $(k, l) = (k_2, l_2)$.

Proof. We first consider the case $2\alpha + \gamma > N$, that is, $p > 2$. By (4.6), a direct computation shows that

$$F'(k) = (2p-2) \frac{\beta^2 - \mu_1 \mu_2}{\beta} k^{2p-3} - \frac{(2-p)(2-2p)}{p\beta} k^{1-2p} \left(\frac{1}{\beta k^{2p-2}} - \frac{\mu_1}{\beta} \right)^{\frac{2}{p}-1}, \quad (4.9)$$

where $0 < k < \mu_1^{-\frac{1}{2p-2}}$. Hence we obtain that $F'(k) < 0$ for $0 < k < \mu_1^{-\frac{1}{2p-2}}$. Combining with (4.7) we obtain that $F(k) = 0$ has unique solution k_0 . This implies that (k_0, l_0) is the unique solution of (1.21). On the other hand, a direct computation shows that for $0 < k \leq \mu_1^{-\frac{1}{2p-2}}$ and $0 < l \leq \mu_2^{-\frac{1}{2p-2}}$

$$\begin{aligned} g_1(0, l) < 0, \quad g_1(k, 0) \leq 0, \quad g_2(0, l) \leq 0, \quad g_2(k, 0) < 0, \\ g'_{1l}(k, l), \quad g'_{1k}(k, l), \quad g'_{2l}(k, l), \quad g'_{2k}(k, l) > 0. \end{aligned}$$

This implies that $g_1(k, l)$ and $g_2(k, l)$ are increasing in k, l respectively. Assume that $k < k_0$. we infer from the monotone property of $g_i(k, l) (i = 1, 2,)$ that

$$0 \leq g_1(k, l_0) < g_1(k_0, l_0) = 0.$$

This is a contradiction. Thus, we get $k \geq k_0$. Similarly, we can prove $l \geq l_0$. Hence we obtain $(k, l) = (k_0, l_0)$. The rest of the conclusion follows from [10, Lemmas 2.2-2.4]. $\square$

Remark 4.3. The condition $0 < \beta \leq \sqrt{\mu_1 \mu_2}$ only is required for the uniqueness of (k_0, l_0) . If (k, l) satisfies (4.8), we knew that (k, l) has unique solution (k_0, l_0) for each $\beta > 0$.

The next lemma states the relations between K and (1.1).

Lemma 4.4. *If K is attained by a couple $(u, v) \in \mathcal{N}$, then (u, v) is a critical point of $\mathcal{J}$ for $-\infty < \beta < 0$.*

Proof. We infer from $(u, v) \in \mathcal{N}$ that

$$|u|_{D^{1,\alpha}}^2 = \mu_1 \int_{\mathbb{R}^N} \phi_u |u|^p - |\beta| \int_{\mathbb{R}^N} \phi_u |v|^p \text{ and } |u|_{D^{1,\alpha}}^2 = \mu_1 \int_{\mathbb{R}^N} \phi_u |u|^p - |\beta| \int_{\mathbb{R}^N} \phi_u |v|^p, \quad (4.10)$$

where $\phi_u(x) = \int_{\mathbb{R}^N} \frac{u(y)^p}{|x-y|^{N-\gamma}} dy$. If $K = \mathcal{J}(u, v)$, then there exists two Lagrangian multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\mathcal{J}'(u, v) + \lambda_1 H_1(u, v) + \lambda_2 H_2(u, v) = 0, \quad (4.11)$$

where $H_1 = \mathcal{J}'(u, v)(u, 0)$ and $H_2 = \mathcal{J}'(u, v)(0, v)$. Using $(u, 0)$ and $(0, v)$ as a test function, we obtain that

$$\begin{aligned} \lambda_1 \left((2p-2)\mu_1 \int_{\mathbb{R}^N} \phi_u |u|^p + (2-p)|\beta| \int_{\mathbb{R}^N} \phi_u |v|^p \right) - \lambda_2 |\beta| p \int_{\mathbb{R}^N} \phi_u |v|^p, \\ \lambda_2 \left((2p-2)\mu_2 \int_{\mathbb{R}^N} \phi_v |v|^p + (2-p)|\beta| \int_{\mathbb{R}^N} \phi_u |v|^p \right) - \lambda_1 |\beta| p \int_{\mathbb{R}^N} \phi_u |v|^p. \end{aligned} \quad (4.12)$$

We infer from $\beta < 0$ and (4.10) that

$$\begin{aligned} \left((2p-2)\mu_1 \int_{\mathbb{R}^N} \phi_u |u|^p + (2-p)|\beta| \int_{\mathbb{R}^N} \phi_u |v|^p \right) \times \\ \left((2p-2)\mu_2 \int_{\mathbb{R}^N} \phi_v |v|^p + (2-p)|\beta| \int_{\mathbb{R}^N} \phi_u |v|^p \right) > \left(|\beta| p \int_{\mathbb{R}^N} \phi_u |v|^p \right)^2. \end{aligned} \quad (4.13)$$

It follows that $\lambda_1 = \lambda_2 = 0$. $\square$

Now we are ready to give the proof of Theorem 1.4. We first consider the case $\beta < 0$.

Proof of Theorem 1.4 (i). Let $w_{\mu_i} := \mu_i^{\frac{\alpha-N}{2(\gamma+\alpha)}} U_1(x) (i = 1, 2)$. It is clear that $w_{\mu_i}(x)$ satisfies the equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \mu_i \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{\frac{N+\gamma}{N-\alpha}}}{|x-y|^{N-\gamma}} dy \right) |u|^{\frac{2\alpha+\gamma-N}{N-\alpha}} u, \quad \text{in } \mathbb{R}^N, \quad i = 1, 2. \quad (4.14)$$

We define $\vec{e}_1 = (1, \dots, 1) \in \mathbb{R}^N$ and

$$(u_R(x), v_R(x)) = (w_{\mu_1}(x), w_{\mu_2}(x + 4R\vec{e}_1)). \quad (4.15)$$

We first claim that

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_R(x)|^{\frac{N+\gamma}{N-\alpha}} |v_R(y)|^{\frac{N+\gamma}{N-\alpha}}}{|x-y|^{N-\gamma}} dx dy = 0, \quad \text{as } R \rightarrow \infty. \quad (4.16)$$

