Research Article

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New asymptotically quadratic conditions for Hamiltonian elliptic systems

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Abstract: This paper is concerned with the following Hamiltonian elliptic system

$$\begin{cases} -\Delta u + V(x)u = W_v(x, u, v), & x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = W_u(x, u, v), & x \in \mathbb{R}^N, \end{cases}$$

where $z = (u, v) : \mathbb{R}^N \rightarrow \mathbb{R}^2$, $V(x)$ and $W(x, z)$ are 1-periodic in x . By making use of variational approach for strongly indefinite problems, we obtain a new existence result of nontrivial solution under new conditions that the nonlinearity $W(x, z) := \frac{1}{2} V_\infty(x)|Az|^2 + F(x, z)$ is general asymptotically quadratic, where $V_\infty(x) \in (\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x and $\inf_{\mathbb{R}^N} V_\infty(x) > \min_{\mathbb{R}^N} V(x)$, and A is a symmetric non-negative definite matrix.

Keywords: Hamiltonian elliptic system; strongly indefinite functional; asymptotically quadratic

MSC: 35J10; 35J20

1 Introduction and main result

Consider the coupled nonlinear Schrödinger system

$$\begin{cases} -i\hbar \frac{\partial \phi_1}{\partial t} = \frac{\hbar^2}{2m} \Delta \phi_1 - a(x)\phi_2 + f(x, \phi)\phi_2, & x \in \mathbb{R}^N, t > 0, \\ -i\hbar \frac{\partial \phi_2}{\partial t} = \frac{\hbar^2}{2m} \Delta \phi_2 - a(x)\phi_1 + f(x, \phi)\phi_1, & x \in \mathbb{R}^N, t > 0, \\ \phi_j = \phi_j(x, t) \in \mathbb{C}, j = 1, 2, \\ \phi_j(x, t) \rightarrow 0, \text{ as } |x| \rightarrow +\infty, t > 0, j = 1, 2, \end{cases} \quad (1.1)$$

where $\phi = (\phi_1, \phi_2)$, i is the imaginary unit, m is the mass of a particle, $\hbar$ is the Planck constant, $a(x)$ is potential function and f is coupled nonlinear function modeling various types of interaction effect among many particles. It is well known that system (1.1) has applications in many physical problems, especially in nonlinear optics and in Bose-Einstein condensates theory for multispecies Bose-Einstein condensates (see [15, 19] and the references therein). Assume that $f(x, e^{i\theta}\phi) = f(x, \phi)$ for $\theta \in [0, 2\pi]$. We will look for standing waves of the form

$$\phi_1(x, t) = e^{i\omega t}u(x), \quad \phi_2(x, t) = e^{i\omega t}v(x),$$

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which propagate without changing their shape and thus have a soliton-like behavior. In general, the above coupled nonlinear Schrödinger system leads to the following elliptic system

$$\begin{cases} -\Delta u + V(x)u = W_v(x, u, v), & x \in \mathbb{R}^N, \\ -\Delta v + V(x)v = W_u(x, u, v), & x \in \mathbb{R}^N, \end{cases} \quad (1.2)$$

where $N \geq 3$, $z := (u, v) : \mathbb{R}^N \rightarrow \mathbb{R} \times \mathbb{R}$, $V : \mathbb{R}^N \rightarrow \mathbb{R}$ and $W : \mathbb{R}^N \times \mathbb{R}^2 \rightarrow \mathbb{R}$. Moreover, according to [5], this type of system is called Hamiltonian elliptic system, which has strongly indefinite variational structure from a viewpoint of variational methods. In the present paper, our purpose is to establish new existence results of nontrivial solutions of system (1.2) under new nonlinear conditions.

In the past few decades, by using variational techniques, a number of important results of the existence and multiplicity of solutions for system (1.2) defined on the bounded domain Ω have been established with W satisfying various conditions, see for example [7, 9, 10]. Recently, many authors began to focus on system (1.2) defined on the whole space $\mathbb{R}^N$. Their most interesting studies were to establish the existence of multiple solutions, ground states and semiclassical states, see [1, 2, 4, 6, 12, 18, 21, 25–29, 31, 32] and their references therein. In these works, a huge machinery is needed to obtain existence and multiplicity of solutions, such as fractional Sobolev spaces, generalized mountain pass theorem, generalized linking theorem, reduction Nehari method and many others. Besides, based on variational arguments, some related problems involving the nonlocal elliptic equations have been received increasingly more attention on mathematical studies. Tang and Chen [20] studied the ground state solution of Nehari-Pohozaev type for the nonlocal Schrödinger-Kirchhoff problems by developing some new analytic techniques. More relevant results and recent developments, we refer the readers to [16] for elliptic problems and the monograph [17] for nonlocal fractional problems.

One of the main difficulties in dealing with system (1.2) relies on the lack of embedding compactness due to the unboundedness of the domain. In some of the above quoted papers this difficulty was overcome by imposing periodicity condition both on the potential V and the nonlinearity W . Along this direction, the papers [8, 13, 14, 32] studied system (1.2) with periodic and global super-quadratic growth, and the existence and multiplicity results are obtained. Subsequently, the authors in [30] weakened the global super-quadratic case to the local super-quadratic case and proved the existence of ground state solutions and infinitely many geometrically distinct solutions by using a new perturbation approach developed by Tang and his collaborators [22, 23].

Motivated by the researches about the Hamiltonian elliptic systems, we continue to study system (1.2) under general conditions, and assume the following basic assumptions.

