Research Article

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Critical nonlocal Schrödinger-Poisson system on the Heisenberg group

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Abstract: In this paper, we are concerned with the following a new critical nonlocal Schrödinger-Poisson system on the Heisenberg group:

$$\begin{cases} -(a - b \int_{\Omega} |\nabla_H u|^2 d\xi) \Delta_H u + \mu \phi u = \lambda |u|^{q-2} u + |u|^2 u, & \text{in } \Omega, \\ -\Delta_H \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where Δ_H is the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$, and $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $a, b > 0$, $1 < q < 2$ or $2 < q < 4$, $\lambda > 0$ and $\mu \in \mathbb{R}$ are some real parameters. Existence and multiplicity of solutions are obtained by an application of the mountain pass theorem, the Ekeland variational principle, the Krasnoselskii genus theorem and the Clark critical point theorem, respectively. However, there are several difficulties arising in the framework of Heisenberg groups, also due to the presence of the non-local coefficient $(a - b \int_{\Omega} |\nabla_H u|^2 dx)$ as well as critical nonlinearities. Moreover, our results are new even on the Euclidean case.

Keywords: Heisenberg group; Schrödinger-Poisson type system; Mountain pass theorem; Krasnoselskii genus theorem; Clark critical point theorem

MSC: 35J20; 35R03; 46E35

1 Introduction and main results

Let $\mathbb{H}^1$ be the Heisenberg Lie group of topological dimension 3, that is the Lie group which has $\mathbb{R}^3$ as a back-ground manifold, endowed with the non-Abelian group law

$$\tau : \mathbb{H}^1 \rightarrow \mathbb{H}^1, \tau_{\xi}(\xi') = \xi \circ \xi',$$

where

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(x'y - y'x)), \quad \forall \xi, \xi' \in \mathbb{H}^1.$$

The inverse is given by $\xi^{-1} = -\xi$, so that $(\xi \circ \xi')^{-1} = (\xi')^{-1} \circ \xi^{-1}$.

For $s > 0$, a natural group of dilation on $\mathbb{H}^1$ is defined by $\delta_s(\xi) = (sx, sy, s^2t)$. Hence, $\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi)$. For all $\xi = (x, y, t) \in \mathbb{H}^1$, it is easy to verify that the Jacobian determinant of dilatations $\delta_s : \mathbb{H}^1 \rightarrow \mathbb{H}^1$ is constant and equal to s^4 . This is why the natural number $Q = 4$ is called homogeneous dimension of $\mathbb{H}^1$.

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and critical exponents $Q^* := \frac{2Q}{Q-2} = 4$. For $\xi \in \mathbb{H}^1$, $|\xi|_H$ is defined as

$$|\xi|_H = \left[(x^2 + y^2)^2 + t^2 \right]^{\frac{1}{4}}, \quad \xi \in \mathbb{H}^1.$$

It is also the Korányi norm. Although the Korányi distance dose not reflect the sub-Riemannian structure of the Heisenberg group, the calculation is relatively simple. The Kohn Laplacian Δ_H on $\mathbb{H}^1$ is defined as

$$\Delta_H u = \operatorname{div}_H(\nabla_H u),$$

where

$$\nabla_H u = (X, Y).$$

The Lie algebra of left invariant vector fields is generated by the following vector fields

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

The left-invariant distance d_H on $\mathbb{H}^1$ is accordingly defined by

$$d_H(\xi_0, \xi) = |\xi^{-1} \circ \xi_0|_H.$$

It is well known that Δ_H is a very degenerate elliptic operator and Bony's maximum principle is satisfied (see [5]).

In the paper, statements involving measure theory are always understood to be with respect to Haar measure on $\mathbb{H}^1$, which coincides with the L^3 dimensional Lebesgue measure, because the Lebesgue measure on $\mathbb{R}^3$ is invariant under left translations. Then

$$|B_H(\xi_0, r)| = \alpha_Q r^Q,$$

where $B_H(\xi_0, r)$ is the Heisenberg ball of radius r centered at ξ_0 , that is

$$B_H(\xi_0, r) = \left\{ \xi \in \mathbb{H}^1 : d_H(\xi_0, \xi) < r \right\}$$

and $\alpha_Q = |B_H(0, 1)|$. It implies $\tau_\xi(B_r(0)) = B_r(\xi)$ and $\delta_r(B_1(0)) = B_r(0)$. As a consequence, for every $0 \leq a < b$ and for every measurable function $f : [a, b] \rightarrow \mathbb{R}$, we have

$$\int_{B_d(0,b) \setminus B_d(0,a)} f(d_0(\xi)) d\xi = q |B_d(0, 1)| \int_b^a f(r) r^{q-1} dr,$$

if at least one of the integrals exists. For a complete treatment on the Heisenberg group functional setting, we refer to [18, 19, 26, 30].

In recent years, the geometric analysis of the Heisenberg group has received the attention of many scholars due to its important applications in quantum mechanics, number theory, partial differential equations and other fields. Analysis on the Heisenberg group is very interesting because this space is topologically Euclidean, but analytically non-Euclidean, so some basic ideas of analysis, such as dilatations, must be developed again (see [32]). The first mathematicians who study subelliptic analysis on the Heisenberg group were Folland and Stein in [15], who consistently created a generalisation of the analysis for more general stratified groups [16]. And it can also be noted that Rothschild and Stein generalised these results for general vector fields satisfying the Hormander's conditions. These results were published in the famous book by Folland and Stein [17] which laid the anisotropic analysis. And it is worth noting that homogeneous Lie group is nilpotent.

In this paper, we focus on the existence and multiplicity of nontrivial solutions for a new nonlocal Schrödinger-Poisson type system on the Heisenberg group of the form:

$$\begin{cases} -(a - b \int_{\Omega} |\nabla_H u|^2 d\xi) \Delta_H u + \mu \phi u = \lambda |u|^{q-2} u + |u|^2 u, & \text{in } \Omega, \\ -\Delta_H \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Δ_H is the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$, and $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $1 < q < 2$ or $2 < q < 4$, $\lambda > 0$ and $\mu \in \mathbb{R}$ are some real parameters.

The paper was motivated by some works appearing in recent years. Note that there are already a number of results related to the existence of solutions to the semilinear equations starting from the pioneering works on the Heisenberg group [6, 39]. However, there are few results for the critical problem on the Heisenberg group. The critical problem on the Euclidean case was initially studied in the seminal paper of Brezis and Nirenberg [8], which treated Laplace equations, and then there have been extensions of [8] in many directions. Elliptic equations involving critical growth are delicate due to the lack of compactness arising in connection with the variational approach. For such problems, the concentration-compactness principles introduced by P.L. Lions in [24, 25] and its variants at infinity [3, 4, 9] have played a decisive role in showing that a minimizing sequence or a Palais-Smale sequence is precompact. Moreover, the concentration-compactness principle on the vectorial Heisenberg context was established in [33], and they consider the following (p, q) critical systems on the Heisenberg group:

$$\begin{cases} -\operatorname{div}(A(|D_H u|_H)|D_H u|) + B(|u|)u = \lambda H_u(u, v) + \frac{\alpha}{\wp} |v|^\beta |u|^{\alpha-2} u, \\ -\operatorname{div}(A(|D_H v|_H)|D_H v|) + B(|v|)v = \lambda H_v(u, v) + \frac{\alpha}{\wp} |v|^\alpha |u|^{\beta-2} v, \end{cases} \quad (1.2)$$

the existence of entire nontrivial solutions are obtained by using the variational methods. In [32], they complete the results in [33] and deal with some kind of elliptic systems involving critical nonlinearities and Hardy terms on the Heisenberg group. For more conclusions, please refer to [36, 37].

In [2], the authors studied the following Schrödinger-Poisson type system on the Heisenberg group:

$$\begin{cases} -\Delta_H u + \mu \phi u = \lambda |u|^{q-2} u + |u|^2 u, & \text{in } \Omega, \\ -\Delta_H \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $1 < q < 2$, $\lambda > 0$ and $\mu \in \mathbb{R}$ are some real parameters. They obtained at least two positive solutions and a positive ground state solution by using the concentration compactness and the critical point theory. On some recent results recovering the Heisenberg group, we refer to [7, 27, 28, 33–35] and the references therein.

On the other hand, the study of Kirchhoff-type problems, which arise in various models of physical and biological systems, have received more and more attention in recent years. In [43], the authors develop the fractional Trudinger-Moser inequality in singular case and use it to study the existence and multiplicity of solutions for a class of perturbed fractional Kirchhoff type problems. In [31], the authors provide an overview of recent results concerning elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators. For more related literature, we refer the reader to [10, 13, 14, 21–23, 42]. In this paper, we mainly consider a new type of nonlocal Schrödinger-poisson equation, that is, the form with a non-local coefficient $(a - b \int_\Omega |\nabla_H u|^2 dx)$. Its background is derived from the negative Young's modulus, when the atoms are pulled apart rather than compressed together and the strain is negative. Recently, the authors in [44] first studied this kind of problem

$$\begin{cases} -(a - b \int_\Omega |\nabla u|^2 dx) \Delta u = |u|^{p-2} u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $2 < p < 2^*$, and they obtained the existence of solutions by using the mountain pass lemma.

In [40], the authors researched the problem:

$$\begin{cases} -(a - b \int_\Omega |\nabla u|^2 dx) \Delta u = |u|^2 u + \mu f(x), & \text{in } \mathbb{R}^4, \\ u \in D^{1,2}(\mathbb{R}^4), \end{cases}$$

where a, b are positive constants, μ is a parameter, $f(x) \in L^{\frac{4}{3}}(\mathbb{R}^4)$ is a non-negative function and they obtained the existence and multiplicity of solutions for the above problem. Furthermore, some interesting results have been obtained for this kind of Kirchhoff-type problem with sublinear terms. We refer the readers to [12, 29, 40] and the references therein.