In fact, it is clear that

$$\begin{aligned}\mathbb{R}^N \times \mathbb{R}^N &= (B_R(0) \times B_R(-4R\vec{e}_1)) \cup \left(B_R(0) \times \left(\mathbb{R}^N \setminus B_R(-4R\vec{e}_1) \right) \right) \cup \left(\left(\mathbb{R}^N \setminus B_R(0) \right) \times B_R(-4R\vec{e}_1) \right) \\ &\quad \cup \left(\left(\mathbb{R}^N \setminus B_R(0) \right) \times \left(\mathbb{R}^N \setminus B_R(-4R\vec{e}_1) \right) \right) \\ &:= \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4,\end{aligned}\quad (4.17)$$

where $B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| \leq R\}$. We shall prove that (4.17) holds in each domain $\Omega_i (i = 1, 2, 3, 4)$. Since $|x - y| \geq 2R$ for $(x, y) \in B_R(0) \times B_R(-4R\vec{e}_1)$, it follows that

$$\begin{aligned}&\int_{B_R(0)} \int_{B_R(-4R\vec{e}_1)} \frac{|u_R(x)|^{\frac{N+\gamma}{N-\alpha}} |v_R(y)|^{\frac{N+\gamma}{N-\alpha}}}{|x-y|^{N-\gamma}} dx dy \\ &\leq \frac{1}{(2R)^{N-\gamma}} \int_{B_R(0)} w_{\mu_1}(x)^{\frac{N+\gamma}{N-\alpha}} dx \int_{B_R(0)} w_{\mu_2}(x)^{\frac{N+\gamma}{N-\alpha}} dx \leq \frac{c}{(2R)^{N-\gamma}}.\end{aligned}\quad (4.18)$$

On the other hand, we infer from Hardy-Littlewood-Sobolev inequality (see Lemma 2.1) that

$$\begin{aligned}&\int_{B_R(0)} \int_{\mathbb{R}^N \setminus B_R(-4R\vec{e}_1)} \frac{|u_R(x)|^{\frac{N+\gamma}{N-\alpha}} |v_R(y)|^{\frac{N+\gamma}{N-\alpha}}}{|x-y|^{N-\gamma}} dx dy \\ &\leq c \left(\int_{B_R(0)} w_{\mu_1}(x)^{\frac{2N}{N-\alpha}} dx \right)^{\frac{N+\gamma}{2N}} \left(\int_{\mathbb{R}^N \setminus B_R(0)} w_{\mu_2}(x)^{\frac{2N}{N-\alpha}} dx \right)^{\frac{N+\gamma}{2N}} \\ &\rightarrow 0 \quad \text{as } R \rightarrow \infty.\end{aligned}\quad (4.19)$$

Similarly, one can prove that

$$\int_{\Omega_3 \cup \Omega_4} \frac{|u_R(x)|^{\frac{N+\gamma}{N-\alpha}} |v_R(y)|^{\frac{N+\gamma}{N-\alpha}}}{|x-y|^{N-\gamma}} dx dy \rightarrow 0 \quad \text{as } R \rightarrow \infty. \quad (4.20)$$

Combining (4.18)-(4.20), we obtain that the claim (4.16) holds.

Next we prove that for each $\beta < 0$, there exists positive (t_R, s_R) such that $(t_R u_R, s_R v_R) \in \mathcal{N}$. Let

$$\begin{aligned}A_1 &= \mu_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_R(x)|^{\frac{N+\gamma}{N-\alpha}} |u_R(y)|^{\frac{N+\gamma}{N-\alpha}}}{|x-y|^{N-\gamma}} dx dy, \quad A_2 = \mu_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_R(x)|^{\frac{N+\gamma}{N-\alpha}} |u_R(y)|^{\frac{N+\gamma}{N-\alpha}}}{|x-y|^{N-\gamma}} dx dy \\ \text{and } A_3 &= \beta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_R(x)|^{\frac{N+\gamma}{N-\alpha}} |v_R(y)|^{\frac{N+\gamma}{N-\alpha}}}{|x-y|^{N-\gamma}} dx dy.\end{aligned}\quad (4.21)$$

Then we infer from (4.16) that

$$A_1 A_2 > A_3^2 \quad \text{for each } \beta < 0 \text{ and } R > 0 \text{ sufficiently large.} \quad (4.22)$$

It is clear that $(t_R u_R, s_R v_R) \in \mathcal{N}$ equals

$$t_R^{2-p} A_1 = t_R^p A_1 + s_R^p A_3 \quad \text{and} \quad s_R^{2-p} A_2 = s_R^p A_2 + t_R^p A_3, \quad t_R, s_R > 0. \quad (4.23)$$

Since $p > 1$, we infer from $s_R^p = (t_R^{2-p} - t_R^p) A_1 / A_3 > 0$ and $A_3 < 0$ that $t > 1$. Substituting this into (4.23), we obtain that

$$G(t_R) := A_2 \left(\frac{A_1}{|A_3|} (1 - t_R^{2-2p}) \right)^{\frac{2-p}{p}} - \frac{A_1 A_2 - |A_3|^2}{|A_3|} t_R^{2p-2} + \frac{A_1 A_2}{|A_3|}, \quad t > 1. \quad (4.24)$$

We infer from $p > 1$ that $G(1) = |A_3| > 0$ and $\lim_{t_R \rightarrow \infty} G(t_R) = -\infty$. Hence we know that there exists $t_R, s_R > 0$ such that $(t_R u_R, s_R v_R) \in \mathcal{N}$. Furthermore, we claim that

$$\lim_{R \rightarrow \infty} (|t_R - 1| + |s_R - 1|) = 0. \quad (4.25)$$

To accomplish this we first prove that t_R, s_R are bounded. Assume that $t_R \rightarrow \infty$ as $R \rightarrow \infty$. We infer from (4.23) that

$$t_R^{2p} A_1 = t_R^{2p} A_1 - s_R^p t_R^p |A_3|, \quad s_R^{2p} A_2 = s_R^{2p} A_2 - s_R^p t_R^p |A_3| \quad (4.26)$$

and

$$t_R^{2p} A_1 - t_R^2 A_1 = s_R^{2p} A_2 - s_R^2 A_2. \quad (4.27)$$

This implies that $s_R \rightarrow \infty$ as $R \rightarrow \infty$. We deduce from $p > 1$ that

$$t_R^p (A_1 - t_R^{2-2p} A_1) \geq \frac{1}{4} t_R^p A_1 \quad \text{and} \quad s_R^p (A_2 - s_R^{2-2p} A_2) \geq \frac{1}{4} t_R^p A_2 \quad \text{for } R \text{ large.} \quad (4.28)$$

Then we obtain

$$|A_3| = \frac{t_R^p - t_R^{2-p}}{s_R^p} A_1 \geq \frac{t_R^p}{4s_R^p} A_1 \quad \text{and} \quad |A_3| = \frac{s_R^p - s_R^{2-p}}{t_R^p} A_2 \geq \frac{s_R^p}{4t_R^p} A_2. \quad (4.29)$$