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(x)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, and $\max_{\mathbb{R}^N} V(x) := \beta \geq V(x) \geq \min_{\mathbb{R}^N} V(x) := \beta_0 > 0$;

(W1) $W \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$, $W(x, z)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, $W(x, z) \geq 0$;

(W2) $|W_z(x, z)| = o(|z|)$, as $|z| \rightarrow 0$ uniformly in $x \in \mathbb{R}^N$.

In the aforementioned references [12, 25–28], the following asymptotically quadratic condition and other technique conditions for the nonlinearity W are commonly assumed:

(W3) $W(x, z) := \frac{1}{2} V_\infty(x) |z|^2 + F(x, z)$, where $V_\infty(x) \in (\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x , and $\inf_{\mathbb{R}^N} V_\infty(x) > \max_{\mathbb{R}^N} V(x) := \beta$.

(W4)

$$\lim_{|z| \rightarrow \infty} \frac{|W(x, z) - \frac{1}{2} V_\infty(x) |z|^2|}{|z|^2} = 0, \text{ uniformly in } x \in \mathbb{R}^N;$$

(W5) $\tilde{W}(x, z) := \frac{1}{2} W_z(x, z)z - W(x, z) \geq 0$, and there exist $\delta_0 > 0$ such that

$$\tilde{W}(x, z) > 0, \text{ if } 0 < |z| \leq \delta_0.$$

Observe that conditions (W4) and (W5) play an important role for showing that any Palais-Smale sequence or Cerami sequence is bounded in the aforementioned works. However, there are many functions do not satisfy these conditions, for example,

$$W(x, u, v) = aV_\infty(x) \left| u + \frac{1}{2}v \right|^2 \left(1 - \frac{1}{\ln^2(e + |u + \frac{1}{2}v|)} \right).$$

In a recent paper [14], making use of some special techniques, Liao, Tang, Zhang and Qin studied the existence of solutions for system (1.2) under more general super-quadratic conditions, that is,

(SQ) there exist $a, b > 0$ such that

$$\lim_{|au+bv| \rightarrow \infty} \frac{|W(x, u, v)|}{|au+bv|^2} = \infty, \text{ a.e. } x \in \mathbb{R}^N.$$

Clearly, this condition is weaker than the usual super-quadratic condition

$$\lim_{|u|+|v| \rightarrow \infty} \frac{|W(x, u, v)|}{|u|^2 + |v|^2} = \infty, \text{ uniformly in } x \in \mathbb{R}^N.$$

Very recently, the singularly perturbed problem with super-quadratic condition (SQ)

$$\begin{cases} -\epsilon^2 \Delta u + u + V(x)v = Q(x)W_v(u, v), & x \in \mathbb{R}^N, \\ -\epsilon^2 \Delta v + v + V(x)u = Q(x)W_u(u, v), & x \in \mathbb{R}^N, \end{cases}$$

has been investigated in [21], where the authors proved the existence of semiclassical ground state solutions and generalized the results in [4].

Inspired by super-quadratic case [14] and [21], we further consider the general periodic asymptotically quadratic case and establish the existence result of solutions. In addition to (V), (W1) and (W2), we introduce the following new asymptotically quadratic conditions for the nonlinearity W :

(W3') there exists symmetric non-negative definite matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2}$$

such that $W(x, z) := \frac{1}{2} V_\infty(x) |Az|^2 + F(x, z)$, where $V_\infty(x) \in (\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x , and $\inf_{\mathbb{R}^N} V_\infty(x) > \beta_0$;

(W4')

$$\lim_{|(a_{11}+a_{12})u+(a_{21}+a_{22})v| \rightarrow \infty} \frac{|W(x, z) - \frac{1}{2} V_\infty(x) |Az|^2|}{|(a_{11}+a_{12})u+(a_{21}+a_{22})v|^2} = 0 \text{ uniformly in } x \in \mathbb{R}^N;$$

(W5') $\tilde{W}(x, z) \geq 0$, and there exists constant $\delta_0 \in (0, \beta_0)$ such that

$$\frac{|(a_{21}+a_{22})W_u(x, z) + (a_{11}+a_{12})W_v(x, z)|}{|z|} \geq \theta \beta_0 \min\{(a_{11}+a_{12}), (a_{21}+a_{22})\}$$

$$\implies \tilde{W}(x, z) \geq \delta_0.$$

It is worth pointing out that conditions (W3'), (W4') and (W5') are different from usual conditions (W3), (W4) and (W5) and weaken these conditions. To the best of our knowledge, it seems that there is no work considered this problem in the literature before. So this result obtained in this paper is new, moreover, it can be viewed as a complement and an extension of [14] and [21]. However, it is difficult for us to prove the linking geometry and boundedness of Cerami sequences under the conditions (W3'), (W4') and (W5') since the arguments as [26, 27] (depending on the behavior of $W(x, z)$ as $|z|^2 = |u|^2 + |v|^2 \rightarrow \infty$) cannot be applied directly. To do this, some new techniques need to be introduced in the proof.

Based on the conditions given above on V and W , we can get the following theorem.

Theorem 1.1. Assume that V and W satisfy (V), (W1), (W2), (W3'), (W4') and (W5'). Then system (1.2) has a nontrivial solution.

Before proceeding to the proof of Theorem 1.1, we give a nonlinear example to illustrate the assumptions.