Inspired by the works in the above references, our main purpose in this paper is to study the existence and multiplicity of solutions for problem (1.1) with sublinear terms or superlinear terms, that is, $1 < q < 2$ or $2 < q < 4$. To the best of our knowledge, this paper is the first to deal with a new critical nonlocal Schrödinger-Poisson type system on the Heisenberg group, and our results are new even on the Euclidean case.

Our main results of this paper involving superlinear and critical terms are stated as follows:

Theorem 1.1. *Let $2 < q < 4$ be satisfied. Then, there exist $\lambda_1 > 0$ and $\mu^* > 0$ such that as $\lambda > \lambda_1$ and $\mu < \mu^*$, problem (1.1) has a positive energy solution.*

Theorem 1.2. *Let $2 < q < 4$ be satisfied. Then, there exists $\mu^* > 0$ such that as $\mu < \mu^*$ and suppose one of the following conditions holds:*

(i) *there exists a positive constant $a^* > 0$ such that for each $a > a^*$, $\lambda > 0$ and $b > 0$;*

(ii) *there exists a positive constant $\lambda_2 > 0$ such that, for all $\lambda > \lambda_2$, $a > 0$ and $b > 0$.*

Problem (1.1) has at least k pairs of nontrivial weak solutions.

We also obtain the following results for problem (1.1) involving sublinear and critical terms.

Theorem 1.3. *Let $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda^* > 0$ such that for $\lambda \in (0, \Lambda^*)$, problem (1.1) has two positive solutions and a positive ground state solution.*

Theorem 1.4. *Let $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda > 0$ such that for $\lambda \in (0, \Lambda)$, problem (1.1) has at least $n - m$ pairs of solutions.*

Remark 1.1. *We point out that problem (1.1) contains nonlocal terms ϕu , nonlocal coefficient $(a - b \int_{\Omega} |\nabla u|^2 dx)$ and critical nonlinearity. There is no doubt that we encounter serious difficulties because of the lack of compactness, which makes it more difficult to give the accurate threshold of c (see Lemmas 3.2 and 3.3). Moreover, although some properties are similar between Kohn Laplacian Δ_H and the classical Laplacian Δ , the similarities may be deceitful; see, e.g., [18]. In addition, the critical exponent Q^* is equal to 4 on $\mathbb{H}^1$ while 2^* is equal to 6 on $\mathbb{R}^3$ which has created us some obstacles in proving the existence of solutions to problem (1.1). In order to overcome these difficulties, we will use some more accurate estimates for related expressions.*

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge in the Heisenberg group functional setting. In Section 3, we prove the Palais-Smale compactness condition. In Section 4, we prove Theorem 1.1 and Theorem 1.2 via the mountain pass theorem and Krasnoselskii's genus theory. In Section 5, we prove Theorem 1.3 and Theorem 1.4 by using the Ekeland's variational principle, mountain pass theorem and Clark critical point theorem.

2 Preliminaries

In this section, we collect some notations and some known facts which will be useful later on. From now on, let $C, C_0, C_1, C_2, \dots$ denote various positive constants, which may vary from line to line.

The Folland-Stein space $S_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2 = \|u\|_{S_0^1(\Omega)}^2 = \int_{\Omega} |\nabla_H u|^2 d\xi.$$

Let

$$\|u\|_p^p = \int_{\Omega} |u|^p d\xi, \quad \forall u \in L^p(\Omega),$$

denotes the usual L^p -norm. We denote by B_ρ the closed ball of radius ρ centered at zero in the Folland-Stein space $S_0^1(\Omega)$, and by S_ρ its relative boundary, that is,

$$B_\rho = \{u \in S_0^1(\Omega) : \|u\| \leq \rho\}, \quad S_\rho = \{u \in S_0^1(\Omega) : \|u\| = \rho\}.$$

By [15], we know that the Folland-Stein space is a Hilbert space and the embedding

$$S_0^1(\Omega) \hookrightarrow L^p(\Omega) \quad \text{for } p \in [1, Q^*),$$

is compact. While it is only continuous if $p = Q^*$, the best Sobolev constant

$$S = \inf_{u \in S_0^1(\mathbb{H}^1) \setminus \{0\}} \frac{\int_{\mathbb{H}^1} |\nabla_H u|^2 d\xi}{\left(\int_{\mathbb{H}^1} |u|^{Q^*} d\xi\right)^{\frac{2}{Q^*}}}, \quad (2.1)$$

is achieved by the C^∞ function

$$U(x, y, t) = \frac{c_0}{\sqrt{(1 + x^2 + y^2)^2 + t^2}},$$

where c_0 is a suitable positive constant (see [20]). On the other hand, the function U is a positive solution of the following equation:

$$-\Delta_H u = u^3, \quad u \in S_0^1(\mathbb{H}^1) \quad (2.2)$$

and satisfies

$$\int_{\mathbb{H}^1} |\nabla_H U|^2 d\xi = \int_{\mathbb{H}^1} |U|^4 d\xi = S^2.$$

Set

$$u_\varepsilon(\xi) = \varphi(\xi) U_\varepsilon(\xi) = \frac{c_0 \varepsilon \varphi(\xi)}{\sqrt{(\varepsilon + x^2 + y^2)^2 + t^2}}, \quad (2.3)$$

where $\varphi \in C_0^\infty(B_H(0, r_0))$, $0 \leq \varphi \leq 1$ and $\varphi = 1$ in $B_H(0, \frac{r_0}{2})$ (see [2]).

Let us first consider the following problem:

$$\begin{cases} -\Delta_H \phi = u^2, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.4)$$

It follows from the Lax-Milgram theorem that, for every $u \in S_0^1(\Omega)$, problem (2.4) has a unique solution $\phi_u \in S_0^1(\Omega)$. In addition, from the maximum principle, we have $\phi_u \geq 0$. Moreover, $\phi_u > 0$ if $u \neq 0$. We give some properties of the solution ϕ_u and the detailed proof process can be found in [2].

Proposition 2.1. *For each solution $\phi_u \in S_0^1(\Omega)$ of problem (2.4), we have*

- (i) $\phi_{tu} = t^2 \phi_u$ for all $t \neq 0$;
- (ii) For all $u \in S_0^1(\Omega)$, there holds that

$$\int_{\Omega} \phi_u u^2 d\xi = \int_{\Omega} |\nabla_H \phi_u|^2 d\xi \leq S^{-1} \|u\|_{\frac{4}{3}}^4 \leq S^{-3} |\Omega|^{\frac{1}{2}} \|u\|^4; \quad (2.5)$$

- (iii) Let $u_n \rightharpoonup u$ in $S_0^1(\Omega)$. Then $\phi_{u_n} \rightarrow \phi_u$ in $S_0^1(\Omega)$ and

$$\int_{\Omega} \phi_{u_n} u_n v d\xi \rightarrow \int_{\Omega} \phi_u u v d\xi, \quad \forall v \in S_0^1(\Omega).$$

Definition 2.1. We say that $u \in S_0^1(\Omega)$ is a weak solution of problem (1.1) if and only if

$$a \int_{\Omega} \nabla_H u \nabla_H v d\xi - b \int_{\Omega} |\nabla_H u|^2 d\xi \int_{\Omega} \nabla_H u \nabla_H v d\xi + \mu \int_{\Omega} \phi_u u v d\xi - \lambda \int_{\Omega} |u|^{q-2} u v d\xi - \int_{\Omega} |u|^2 u v d\xi = 0$$

for any $v \in S_0^1(\Omega)$.

The corresponding energy functional $I_\lambda(u) : S_0^1(\Omega) \rightarrow \mathbb{R}$ to problem (1.1) is defined by

$$I_\lambda(u) = \frac{a}{2} \|u\|^2 - \frac{b}{4} \|u\|^4 + \frac{\mu}{4} \int_{\Omega} \phi_u u^2 d\xi - \frac{\lambda}{q} \int_{\Omega} |u|^q d\xi - \frac{1}{4} \int_{\Omega} |u|^4 d\xi. \quad (2.6)$$

From Proposition 2.1, we know that the functional I_λ is well defined and $I_\lambda \in C^1(S_0^1(\Omega), \mathbb{R})$ (see [38]). Moreover, the Fréchet derivative of I_λ is given by

$$\begin{aligned} \langle I'_\lambda(u), v \rangle = & a \int_{\Omega} \nabla_H u \nabla_H v d\xi - b \int_{\Omega} |\nabla_H u|^2 d\xi \int_{\Omega} \nabla_H u \nabla_H v d\xi + \mu \int_{\Omega} \phi_u u v d\xi \\ & - \lambda \int_{\Omega} |u|^{q-2} u v d\xi - \int_{\Omega} |u|^2 u v d\xi, \end{aligned} \quad (2.7)$$

for all $u, v \in S_0^1(\Omega)$. Thus, the (weak) solutions of problem (1.1) coincide with the critical points of I_λ .

3 The Palais–Smale condition

In this section, we will prove that the functional I_λ satisfies the Palais-Smale condition in the cases $1 < q < 2$ and $2 < q < 4$, respectively. First, we recall that a C^1 functional I_λ on Banach space $S_0^1(\Omega)$ is said to satisfy the Palais-Smale condition at level c ($(PS)_c$ in short) if every sequence $\{u_n\}_n \subset S_0^1(\Omega)$ satisfying $I_\lambda(u_n) \rightarrow c$ and $I'_\lambda(u_n) \rightarrow 0$ ($n \rightarrow \infty$) has a convergent subsequence.

