Research Article

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Weighted $W^{1,p(\cdot)}$ -Regularity for Degenerate Elliptic Equations in Reifenberg Domains<https://doi.org/10.1515/anona-2021-0206>

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Abstract: Let w be a Muckenhoupt $A_2(\mathbb{R}^n)$ weight and Ω a bounded Reifenberg flat domain in $\mathbb{R}^n$. Assume that $p(\cdot) : \Omega \rightarrow (1, \infty)$ is a variable exponent satisfying the log-Hölder continuous condition. In this article, the authors investigate the weighted $W^{1,p(\cdot)}(\Omega, w)$ -regularity of the weak solutions of second order degenerate elliptic equations in divergence form with Dirichlet boundary condition, under the assumption that the degenerate coefficients belong to weighted BMO spaces with small norms.

Keywords: Reifenberg flat domain, degenerate elliptic equation, Muckenhoupt weight, variable Lebesgue space, BMO

MSC: Primary 35J70; Secondary 35B65, 46E35, 42B35, 42B37

1 Introduction

In this article, we consider the following second order degenerate elliptic equation in divergence form with Dirichlet boundary condition:

$$\begin{cases} \operatorname{div}(A \nabla u) = \operatorname{div} \vec{F} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain of $\mathbb{R}^n$ and $\partial\Omega$ denotes its boundary, $\vec{F} := (f_1, \dots, f_n) : \Omega \rightarrow \mathbb{R}^n$ is a given vector field satisfying $|\vec{F}|/w \in L^2(\Omega, w)$ [see (1.3) below] with w being a Muckenhoupt $A_2(\mathbb{R}^n)$ weight, and $A : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is a symmetric matrix of measurable functions $\{a_{ij}\}_{i,j=1}^n$ on $\mathbb{R}^n$ satisfying the *degenerate elliptic condition*, namely, there exists a positive constant $\Lambda \in [1, \infty)$ such that, for any $\xi \in \mathbb{R}^n$ and almost every $x \in \mathbb{R}^n$,

$$\Lambda^{-1} w(x) |\xi|^2 \leq \langle A(x) \xi, \xi \rangle \leq \Lambda w(x) |\xi|^2. \quad (1.2)$$

In this article, we *always* let Λ be as in (1.2) and A a symmetric matrix. Recall that a non-negative locally integrable function w on $\mathbb{R}^n$ is said to belong to the *Muckenhoupt class* $A_p(\mathbb{R}^n)$, denoted by $w \in A_p(\mathbb{R}^n)$, if, when $p \in (1, \infty)$,

$$[w]_{A_p(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B w(x) dx \left\{ \frac{1}{|B|} \int_B [w(x)]^{-\frac{1}{p-1}} dx \right\}^{p-1} < \infty$$

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and, when $p = 1$,

$$[w]_{A_1(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B w(x) dx \left\{ \operatorname{ess\,sup}_{x \in B} [w(x)]^{-1} \right\} < \infty,$$

where the suprema are taken over all balls B of $\mathbb{R}^n$. We also let

$$A_\infty(\mathbb{R}^n) := \bigcup_{p \in [1, \infty)} A_p(\mathbb{R}^n).$$

Let Ω be a measurable subset of $\mathbb{R}^n$. For any $k \in \mathbb{N}$, $p \in [1, \infty)$, and $w \in A_p(\mathbb{R}^n)$, the *weighted Lebesgue space* $L^p(\Omega, w)$ and the *weighted Sobolev space* $W^{1,p}(\Omega, w)$ are defined, respectively, to be the sets of all measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega, w)} := \left[\int_\Omega |f(x)|^p w(x) dx \right]^{1/p} < \infty \quad (1.3)$$

and

$$\|f\|_{W^{1,p}(\Omega, w)} := \left\{ \int_\Omega [|f(x)|^p + |\nabla f(x)|^p] w(x) dx \right\}^{1/p} < \infty, \quad (1.4)$$

where $\nabla f := (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ denotes the *gradient* of f and $\{\frac{\partial f}{\partial x_i}\}_{i=1}^n$ are the *distributional derivatives* of f . If $w \equiv 1$, we denote $L^p(\Omega, w)$ and $W^{1,p}(\Omega, w)$ simply, respectively, by $L^p(\Omega)$ and $W^{1,p}(\Omega)$. Moreover, we define $W_0^{1,p}(\Omega, w)$ to be the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega, w)}$. Here and thereafter, $C_c^\infty(\Omega)$ denotes the *set of all infinitely differential functions with compact support in Ω* .