Example 1.2. For example, let

$$W(x, u, v) = aV_\infty(x) \left| u + \frac{1}{2}v \right|^2 \left(1 - \frac{1}{\ln^2(e + |u + \frac{1}{2}v|)} \right),$$

where $V_\infty(x) \in (\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, and $\inf_{\mathbb{R}^N} V_\infty(x) > \beta_0$. By a straightforward computation, we can see that all conditions (W1), (W2), (W3'), (W4') and (W5') are satisfied with $a = \frac{5}{2}$ and $A = \begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix}$. Note that $W(x, u, v) = \tilde{W}(x, u, v) = 0$ for $u = -\frac{1}{2}v$, $v \in \mathbb{R}$, thus W does not satisfy (W4) and (W5).

The remainder of this paper is organized as follows. In Sect. 2, we introduce the variational setting of system (1.1). In Sect. 3, we analyze the geometry structure of the functional and property of Cerami sequence, and give the proof of Theorem 1.1.

2 Variational setting

Throughout this paper, we make use of the following notations. $\|\cdot\|_s$ denotes the usual norm of the space L^s , $1 \leq s \leq \infty$; $(\cdot, \cdot)_2$ denotes the usual L^2 inner product; c or c_i ($i = 1, 2, \dots$) are some different positive constants.

In the following, we establish the variational setting of system (1.2). According to condition (V), we define the following Hilbert space

$$H := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx < \infty \right\}$$

with the inner product

$$(u, v)_H = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) dx,$$

and the reduced norm

$$\|u\|_H^2 = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx \right)^{\frac{1}{2}}.$$

Let the working space $E = H \times H$. Then E is a Hilbert space with the standard inner product

$$(z_1, z_2) = ((u_1, v_1), (u_2, v_2))_H = (u_1, u_2)_H + (v_1, v_2)_H$$

for $z_i = (u_i, v_i) \in E$, $i = 1, 2$, and the corresponding norm

$$\|z\| = \left(\|u\|_H^2 + \|v\|_H^2 \right)^{\frac{1}{2}}, \quad \forall z = (u, v) \in E.$$

Observe that, the natural functional associated with system (1.2) is given by

$$\Phi(z) = \frac{1}{2} \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) dx - \int_{\mathbb{R}^N} W(x, u, v) dx, \quad \forall z = (u, v) \in E. \quad (2.1)$$

Moreover, according to conditions (V), (W1) and (W2), it is easy to prove that $\Phi \in C^1(E, \mathbb{R})$, and for any $z = (u, v)$ and $\eta = (\varphi, \psi) \in E$, there holds

$$\begin{aligned} \langle \Phi'(z), \eta \rangle &= \int_{\mathbb{R}^N} [\nabla u \cdot \nabla \psi + \nabla v \cdot \nabla \varphi + V(x)(u\psi + v\varphi)] dx - \int_{\mathbb{R}^N} W_z(x, u, v)\eta dx \\ &= \int_{\mathbb{R}^N} [\nabla u \cdot \nabla \psi + \nabla v \cdot \nabla \varphi + V(x)(u\psi + v\varphi)] dx \\ &\quad - \int_{\mathbb{R}^N} [W_u(x, u, v)\varphi + W_v(x, u, v)\psi] dx \end{aligned} \quad (2.2)$$

Following the idea of De Figueiredo and Felmer [7] (see also Hulshof and Van Der Vorst [9]), for any $z = (u, v)$ and $w = (w_1, w_2) \in E$, we introduce a bilinear form on $E \times E$ as

$$\mathcal{B}[z, w] = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla w_2 + V(x)uw_2 + \nabla v \cdot \nabla w_1 + V(x)vw_1) dx.$$

It is clear that $\mathcal{B}[z, w]$ is continuous and symmetric, and hence $\mathcal{B}$ induces a self-adjoint bounded linear operator $\mathcal{L} : E \rightarrow E$ such that

$$\mathcal{B}[z, w] = (\mathcal{L}z, w), \quad \forall z, w \in E.$$

By a direct computation, we can deduce that

$$\mathcal{L}z = (v, u), \quad \forall z = (u, v) \in E.$$

Moreover, it is easy to see that 1 and -1 are two eigenvalues of the operator $\mathcal{L}$, and the corresponding eigenspaces are

$$E^+ = \{(u, u) : u \in H\} \text{ for } \lambda = 1, \quad E^- = \{(u, -u) : u \in H\} \text{ for } \lambda = -1.$$

Hence, based on the above fact, the working space E has the following decomposition

$$E^+ = \{(u, u) : u \in H\}, \quad E^- = \{(u, -u) : u \in H\}.$$

Clearly, $E = E^- \oplus E^+$. Furthermore, for $z = (u, v) \in E$, set

$$z^+ = \left(\frac{u+v}{2}, \frac{u+v}{2} \right) \text{ and } z^- = \left(\frac{u-v}{2}, \frac{v-u}{2} \right).$$

Then we have

$$\mathcal{B}[z^+, z^-] = (\mathcal{L}z^+, z^-) = 0, \quad \forall z^\pm \in E^\pm.$$

Now we define the functional $\mathcal{F} : E \rightarrow \mathbb{R}$ as

$$\mathcal{F}(z) = \mathcal{F}(u, v) = \frac{1}{2} \mathcal{B}[z, z] = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) dx.$$

Computing directly, we get

$$\begin{aligned} \mathcal{F}(z) &= \frac{1}{2} \mathcal{B}[z, z] = \frac{1}{2} \mathcal{B}[z^+ + z^-, z^+ + z^-] \\ &= \frac{1}{2} (\mathcal{B}[z^+, z^+] + \mathcal{B}[z^-, z^-]) \\ &= \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2). \end{aligned}$$

Therefore, the functional Φ defined by (2.1) can be rewritten the following form

$$\Phi(z) = \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) - \int_{\mathbb{R}^N} W(x, z) dx, \quad \text{for } z = z^+ + z^- \in E, \quad (2.3)$$