Lemma 3.1. Assume that $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $2 < q < 4$ are satisfied. Then for each $\lambda > 0$, there exists a positive constant M which is independent of λ such that

$$\limsup_{n \rightarrow \infty} \|u_n\| \leq M.$$

Proof. Let $\{u_n\}_n \subset S_0^1(\Omega)$ be a $(PS)_c$ sequence associated with the functional I_λ , that is

$$c + o(1) = I_\lambda(u_n) = \frac{a}{2} \|u_n\|^2 - \frac{b}{4} \|u_n\|^4 + \frac{\mu}{4} \int_{\Omega} \phi_{u_n} u_n^2 d\xi - \frac{\lambda}{q} \int_{\Omega} |u_n|^q d\xi - \frac{1}{4} \int_{\Omega} |u_n|^4 d\xi \quad (3.1)$$

and

$$\begin{aligned} o(1)\|u_n\| &= \langle I'_\lambda(u_n), v \rangle = a \int_{\Omega} \nabla_H u_n \nabla_H v d\xi - b \int_{\Omega} |\nabla_H u_n|^2 d\xi \int_{\Omega} \nabla_H u_n \nabla_H v d\xi \\ &\quad + \mu \int_{\Omega} \phi_{u_n} u_n v d\xi - \lambda \int_{\Omega} |u_n|^{q-2} u_n v d\xi - \int_{\Omega} |u_n|^2 u_n v d\xi. \end{aligned} \quad (3.2)$$

It follows from (2.5), (3.1) and (3.2) that

$$\begin{aligned} c + o(1)\|u_n\| &= I_\lambda(u_n) - \frac{1}{q} I'_\lambda(u_n)(u_n) \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) a \|u_n\|^2 + \left(\frac{1}{q} - \frac{1}{4}\right) b \|u_n\|^4 + \left(\frac{\mu}{4} - \frac{\mu}{q}\right) \int_{\Omega} \phi_{u_n} u_n^2 d\xi + \left(\frac{1}{q} - \frac{1}{4}\right) \int_{\Omega} |u_n|^4 d\xi \\ &\geq \frac{(q-2)a}{2q} \|u_n\|^2 + \frac{(4-q)}{4q} (b - \mu S^{-3} |\Omega|^{\frac{1}{2}}) \|u_n\|^4 + \frac{4-q}{4q} \int_{\Omega} |u_n|^4 d\xi \\ &\geq \frac{(q-2)a}{2q} \|u_n\|^2. \end{aligned} \quad (3.3)$$

This means that $\{u_n\}$ is also bounded in $S_0^1(\Omega)$ since $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $2 < q < 4$. Thus for each $\lambda > 0$, there exists a positive constant M which is independent of λ such that

$$\limsup_{n \rightarrow \infty} \|u_n\| \leq M.$$

This completes the proof of Lemma 3.1. □

For notational convenience, we denote by

$$\mu^* := \min \left\{ bS^3|\Omega|^{-\frac{1}{2}}, \frac{a^2S^5}{(bS^2+1)|\Omega|^{\frac{1}{2}}M^4} \right\},$$

where S and M are given by (2.1) and Lemma 3.1, respectively.

Lemma 3.2. Assume that $\mu < \mu^*$ and $2 < q < 4$ are satisfied. Then for each $\lambda > 0$, the functional I_λ satisfies the $(PS)_c$ condition with

$$c \in \left(0, \frac{a^2S^2}{4(bS^2+1)} - \frac{\mu}{4}D_1 \right),$$

where

$$D_1 = S^{-3}|\Omega|^{\frac{1}{2}}M^4.$$

Proof. Let $\{u_n\}_n$ be a $(PS)_c$ sequence. By Lemma 3.1, $\{u_n\}_n$ is bounded in $S_0^1(\Omega)$. The limit in (3.3) shows that $c \geq 0$. Since $S_0^1(\Omega)$ is reflexive, without loss of generality, we may assume that $u_n \rightharpoonup u_0$ weakly in $S_0^1(\Omega)$ and $u_n \rightarrow u_0$ strongly in $L^p(\Omega)$ with $1 \leq p < 4$.

By (3.2), we have

$$\begin{aligned} o(1)\|u_n\| &= \langle I'_\lambda(u_n), u_n - u_0 \rangle = (a - b\|u_n\|^2) \int_{\Omega} \nabla_H u_n \nabla_H (u_n - u_0) d\xi \\ &\quad + \mu \int_{\Omega} \phi_{u_n} u_n (u_n - u_0) d\xi - \lambda \int_{\Omega} |u_n|^{q-1} (u_n - u_0) d\xi - \int_{\Omega} |u_n|^3 (u_n - u_0) d\xi. \end{aligned}$$

By the Hölder's inequality, one has

$$\int_{\Omega} \phi_{u_n} u_n (u_n - u_0) d\xi \leq \left(\int_{\Omega} |\phi_{u_n} u_n|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\Omega} |u_n - u_0|^2 d\xi \right)^{\frac{1}{2}}$$

and

$$\int_{\Omega} |u_n|^{q-1} (u_n - u_0) d\xi \leq \left(\int_{\Omega} |u_n|^4 d\xi \right)^{\frac{q-1}{4}} \left(\int_{\Omega} |u_n - u_0|^{\frac{4}{5-q}} d\xi \right)^{\frac{5-q}{4}}$$

which shows that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \phi_{u_n} u_n (u_n - u_0) d\xi = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{q-1} (u_n - u_0) d\xi = 0. \quad (3.4)$$

Next, let $w_n := u_n - u_0$, and then $\|w_n\| \rightarrow 0$ as $n \rightarrow \infty$. Otherwise $\|w_n\| \not\rightarrow 0$. Through contradiction, we can assume there is a subsequence (still denoted by w_n) such that $\lim_{n \rightarrow \infty} \|w_n\| = l > 0$. By the Brézis-Lieb lemma [8] and (3.4), we have

$$\begin{aligned} o(1)\|u_n\| &= \langle I'_\lambda(u_n), u_n - u_0 \rangle \\ &= (a - b\|u_n\|^2) \int_{\Omega} \nabla_H u_n \nabla_H (u_n - u_0) d\xi - \int_{\Omega} |u_n|^3 (u_n - u_0) d\xi \\ &= (a - bl^2 - b\|u\|^2)\|w_n\|^2 - \int_{\Omega} |w_n|^4 d\xi. \end{aligned}$$

So

$$a\|w_n\|^2 - b\|w_n\|^4 - b\|u_0\|^2\|w_n\|^2 = \int_{\Omega} |w_n|^4 d\xi + o(1). \quad (3.5)$$

Noting that

$$\int_{\Omega} |w_n|^4 d\xi \leq S^{-2} \left(\int_{\Omega} |\nabla_H w_n|^2 d\xi \right)^2.$$

By this fact together with (3.5), we obtain

$$0 \leq l^2(a - bl^2 - b\|u_0\|^2) \leq S^{-2}l^4.$$

Then

$$0 < \frac{S^2(a - b\|u_0\|^2)}{bS^2 + 1} \leq l^2. \quad (3.6)$$

On the one hand, by (3.6) and the Brézis-Lieb lemma, it holds that

$$\begin{aligned} I_{\lambda}(u_0) &= \frac{a}{2}\|u_0\|^2 - \frac{b}{4}\|u_0\|^4 + \frac{\mu}{4} \int_{\Omega} \phi_{u_0} u_0^2 d\xi - \frac{\lambda}{q} \int_{\Omega} |u_0|^q d\xi - \frac{1}{4} \int_{\Omega} |u_0|^4 d\xi \\ &= c - \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u_0\|^2 + \frac{1}{4} \int_{\Omega} |w_n|^4 d\xi \\ &= c - \frac{a}{2}l^2 + \frac{b}{4}l^4 + \frac{b}{2}l^2\|u_0\|^2 + \frac{1}{4}(al^2 - bl^4 - bl^2\|u_0\|^2) \\ &= c - \frac{a - b\|u_0\|^2}{4}l^2 \\ &\leq c - \frac{a^2S^2}{4(bS^2 + 1)} + \frac{abS^2}{2(bS^2 + 1)}\|u_0\|^2 - \frac{b^2S^2}{4(bS^2 + 1)}\|u_0\|^4 \\ &< -\frac{\mu}{4}D_1 + \frac{abS^2}{2(bS^2 + 1)}\|u_0\|^2 - \frac{b^2S^2}{4(bS^2 + 1)}\|u_0\|^4. \end{aligned} \quad (3.7)$$

On the other hand, by (3.2), one has

$$a\|u_0\|^2 = b\|u_0\|^4 + bl^2\|u_0\|^2 + \lambda \int_{\Omega} |u_0|^q d\xi + \int_{\Omega} |u_0|^4 d\xi - \mu \int_{\Omega} \phi_{u_0} u_0^2 d\xi. \quad (3.8)$$