Combining (4.28)-(4.29) we get

$$0 < \frac{1}{16} A_1 A_2 \leq |A_3|^2 \rightarrow 0, \quad \text{as } R \rightarrow \infty. \quad (4.30)$$

This is a contradiction. Therefore s_R, t_R are bounded. We infer from (4.26) that (4.25) holds. Hence we infer from (1.17) that

$$\begin{aligned} K &\leq \mathcal{J}(t_R u_R, s_R v_R) = \frac{\gamma + \alpha}{2(N + \gamma)} \left(t_R^2 |u_R|_{D^{1,\alpha}}^2 + s_R^2 |v_R|_{D^{1,\alpha}}^2 \right) \\ &= \frac{\gamma + \alpha}{2(N + \gamma)} \left(t_R^2 \mu_1^{\frac{\alpha-N}{\gamma+\alpha}} + s_R^2 \mu_2^{\frac{\alpha-N}{\gamma+\alpha}} \right) S^{\frac{N+\gamma}{\gamma+\alpha}}. \end{aligned} \quad (4.31)$$

Thus, as $R \rightarrow \infty$ in (4.31), we get that $K \leq \frac{\gamma+\alpha}{2(N+\gamma)} \left(\mu_1^{\frac{\alpha-N}{\gamma+\alpha}} + \mu_2^{\frac{\alpha-N}{\gamma+\alpha}} \right) S^{\frac{N+\gamma}{\gamma+\alpha}}$. On the other hand, we infer from $\beta < 0$ and (1.16) that

$$|u|_{D^{1,\alpha}}^2 \leq \mu_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\gamma}} dx dy \leq \mu_1 S^{-p} \left(|u|_{D^{1,\alpha}}^2 \right)^p. \quad (4.32)$$

This implies that $|u|_{D^{1,\alpha}}^2 \geq \mu_1^{\frac{\alpha-N}{\gamma+\alpha}} S^{\frac{N+\gamma}{\gamma+\alpha}}$. Similarly, we can deduce that $|v|_{D^{1,\alpha}}^2 \geq \mu_2^{\frac{\alpha-N}{\gamma+\alpha}} S^{\frac{N+\gamma}{\gamma+\alpha}}$. Hence we infer from (4.23) that

$$K \geq \frac{\gamma + \alpha}{2(N + \gamma)} \left(\mu_1^{\frac{\alpha-N}{\gamma+\alpha}} + \mu_2^{\frac{\alpha-N}{\gamma+\alpha}} \right) S^{\frac{N+\gamma}{\gamma+\alpha}}. \quad (4.33)$$

Then we get

$$K = \frac{\gamma + \alpha}{2(N + \gamma)} \left(\mu_1^{\frac{\alpha-N}{\gamma+\alpha}} + \mu_2^{\frac{\alpha-N}{\gamma+\alpha}} \right) S^{\frac{N+\gamma}{\gamma+\alpha}}. \quad (4.34)$$

If K is attained by $(u, v) \in \mathcal{N}$, then by Lemma 4.4 we know that (u, v) is a nontrivial solution of (1.1). Thus we get $\int_{\mathbb{R}^N} \phi_u |u|^p > 0$. Hence we infer that

$$\begin{aligned} |u|_{D^{1,\alpha}}^2 &< \mu_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\gamma}} dx dy \leq \mu_1 S^{-p} \left(|u|_{D^{1,\alpha}}^2 \right)^p, \\ \text{and} \quad |v|_{D^{1,\alpha}}^2 &< \mu_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)|^p |v(y)|^p}{|x-y|^{N-\gamma}} dx dy \leq \mu_2 S^{-p} \left(|v|_{D^{1,\alpha}}^2 \right)^p. \end{aligned} \quad (4.35)$$

Then we infer from (1.20) that

$$K = \mathcal{J}(u, v) = \frac{\gamma + \alpha}{2(N + \gamma)} \left(|u|_{D^{1,\alpha}}^2 + |v|_{D^{1,\alpha}}^2 \right) > \frac{\gamma + \alpha}{2(N + \gamma)} \left(\mu_1^{\frac{\alpha-N}{\gamma+\alpha}} + \mu_2^{\frac{\alpha-N}{\gamma+\alpha}} \right) S^{\frac{N+\gamma}{\gamma+\alpha}}. \quad (4.36)$$

This is a contradiction. Then we finish the proof. $\square$

Next we prove that the system (1.1) has a ground state solution for each $\beta > 0$. To this purpose we define

$$K_1 := \inf_{(u,v) \in \mathcal{N}_1} \mathcal{J}(u, v), \quad (4.37)$$

where

$$\mathcal{N}_1 = \{(u, v) \in D^\alpha \setminus (0, 0) : \mathcal{J}'(u, v)(u, v) = 0\}. \quad (4.38)$$

It is clear that $\mathcal{N} \subset \mathcal{N}_1$ and $K_1 \leq K$. Furthermore, one can check that $K_1 > 0$. In order to study the existence of ground state solution, we shall consider the system (1.1) in the bounded domain $B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$. Set $D_R^\alpha := D^\alpha(0, R) \times D^\alpha(0, R)$, where $D^\alpha(0, R)$ is similar to (1.14)-(1.17) when $\mathbb{R}^N$ is replaced by $B_R(0)$. Moreover, we require $u = 0$ if $u \in \partial B_R(0)$. Next we consider the system

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \mu_1 \phi_{u,R} |u|^{p-2} u + \beta \phi_{v,R} |u|^{p-2} u, & x \in B_R(0), \\ (-\Delta)^{\frac{\alpha}{2}} v = \mu_2 \phi_{v,R} |v|^{p-2} v + \beta \phi_{u,R} |v|^{p-2} v, & x \in B_R(0), \\ (u, v) \in D_R^\alpha, \end{cases} \quad (4.39)$$

where $\phi_{u,R}(x) = \int_{B_R(0)} \frac{|u(y)|^p}{|x-y|^{N-\gamma}} dy$ and $p = \frac{N+\gamma}{N-\alpha}$. We define

$$K(R) := \inf_{(u,v) \in \mathcal{N}(R)} \mathcal{J}(u, v), \quad (4.40)$$

where

$$\mathcal{N}(R) = \{(u, v) \in D_R^\alpha \setminus (0, 0) : G(u, v) = \mathcal{J}'(u, v)(u, v) = 0\}. \quad (4.41)$$

The next lemma proves $K(R) = K_1$ for each $R > 0$.