Obviously, Φ is strongly indefinite and the critical points of Φ are solutions of system (1.2) (see [3]), and for $z, \varphi \in E$ we have

$$\langle \Phi'(z), \varphi \rangle = (z^+, \varphi^+) - (z^-, \varphi^-) - \int_{\mathbb{R}^N} W_z(x, z) \varphi dx. \quad (2.4)$$

On the other hand, according to the embedding theorem and condition (V), H embeds continuously into $L^p(\mathbb{R}^N)$ for all $p \in [2, 2^*]$ and compactly into $L^p_{loc}(\mathbb{R}^N)$ for all $p \in [1, 2^*]$. Therefore, it is easy to see that E embeds continuously into $L^p := L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for all $p \in [2, 2^*]$ and compactly into $L^p_{loc} := L^p_{loc}(\mathbb{R}^N) \times L^p_{loc}(\mathbb{R}^N)$ for all $p \in [1, 2^*]$.

3 Proof of main result

In this section, we will in the sequel focus on the proof of Theorem 1.1. Firstly, we need verify the linking geometry structure of the functional Φ .

Lemma 2.1. Assume that (V), (W1) and (W2) hold, then there exists a constant $\rho > 0$ such that $\kappa_0 := \inf \Phi(S^+_\rho) > 0$, where $S^+_\rho = \partial B_\rho \cap E^+$ and $B_\rho = \{z \in E : \|z\| \leq \rho\}$.

Applying the embedding theorem and some standard arguments (see [3] and [21]), one can check easily the Lemma 2.1, and omit the details of the proof.

Lemma 2.2 Assume that (V), (W1), (W2), (W3') and (W4') hold. Let $e = (e_0, e_0) \in E^+$ with $\|e\| = 1$. Then, there exists a constant $r_0 > 0$ such that $\sup \Phi(\partial Q) \leq 0$, where

$$Q = \{w + se : w = (u, -u) \in E^-, s \geq 0, \|w + se\| \leq r_0\}. \quad (3.1)$$

Proof: Obviously, it follows from (W1) that $\Phi(z) \leq 0$ for $z \in E^-$. Next, it is sufficient to show that $\Phi(z) \rightarrow -\infty$ as $z \in E^- \oplus \mathbb{R}e$ and $\|z\| \rightarrow \infty$. Arguing indirectly, assume that for some sequence $\{w_n + s_n e\} \subset E^- \oplus \mathbb{R}e$ with $\|w_n + s_n e\| \rightarrow \infty$ and $\Phi(w_n + s_n e) \geq 0, \forall n \in \mathbb{N}$. Set $\xi_n = (w_n + s_n e) / \|w_n + s_n e\| = \xi_n^- + \tau_n e$, then $\|\xi_n^- + \tau_n e\| = 1$. Up to a subsequence, we may assume that $\xi_n \rightharpoonup \xi$ in E , then $\xi_n^- \rightharpoonup \xi^-$ in E , $\xi_n \rightarrow \xi$ a.e. on $\mathbb{R}^N$ and $\tau_n \rightarrow \tau$. For convenience of notation, we write $\xi_n^- = (v_n, -v_n)$ and $\xi^- = (v, -v)$. According to the fact $\Phi(w_n + s_n e) \geq 0$ we get

$$0 \leq \frac{\Phi(w_n + s_n e)}{\|w_n + s_n e\|^2} = \frac{\tau_n^2}{2} \|e\|^2 - \frac{1}{2} \|\xi_n^-\|^2 - \int_{\mathbb{R}^N} \frac{W(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx. \quad (3.2)$$

By (3.2) we can deduce that $\tau > 0$. Observe that, since $e \in E^+$, there is a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$\tau^2 \|e\|^2 - \|\xi^-\|^2 - \int_{\Omega} V_\infty(x) |A(\xi^- + \tau e)|^2 dx < 0. \quad (3.3)$$

Moreover, by (W3') and (3.2) we obtain

$$\begin{aligned} 0 &\leq \frac{\tau_n^2}{2} \|e\|^2 - \frac{1}{2} \|\xi_n^-\|^2 - \int_{\Omega} \frac{W(x, w_n + s_n e)}{\|w_n + s_n e\|^2} \\ &= \frac{\tau_n^2}{2} \|e\|^2 - \frac{1}{2} \|\xi_n^-\|^2 - \frac{1}{2} \int_{\Omega} V_\infty(x) |A(w_n + s_n e)|^2 dx - \int_{\Omega} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx. \end{aligned} \quad (3.4)$$

Clearly, according to (W3') and (W4'), we know that

$$|F(x, z)| \leq c_0 |(a_{11} + a_{12})u + (a_{21} + a_{22})v|^2$$

for some $c_0 > 0$ and any $z = (u, v) \in E$, and

$$\frac{|F(x, z)|}{|(a_{11} + a_{12})u + (a_{21} + a_{22})v|^2} \rightarrow 0, \text{ as } |(a_{11} + a_{12})u + (a_{21} + a_{22})v|^2 \rightarrow \infty.$$

Since $\xi_n^- + \tau_n e \rightharpoonup \xi^- + \tau e$ in E , then $\xi_n^- + \tau_n e \rightarrow \xi^- + \tau e$ in $L^2(\Omega)$. Next we claim that

$$\int_{\Omega} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx = o(1). \quad (3.5)$$