Therefore, from (3.8), we get

$$\begin{aligned} I_{\lambda}(u_0) &= \frac{b\|u_0\|^4}{2} + \frac{bl^2\|u_0\|^2}{2} + \frac{\lambda}{2} \int_{\Omega} |u_0|^q d\xi + \frac{1}{2} \int_{\Omega} |u_0|^4 d\xi - \frac{\mu}{2} \int_{\Omega} \phi_{u_0} u_0^2 d\xi \\ &\quad + \frac{\mu}{4} \int_{\Omega} \phi_{u_0} u_0^2 d\xi - \frac{b\|u_0\|^4}{4} - \frac{\lambda}{q} \int_{\Omega} |u_0|^q d\xi - \frac{1}{4} \int_{\Omega} |u_0|^4 d\xi \\ &= \frac{bl^2\|u_0\|^2}{2} + \frac{b\|u_0\|^4}{4} - \frac{\mu}{4} \int_{\Omega} \phi_{u_0} u_0^2 d\xi + \frac{1}{4} \int_{\Omega} |u_0|^4 d\xi + \left(\frac{\lambda}{2} - \frac{\lambda}{q} \right) \int_{\Omega} |u_0|^q d\xi \\ &\geq \frac{b\|u_0\|^2}{2} \cdot \frac{S^2(a - b\|u_0\|^2)}{bS^2 + 1} + \frac{b\|u_0\|^4}{4} - \frac{\mu}{4} \int_{\Omega} \phi_{u_0} u_0^2 d\xi \\ &\geq \frac{abS^2}{2(bS^2 + 1)}\|u_0\|^2 - \frac{b^2S^2}{4(bS^2 + 1)}\|u_0\|^4 - \frac{\mu}{4}S^{-3}|\Omega|^{\frac{1}{2}}\|u_0\|^4 \\ &= \frac{abS^2}{2(bS^2 + 1)}\|u_0\|^2 - \frac{b^2S^2}{4(bS^2 + 1)}\|u_0\|^4 - \frac{\mu}{4}D_1, \end{aligned} \quad (3.9)$$

which is a contradiction from (3.7) and (3.9), hence $l = 0$. As a consequence, we get $u_n \rightarrow u_0$ in $S_0^1(\Omega)$. The proof is complete. $\square$

For the sublinear case, that is $1 < q < 2$, we have the following compactness results.

Lemma 3.3. Assume that $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ are satisfied. Then for each $\lambda > 0$, there exists a positive constant $\widehat{M}$ which is independent of λ such that

$$\limsup_{n \rightarrow \infty} \|u_n\| \leq \widehat{M}.$$

Moreover, the functional I_λ satisfies the $(PS)_c$ condition, where precisely

$$c < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{(2-q)\lambda}{2q} D_2 \quad (3.10)$$

and

$$D_2 = S^{-\frac{q}{2}} |\Omega|^{\frac{4-q}{4}} \widehat{M}^q.$$

Proof. First, by (2.5), (3.1) and (3.2), we have

$$\begin{aligned} c + o(1)\|u_n\| &= I_\lambda(u_n) - \frac{1}{q} \langle I'_\lambda(u_n), u_n \rangle = -\frac{(2-q)a}{2q} \|u_n\|^2 + \frac{(4-q)b}{4q} \|u_n\|^4 \\ &\quad - \frac{(4-q)\mu}{4q} \int_{\Omega} \phi_{u_n} u_n^2 d\xi + \frac{4-q}{4q} \int_{\Omega} |u_n|^4 d\xi \\ &\geq -\frac{(2-q)a}{2q} \|u_n\|^2 + \frac{(4-q)}{4q} (b - \mu S^{-3} |\Omega|^{\frac{1}{2}}) \|u_n\|^4 + \frac{4-q}{4q} \int_{\Omega} |u_n|^4 d\xi. \end{aligned} \quad (3.11)$$

From the Young's inequality, we have

$$\frac{(2-q)a}{2q} \|u_n\|^2 \leq \frac{1}{2\varepsilon} \left[\frac{(2-q)a}{2q} \right]^2 + \frac{\varepsilon}{2} \|u_n\|^4.$$

Let $\varepsilon = \frac{(4-q)}{4q} (b - \mu S^{-3} |\Omega|^{\frac{1}{2}}) > 0$. It follows from (3.11) that

$$c + o(1)\|u_n\| \geq \frac{(4-q)}{8q} (b - \mu S^{-3} |\Omega|^{\frac{1}{2}}) \|u_n\|^4 - \frac{2q}{4-q} (b - \mu S^{-3} |\Omega|^{\frac{1}{2}})^{-1} \left[\frac{(2-q)a}{2q} \right]^2. \quad (3.12)$$

Thus for each $\lambda > 0$, there exists a positive constant $\widehat{M}$ which is independent of λ such that

$$\limsup_{n \rightarrow \infty} \|u_n\| \leq \widehat{M}.$$

Therefore, there exists $u_0 \in S_0^1(\Omega)$ such that up to a subsequence $u_n \rightharpoonup u_0$ weakly in $S_0^1(\Omega)$ and $u_n \rightarrow u_0$ strongly in $L^p(\Omega)$ with $1 \leq p < 4$.

Next, set $w_n = u_n - u_0$, we claim that $\|w_n\| \rightarrow 0$. If not, we have $\lim_{n \rightarrow \infty} \|w_n\| = l$. Using a similar discussion as in (3.7), we have

$$I_\lambda(u_0) < -\frac{(2-q)\lambda}{2q} D_2 + \frac{abS^2}{2(bS^2 + 1)} \|u_0\|^2 - \frac{b^2 S^2}{4(bS^2 + 1)} \|u_0\|^4. \quad (3.13)$$

On the other hand, we also have

$$\begin{aligned} I_\lambda(u_0) &\geq \frac{b\|u_0\|^2}{2} \cdot \frac{S^2(a - b\|u_0\|^2)}{bS^2 + 1} + \frac{b\|u_0\|^4}{4} - \frac{\mu}{4} \int_{\Omega} \phi_{u_0} u_0^2 d\xi + \left(\frac{\lambda}{2} - \frac{\lambda}{q} \right) \int_{\Omega} |u_0|^q d\xi \\ &\geq \frac{abS^2}{2(bS^2 + 1)} \|u_0\|^2 - \frac{b^2 S^2}{4(bS^2 + 1)} \|u_0\|^4 - \frac{(b-\mu)}{4} S^{-3} |\Omega|^{\frac{1}{2}} \|u_0\|^4 - \frac{(2-q)\lambda}{2q} S^{-\frac{q}{2}} |\Omega|^{\frac{4-q}{4}} \|u\|^q \\ &\geq \frac{abS^2}{2(bS^2 + 1)} \|u_0\|^2 - \frac{b^2 S^2}{4(bS^2 + 1)} \|u_0\|^4 - \frac{(2-q)\lambda}{2q} D_2. \end{aligned} \quad (3.14)$$

We know that this case is impossible by (3.13) and (3.14). So, we still have $l = 0$. As a consequence, we get $u_n \rightarrow u_0$ in $S_0^1(\Omega)$. The proof is complete. $\square$

4 Superlinear case: $2 < q < 4$

In order to prove Theorem 1.1 and Theorem 1.2, we shall use the mountain pass theorem in [1] and the Krasnoselskii genus in [38], respectively.

Lemma 4.1. Assume that $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $2 < q < 4$ are satisfied, the functional I_λ satisfies the mountain pass geometry, that is,

- (i) there exist $\alpha, \rho > 0$ such that $I_\lambda(u) \geq \alpha$ for any $u \in S_0^1(\Omega)$ such that $\|u\| = \rho$;
- (ii) there exists $e \in S_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.

Proof. On the one hand, we divide the proof of Lemma 4.1 (i) into two cases.

Case I: $\mu \leq 0$.

By the Hölder inequality, (2.1) and (2.5), we have

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 + \frac{\mu}{4} \int_{\Omega} \phi_u u^2 d\xi - \frac{\lambda}{q} \int_{\Omega} |u|^q d\xi - \frac{1}{4} \int_{\Omega} |u|^4 d\xi \\ &\geq \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 + \frac{\mu}{4} S^{-3} |\Omega|^{\frac{1}{2}} \|u\|^4 d\xi - \frac{\lambda}{q} S^{-\frac{q}{2}} |\Omega|^{\frac{4-q}{4}} \|u\|^q - \frac{1}{4} S^{-2} \|u\|^4 \\ &= \frac{a}{2}\|u\|^2 - \frac{d_0}{4}\|u\|^4 - \frac{\lambda}{q} S^{-\frac{q}{2}} |\Omega|^{\frac{4-q}{4}} \|u\|^q, \end{aligned}$$

where $d_0 = \frac{b - \mu S^{-3} |\Omega|^{\frac{1}{2}} + S^{-2}}{4}$. Since $2 < q < 4$, then we can choose $\rho, \alpha > 0$ such that $I_\lambda(u) \geq \alpha$ for $\|u\| = \rho$.

Case II: $0 < \mu < bS^3|\Omega|^{-\frac{1}{2}}$.

In this case, it is easy to prove that condition (i) of Lemma 4.1 holds. Hence (i) of Lemma 4.1 holds.