Lemma 4.5. *The conclusion $K(R) = K_1$ holds for each $R > 0$.*

Proof. We first prove that $K(R_1) = K(R_2)$ for each $R_1 > R_2$. In fact, it is easy to see that $\mathcal{N}(R_2) \subset \mathcal{N}(R_1)$. This implies $K(R_1) \leq K(R_2)$. Next we shall prove the reverse inequality. For each $(u_1, v_1) \in \mathcal{N}(R_1)$ we set

$$(u_2(x), v_2(x)) = \left(\frac{R_1}{R_2} \right)^{\frac{N-\alpha}{2}} \left(u_1 \left(\frac{R_1}{R_2} x \right), v_1 \left(\frac{R_1}{R_2} x \right) \right). \quad (4.42)$$

Then it follows that $(u_2, v_2) \in \mathcal{N}(R_2)$ and

$$K(R_2) \leq \mathcal{J}(u_2, v_2) = \mathcal{J}(u_1, v_1) \quad \text{for each } (u_1, v_1) \in \mathcal{N}(R_1). \quad (4.43)$$

That is, we obtain $K(R_2) \leq K(R_2)$ and $K(R_1) = K(R_2)$. Next we prove $K(R) = K_1$. It is easy to see that $K_1 \leq K(R)$. Let $\{(u_n, v_n)\} \subset \mathcal{N}_1$ be the minimizing sequence of K_1 . Without loss of generality we can assume that $(u_n, v_n) \in D_{R_n}^\alpha$ for some $R_n \rightarrow \infty$, as $n \rightarrow \infty$. Then we have $(u_n, v_n) \in \mathcal{N}(R_n)$ and

$$K_1 = \lim_{n \rightarrow \infty} \mathcal{J}(u_n, v_n) \geq \lim_{n \rightarrow \infty} K(R_n) = K(R). \quad (4.44)$$

This gives the conclusion $K(R) = K_1$ for each $R > 0$. $\square$

In order to overcome the lack of compactness we consider the following perturbation problem

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \mu_1 \phi_{u,R,\epsilon} |u|^{p-2-\epsilon} u + \beta \phi_{v,R,\epsilon} |u|^{p-2-\epsilon} u, & x \in B_R(0), \\ (-\Delta)^{\frac{\alpha}{2}} v = \mu_2 \phi_{v,R,\epsilon} |v|^{p-2-\epsilon} v + \beta \phi_{u,R,\epsilon} |v|^{p-2-\epsilon} v, & x \in B_R(0), \\ (u, v) \in D_R^\alpha, \end{cases} \quad (4.45)$$

where $\phi_{u,R,\epsilon}(x) = \int_{B_R(0)} \frac{|u(y)|^{p-\epsilon}}{|x-y|^{N-\gamma}} dy$ and $0 < \epsilon < p - 1$. Correspondingly, we define

$$K_\epsilon = \inf_{(u,v) \in \mathcal{N}_\epsilon} \mathcal{J}_\epsilon(u, v), \quad (4.46)$$

where

$$\begin{aligned} \mathcal{J}_\epsilon(u, v) = & \frac{1}{2} \left(|u|_{D^a(0, R)}^2 + |v|_{D^a(0, R)}^2 \right) \\ & - \frac{1}{2p - 2\epsilon} \int_{\mathbb{R}^N} (\mu_1 \phi_{u, R, \epsilon} |u|^{p-\epsilon} + \mu_2 \phi_{v, R, \epsilon} |v|^{p-\epsilon} + 2\beta \phi_{u, R, \epsilon} |v|^{p-\epsilon}), \end{aligned} \quad (4.47)$$

$$\mathcal{N}_\epsilon = \{(u, v) \in D_R^\alpha \setminus (0, 0) : \mathcal{J}'_\epsilon(u, v)(u, v) = 0\}.$$

In the next lemma we follow the idea of [10, 50] to give the estimates for K_ϵ .

Lemma 4.6. *For each $\epsilon < p - 1$, we have*

$$K_\epsilon < \min \left\{ \inf_{(u, 0) \in \mathcal{N}_\epsilon} \mathcal{J}_\epsilon(u, 0), \inf_{(0, v) \in \mathcal{N}_\epsilon} \mathcal{J}_\epsilon(0, v) \right\}. \quad (4.48)$$

Proof. For any $0 < \epsilon < p - 1$, we know that the equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \mu_i \phi_{u, R, \epsilon} |u|^{p-2-\epsilon} u, \quad u \in D^\alpha(0, R) \quad (4.49)$$

has a least energy solution u_i ($i = 1, 2$). Hence we know that

$$\mathcal{J}_\epsilon(u_1, 0) = d_1 = \inf_{(u, 0) \in \mathcal{N}_\epsilon} \mathcal{J}_\epsilon(u, 0) \quad \text{and} \quad \mathcal{J}_\epsilon(0, u_2) = d_2 = \inf_{(0, v) \in \mathcal{N}_\epsilon} \mathcal{J}_\epsilon(0, v). \quad (4.50)$$

We divide into the following two cases to prove our results. If $1 + \epsilon < p \leq 2$, the conclusion follows from [10, Lemma 2.7]. If $p > 2$, we define

$$\tilde{\mathcal{J}}_\epsilon(u, v) = \frac{\left(|u|_{D^a(0, R)}^2 + |v|_{D^a(0, R)}^2 \right)^2}{\int_{\mathbb{R}^N} (\mu_1 \phi_{u, R, \epsilon} |u|^{p-\epsilon} + \mu_2 \phi_{v, R, \epsilon} |v|^{p-\epsilon} + 2\beta \phi_{u, R, \epsilon} |v|^{p-\epsilon})}. \quad (4.51)$$

As in [50, Lemma 3.3], we know that

$$K_\epsilon = \inf_{(u, v) \in D_R^\alpha \setminus \{(0, 0)\}} \tilde{\mathcal{J}}_\epsilon(u, v). \quad (4.52)$$

Now we consider the function

$$\begin{aligned} f(s, t) &= \tilde{\mathcal{J}}_\epsilon(su_1, tu_2) \\ &= \frac{\left(s^2 |u_1|_{D^a(0, R)}^2 + t^2 |u_2|_{D^a(0, R)}^2 \right)^2}{\int_{\mathbb{R}^N} (\mu_1 s^{2p} \phi_{u_1, R, \epsilon} |u_1|^{p-\epsilon} + \mu_2 t^{2p} \phi_{u_2, R, \epsilon} |u_2|^{p-\epsilon} + 2s^p s^p \beta \phi_{u_1, R, \epsilon} |u_2|^{p-\epsilon})}, \end{aligned} \quad (4.53)$$

where $(s, t) \in \Lambda = \{(s, t) : s \geq 0, t \geq 0, (s, t) \neq (0, 0)\}$. Then it is sufficient to show that f does not attain its minimum over Λ on the lines $s = 0$ or $t = 0$. We infer from $p > 2$ that

$$f'_s(s, 0) < 0, \quad f'_s(0, t) = 0, \quad f'_t(0, t) < 0 \quad \text{and} \quad f'_t(s, 0) = 0. \quad (4.54)$$