Definition 1.1. Let $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^n)$, A satisfy (1.2), and $|\vec{F}|/w \in L^p(\Omega, w)$. A function $u \in W_0^{1,p}(\Omega, w)$ is said to be a *weak solution* of (1.1) if, for any $\varphi \in C_c^\infty(\Omega)$, it holds true that

$$\int_\Omega [A(x) \nabla u(x)] \cdot \nabla \varphi(x) dx = \int_\Omega \vec{F}(x) \cdot \nabla \varphi(x) dx.$$

When $w \in A_2(\mathbb{R}^n)$ and $|\vec{F}|/w \in L^2(\Omega, w)$, Fabes et al. [30] first established the existence, the uniqueness, the Harnack inequality, and the Hölder regularity of the weak solution of (1.1). In particular, if $w(\cdot) \equiv 1$ in (1.2), namely, the matrix A is uniformly elliptic, then the study of the regularity of the weak solution of (1.1) has been a classical topic in PDEs. By the De Giorgi–Nash–Möser theory [45], it is well known that the weak solution of (1.1) satisfies the Hölder regularity when the nonhomogeneous term $\vec{F}$ is sufficiently regular. Moreover, a lot of attention has been paid to the study of the Sobolev type regularity of the weak solution of (1.1). If $w(\cdot) \equiv 1$ in (1.2), the coefficients of A are continuous and Ω is smooth, it is well known (see, for instance, [33, 44]) that the weak solution u of (1.1) satisfies the following $W^{1,p}$ -regularity, namely, there exists a positive constant C , independent of u and $\vec{F}$, such that

$$\|\nabla u\|_{L^p(\Omega)} \leq C \left\| \vec{F} \right\|_{L^p(\Omega)}. \quad (1.5)$$

This $W^{1,p}$ -regularity was achieved via the theory of Calderón–Zygmund operators in [17]. By a counterexample constructed by Meyer [42], it is known that, if A is only assumed to be uniformly elliptic, it is not sufficient to guarantee that (1.5) holds true. Di Fazio [22] showed that (1.5) holds true under the assumptions that $A \in \text{VMO}$ (the vanishing mean oscillation space) and $\partial\Omega \in C^{1,1}$ which was further weakened to $\partial\Omega \in C^1$ by Auscher and Qafsaoui [5]. Moreover, Byun and Wang [15] generalized (1.5) to the case that $A \in \text{BMO}$ (the bounded mean oscillation space) with a small norm and Ω is a Reifenberg flat domain. Dong and Kim [26–28] established (1.5) under the assumptions that A has partial small BMO coefficients and Ω is a bounded Lipschitz domain with small Lipschitz constant or a bounded Reifenberg flat domain. Furthermore, Shen [56]

proved that (1.5) holds true for any given $p \in (\frac{3}{2} - \varepsilon, 3 + \varepsilon)$ when $n \geq 3$, or $p \in (\frac{4}{3} - \varepsilon, 4 + \varepsilon)$ when $n = 2$ if $A \in \text{VMO}$ and Ω is a bounded Lipschitz domain, where $\varepsilon \in (0, \infty)$ is a positive constant depending only on the Lipschitz constant of Ω and n (see also [31, 32]). Note that C^1 domains are Lipschitz domains with small Lipschitz constants and Lipschitz domains with small Lipschitz constants are Reifenberg flat domains [see Remark 1.4(i) below]. The main approach used in [15, 56] is the approximation method introduced by Caffarelli and Peral [16], which is different from that used in [5, 26]. Recently, using the same method of Caffarelli and Peral [16], Cao et al. [18, Theorem 2.7] established the existence and the weighted $W^{1,p}$ -regularity of the weak solution of the degenerate elliptic equation (1.1) when Ω is a Reifenberg flat domain and $A \in \text{BMO}_R(\Omega, w)$ with a small norm (see Definition 1.5 below), where $R \in (0, \infty)$ is a constant. Precisely, for any given $p \in (1, \infty)$ and $w \in [A_2(\mathbb{R}^n) \cap A_p(\mathbb{R}^n)]$, it holds true that

$$\|\nabla u\|_{L^p(\Omega, w)} \leq C_0 \left\| \frac{\vec{F}}{w} \right\|_{L^p(\Omega, w)}, \quad (1.6)$$

where C_0 is a positive constant depending only on $n, [w]_{A_2(\mathbb{R}^n) \cap A_p(\mathbb{R}^n)}, p, \Lambda, R$, and $\text{diam}(\Omega)$. Here and thereafter, $\text{diam}(\Omega) := \sup_{x, y \in \Omega} |x - y|$ denotes the *diameter* of Ω .