On the other hand, if $0 \leq \mu < bS^3|\Omega|^{-\frac{1}{2}}$, fixed $u \in C_0^\infty(\Omega)$ with $u \neq 0$, $t \in \mathbb{R}$, by (2.5), we have

$$\begin{aligned} I_\lambda(tu) &= \frac{a}{2}\|u\|^2 t^2 - \frac{b}{4}\|u\|^4 t^4 + \frac{\mu t^4}{4} \int_{\Omega} \phi_u u^2 d\xi - \frac{t^q \lambda}{q} \int_{\Omega} |u|^q d\xi - \frac{t^4}{4} \int_{\Omega} |u|^4 d\xi \\ &\leq \frac{a}{2}\|u\|^2 t^2 - \frac{b}{4}\|u\|^4 t^4 + \frac{\mu t^4}{4} S^{-3} |\Omega|^{\frac{1}{2}} \|u\|^4 d\xi - \frac{t^q \lambda}{q} \int_{\Omega} |u|^q d\xi - \frac{t^4}{4} \int_{\Omega} |u|^4 d\xi \\ &= \frac{a}{2}\|u\|^2 t^2 - \left(\frac{b - \mu S^{-3} |\Omega|^{\frac{1}{2}}}{4} \right) \|u\|^4 t^4 d\xi - \frac{t^q \lambda}{q} \int_{\Omega} |u|^q d\xi - \frac{t^4}{4} \int_{\Omega} |u|^4 d\xi. \end{aligned} \quad (4.1)$$

As $2 < q < 4$, it follows from (4.1) that

$$I_\lambda(tu) \rightarrow -\infty \quad \text{as } t \rightarrow \infty. \quad (4.2)$$

If $\mu < 0$, obviously, (4.2) is also true. This completes the condition (ii) of Lemma 4.1. $\square$

Proof of Theorem 1.1. We claim that

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_\lambda(\gamma(t)) < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{\mu}{4} D_1. \quad (4.3)$$

In fact, we choose $v_0 \in S_0^1(\Omega) \setminus \{0\}$ such that $\lim_{t \rightarrow \infty} I_\lambda(tv_0) = -\infty$. Then

$$\sup_{t \geq 0} I_\lambda(tv_0) = I_\lambda(t_\lambda v_0)$$

for some $t_\lambda > 0$. Hence t_λ satisfies

$$t_\lambda^q \lambda \int_{\Omega} |v_0|^q d\xi = t_\lambda^2 \|v_0\|^2 + t_\lambda^4 \left(\mu \int_{\Omega} \phi_{v_0} v_0^2 - b \|v_0\|^4 - \int_{\Omega} |v_0|^4 d\xi \right). \quad (4.4)$$

Since $\mu < \mu^*$ and (2.5), we have

$$\mu \int_{\Omega} \phi_{v_0} v_0^2 d\xi - b \|v_0\|^4 - \int_{\Omega} |v_0|^4 d\xi < 0.$$

Thus, (4.4) implies that $\{t_\lambda\}_{\lambda>0}$ is bounded due to $2 < q < 4$.

In the following, we prove that

$$t_\lambda \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty. \quad (4.5)$$

Arguing by contradiction, we can assume that there exists $t_0 > 0$ and a sequence $\{\lambda_n\}_n$, with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $t_{\lambda_n} \rightarrow t_0$ as $n \rightarrow \infty$. By the Lebesgue dominated convergence theorem, we deduce that

$$\int_{\Omega} |t_{\lambda_n} v_0|^q d\xi \rightarrow \int_{\Omega} |t_0 v_0|^q d\xi \quad \text{as } n \rightarrow \infty. \quad (4.6)$$

From which it follows that

$$\lambda_n \int_{\Omega} |t_{\lambda_n} v_0|^q d\xi \rightarrow +\infty \quad \text{as } n \rightarrow \infty. \quad (4.7)$$

However, (4.4) implies that this fact is absurd. Hence (4.5) holds.

From (4.4), (4.5) and (4.6), we get

$$\lim_{\lambda \rightarrow \infty} \left(\sup_{t \geq 0} I_\lambda(tv_0) \right) = \lim_{\lambda \rightarrow \infty} I_\lambda(t_\lambda v_0) = 0.$$

Then, there exists $\lambda_1 > 0$ such that for any $\lambda > \lambda_1$ we have

$$\sup_{t \geq 0} I_\lambda(tv_0) < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{\mu}{4} D_1. \quad (4.8)$$

If we take $e = \eta v_0$, with η large enough to verify $I_\lambda(e) < 0$, then we obtain $c_\lambda \leq \max_{t \in [0,1]} I_\lambda(\gamma(t))$ by taking $\gamma(t) = t\eta v_0$. Therefore

$$c_\lambda \leq \sup_{t \geq 0} I_\lambda(tv_0) < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{\mu}{4} D_1$$

for λ large enough. Clearly, Lemma 3.2 and Lemma 4.1 give the existence of nontrivial critical points of I_λ and this concludes the proof. $\square$

Theorem 4.1. (see [38]) Let X be an infinite dimensional Banach space and let $J \in C^1(X, \mathbb{R})$ be an even functional, with $J(0) = 0$. Suppose that $X = Y \oplus Z$, where Y is finite dimensional, and that J satisfies

- (I₁) There exists constant $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in \partial B_\rho \cap Z$;
- (I₂) There exists $\Theta > 0$ such that J satisfies the (PS)_c condition for all c , with $c \in (0, \Theta)$;
- (I₃) For any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $J(u) \leq 0$ on $\tilde{X} \setminus B_R$.

Assume furthermore that Y is k dimensional and $Y = \text{span}\{v_1, \dots, v_k\}$. For $n \geq k$, inductively choose $v_{n+1} \notin E_n = \text{span}\{v_1, \dots, v_n\}$. Let $R_n = R(E_n)$ and $\Omega_n = B_{R_n} \cap E_n$. Define

$$G_n = \{\psi \in C(\Omega_n, X) : \psi|_{\partial B_{R_n} \cap E_n} = \text{id} \text{ and } \psi \text{ is odd}\}$$

and

$$\Gamma_j = \left\{ \psi(\overline{\Omega_n \setminus V}) : \psi \in G_n, n \geq j, V \in \Lambda, \gamma(V) \leq n - j \right\},$$

where $\gamma(V)$ is the Krasnoselskii genus of V . For $j \in \mathbb{N}$, set

$$c_j = \inf_{E \in \Gamma_j} \max_{u \in E} J(u).$$

Thus, $0 \leq c_j \leq c_{j+1}$ and $c_j < \Theta$ for $j > k$, and then we get c_j is a critical value of J . Furthermore, if $c_j = c_{j+1} = \dots = c_{j+m} = c < \Theta$ for $j > k$, then $\gamma(K_c) \geq m + 1$, where

$$K_c = \{u \in X : J(u) = c \text{ and } J'(u) = 0\}.$$

Proof of Theorem 1.2. We shall apply Theorem 4.1 to I_λ . First, we know that $S_0^1(\Omega)$ is a Banach space and $I_\lambda \in C^1(S_0^1(\Omega), \mathbb{R})$. The functional I_λ satisfies $I_\lambda(0) = 0$, $I_\lambda(-u) = I_\lambda(u)$. Using a similar discussion as in Lemma 4.1, it is easy to prove the functional I_λ satisfies conditions (I_1) and (I_3) of Theorem 4.1.

Next, we use an argument given in [41]. In view of the definition of c_n^λ , we get

$$\begin{aligned} c_n^\lambda &= \inf_{E \in \Gamma_n} \max_{u \in E} I_\lambda(u) \\ &\leq \inf_{E \in \Gamma_n} \max_{u \in E} \left\{ \frac{a}{2} \|u\|^2 - \frac{\lambda}{q} \int_{\Omega} |u|^q d\xi - \frac{1}{4} \int_{\Omega} |u|^4 d\xi \right\}. \end{aligned} \quad (4.9)$$

Set

$$\Psi_n = \inf_{E \in \Gamma_n} \max_{u \in E} \left\{ \frac{a}{2} \|u\|^2 - \frac{\lambda}{q} \int_{\Omega} |u|^q d\xi - \frac{1}{4} \int_{\Omega} |u|^4 d\xi \right\}.$$

So $\Psi_n < \infty$. By the definition of Γ_n , we have $\Psi_n \leq \Psi_{n+1}$.

To this end, we distinguish two cases:

Case I: Fix $\lambda > 0$ and $b > 0$. Let us choose $a^* > 0$ large enough such that for any $a > a^*$ yields

$$\sup_n \Psi_n < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{\mu}{4} D_1.$$

Case II: Fix $\lambda > 0$. Using a similar discussion as in (4.3), there exists $\lambda_2 > 0$ such that $\lambda > \lambda_2$, we have

$$0 < c_n^\lambda \leq \Psi_n < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{\mu}{4} D_1. \quad (4.10)$$

In any case, we have

$$0 < c_1^\lambda \leq c_2^\lambda \leq c_3^\lambda \leq \dots \leq c_n^\lambda < \Psi_n < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{\mu}{4} D_1.$$

An application of Proposition 9.30 in [38] guarantees that the levels $c_1^\lambda \leq \dots \leq c_n^\lambda$ are critical values of I_λ .

If $c_j^\lambda = c_{j+1}^\lambda$ for some $j = 1, 2, \dots, m - k$, then by Theorem 2 and Remark 2.12 in [1], the set $K_{c_j^\lambda}$ contains infinitely many distinct points and therefore problem (1.1) has infinitely many weak solutions. Consequently, problem (1.1) has at least k pairs of solutions. $\square$

5 Sublinear case: $1 < q < 2$

In this section, we apply Ekeland's variational principle, mountain pass theorem (see [1]) and Clark critical theorem (see [11]) to prove Theorem 1.3 and Theorem 1.4.

Lemma 5.1. Assume that $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ are satisfied. Then there exists $\Lambda_1 > 0$ such that for each $\lambda \in (0, \Lambda_1)$, the functional I_λ satisfies the mountain pass geometry, that is,

- (i) there exist $\alpha, \rho > 0$ such that $I_\lambda(u) \geq \alpha$ for any $u \in S_0^1(\Omega)$ such that $\|u\| = \rho$;
- (ii) there exists $e \in S_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.

Proof. On the one hand, we divide the proof of Lemma 5.1 (i) into two cases.

Case I: $\mu \leq 0$.