Hence we know that f can not attain its minimum over Λ on the lines $s = 0$ or $t = 0$. This finishes the proof. $\square$

Recall that $(w_{\mu_1}, w_{\mu_2}) = \left(\mu_1^{\frac{\alpha-N}{2(\gamma+\alpha)}} U_1(x), \mu_2^{\frac{\alpha-N}{2(\gamma+\alpha)}} U_1(x) \right)$. By using the same arguments as in Lemma 4.6, we deduce that

$$\begin{aligned} K_1 &< \min \left\{ \inf_{(u, 0) \in \mathcal{N}_1} \mathcal{J}(u, 0), \inf_{(0, v) \in \mathcal{N}_1} \mathcal{J}(0, v) \right\} = \min \{ \mathcal{J}(w_{\mu_1}, 0), \mathcal{J}(0, w_{\mu_2}) \} \\ &= \min \left\{ \frac{\gamma + \alpha}{2(N + \gamma)} \mu_1^{\frac{\alpha-N}{\gamma+\alpha}} S^{\frac{N+\gamma}{\gamma+\alpha}}, \frac{\gamma + \alpha}{2(N + \gamma)} \mu_2^{\frac{\alpha-N}{\gamma+\alpha}} S^{\frac{N+\gamma}{\gamma+\alpha}} \right\}. \end{aligned} \quad (4.55)$$

Next we prove the existence of nontrivial solution of (4.45).

Lemma 4.7. For each $\epsilon < p - 1$, we know that (4.45) has a classical least energy solution (u_ϵ, v_ϵ) .

Proof. Let $\{(u_n, v_n)\} \subset \mathcal{N}_\epsilon$ be the minimizing sequence of K_ϵ . That is, $\mathcal{J}_\epsilon(u_n, v_n) \rightarrow K_\epsilon$ as $n \rightarrow \infty$. Hence we obtain that

$$K_\epsilon = \lim_{n \rightarrow \infty} \mathcal{J}_\epsilon(u_n, v_n) = \lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{2p - 2\epsilon} \right) \left(|u_n|_{D^\alpha(0, R)}^2 + |v_n|_{D^\alpha(0, R)}^2 \right). \quad (4.56)$$

Then $\{(u_n, v_n)\}$ is bounded in D_R^α . Passing to a subsequence, we may assume that $u_n \rightharpoonup u_\epsilon$, $v_n \rightharpoonup v_\epsilon$ weakly in $D^\alpha(0, R)$. By the compactness of the embedding $D^\alpha(0, R) \hookrightarrow L^{2p-2\epsilon}(B_R(0))$ (see [15, Corollary 7.2]), we infer from Hardy-Littlewood-Sobolev inequality (see Lemma 2.1) that

$$\begin{aligned} \frac{2p-2\epsilon}{p-1-\epsilon} K_\epsilon &= \frac{2p-2\epsilon}{p-1-\epsilon} \lim_{n \rightarrow \infty} \mathcal{J}_\epsilon(u_n, v_n) \\ &= \lim_{n \rightarrow \infty} \int_{B_R(0)} (\mu_1 \phi_{u_n, R, \epsilon} |u_n|^{p-\epsilon} + \mu_2 \phi_{v_n, R, \epsilon} |v_n|^{p-\epsilon} + 2\beta \phi_{u_n, R, \epsilon} |v_n|^{p-\epsilon}) \\ &= \int_{B_R(0)} (\mu_1 \phi_{u_\epsilon, R, \epsilon} |u_\epsilon|^{p-\epsilon} + \mu_2 \phi_{v_\epsilon, R, \epsilon} |v_\epsilon|^{p-\epsilon} + 2\beta \phi_{u_\epsilon, R, \epsilon} |v_\epsilon|^{p-\epsilon}). \end{aligned} \quad (4.57)$$

This implies that $(u_\epsilon, v_\epsilon) \neq (0, 0)$. Furthermore, we infer from Fatou's lemma that

$$\begin{aligned} (|u_\epsilon|_{D^\alpha(0, R)}^2 + |v_\epsilon|_{D^\alpha(0, R)}^2) &\leq \lim_{n \rightarrow \infty} (|u_n|_{D^\alpha(0, R)}^2 + |v_n|_{D^\alpha(0, R)}^2) \\ &= \int_{B_R(0)} (\mu_1 \phi_{u_\epsilon, R, \epsilon} |u_\epsilon|^{p-\epsilon} + \mu_2 \phi_{v_\epsilon, R, \epsilon} |v_\epsilon|^{p-\epsilon} + 2\beta \phi_{u_\epsilon, R, \epsilon} |v_\epsilon|^{p-\epsilon}). \end{aligned} \quad (4.58)$$

Therefore, it is easy to check that there exists $0 < t_\epsilon \leq 1$ such that $(t_\epsilon u_\epsilon, t_\epsilon v_\epsilon) \in \mathcal{N}_\epsilon$. Then we infer that

$$\begin{aligned} K_\epsilon &\leq \mathcal{J}_\epsilon(t_\epsilon u_\epsilon, t_\epsilon v_\epsilon) = \frac{p-1-\epsilon}{2p-2\epsilon} t_\epsilon^2 (|u_\epsilon|_{D^\alpha(0, R)}^2 + |v_\epsilon|_{D^\alpha(0, R)}^2) \\ &\leq \lim_{n \rightarrow \infty} (|u_n|_{D^\alpha(0, R)}^2 + |v_n|_{D^\alpha(0, R)}^2) = \lim_{n \rightarrow \infty} \mathcal{J}_\epsilon(u_n, v_n) = K_\epsilon. \end{aligned} \quad (4.59)$$

Then we get that $t_\epsilon = 1$ and $\mathcal{J}_\epsilon(u_\epsilon, v_\epsilon) = K_\epsilon$. Furthermore, we have

$$\lim_{n \rightarrow \infty} (|u_n|_{D^\alpha(0, R)}^2 + |v_n|_{D^\alpha(0, R)}^2) = (|u_\epsilon|_{D^\alpha(0, R)}^2 + |v_\epsilon|_{D^\alpha(0, R)}^2). \quad (4.60)$$

That is, $(u_n, v_n) \rightarrow (u_\epsilon, v_\epsilon)$ in D_R^α . As in [53, Lemma 2.3], one can prove that the Nehari type constraint set $\mathcal{N}(R)$ is a natural constraint. That is, (u_ϵ, v_ϵ) is a solution of (4.45). Furthermore, we infer from Lemma 4.6 that $u_\epsilon \neq 0$ and $v_\epsilon \neq 0$. By regularity theory (see [25]), we see that $u_\epsilon, v_\epsilon \in C^\alpha(B_R(0))$. $\square$

Now we are ready to give the proof Theorem 1.4 (ii).