First, according to $\tau > 0$, we need to show $(a_{11} + a_{12})(\tau e_0 + v) + (a_{21} + a_{22})(\tau e_0 - v) \neq 0$. Arguing indirectly, we assume that $(a_{11} + a_{12})(\tau e_0 + v) + (a_{21} + a_{22})(\tau e_0 - v) = 0$, that is $((a_{11} + a_{12}) + (a_{11} + a_{12})) \tau e_0 + ((a_{11} + a_{12}) - (a_{11} + a_{12})) v = 0$, then $a_{11} + a_{12} \neq a_{21} + a_{22}$ and by (3.2) we get

$$\begin{aligned} \tau^2 &\geq \liminf_{n \rightarrow \infty} \|\xi_n^-\|^2 \geq \|\xi^-\|^2 \\ &= \int_{\mathbb{R}^N} |\nabla \xi^-|^2 + V(x) |\xi^-|^2 dx \\ &= \frac{(a_{11} + a_{12} + a_{21} + a_{22})^2}{(a_{11} + a_{12} - a_{21} - a_{22})^2} \tau^2 \int_{\mathbb{R}^N} |\nabla e|^2 + V(x) |e|^2 dx \\ &> \tau^2 \|e\|^2 = \tau^2, \end{aligned}$$

which yields a contradiction. Therefore, from the above fact we can deduce that

$$\begin{aligned} &|(a_{11} + a_{12})(s_n e_0 + u_n) + (a_{21} + a_{22})(s_n e_0 - u_n)| \\ &= \|w_n + s_n e\| |(a_{11} + a_{12})(\tau_n e_0 + v_n) + (a_{21} + a_{22})(\tau_n e_0 - v_n)| \rightarrow +\infty. \end{aligned}$$

Let $\phi_n = (a_{11} + a_{12})(\tau_n e_0 + v_n) + (a_{21} + a_{22})(\tau_n e_0 - v_n)$. By Lebesgue dominated convergence theorem and (W4) we have

$$\int_{\Omega} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx = \int_{\Omega} \frac{F(x, s_n e_0 + u_n, s_n e_0 - u_n)}{|(a_{11} + a_{12})(s_n e_0 + u_n) + (a_{21} + a_{22})(s_n e_0 - u_n)|^2} |\phi_n|^2 dx = o(1).$$

Moreover, by the embedding theorem, (3.4) and (3.5) we deduce that

$$0 \leq \tau^2 \|e\|^2 - \|\xi^-\|^2 - \int_{\Omega} V_{\infty}(x) |A(\xi^- + \tau e)|^2 dx,$$

which implies a contradiction to (3.3). $\square$

In order to end the proof of Theorem 1.1, we will use the following generalized linking theorem developed by Li and Szulkin [11].

Lemma 2.3 Assume that Z is a Hilbert space with the decomposition $Z = Z^- \oplus Z^+$, and let $\Phi \in C^1(Z, \mathbb{R})$ be of the form

$$\Phi(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - \Psi(u), \quad u = u^- + u^+ \in Z^- \oplus Z^+.$$

Assume that the following assumptions are satisfied:

- (Φ_1) $\Psi \in C^1(Z, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;
- (Φ_2) Ψ' is weakly sequentially continuous;
- (Φ_3) there exist $R > \rho > 0$ and $e \in Z^+$ with $\|e\| = 1$ such that

$$\kappa := \inf \Phi(S_{\rho}^+) > \sup \Phi(\partial Q),$$

where

$$S_{\rho}^+ = \{u \in Z^+ : \|u\| = \rho\}, \quad Q = \{v + se : v \in Z^-, s \geq 0, \|v + se\| \leq R\}.$$

Then there exist a constant $c \in [\kappa, \sup \Phi(Q)]$ and a sequence $\{u_n\} \subset X$ satisfying

$$\Phi(u_n) \rightarrow c, \quad (1 + \|u_n\|) \|\Phi'(u_n)\| \rightarrow 0.$$

For the sake of convenience, let

$$\Psi(z) = \int_{\mathbb{R}^N} W(x, z) dx = \int_{\mathbb{R}^N} W(x, u, v) dx.$$

Using some standard arguments, we can verify Ψ is nonnegative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous. Moreover, applying Lemma 2.1, Lemma 2.2 and Lemma 2.3, we can demonstrate the following result.

Lemma 2.4. Assume that (V), (W1), (W2), (W3'), (W4') hold. Then there exists a constant $c_* \in [\kappa_0, \sup \Phi(Q)]$ and a sequence $\{z_n\} = \{(u_n, v_n)\} \subset E$ satisfying

$$\Phi(z_n) \rightarrow c_*, \quad (1 + \|z_n\|)\|\Phi'(z_n)\| \rightarrow 0. \quad (3.6)$$

Lemma 2.5. Assume that (V), (W1), (W2), (W3'), (W4') and (W5') are satisfied. Then any sequence $\{z_n\} = \{(u_n, v_n)\} \subset E$ satisfying (3.6) is bounded in E .