On the other hand, the study of variable Lebesgue spaces originated from Orlicz [49] in 1931, which was further developed by Nakano [47, 48]. The next major step in the investigation of variable function spaces was made in the article of Kováčik and Rákosník [40] in 1991. In [40], many basic properties of variable Lebesgue and Sobolev spaces were established. It was shown, in [51, 52, 54, 55], that the theory of variable function spaces has a close connection with modelling of electrorheological fluids, the study of which is an important issue in physics and engineering [23, 55]. Moreover, the variable function spaces have many applications in, for instance, elastic mechanics [51, 66], fluid mechanics [1, 55], and image processing [19]. Inspired by [2, 3], where Acerbi and Mingione established the Calderón–Zygmund type estimates for a non-homogeneous p [or $p(\cdot)$] Laplacian system (see also [6, 7, 20, 29, 43]), Byun et al. [14] extended the Sobolev type regularity (1.5) to the setting of variable Sobolev spaces. Moreover, Bui et al. [8, 10–13] did a lot of nice works on the variable Sobolev type estimates for the solutions of elliptic or parabolic equations. Very recently, we [65] established the variable Lorentz regularity of the weak solution of (1.1) in Reifenberg domains. Let Ω be a measurable subset of $\mathbb{R}^n$ and $\mathcal{P}(\Omega)$ the set of all measurable functions $p(\cdot) : \Omega \rightarrow (0, \infty)$ satisfying

$$p_- := \text{ess inf}_{x \in \Omega} p(x) > 0 \text{ and } p_+ := \text{ess sup}_{x \in \Omega} p(x) < \infty. \quad (1.7)$$

Recall that a function $p(\cdot) \in \mathcal{P}(\Omega)$ is called a *variable exponent function on Ω* . Let $w \in A_\infty(\mathbb{R}^n)$, Ω be a measurable subset of $\mathbb{R}^n$, and $p(\cdot) \in \mathcal{P}(\Omega)$. The *weighted variable Lebesgue space* $L^{p(\cdot)}(\Omega, w)$ is defined to be the set of all measurable functions f on Ω such that, for some $\lambda \in (0, \infty)$,

$$\int_{\Omega} [|f(x)|/\lambda]^{p(x)} w(x) dx < \infty,$$

equipped with the *Luxemburg* (or called the *Luxemburg–Nakano*) (*quasi*-)norm

$$\|f\|_{L^{p(\cdot)}(\Omega, w)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\Omega} \left[\frac{|f(x)|}{\lambda} \right]^{p(x)} w(x) dx \leq 1 \right\}. \quad (1.8)$$

The *weighted variable Sobolev space* $W^{1,p(\cdot)}(\Omega, w)$ is defined to be the set of all measurable functions f on Ω such that

$$\|f\|_{W^{1,p(\cdot)}(\Omega, w)} := \|f\|_{L^{p(\cdot)}(\Omega, w)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega, w)} < \infty,$$

where ∇f is as in (1.4).

Remark 1.2. Let Ω be a measurable subset of $\mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$, and $w \in A_\infty(\mathbb{R}^n)$.

(i) By (1.8), it is easy to see that, for any $f \in L^{p(\cdot)}(\Omega, w)$,

$$\|f\|_{L^{p(\cdot)}(\Omega, w)} \leq 1 \quad \text{if and only if} \quad \int_{\Omega} |f(x)|^{p(x)} w(x) dx \leq 1.$$

(ii) By [24, Theorem 3.4.12], we find that $C_c^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega, w)$.

(iii) A Banach space X is said to be continuously imbedded into a Banach space Y , denoted by $X \hookrightarrow Y$, if $X \subset Y$ and there exists a positive constant C such that, for any $x \in X$, $\|x\|_Y \leq C\|x\|_X$. If $q(\cdot) \in \mathcal{P}(\Omega)$ and Ω is bounded, then, from [24, Corollary 3.3.4], it follows that $L^{p(\cdot)}(\Omega, w) \hookrightarrow L^{q(\cdot)}(\Omega, w)$ if and only if, for almost every $x \in \Omega$, $q(x) \leq p(x)$. In particular, we have

$$L^{p^+}(\Omega, w) \hookrightarrow L^{p(\cdot)}(\Omega, w) \hookrightarrow L^{p^-}(\Omega, w).$$

Motivated by [8, 11–14, 18], in this article, we extend (1.6) to the case that p is a variable exponent function [see (1.17) below]. To state the main result of this article, we first introduce some notions and notation.