Proof of Theorem 1.4 (ii). It is clear that for each $(u, v) \in \mathcal{N}(1)$, there exists $t_\epsilon > 0$ such that $(t_\epsilon u, t_\epsilon v) \in \mathcal{N}_\epsilon$ and $t_\epsilon \rightarrow 1$ as $\epsilon \rightarrow 0$. Hence we infer that

$$\limsup_{\epsilon \rightarrow 0} K_\epsilon \leq \limsup_{\epsilon \rightarrow 0} \mathcal{J}_\epsilon(t_\epsilon u, t_\epsilon v) = \mathcal{J}(u, v). \quad (4.61)$$

Then we infer from Lemma 4.5 that

$$\limsup_{\epsilon \rightarrow 0} K_\epsilon \leq K(1) = K_1. \quad (4.62)$$

Let (u_ϵ, v_ϵ) be the least energy solution of (1.1). Then we have $\mathcal{J}'_\epsilon(u_\epsilon, v_\epsilon)(u_\epsilon, v_\epsilon) = 0$. From Hardy-Littlewood-Sobolev inequality and Sobolev inequality, we deduce that

$$\frac{2p-2\epsilon}{p-1-\epsilon} K_\epsilon = |u_\epsilon|_{D^\alpha(0, R)}^2 + |v_\epsilon|_{D^\alpha(0, R)}^2 \geq \sigma_0 \quad \text{for each } 0 < \epsilon < \frac{p-1}{4}, \quad (4.63)$$

where σ_0 is a positive constant independent of ϵ . Then we know that (u_ϵ, v_ϵ) is uniformly bounded in $D^\alpha(0, R)$. Passing to a subsequence, we may assume that $u_\epsilon \rightharpoonup u$ and $v_\epsilon \rightharpoonup v$ weakly in $D^\alpha(0, R)$. Then (u, v) is a solution of

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \mu_1 \phi_{u,1} |u|^{p-2} u + \beta \phi_{v,1} |u|^{p-2} u, & x \in B_1(0), \\ (-\Delta)^{\frac{\alpha}{2}} v = \mu_2 \phi_{v,1} |v|^{p-2} v + \beta \phi_{u,1} |v|^{p-2} v, & x \in B_1(0), \\ (u, v) \in D_1^\alpha. \end{cases} \quad (4.64)$$

Next we divide into the following two cases to prove the results. That is, $|u_\epsilon|_\infty + |v_\epsilon|_\infty$ is bounded, or $|u_\epsilon|_\infty + |v_\epsilon|_\infty \rightarrow \infty$ as $\epsilon \rightarrow 0$.

If $|u_\epsilon|_\infty + |v_\epsilon|_\infty$ is bounded, we infer from the Dominated Convergence Theorem that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{B_1(0)} \phi_{u_\epsilon,1,\epsilon} |u_\epsilon|^{p-\epsilon} &= \int_{B_1(0)} \phi_{u,1} |u|^p, \quad \lim_{\epsilon \rightarrow 0} \int_{B_1(0)} \phi_{v_\epsilon,1,\epsilon} |v_\epsilon|^{p-\epsilon} = \int_{B_1(0)} \phi_{v,1} |v|^p, \\ \text{and } \lim_{\epsilon \rightarrow 0} \int_{B_1(0)} \phi_{u_\epsilon,1,\epsilon} |v_\epsilon|^{p-\epsilon} &= \int_{B_1(0)} \phi_{u,1} |v|^p. \end{aligned} \quad (4.65)$$

Moreover, since $\mathcal{J}'_\epsilon(u_\epsilon, v_\epsilon) = \mathcal{J}'(u, v) = 0$, it follows that $u_\epsilon \rightarrow u$ and $v_\epsilon \rightarrow v$ in $D^\alpha(0, R)$. Moreover, we infer from (4.63) that $(u, v) \neq (0, 0)$. Without loss of generality we assume that $u \neq 0$. From [41, Proposition 3.1] and [45, Proposition 1.6], we infer that the system (4.39) has following Pohozaev identity

$$\begin{aligned} & \frac{\alpha - N}{2} \int_{B_1(0)} (\mu_1 \phi_{u,1} |u|^p + \mu_2 \phi_{u,1} |u|^p + 2\beta \phi_{u,1} |u|^p) \\ & - \frac{\Gamma(\frac{\alpha+2}{2})}{2} \int_{\partial B_1(0)} \left(\left(\frac{u}{\delta} \right)^2 + \left(\frac{v}{\delta} \right)^2 \right) x \cdot \nu d\sigma \\ & = -\frac{N+\gamma}{2p} \int_{B_1(0)} (\mu_1 \phi_{u,1} |u|^p + \mu_2 \phi_{u,1} |u|^p + 2\beta \phi_{u,1} |u|^p), \end{aligned} \quad (4.66)$$

where $\delta(s) = \text{dist}(x, \partial B_1(0))$, ν is the unit outward normal vector to $\partial B_R(0)$ at x , and Γ is the Gamma function. Hence we infer that

$$0 < \frac{\Gamma(\frac{\alpha+2}{2})}{2} \int_{\partial B_1(0)} \left(\left(\frac{u}{\delta} \right)^2 + \left(\frac{v}{\delta} \right)^2 \right) x \cdot \nu d\sigma = 0. \quad (4.67)$$

This is a contradiction. Hence we have $|u_\epsilon|_\infty + |v_\epsilon|_\infty \rightarrow \infty$ as $\epsilon \rightarrow 0$. In the following we shall use the blowup arguments. To this purpose we denote $\lambda_\epsilon = \{|u_\epsilon|_\infty, |v_\epsilon|_\infty\}$. Then $\lambda_\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0$. We define

$$U_\epsilon(x) = \lambda_\epsilon^{-1} u_\epsilon(\lambda_\epsilon^{-\chi_\epsilon} x) \quad \text{and} \quad V_\epsilon(x) = \lambda_\epsilon^{-1} v_\epsilon(\lambda_\epsilon^{-\chi_\epsilon} x), \quad \text{where } \chi_\epsilon = \frac{2(p-1-\epsilon)}{\gamma+\alpha}. \quad (4.68)$$