Proof: In order to prove the boundedness of $\{z_n\} = \{(u_n, v_n)\}$, arguing by contradiction we suppose that $\|z_n\| \rightarrow \infty$ as $n \rightarrow +\infty$. Let

$$\begin{aligned} \xi_n &= \frac{z_n}{\|z_n\|} = (\varphi_n, \psi_n), \\ \hat{z}_n &= (\hat{u}_n, \hat{v}_n) := \left(\frac{(a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n}{2(a_{11} + a_{12})}, \frac{(a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n}{2(a_{21} + a_{22})} \right), \\ \hat{\xi}_n &= (\hat{\varphi}_n, \hat{\psi}_n) := \frac{\hat{z}_n}{\|z_n\|} = \left(\frac{(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n}{2(a_{11} + a_{12})}, \frac{(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n}{2(a_{21} + a_{22})} \right). \end{aligned}$$

By (W1), (2.3) and (3.6) we obtain

$$2c_* + o(1) = \|z_n^+\|^2 - \|z_n^-\|^2 - 2 \int_{\mathbb{R}^N} W(x, z_n) dx \leq \|z_n^+\|^2 - \|z_n^-\|^2, \quad (3.7)$$

and

$$c_* + o(1) = \int_{\mathbb{R}^N} \tilde{W}(x, z_n) dx. \quad (3.8)$$

Computing directly, we have

$$\begin{aligned} & \frac{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \|\hat{z}_n\|^2 \\ &= \|(a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n\|_H^2 \\ &= (a_{11} + a_{12})^2 \|u_n\|_H^2 + (a_{21} + a_{22})^2 \|v_n\|_H^2 \\ &\quad + 2(a_{11} + a_{12})(a_{21} + a_{22}) \int_{\mathbb{R}^N} (\nabla u_n \nabla v_n + V(x)u_n v_n) \\ &= (a_{11} + a_{12})^2 \|u_n\|_H^2 + (a_{21} + a_{22})^2 \|v_n\|_H^2 \\ &\quad + 2(a_{11} + a_{12})(a_{21} + a_{22}) \left(\Phi(z_n) + \int_{\mathbb{R}^N} W(x, u_n, v_n) dx \right) \\ &\geq \min\{(a_{11} + a_{12})^2, (a_{21} + a_{22})^2\} \|z_n\|^2 + 2(a_{11} + a_{12})(a_{21} + a_{22})(c_* + o(1)), \end{aligned} \quad (3.9)$$

which implies that

$$\|z_n\| \leq \frac{2(a_{11} + a_{12})(a_{21} + a_{22})}{\sqrt{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \min\{a_{11} + a_{12}, a_{21} + a_{22}\}} \|\hat{z}_n\|. \quad (3.10)$$

Note that

$$\begin{aligned}\|\hat{\xi}_n\|^2 &= \frac{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2}{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2} \|(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n\|_H^2 \\ &\leq \frac{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2}{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2} ((a_{11} + a_{12})\|\varphi_n\|_H + (a_{21} + a_{22})\|\psi_n\|_H)^2 \\ &\leq \frac{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2}{2(a_{11} + a_{12})^2(a_{21} + a_{22})^2} ((a_{11} + a_{12})^2\|\varphi_n\|_H^2 + (a_{21} + a_{22})^2\|\psi_n\|_H^2) \\ &\leq \frac{[(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2]^2}{2(a_{11} + a_{12})^2(a_{21} + a_{22})^2} (\|\varphi_n\|_H^2 + \|\psi_n\|_H^2) \\ &= \frac{[(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2]^2}{2(a_{11} + a_{12})^2(a_{21} + a_{22})^2} \|\xi_n\|^2 = \frac{[(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2]^2}{2(a_{11} + a_{12})^2(a_{21} + a_{22})^2},\end{aligned}$$

which implies that $\{\hat{\xi}_n\}$ is bounded in E . So there are two cases need to discuss: vanishing case and non-vanishing case. If $\{\hat{\xi}_n\}$ is vanishing, then

$$\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |\hat{\xi}_n|^2 dx = 0,$$

then by Lions's concentration compactness principle [24, Lemma 1.21] we have $(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n \rightarrow 0$ in L^s for $2 < s < 2^*$. Set $\theta \in (0, 1)$ and

$$\begin{aligned}\Omega_n &:= \left\{ x \in \mathbb{R}^N : \frac{|(a_{21} + a_{22})W_u(x, z_n) + (a_{11} + a_{12})W_v(x, z_n)|}{|z_n|} \right. \\ &\quad \left. \leq \theta\beta_0 \min\{(a_{11} + a_{12}), (a_{21} + a_{22})\} \right\}.\end{aligned}$$

Hence, by (W1), (W2), (W3') and (W4'), it follows from (3.9) and Hölder inequality that

$$\begin{aligned}&\int_{\Omega_n} |(a_{21} + a_{22})W_u(x, z_n) + (a_{11} + a_{12})W_v(x, z_n)| |(a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n| dx \\ &\leq \int_{\Omega_n} \frac{|(a_{21} + a_{22})W_u(x, z_n) + (a_{11} + a_{12})W_v(x, z_n)|}{|z_n|} |z_n| |(a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n| dx \\ &\leq \theta\beta_0 \min\{(a_{11} + a_{12}), (a_{21} + a_{22})\} \int_{\Omega_n} |z_n| |(a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n| dx \\ &\leq \theta\beta_0 \min\{(a_{11} + a_{12})^2, (a_{21} + a_{22})^2\} \left(\int_{\mathbb{R}^N} |z_n|^2 dx \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}^N} |(a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n|^2 dx \right)^{1/2} \\ &\leq \theta \min\{(a_{11} + a_{12}), (a_{21} + a_{22})\} \|z_n\| \|(a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n\|_H \\ &= \left(\theta \frac{2(a_{11} + a_{12})(a_{21} + a_{22}) \min\{(a_{11} + a_{12}), (a_{21} + a_{22})\}}{\sqrt{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2}} \right) \|z_n\| \|\hat{z}_n\| \\ &\leq \left(\theta \frac{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \right) \|\hat{z}_n\|^2.\end{aligned}\tag{3.11}$$