Notation. For any $r \in (0, \infty)$ and $x_0 \in \mathbb{R}^n$, $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ always denotes a ball centered at x_0 with radius r . Define

$$B_r^+(x_0) := B_r(x_0) \cap \{x \in \mathbb{R}^n : x_n > 0\}.$$

Let $\vec{0}_n := (0, \dots, 0)$ denote the origin of $\mathbb{R}^n$. If $x_0 = \vec{0}_n$, we simply write B_r and B_r^+ instead of $B_r(\vec{0}_n)$ and $B_r^+(\vec{0}_n)$, respectively. For any subset Ω of $\mathbb{R}^n$, $r \in (0, \infty)$, and $x \in \mathbb{R}^n$, let $\mathbf{1}_\Omega$ denote its characteristic function and $\Omega_r(x_0) := \Omega \cap B_r(x_0)$. If $x_0 = \vec{0}_n$, we denote $\Omega_r(\vec{0}_n)$ simply by Ω_r . Assume that $w \in A_\infty(\mathbb{R}^n)$. For any measurable subset E of $\mathbb{R}^n$, define

$$w(E) := \int_E w(x) dx. \quad (1.9)$$

In this article, let $\mathbb{N} := \{1, 2, \dots\}$. We always use C to denote a positive constant which may change from line to line, but is independent of the main parameters, and use $C_{(\alpha, \beta, \dots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \beta, \dots$. The symbol $f \lesssim g$ means that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If $f \leq Cg$ and $g = h$ or $g \leq h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \leq h$. For any $p \in [1, \infty]$, p' denotes its conjugate index, namely, $1/p + 1/p' = 1$.

Definition 1.3. Let $\delta, R \in (0, \infty)$ and Ω be a bounded domain of $\mathbb{R}^n$. Then Ω is said to be a (δ, R) -Reifenberg flat domain if, for any $x \in \partial\Omega$ and $r \in (0, R)$, there exists a Y -coordinate system $\{\vec{e}_{Y,1}, \dots, \vec{e}_{Y,n}\}$, which may depends on x and r , such that, in this coordinate system, $x = \vec{0}_Y$ and

$$\left[B_r(\vec{0}_Y) \cap \{y \in \mathbb{R}^n : y_n > \delta r\} \right] \subset \left[B_r(\vec{0}_Y) \cap \Omega \right] \subset \left[B_r(\vec{0}_Y) \cap \{y \in \mathbb{R}^n : y_n > -\delta r\} \right],$$

where $\vec{0}_Y$ denotes the origin of the Y -coordinate system and $y := (y_1, \dots, y_n) := \sum_{i=1}^n y_i \vec{e}_{Y,i}$.

Remark 1.4. (i) Reifenberg flat domains were first introduced by Reifenberg [53], which appear naturally in the theory of minimal surfaces and free boundary problems. Intuitively, a Reifenberg flat domain is a domain such that, at its any boundary point and at any scale locally, the boundary can be placed between two hyperplanes, but does not require any smoothness on the boundary. It is known (see [59]) that Lipschitz domains with sufficiently small Lipschitz constants are Reifenberg flat domains, but generally Lipschitz domains may not be Reifenberg flat domains. Moreover, Reifenberg flat domains are $W^{1,p}$, with $p \in [1, \infty]$, extension domains (see, for instance, [36]), which are very important to the study on the $W^{1,p}$ -regularity (1.5). In recent years, boundary value problems of elliptic or parabolic equations on Reifenberg flat domains have been widely studied (see, for instance, [8, 9, 14, 15, 27, 28, 62–64]).

(ii) By [41, Remark 3.2], we know that, if Ω is a (δ, R) -Reifenberg flat domain for some $\delta \in (0, 1)$ and $R \in (0, \infty)$, then, for any $x \in \partial\Omega$ and $r \in (0, (1 - \delta)R)$, there exists a Z -coordinate system of basis $\{\vec{e}_{Z,1}, \dots, \vec{e}_{Z,n}\}$ such that, in this coordinate system, $\vec{0}_Z \in \dot{\Omega}$ (the set of all interior points of Ω),

$x = -\delta r \vec{e}_{Z,n}$, and

$$B_r^+(\vec{0}_Z) \subset \Omega_r(\vec{0}_Z) \subset \left[B_r(\vec{0}_Z) \cap \{z \in \mathbb{R}^n : z_n > -2r\tilde{\delta}\} \right],$$

where $\tilde{\delta} := \frac{\delta}{1-\delta}$.