Thus, we have that

$$1 = \max\{|U_\epsilon|_\infty, |V_\epsilon|_\infty\} \quad (4.69)$$

and $(U_\epsilon(x), V_\epsilon(x))$ satisfies

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U_\epsilon = \mu_1 \phi_{U_\epsilon,R,\epsilon} |U_\epsilon|^{p-2-\epsilon} U_\epsilon + \beta \phi_{V_\epsilon,R,\epsilon} |U_\epsilon|^{p-2-\epsilon} U_\epsilon, & x \in B_{\lambda_\epsilon^{\chi_\epsilon}}(0), \\ (-\Delta)^{\frac{\alpha}{2}} V_\epsilon = \mu_2 \phi_{V_\epsilon,R,\epsilon} |V_\epsilon|^{p-2-\epsilon} V_\epsilon + \beta \phi_{U_\epsilon,R,\epsilon} |V_\epsilon|^{p-2-\epsilon} V_\epsilon, & x \in B_{\lambda_\epsilon^{\chi_\epsilon}}(0). \end{cases} \quad (4.70)$$

A direct computation shows that

$$|U_\epsilon|_{D^{1,\alpha}}^2 = \lambda_\epsilon^{-(N-\alpha)\epsilon} |u_\epsilon|_{D^{1,\alpha}}^2 \leq |u_\epsilon|_{D^{1,\alpha}}^2. \quad (4.71)$$

Thus, we infer from (4.62)-(4.63) that (U_ϵ, V_ϵ) is bounded in D^α . By regularity results for fractional nonlocal problems(see [25]), for a subsequence we have $U_\epsilon \rightarrow U$ and $V_\epsilon \rightarrow V$ in C_{loc}^α as $\epsilon \rightarrow 0$. Moreover, since $|U_\epsilon|_\infty, |V_\epsilon|_\infty \leq 1$, it follows that $\phi_{U_\epsilon,R,\epsilon}(x) \rightarrow \phi_{U,R}(x)$ and $\phi_{V_\epsilon,R,\epsilon}(x) \rightarrow \phi_{V,R}(x)$ in C_{loc}^α as $\epsilon \rightarrow 0$. Thus we

know that (U, V) satisfies (1.1). From (4.69) we infer that $(U, V) \neq (0, 0)$ and $(U, V) \in \mathcal{N}_1$. Then we infer from (4.61) that

$$\begin{aligned} K_1 &\leq \mathcal{J}(U, V) = \frac{p-1}{2p} \left(|U|_{D^{1,\alpha}}^2 + |U|_{D^{1,\alpha}}^2 \right) \\ &\leq \liminf_{\epsilon \rightarrow 0} \frac{p-1-\epsilon}{2p-2\epsilon} \left(|U_n|_{D^{\alpha}(0, \lambda_\epsilon^\epsilon)}^2 + |V_n|_{D^{\alpha}(0, \lambda_\epsilon^\epsilon)}^2 \right) \\ &\leq \liminf_{\epsilon \rightarrow 0} \frac{p-1-\epsilon}{2p-2\epsilon} \left(|u_n|_{D^{\alpha}(0,1)}^2 + |u_n|_{D^{\alpha}(0,1)}^2 \right) = \liminf_{\epsilon \rightarrow 0} K_\epsilon \leq K_1. \end{aligned} \quad (4.72)$$

This together with (4.55) imply that $K_1 = \mathcal{J}(U, V)$ and $U, V \neq 0$. Also, we know that $(U, V) \in \mathcal{N}$ and $\mathcal{J}(U, V) \geq K$. Thus, we have $K_1 = \mathcal{J}(U, V) = K$ and (U, V) is nontrivial ground state solution of (1.1). $\square$

Next we find the unique solution of (1.21) for $|\beta|$ small.

Lemma 4.8. *There exists $\tilde{\beta}_0 > 0$ small such that $(k(\beta), l(\beta))$ is the unique solution of (1.21) for $0 < \beta < \tilde{\beta}_0$. Moreover, we have*

$$\lim_{\beta \rightarrow 0} (k^2(\beta) + l^2(\beta)) = k^2(0) + l^2(0) = \mu_1^{-\frac{N-\alpha}{\gamma+\alpha}} + \mu_2^{-\frac{N-\alpha}{\gamma+\alpha}}.$$

Proof. We shall use the implicit function theorem to prove the existence of $(k(\beta), l(\beta))$ for $\beta > 0$ small. For convenience we let $g_i(k, l, \beta)$ to denote $g_i(k, l)$ ($i = 1, 2$). We define $k(0) = \mu_1^{-\frac{1}{2(p-1)}}$ and $l(0) = \mu_2^{-\frac{1}{2(p-1)}}$. Thus we have $g_i(k(0), l(0), 0) = 0$ ($i = 1, 2$). Moreover, a direct computation shows that

$$\begin{aligned} \partial_k g_1(k(0), l(0), 0) &= 2(p-1)\mu_1 k(0)^{2p-3} > 0, \quad \partial_l g_2(k(0), l(0), 0) = 2(p-1)\mu_2 l(0)^{2p-3} > 0, \\ \partial_l g_1(k(0), l(0), 0) &= \partial_k g_2(k(0), l(0), 0) = 0. \end{aligned} \quad (4.73)$$

This implies that

$$\det \begin{pmatrix} \partial_k g_1(k(0), l(0), 0) & \partial_l g_1(k(0), l(0), 0) \\ \partial_k g_2(k(0), l(0), 0) & \partial_l g_2(k(0), l(0), 0) \end{pmatrix} > 0. \quad (4.74)$$

Then we apply the implicit function theorem to obtain that $k(\beta), l(\beta)$ are well defined and of class C^1 on $(-\tilde{\beta}_0, \tilde{\beta}_0)$ for some $\tilde{\beta}_0 > 0$, and $g_i(k(\beta), l(\beta), \beta) = 0$ ($i = 1, 2$). Hence we know that $(k(\beta)U_1, l(\beta)U_1)$ is a positive solution of (1.1). Furthermore, we get

$$\lim_{\beta \rightarrow 0} (k^2(\beta) + l^2(\beta)) = k^2(0) + l^2(0) = \mu_1^{-\frac{N-\alpha}{\gamma+\alpha}} + \mu_2^{-\frac{N-\alpha}{\gamma+\alpha}}. \quad (4.75)$$

Thus, there exists $0 < \beta_0 < \tilde{\beta}_0$ such that

$$k^2(\beta) + l^2(\beta) > \min \left\{ \mu_1^{-\frac{N-\alpha}{\gamma+\alpha}}, \mu_2^{-\frac{N-\alpha}{\gamma+\alpha}} \right\} \quad \text{for } \beta \in (0, \beta_0). \quad (4.76)$$

We infer from (4.79) and (4.55) that

$$\mathcal{J}(U, V) = K_1 = K < \mathcal{J}(k(\beta)U_1, l(\beta)U_1), \quad \forall \beta \in (0, \beta_0). \quad (4.77)$$

This implies that $(k(\beta)U_1, l(\beta)U_1)$ is different positive solution of (1.1) from (U, V) . This completes the proof. $\square$

Then from Lemma 4.8 we know that Theorem 1.4 (iii) holds. Finally, we prove the Theorem 1.4 (iv).