First, by the Hölder inequality, (2.1) and (2.5), we have

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2} \|u\|^2 - \frac{b}{4} \|u\|^4 + \frac{\mu}{4} \int_{\Omega} \phi_u u^2 d\xi - \frac{\lambda}{q} \int_{\Omega} |u|^q d\xi - \frac{1}{4} \int_{\Omega} |u|^4 d\xi \\ &\geq \frac{a}{2} \|u\|^2 - \frac{b}{4} \|u\|^4 + \frac{\mu}{4} S^{-3} |\Omega|^{\frac{1}{2}} \|u\|^4 d\xi - \frac{\lambda}{q} S^{-\frac{q}{2}} \|u\|^q |\Omega|^{\frac{4-q}{4}} - \frac{1}{4} S^{-2} \|u\|^4 \end{aligned}$$

$$:= g(u). \quad (5.1)$$

Note that $1 < q < 2$, we have

$$g(u) \geq \|u\|^q \left[\frac{a}{2} \|u\|^{2-q} - d_0 \|u\|^{4-q} - \frac{\lambda}{q} S^{-\frac{q}{2}} |\Omega|^{\frac{4-q}{4}} \right],$$

where $d_0 = \frac{b-\mu S^{-3}|\Omega|^{\frac{1}{2}+S^{-2}}}{4}$.

Let

$$h(t) = \frac{a}{2} t^{2-q} - d_0 t^{4-q},$$

then there exists

$$t_0 = \left[\frac{a(2-q)}{2d_0(4-q)} \right]^{\frac{1}{2}}$$

being the maximum value point of $h(t)$, that is,

$$h(t_0) = \max_{t>0} h(t) = \frac{a}{4-q} \left[\frac{a(2-q)}{2d_0(4-q)} \right]^{\frac{2-q}{2}} > 0.$$

Set

$$\Lambda_1 = \frac{1}{2} q S^{\frac{q}{2}} |\Omega|^{\frac{q-4}{4}} h(t_0).$$

Then, for all $\lambda \in (0, \Lambda_1)$, we have

$$I_\lambda(u) \geq \frac{h(t_0)}{2} \rho_0^q > 0, \quad \forall u \in S_{\rho_0},$$

where ρ_0 small enough.

Case II: $0 < \mu < bS^3|\Omega|^{-\frac{1}{2}}$.

In this case, it is easy to prove that Lemma 5.1 (i) holds.

On the other hand, using a similar discussion as in (4.2), it is easy to prove Lemma 5.1 (ii) holds. This completes the proof of Lemma 5.1. $\square$

Lemma 5.2. *Let $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda_1 > 0$ such that $\lambda \in (0, \Lambda_1)$, problem (1.1) has a positive solution.*

Proof. On the one hand, for any $u \in S_0^1(\Omega) \setminus \{0\}$, we have

$$\lim_{s \rightarrow 0^+} \frac{I_\lambda(su)}{s^q} = -\frac{\lambda}{q} \int_{\Omega} |u|^q d\xi < 0 \quad (5.2)$$

which means that there exists $u \in B_{\rho_0}$ such that $I_\lambda(u) < 0$, that is,

$$\inf_{u \in B_{\rho_0}} I_\lambda(u) < 0.$$

On the other hand, from Lemma 5.1, we have

$$\inf_{u \in B_{\rho_0}} I_\lambda(u) < 0 < \inf_{u \in S_{\rho_0}} I_\lambda(u).$$

Noting that $I_\lambda(|u|) = I_\lambda(u)$, by applying the Ekeland's variational principle in B_{ρ_0} , there exists a minimizing sequence $u_n \subset B_{\rho_0}$ such that

$$I_\lambda(u_n) < \inf_{u \in B_{\rho_0}} I_\lambda(u) + \frac{1}{n} \quad \text{and} \quad I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n} \|v - u_n\| \quad \text{for all } v \in B_{\rho_0}.$$

Therefore

$$I_\lambda(u_n) \rightarrow c \quad \text{and} \quad I'_\lambda(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\|u_n\| \leq \rho_0$ and $u_n \geq 0$, there exists $u_\lambda \in B_{\rho_0}$ with $u_\lambda \geq 0$ such that $u_n \rightharpoonup u_\lambda$ in $S_0^1(\Omega)$ as $n \rightarrow \infty$. By Lemma 3.3, we can obtain $u_n \rightarrow u_\lambda$ in $S_0^1(\Omega)$ and

$$d = \lim_{n \rightarrow \infty} I_\lambda(u_n) = I_\lambda(u_\lambda) < 0.$$

Hence, we have $u_\lambda \geq 0$ with $u_\lambda \neq 0$ being a solution of problem (1.1). By the maximum principle [5], we can know that $u_\lambda > 0$ in Ω . $\square$

Lemma 5.3. *Let $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda^* > 0$ such that for $\lambda \in (0, \Lambda^*)$, problem (1.1) has a positive mountain pass solution.*

Proof. We claim that

$$c \leq \sup_{t \geq 0} I_\lambda(u_\lambda + tu_\varepsilon) < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{(2-q)\lambda}{2q} D_2, \quad 1 < q < 2. \quad (5.3)$$

Let u_ε be defined as in (2.3). Since u_λ is a positive solution of problem (1.1) and $I_\lambda(u_\lambda) < 0$, then if $0 < \mu < bS^3|\Omega|^{-\frac{1}{2}}$, it holds that

$$\begin{aligned} I_\lambda(u_\lambda + tu_\varepsilon) &= \frac{a}{2} \|u_\lambda + tu_\varepsilon\|^2 - \frac{b}{4} \|u_\lambda + tu_\varepsilon\|^4 - \frac{\lambda}{q} \int_\Omega |u_\lambda + tu_\varepsilon|^q d\xi \\ &\quad + \frac{\mu}{4} \int_\Omega \phi_{u_\lambda + tu_\varepsilon} (u_\lambda + tu_\varepsilon)^2 d\xi - \frac{1}{4} \int_\Omega |u_\lambda + tu_\varepsilon|^4 d\xi \\ &= \frac{a}{2} \|u_\lambda\|^2 + at \int_\Omega (\nabla_H u_\lambda \nabla_H u_\varepsilon) d\xi + \frac{at^2}{2} \|u_\varepsilon\|^2 - \frac{b}{4} \|u_\lambda\|^4 - \frac{bt^4}{4} \|u_\varepsilon\|^4 \\ &\quad - bt^3 \|u_\varepsilon\|^2 \int_\Omega (\nabla_H u_\lambda \nabla_H u_\varepsilon) d\xi - bt \|u_\lambda\|^2 \int_\Omega (\nabla_H u_\lambda \nabla_H u_\varepsilon) d\xi \\ &\quad - bt^2 \left(\int_\Omega (\nabla_H u_\lambda \nabla_H u_\varepsilon) d\xi \right)^2 - \frac{bt^2}{2} \|u_\lambda\|^2 \|u_\varepsilon\|^2 - \frac{\lambda}{q} \int_\Omega |u_\lambda + tu_\varepsilon|^q d\xi \\ &\quad + \frac{\mu}{4} \int_\Omega \phi_{u_\lambda + tu_\varepsilon} (u_\lambda + tu_\varepsilon)^2 d\xi - \frac{1}{4} \int_\Omega |u_\lambda + tu_\varepsilon|^4 d\xi \\ &\leq I_\lambda(u_\lambda) + \frac{at^2}{2} \|u_\varepsilon\|^2 - \frac{bt^4}{4} \|u_\varepsilon\|^4 + \frac{\mu}{4} \left(\int_\Omega \phi_{u_\lambda + tu_\varepsilon} (u_\lambda + tu_\varepsilon)^2 - \phi_{u_\lambda} u_\lambda^2 - 4t \phi_{u_\lambda} u_\lambda u_\varepsilon d\xi \right) \\ &\quad - \frac{\lambda}{q} \int_\Omega [|u_\lambda + tu_\varepsilon|^q - |u_\lambda|^q - qtu_\lambda^{q-1} u_\varepsilon] d\xi - \frac{1}{4} \int_\Omega [|u_\lambda + tu_\varepsilon|^4 - |u_\lambda|^4 - 4tu_\lambda^3 u_\varepsilon] d\xi \\ &\quad + \left| bt^3 \|u_\varepsilon\|^2 \int_\Omega (\nabla_H u_\lambda \nabla_H u_\varepsilon) d\xi \right| \\ &\leq \frac{at^2}{2} \|u_\varepsilon\|^2 - \frac{bt^4}{4} \|u_\varepsilon\|^4 - \frac{1}{4} t^4 \int_\Omega |u_\varepsilon|^4 d\xi - C_1 t^3 \int_\Omega |u_\varepsilon|^3 d\xi + \left| bt^3 \|u_\varepsilon\|^2 \int_\Omega (\nabla_H u_\lambda \nabla_H u_\varepsilon) d\xi \right| \\ &\quad + \frac{\mu}{4} \int_\Omega \left(\phi_{u_\lambda + tu_\varepsilon} (u_\lambda + tu_\varepsilon)^2 - \phi_{u_\lambda} u_\lambda^2 - 4t \phi_{u_\lambda} u_\lambda u_\varepsilon \right) d\xi. \end{aligned} \quad (5.4)$$

On the other hand, from Lemma 3.8 in [2], we have

$$\begin{aligned} &\int_\Omega (\phi_{u_\lambda + tu_\varepsilon} (u_\lambda + tu_\varepsilon)^2 - \phi_{u_\lambda} u_\lambda^2 - 4t \phi_{u_\lambda} u_\lambda u_\varepsilon) d\xi \\ &\leq \int_\Omega (6t^2 \phi_{u_\lambda} u_\varepsilon^2 + 4t^3 \phi_{u_\varepsilon} u_\lambda u_\varepsilon + t^4 \phi_{u_\varepsilon} u_\varepsilon^2) d\xi. \end{aligned} \quad (5.5)$$