On the other hand, set $q' = q/(q-1)$, then $2 < 2q' < 2^*$, by (W5'), (3.8), (3.9) and Hölder inequality we obtain

$$\begin{aligned}
 & \int_{\mathbb{R}^N \setminus \Omega_n} |(a_{21} + a_{22})W_u(x, z_n) + (a_{11} + a_{12})W_v(x, z_n)| |(a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n| dx \\
 &= \int_{\mathbb{R}^N \setminus \Omega_n} \frac{|(a_{21} + a_{22})W_u(x, z_n) + (a_{11} + a_{12})W_v(x, z_n)|}{|z_n|} \\
 & \quad \times |\xi_n| |(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n| \|z_n\|^2 dx \\
 &\leq c_1 \left(\int_{\mathbb{R}^N \setminus \Omega_n} \tilde{W}(x, z_n) dx \right)^{1/q} \|\xi_n\|_{2q'} \|(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n\|_{2q'} \|z_n\|^2 \\
 &\leq c_1 (c^* + o(1))^{1/q} \|\xi_n\|_{2q'} \|(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n\|_{2q'} \|z_n\|^2 \\
 &\leq c_1 (c^* + o(1))^{1/q} \|\xi_n\|_{2q'} \|(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n\|_{2q'} \\
 & \quad \times \frac{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{((a_{11} + a_{12})^2 + (a_{21} + a_{22})^2) \min\{(a_{11} + a_{12})^2, (a_{21} + a_{22})^2\}} \|\hat{z}_n\|^2 \\
 &\leq \left(\frac{1-\theta}{2} \right) \frac{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \|\hat{z}_n\|^2.
 \end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12), and using (2.2) and (3.9), we have

$$\begin{aligned}
 & \frac{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \|\hat{z}_n\|^2 + o(1) \\
 &= \frac{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \|\hat{z}_n\|^2 - 2(a_{11} + a_{12})(a_{21} + a_{22}) \langle \Phi'(z_n), \hat{z}_n \rangle \\
 &= \int_{\mathbb{R}^N} [(a_{21} + a_{22})W_u(x, z_n) + (a_{11} + a_{12})W_v(x, z_n)] ((a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n) dx \\
 &= \int_{\Omega_n} [(a_{21} + a_{22})W_u(x, z_n) + (a_{11} + a_{12})W_v(x, z_n)] ((a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n) dx \\
 & \quad + \int_{\mathbb{R}^N \setminus \Omega_n} [(a_{21} + a_{22})W_u(x, z_n) + (a_{11} + a_{12})W_v(x, z_n)] ((a_{11} + a_{12})u_n + (a_{21} + a_{22})v_n) dx \\
 &\leq \left[\theta \frac{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} + \left(\frac{1-\theta}{2} \right) \frac{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \right] \|\hat{z}_n\|^2 \\
 &= \left(\frac{1+\theta}{2} \right) \times \frac{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \|\hat{z}_n\|^2.
 \end{aligned} \tag{3.13}$$

This contradiction shows that the vanishing case does not occur. Therefore, we get $\delta \neq 0$.

If necessary going to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{N}}(k_n)} |\hat{\xi}_n|^2 dx > \frac{\delta}{2}.$$

Since

$$|\hat{\xi}_n|^2 = \frac{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2}{4(a_{11} + a_{12})^2(a_{21} + a_{22})^2} |(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n|^2,$$

we can obtain

$$\int_{B_{1+\sqrt{N}}(k_n)} |(a_{11} + a_{12})\varphi_n + (a_{21} + a_{22})\psi_n|^2 dx > \frac{2(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \delta.$$

Let us define $\tilde{\varphi}_n(x) = \varphi_n(x + k_n)$, $\tilde{\psi}_n(x) = \psi_n(x + k_n)$, then

$$\int_{B_{1+\sqrt{N}}(0)} |(a_{11} + a_{12})\tilde{\varphi}_n + (a_{21} + a_{22})\tilde{\psi}_n|^2 dx > \frac{2(a_{11} + a_{12})^2(a_{21} + a_{22})^2}{(a_{11} + a_{12})^2 + (a_{21} + a_{22})^2} \delta. \quad (3.14)$$

Now we define $\tilde{u}_n(x) = u_n(x + k_n)$, $\tilde{v}_n(x) = v_n(x + k_n)$, then $\tilde{\varphi}_n = \tilde{u}_n/\|z_n\|$, $\tilde{\psi}_n = \tilde{v}_n/\|z_n\|$. After passing to a subsequence, we get

$$(a_{11} + a_{12})\tilde{\varphi}_n(x) + (a_{21} + a_{22})\tilde{\psi}_n(x) \rightharpoonup (a_{11} + a_{12})\tilde{\varphi}(x) + (a_{21} + a_{22})\tilde{\psi}(x)$$

in E ,

$$(a_{11} + a_{12})\tilde{\varphi}_n(x) + (a_{21} + a_{22})\tilde{\psi}_n(x) \rightarrow (a_{11} + a_{12})\tilde{\varphi}(x) + (a_{21} + a_{22})\tilde{\psi}(x)$$

in L^s_{loc} for $2 \leq s < 2^*$ and

$$(a_{11} + a_{12})\tilde{\varphi}_n(x) + (a_{21} + a_{22})\tilde{\psi}_n(x) \rightarrow (a_{11} + a_{12})\tilde{\varphi}(x) + (a_{21} + a_{22})\tilde{\psi}(x)$$

a.e. on $\mathbb{R}^N$. Obviously, it follows from (3.14) that $(a_{11} + a_{12})\tilde{\varphi}(x) + (a_{21} + a_{22})\tilde{\psi}(x) \neq 0$. For a.e. $x \in \{y \in \mathbb{R}^N : (a_{11} + a_{12})\tilde{\varphi}(y) + (a_{21} + a_{22})\tilde{\psi}(y) \neq 0\}$, we have

$$\lim_{n \rightarrow \infty} |(a_{11} + a_{12})\tilde{u}_n(x) + (a_{21} + a_{22})\tilde{v}_n(x)| = \infty. \quad (3.15)$$