Proof of Theorem 1.4 (iv). We first consider the case $2\alpha + \gamma > N$. It is clear that $(k_0(\beta), l_0(\beta)) = (k(\beta), l(\beta))$ for $\beta > 0$ small. From (4.2) we infer that $k(\beta), l(\beta)$ are strictly decreasing for $\beta > 0$. Hence we know that there exists $\beta_1 > \beta_0$ such that

$$k^2(\beta) + l^2(\beta) < \min \left\{ \mu_1^{-\frac{N-\alpha}{\gamma+\alpha}}, \mu_2^{-\frac{N-\alpha}{\gamma+\alpha}} \right\} \quad \text{for } \beta \in (\beta_1, +\infty). \quad (4.78)$$

Next we prove that the solution $(k_0 U_1, l_0 U_1)$ is a ground state solution of (1.1) for $\beta > \beta_1$. In fact, since $(k_0 U_1, l_0 U_1)$ is a nontrivial solution of (1.1), it follows that

$$K \leq \mathcal{J}(k_0 U_1, l_0 U_1) = \frac{\gamma + \alpha}{2(N + \gamma)} \left(k_0^2 + l_0^2 \right) S^{\frac{N+\gamma}{\gamma+\alpha}}. \quad (4.79)$$

Let $\{(u_n, v_n)\}$ be a minimizing sequence for K . Then we infer from (1.16) that

$$\begin{aligned} S e_n &\leq |u_n|_{D^{1,\alpha}}^2 = \mu_1 \int_{\mathbb{R}^N} \phi_{u_n} |u_n|^p + \beta \int_{\mathbb{R}^N} \phi_{u_n} |v_n|^p \leq \mu_1 e_n^p + \beta e_n^{\frac{p}{2}} f_n^{\frac{p}{2}}, \\ S f_n &\leq |v_n|_{D^{1,\alpha}}^2 = \mu_2 \int_{\mathbb{R}^N} \phi_{v_n} |v_n|^p + \beta \int_{\mathbb{R}^N} \phi_{v_n} |u_n|^p \leq \mu_2 f_n^p + \beta e_n^{\frac{p}{2}} f_n^{\frac{p}{2}}, \end{aligned} \quad (4.80)$$

where

$$e_n = \left(\int_{\mathbb{R}^N} \phi_{u_n} |u_n|^p \right)^{\frac{1}{p}} \quad \text{and} \quad f_n = \left(\int_{\mathbb{R}^N} \phi_{v_n} |v_n|^p \right)^{\frac{1}{p}}. \quad (4.81)$$

This implies that

$$\begin{aligned} S(e_n + f_n) &\leq \frac{2(N + \gamma)}{\gamma + \alpha} \mathcal{J}(u_n, v_n) \leq \left(k_0^2 + l_0^2 \right) S^{\frac{N+\gamma}{\gamma+\alpha}}, \\ S &\leq \mu_1 e_n^{p-1} + \beta e_n^{\frac{p}{2}-1} f_n^{\frac{p}{2}}, \quad S \leq \mu_2 f_n^{p-1} + \beta f_n^{\frac{p}{2}-1} e_n^{\frac{p}{2}}. \end{aligned} \quad (4.82)$$

The first inequality implies that e_n, f_n are bounded. Then we can assume that $e_n \rightarrow e, f_n \rightarrow f$, as $n \rightarrow \infty$. From (1.20), we deduce that $\mu_1 e^p + 2\beta e^{\frac{p}{2}} f^{\frac{p}{2}} + \mu_2 f^p \geq \frac{2(N+\gamma)}{\gamma+\alpha} K > 0$. Thus, at least one of e, f is nonzero. Assume that $e > 0$ and $f = 0$. Since $2\alpha + \gamma > N$, we know that $p > 2$. Hence we infer from the third inequality of (4.82) that $S \leq 0$. This is a contradiction. Thus, we have $e > 0$ and $f > 0$. Set $k = \left(e / S^{\frac{N-\alpha}{\gamma+\alpha}} \right)^{\frac{1}{2}}$ and $l = \left(f / S^{\frac{N-\alpha}{\gamma+\alpha}} \right)^{\frac{1}{2}}$. Let $n \rightarrow \infty$, we know that (4.82) becomes (4.8). This implies that $e_n \rightarrow k_0^2 S^{\frac{N-\alpha}{\gamma+\alpha}}$ and $f_n \rightarrow l_0^2 S^{\frac{N-\alpha}{\gamma+\alpha}}$, as $n \rightarrow \infty$. We infer from (4.82) that

$$\lim_{n \rightarrow \infty} \mathcal{J}(u_n, v_n) \geq \left(k_0^2 + l_0^2 \right) S^{\frac{N+\gamma}{\gamma+\alpha}}. \quad (4.83)$$

Combining (4.79) and (4.83), we know that

$$K = \frac{\gamma + \alpha}{2(N + \gamma)} \left(k_0^2 + l_0^2 \right) S^{\frac{N+\gamma}{\gamma+\alpha}}. \quad (4.84)$$

This finishes the proof of the case $2\alpha + \gamma > N$ and $0 < \beta \leq \sqrt{\mu_1 \mu_2}$.

For the case $2\alpha + \gamma = N$ (i.e., $p = 2$) and $\beta \in (0, \min\{\mu_1, \mu_2\}) \cap (\max\{\mu_1, \mu_2\}, \infty)$, by using the same arguments as in Lemma 4.2 we know that $\left(\sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}}, \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}} \right)$ is the unique solution of (1.21). Hence as in the above one can prove that $\left(\sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}} U_1, \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}} U_1 \right)$ is a ground state solution of (1.1). Similarly, one can prove the conclusions for the case $2\alpha + \gamma < N$ and $\beta \geq (p - 1) \max\{\mu_1, \mu_2\}$. $\square$

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