From (5.4) and (5.5), one has

$$\begin{aligned} I_\lambda(u_\lambda + tu_\varepsilon) &\leq \frac{at^2}{2} \|u_\varepsilon\|^2 - \frac{bt^4}{4} \|u_\varepsilon\|^4 - \frac{1}{4} t^4 \int_{\Omega} |u_\varepsilon|^4 d\xi - C_1 t^3 \int_{\Omega} |u_\varepsilon|^3 d\xi + \left| bt^3 \|u_\varepsilon\|^2 \int_{\Omega} (\nabla_H u_\lambda \nabla_H u_\varepsilon) d\xi \right| \\ &\quad + \frac{\mu}{4} \left(\int_{\Omega} (6t^2 \phi_{u_\lambda} u_\varepsilon^2 + 4t^3 \phi_{u_\varepsilon} u_\lambda u_\varepsilon + t^4 \phi_{u_\varepsilon} u_\varepsilon^2) d\xi \right). \end{aligned} \quad (5.6)$$

It follows from the Hölder inequality and Lemma 2.1 of [2] that

$$\begin{aligned} \|u_\varepsilon\|^2 &= S^2 + O(\varepsilon^2), \quad \|u_\varepsilon\|_4^4 = S^2 + O(\varepsilon^4), \\ \int_{\Omega} \phi_{u_\lambda} u_\varepsilon^2 d\xi &\leq \|\phi_{u_\lambda}\|_4 \|u_\varepsilon\|_{\frac{8}{3}}^2 \leq C\varepsilon, \\ \int_{\Omega} \phi_{u_\varepsilon} u_\lambda u_\varepsilon d\xi &\leq \|\phi_{u_\varepsilon}\|_4 \|u_\lambda\|_{\frac{8}{3}} \|u_\varepsilon\|_{\frac{8}{3}} \leq C\varepsilon^{\frac{3}{2}}, \\ \int_{\Omega} \phi_{u_\varepsilon} u_\varepsilon^2 d\xi &\leq \|\phi_{u_\varepsilon}\|_4 \|u_\lambda\|_{\frac{8}{3}}^2 \leq C\varepsilon^2. \end{aligned}$$

Noting that

$$\int_{\Omega} \phi_{u_\varepsilon} u_\varepsilon^2 \leq S^{-3} |\Omega|^{\frac{1}{2}} \|u_\varepsilon\|^4.$$

Thus, it follows from (5.6) that

$$\begin{aligned} I_\lambda(u_\lambda + tu_\varepsilon) &\leq at^2 \left(\frac{S^2}{2} + C\varepsilon \right) - \frac{a_0 t^4 (S^2 + O(\varepsilon^4))}{4} \\ &\quad - \frac{1}{4} t^4 \int_{\Omega} |u_\varepsilon|^4 d\xi - C_1 t^3 \int_{\Omega} |u_\varepsilon|^3 d\xi + Ct^3 \varepsilon^{\frac{3}{2}} + C_2 t^3 (S^2 + O(\varepsilon^2)) \varepsilon, \end{aligned} \quad (5.7)$$

where

$$a_0 = b - \mu S^{-3} |\Omega|^{\frac{1}{2}} > 0.$$

Define

$$g(t) = at^2 \left(\frac{S^2}{2} + C\varepsilon \right) - \frac{a_0 t^4 (S^2 + O(\varepsilon^4))}{4} - \frac{1}{4} t^4 \int_{\Omega} |u_\varepsilon|^4 d\xi - C_1 t^3 \int_{\Omega} |u_\varepsilon|^3 d\xi + Ct^3 \varepsilon^{\frac{3}{2}} + C_2 t^3 (S^2 + O(\varepsilon^2)) \varepsilon.$$

Since

$$\lim_{t \rightarrow 0^+} g(t) > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} g(t) = -\infty,$$

thus, there exists $t_\varepsilon > 0$ such that

$$\sup_{t \geq 0} g(t) = g(t_\varepsilon), \quad \frac{dg}{dt} \Big|_{t=t_\varepsilon} = 0.$$

Therefore, there exist two positive constants t_1, t_2 independent of ε, λ such that

$$0 < t_1 < t_\varepsilon < t_2 < \infty.$$

Moreover, it follows from (5.5) and (5.7) that

$$\begin{aligned} \sup_{t \geq 0} I_\lambda(u_\lambda + tu_\varepsilon) &\leq \sup_{t \geq 0} g(t) \\ &\leq \sup_{t \geq 0} \left(\frac{at^2}{2} \|u_\varepsilon\|^2 - \frac{bt^4}{4} \|u_\varepsilon\|^4 - \frac{1}{4} t^4 \int_{\Omega} |u_\varepsilon|^4 d\xi \right) - C_1 t^3 \int_{\Omega} |u_\varepsilon|^3 d\xi \end{aligned}$$

$$\begin{aligned}
& + CC_0^2 \varepsilon + CC_0^3 \varepsilon^{\frac{3}{2}} + CC_0^4 \varepsilon^2 + C_2 t^3 (S^2 + O(\varepsilon^2)) \varepsilon \\
\leq & \sup \left(\frac{at^2}{2} \|u_\varepsilon\|^2 - \frac{bt^4}{4} \|u_\varepsilon\|^4 - \frac{1}{4} t^4 \int_{\Omega} |u_\varepsilon|^4 d\xi \right) - C_1 t^3 \int_{\Omega} |u_\varepsilon|^3 d\xi + C_4 \varepsilon. \quad (5.8)
\end{aligned}$$

Since

$$\int_{\Omega} |u_\varepsilon|^3 d\xi = C\varepsilon^{\frac{1}{2}} + O(\varepsilon),$$

it follows from (5.8) that

$$\begin{aligned}
\sup_{t \geq 0} I_\lambda(u_\lambda + tu_\varepsilon) & \leq \sup_{t \geq 0} g(t) \\
& \leq \sup \left(\frac{at^2}{2} S^2 - \frac{bt^4}{4} S^4 - \frac{1}{4} t^4 S^2 \right) - C_1 t^3 \int_{\Omega} |u_\varepsilon|^3 d\xi + C_4 \varepsilon \\
& \leq \frac{a^2 S^2}{4(bS^2 + 1)} - C_1 t^3 \int_{\Omega} |u_\varepsilon|^3 d\xi + C_4 \varepsilon \\
& = \frac{a^2 S^2}{4(bS^2 + 1)} - C_3 \varepsilon^{\frac{1}{2}} + C_4 \varepsilon.
\end{aligned}$$

Let

$$\varepsilon = \lambda \quad \text{and} \quad \Lambda_2 := \left(\frac{C_4 + \frac{(2-q)}{2q} D_2}{C_3} \right)^{-2}.$$

Then

$$-C_3 \lambda^{\frac{1}{2}} + C_4 \lambda = \lambda(C_4 - C_3 \lambda^{-\frac{1}{2}}) < -\frac{(2-q)\lambda}{2q} D_2$$

for all $0 < \lambda < \Lambda_2$. So, we get

$$\sup_{t \geq 0} I_\lambda(u_\lambda + tu_\varepsilon) < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{(2-q)\lambda}{2q} D_2.$$

On the other hand, it is easy to verify that

$$\sup_{t \geq 0} I_\lambda(u_\lambda + tu_\varepsilon) < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{(2-q)\lambda}{2q} D_2 \quad \text{for } \mu \leq 0.$$

In either case, we have

$$\sup_{t \geq 0} I_\lambda(u_\lambda + tu_\varepsilon) < \frac{a^2 S^2}{4(bS^2 + 1)} - \frac{(2-q)\lambda}{2q} D_2.$$

Let

$$\Lambda^* = \min \left\{ \Lambda_1, \Lambda_2, \frac{2qa^2 S^2}{4(bS^2 + 1)(2-q)D_2} \right\}. \quad (5.9)$$

Applying the mountain pass lemma (see [1]), there exists $u_n \in S_0^1(\Omega)$ such that

$$I_\lambda(u_n) \rightarrow c \quad \text{and} \quad I'_\lambda(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_\lambda(\gamma(t))$$

and

$$\Gamma = \left\{ \gamma \in C([0, 1], S_0^1(\Omega)) : \gamma(0) = u_\lambda, \gamma(1) = e \right\}.$$

It follows from Lemma 3.3 that $\{u_n\}$ has a convergent subsequence (still denoted by $\{u_n\}$) such that $u_n \rightarrow u_1$ in $S_0^1(\Omega)$. Moreover, we obtain u_1 is a non-negative weak solution of problem (1.1) and

$$I_\lambda(u_1) = \lim_{n \rightarrow \infty} I_\lambda(u_n) = c > 0.$$

It follows that $u_\lambda \neq u_1$ and $u_1 \neq 0$. In fact, similar to the proof of Lemma 5.1, we also have $u_1 > 0$. Hence, the proof of Theorem 1.3 is complete. $\square$

Lemma 5.4. *Let $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $1 < q < 2$ be satisfied. Then, there exists $\Lambda^* > 0$ such that for $\lambda \in (0, \Lambda^*)$, problem (1.1) has a positive ground state solution.*

Proof. In this proof, we will use the Fatou theorem to prove that problem (1.1) has a positive ground state solution. To this end, we define

$$\psi = \inf_{u \in \mathcal{N}} I_\lambda(u), \quad \mathcal{N} = \left\{ u \in S_0^1(\Omega) : u \neq 0, \langle I'_\lambda(u), u \rangle = 0 \right\}. \quad (5.10)$$

Obviously, if $u \in \mathcal{N}$, one has $I_\lambda(|u|) = I_\lambda(u)$. Therefore we can consider a nonnegative minimizing sequence $\{u_n\} \subset \mathcal{N}$ such that

$$I_\lambda(u_n) \rightarrow \psi \quad \text{as } n \rightarrow \infty. \quad (5.11)$$

By $I_\lambda(u_\lambda) < 0$ and Lemma 3.3, we can see that $\psi < 0$ and $\{u_n\}$ is bounded in $S_0^1(\Omega)$. We may assume that $u_n \rightharpoonup u_2$ weakly in $S_0^1(\Omega)$ and $u_n \rightarrow u_2$ strongly in $L^p(\Omega)$ with $1 \leq p < 4$. Then $u_2 \neq 0$. In fact, if $u_2 \equiv 0$ and $\lim_{n \rightarrow \infty} \|u_n\|^2 = l$, we have

$$\langle I'_\lambda(u_n), u_n \rangle = a\|u_n\|^2 - b\|u_n\|^4 + \mu \int_{\Omega} \phi_{u_n} u_n^2 d\xi - \lambda \int_{\Omega} |u_n|^q d\xi - \int_{\Omega} |u_n|^4 d\xi. \quad (5.12)$$

So, we get

$$al - bl^2 = 0.$$

From this fact, we obtain $l = 0$ or $l = \frac{a}{b}$.