Set $\eta_n = \eta(x + k_n)$ for each $\eta = (\mu, v) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$. According to (2.4) and (W3'), it is easy to show that

$$\begin{aligned} \langle \Phi'(z_n), \eta_n \rangle &= (z_n^+ - z_n^-, \eta_n) - (V_\infty A z_n, \eta_n)_2 - \int_{\mathbb{R}^N} F_z(x, z_n) \eta_n dx \\ &= (z_n^+ - z_n^-, \eta_n) - (V_\infty A z_n, \eta_n)_2 - \int_{\mathbb{R}^N} F_z(x, z_n) \eta_n dx \\ &= (z_n^+ - z_n^-, \eta_n) - (V_\infty A z_n, \eta_n)_2 - \int_{\mathbb{R}^N} F_z(x, z_n) \eta_n dx \\ &= \|z_n\| \left[(\tilde{\xi}_n^+ - \tilde{\xi}_n^-, \eta) - (V_\infty A \tilde{\xi}_n, \eta)_2 - \int_{\mathbb{R}^N} F_z(x, \tilde{z}_n) \eta \frac{|\tilde{\xi}_n|}{|\tilde{z}_n|} dx \right]. \end{aligned}$$

and hence

$$(\tilde{\xi}_n^+ - \tilde{\xi}_n^-, \eta) - (V_\infty A \tilde{\xi}_n, \eta)_2 - \int_{\mathbb{R}^N} F_z(x, \tilde{z}_n) \eta \frac{|\tilde{\xi}_n|}{|\tilde{z}_n|} dx = o(1). \quad (3.16)$$

Observe that

$$\int_{\mathbb{R}^N} F_z(x, \tilde{z}_n) \eta \frac{|\tilde{\xi}_n|}{|\tilde{z}_n|} dx \leq \int_{\mathbb{R}^N} F_z(x, \tilde{z}_n) \eta \frac{|\tilde{\xi}_n - \tilde{\xi}|}{|\tilde{z}_n|} dx + \int_{\mathbb{R}^N} F_z(x, \tilde{z}_n) \eta \frac{|\tilde{\xi}|}{|\tilde{z}_n|} dx := I_1 + I_2.$$

Using the embedding theorem and Hölder inequality we have

$$I_1 \leq c_2 \left| \int_{\text{supp} \eta} \eta |\tilde{\xi}_n - \tilde{\xi}| dx \right| \leq c_3 \|\eta\|_2 \|\tilde{\xi}_n - \tilde{\xi}\|_{L^2(\text{supp} \eta)} \rightarrow 0.$$

Moreover, it follows from (W4') and (3.15) that

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^N} F_z(x, \tilde{z}_n) \eta \frac{|\tilde{\xi}|}{|\tilde{z}_n|} dx \\ &= \int_{\mathbb{R}^N} \frac{F_z(x, \tilde{z}_n)}{|(a_{11} + a_{12})\tilde{u}_n + (a_{21} + a_{22})\tilde{v}_n|} \eta |(a_{11} + a_{12})\tilde{\varphi}(x) + (a_{21} + a_{22})\tilde{\psi}(x)| dx \rightarrow 0. \end{aligned}$$

Combining the above fact, and letting $n \rightarrow \infty$ in (3.16) we have

$$(\tilde{\xi}^+ - \tilde{\xi}^-, \eta) - (V_\infty(x)A\tilde{\xi}, \eta)_2 = 0, \quad (3.17)$$

for each $\eta = (\mu, \nu) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$, which implies that

$$\begin{pmatrix} 0 & -\Delta + V \\ -\Delta + V & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = V_\infty(x) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

Set

$$\mathcal{A} := \begin{pmatrix} 0 & -\Delta + V \\ -\Delta + V & 0 \end{pmatrix}.$$

Hence, from (3.17) we can deduce that (φ, ψ) is an eigenfunction of $\mathcal{J} := \mathcal{A} - V_\infty(x)A$, which contradicts with the fact that $\mathcal{J}$ has only continuous spectrum since the matrix A is symmetric non-negative, $V(x)$ and $V_\infty(x)$ are 1-periodic in x . Therefore, this shows that $\{z_n\}$ is bounded in E .

Proof of Theorem 1.1. Employing Lemma 2.4, there exists a sequence $\{z_n\} = \{(u_n, v_n)\} \subset E$ of Φ such that

$$\Phi(z_n) \rightarrow \tilde{c}_* \text{ and } (1 + \|z_n\|)\|\Phi'(z_n)\| \rightarrow 0.$$

By virtue of Lemma 2.5, we know that $\{z_n\}$ is bounded in E , namely, there exists a positive constant c such that $\|z_n\| \leq c$. Moreover, by using the Lion's concentration compactness principle [24, Lemma 1.21] and some standard arguments, we can show that the vanishing does not occur. So $\{z_n\}$ is non-vanishing. Finally, using a standard translation argument, we can see that $\{z_n\}$ converges weakly (up to a subsequence) to some $z_0 \neq 0$, and $\Phi'(z_0) = 0$. This shows z_0 is a nontrivial solution of system (1.2). The proof of Theorem is completed. $\square$

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