If $l = 0$, we have $I_\lambda(u_n) \rightarrow 0$ as $n \rightarrow \infty$. This is a contradiction from (5.2).

If $l = \frac{a}{b}$, we have $I_\lambda(u_n) \rightarrow \frac{a^2}{4b}$. This is a contradiction from Lemma 3.3. Therefore, we have $u_2 \neq 0$ in $S_0^1(\Omega)$.

It follows from Lemma 3.3 that $u_n \rightarrow u_2$ in $S_0^1(\Omega)$. It means that u_2 is a positive solution of problem (1.1) and $I_\lambda(u_2) \geq \psi$.

Next, we will prove $I_\lambda(u_2) \leq \psi$. By the Fatou's Lemma, we get

$$\begin{aligned} \psi &= \lim_{n \rightarrow \infty} \left\{ I_\lambda(u_n) - \frac{1}{q} \langle I'_\lambda(u_n), u_n \rangle \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{q-2}{2q} \|u_n\|^2 - \frac{b(4-q)}{4q} \|u_n\|^4 - \frac{\mu(4-q)}{4q} \int_{\Omega} \phi_{u_n} u_n^2 d\xi - \frac{4-q}{4q} \int_{\Omega} |u_n|^4 d\xi \right\} \\ &\geq \frac{q-2}{2q} \|u_2\|^2 - \frac{b(4-q)}{4q} \|u_2\|^4 - \frac{\mu(4-q)}{4q} \int_{\Omega} \phi_{u_2} u_2^2 d\xi - \frac{4-q}{4q} \int_{\Omega} |u_2|^4 d\xi. \end{aligned} \quad (5.13)$$

In addition, since u_2 is a positive solution of problem (1.1), one has

$$\begin{aligned} I_\lambda(u_2) &= I_\lambda(u_2) - \frac{1}{q} \langle I'_\lambda(u_2), u_2 \rangle \\ &= \frac{q-2}{2q} \|u_2\|^2 - \frac{b(4-q)}{4q} \|u_2\|^4 - \frac{\mu(4-q)}{4q} \int_{\Omega} \phi_{u_2} u_2^2 d\xi - \frac{4-q}{4q} \int_{\Omega} |u_2|^4 d\xi. \end{aligned} \quad (5.14)$$

It follows from (5.13) that $I_\lambda(u_2) \leq \psi$. Thus $I_\lambda(u_2) = \psi$ and $\psi \neq 0$. This proves that u_2 is a positive ground state solution of problem (1.1). $\square$

Proof of Theorem 1.3. According to the (5.9), Lemmas 5.1, 5.2, 5.3 and 5.4, it is easy to prove Theorem 1.3. $\square$

Next, we apply the Clark critical point theorem (see [11]) to prove that problem (1.1) has at least $m - n$ pairs of negative energy solutions. Note that Y is a closed, symmetric subset of $X \setminus \{0\}$ at the origin. If $A \in Y$, define genus

$$\gamma(A) = \min\{k \in \mathbb{N} \mid \exists \phi \in C(A, \mathbb{R}^k \setminus \{0\}), \phi(-x) = \phi(x)\}$$

and

$$\Gamma(A) = \{A \in Y; \gamma(A) \geq k\}.$$

Theorem 5.1. (see [11]) Suppose $I_\lambda \in C^1(S_0^1(\Omega), \mathbb{R})$ is even, $I_\lambda(0) = 0$, I_λ satisfies $(PS)_c$ condition and the following conditions:

(I₁) There is an m dimensional subspace X_m and a constant $r > 0$, $S_r(0) = \{u \in S_0^1(\Omega) \mid \|u\| = r\}$ such that

$$\sup_{u \in S_r(0) \cap X_m} I_\lambda(u) < 0.$$

(I₂) If there exist an n dimensional subspace X_n ($n < m$) such that $\inf_{u \in X_n^\perp} I_\lambda(u) > -\infty$.

Then, I_λ has at least $m - n$ pairs of critical points with negative critical value, where

$$c_j = \inf_{A \in \Gamma_j} \max_{u \in A} I_\lambda(u).$$

Remark 5.1. Assume that $1 < q < 2$, $\mu < bS^3|\Omega|^{-\frac{1}{2}}$ and $c < 0$, then for each $\lambda \in \left(0, \frac{2qa^2S^2}{4(bS^2+1)(2-q)D_2}\right)$, the functional I_λ satisfies the $(PS)_c$ condition, where D is given by Lemma 3.3.

Proof. If $\lambda \in \left(0, \frac{2qa^2S^2}{4(bS^2+1)(2-q)D_2}\right)$, we have

$$\frac{b^2S^2}{4(bS^2+1)} - \frac{(2-q)\lambda}{2q}D_2 > 0 > c.$$

Then, the conclusion of Remark 5.1 is the corollary of Lemma 3.2. $\square$

Proof of Theorem 1.4. To prove Theorem 1.4, we shall use Theorem 5.1 and Remark 5.1. Note that $S_0^1(\Omega)$ is a Banach space, $I_\lambda \in C^1$ is an even functional and $I_\lambda(0) = 0$.

In the following, we divide the proof of Theorem 1.4 into two steps.

Step I: Let $0 \neq u \in X_m \subset S_0^1(\Omega)$, where X_m is an m dimensional subspace. We define $u = \tau_m v$ with $\|v\| = 1$ and $\tau_m = \|u\|$.

If $0 < \mu < bS^3|\Omega|^{-\frac{1}{2}}$, then

$$\begin{aligned} I_\lambda(u) &= \frac{\tau_m^2 a}{2} - \frac{\tau_m^4 b}{4} + \frac{\tau_m^4 \mu}{4} \int_{\Omega} \phi_v v^2 d\xi - \frac{\tau_m^q \lambda}{q} \int_{\Omega} |v|^q d\xi - \frac{\tau_m^4}{4} \int_{\Omega} |v|^4 d\xi \\ &\leq \frac{\tau_m^2 a}{2} - \frac{\tau_m^q \lambda}{q} \int_{\Omega} |v|^q d\xi - \frac{\tau_m^4}{4} \int_{\Omega} |v|^4 d\xi. \end{aligned}$$

Since $1 < q < 2$ and the finite dimensions space all norms are equivalent, there exists $\tau_0 > 0$ small enough such that

$$I_\lambda(u) \leq \frac{\tau_m^2 a}{2} - \frac{\tau_m^q \lambda}{q} \int_{\Omega} |v|^q d\xi - \frac{\tau_m^4}{4} \int_{\Omega} |v|^4 d\xi < 0 \text{ for all } \tau \in (0, \tau_0). \quad (5.15)$$

Besides, if $\mu \leq 0$, it is obvious that (5.15) still holds. Thus

$$\sup_{u \in S_r(0) \cap X_m} I_\lambda(u) < 0,$$

where $S_r(0) = \{u \in S_0^1(\Omega) \mid \|u\|_{S_0^1(\Omega)} = r\}$. This fact means that (I₁) in Theorem 5.1 holds.

Step II: If $-\infty < \mu \leq 0$, by (2.5) and the Hölder inequality, we get

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2} \|u\|^2 - \frac{b}{4} \|u\|^4 + \frac{\mu}{4} \int_{\Omega} \phi_u u^2 d\xi - \frac{\lambda}{q} \int_{\Omega} |u|^q d\xi - \frac{1}{4} \int_{\Omega} |u|^4 d\xi \\ &\geq \frac{a}{2} \|u\|^2 - \frac{b}{4} \|u\|^4 + \frac{\mu}{4} S^{-3} |\Omega|^{\frac{1}{2}} \|u\|^4 - \lambda S^{-\frac{q}{2}} |\Omega|^{\frac{4-q}{4}} \|u\|^q - S^{-2} \|u\|^4 \\ &> \frac{a}{2} \|u\|^2 - C_1 \|u\|^q - S^{-2} \|u\|^4, \end{aligned} \quad (5.16)$$

where $C_1 = b + \mu + \lambda S^{-\frac{q}{2}} |\Omega|^{\frac{4-q}{4}}$. Since $1 < q < 2$, we can choose an n dimensional subspace $X_n \subset X_m$ ($n < m$) such that

$$\inf_{u \in X_n^\perp} I_\lambda(u) > -\infty.$$

Besides, if $0 < \mu \leq bS^3 |\Omega|^{-\frac{1}{2}}$, it is easy to verify that (5.16) still holds. This fact means that (I_2) in Theorem 5.1 holds.

Let

$$\Lambda := \left\{ \Lambda_1, \frac{2qa^2S^2}{4(bS^2 + 1)(2 - q)D_2} \right\}.$$

It follows from Remark 5.1 and Lemma 3.3 that I_λ satisfies the $(PS)_c$ condition at all levels $c < 0$. Consequently, by Theorem 5.1, we know that the problem (1.1) has at least $m - n$ pairs of solutions. $\square$

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