- (ii) Let $\delta, R \in (0, \infty)$ and Ω be a (δ, R) -Reifenberg flat domain. It is easy to see that, for any $x_0 \in \mathbb{R}^n$ and $\tau \in (0, \infty)$, $\Omega_{\tau, x_0} := \{(x - x_0)/\tau : x \in \Omega\}$ is a $(\delta, R/\tau)$ -Reifenberg flat domain.
- (iii) Equation (1.1) is translation and scaling invariant in the following sense. Let A and $\vec{F}$ be as in (1.1). Assume that $u \in W_0^{1,2}(\Omega, w)$ is a weak solution of (1.1). For any given $x_0 \in \mathbb{R}^n$ and $\tau \in (0, \infty)$, define $A_{\tau, x_0}(\cdot) := A(\tau \cdot + x_0)$, $u_{\tau, x_0}(\cdot) := u(\tau \cdot + x_0)$, and $\vec{F}_{\tau, x_0}(\cdot) := \vec{F}(\tau \cdot + x_0)$. Then u_{τ, x_0} solves

$$\begin{cases} \operatorname{div}(A_{\tau, x_0} \nabla u_{\tau, x_0}) = \operatorname{div} \vec{F}_{\tau, x_0} & \text{in } \Omega_{\tau, x_0}, \\ u_{\tau, x_0} = 0 & \text{on } \partial\Omega_{\tau, x_0}. \end{cases}$$

In what follows, we use $L_{\text{loc}}^1(\mathbb{R}^n)$ to denote the set of all locally integrable functions on $\mathbb{R}^n$.

Definition 1.5 ([18]). Let Ω be a measurable subset of $\mathbb{R}^n$, $R \in (0, \infty)$, and $w \in A_\infty(\mathbb{R}^n)$. The space of functions of weighted bounded mean oscillation, denoted by $\text{BMO}_R(\Omega, w)$, is defined to be the set of all $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ such that

$$\|f\|_{\text{BMO}_R(\Omega, w)} := \sup_{x \in \Omega} \sup_{r \in (0, R)} \frac{1}{w(B_r(x))} \int_{B_r(x)} |f(y) - \langle f \rangle_{B_r(x)}| dy < \infty.$$

Here and thereafter,

$$\langle f \rangle_{B_r(x)} := \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy \quad (1.10)$$

denotes the average of f over the ball $B_r(x)$, and $w(B_r(x))$ is as in (1.9) with E therein replaced by $B_r(x)$.

Remark 1.6. The weighted bounded mean oscillation space $\text{BMO}_R(\Omega, w)$ when $\Omega := \mathbb{R}^n$ and $R := \infty$ was first introduced by Muckenhoupt and Wheeden [46]. The study of $\text{BMO}_R(\Omega, w)$ is motivated by the behavior of the Hilbert transform of any measurable function bounded by a multiple of the weighted function. Let $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. Then, for any given $r \in [1, p']$, by [46, Theorem 4], we know that $f \in \text{BMO}_R(\Omega, w)$ if and only if $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ and

$$\|f\|_{\text{BMO}_R^r(\Omega, w)} := \sup_{x \in \Omega} \sup_{r \in (0, R)} \left\{ \frac{1}{w(B_r(x))} \int_{B_r(x)} |f(y) - \langle f \rangle_{B_r(x)}|^r [w(y)]^{1-r} dy \right\}^{1/r} < \infty.$$

Moreover, $\|\cdot\|_{\text{BMO}_R(\Omega, w)}$ is equivalent to $\|\cdot\|_{\text{BMO}_R^r(\Omega, w)}$ with the positive equivalence constants depending only on n, r , and $[w]_{A_p(\mathbb{R}^n)}$. Recalling that the space $\text{BMO}_R(\Omega, w)$ in [18, Definition 2.3] was defined via the norm $\|\cdot\|_{\text{BMO}_R^2(\Omega, w)}$, by the aforementioned fact, we know that Definition 1.5 of this article coincides with [18, Definition 2.3].

In what follows, for any given symmetric matrix A of measurable functions $\{a_{ij}\}_{i,j=1}^n$, and for any ball B of $\mathbb{R}^n$ and any $x \in \mathbb{R}^n$, let

$$|A(x)| = \left[\sum_{i,j=1}^n |a_{ij}(x)|^2 \right]^{1/2}, \quad \langle A \rangle_B := \left\{ \langle a_{ij} \rangle_B \right\}_{i,j=1}^n,$$

and

$$\|A\|_{\text{BMO}_R(\Omega, w)} := \left\{ \sum_{i,j=1}^n \|a_{ij}\|_{\text{BMO}_R(\Omega, w)}^2 \right\}^{1/2}.$$

In this article, let Ω be a bounded domain of $\mathbb{R}^n$. We always assume that $p(\cdot) \in \mathcal{P}(\Omega)$,

$$1 < p_- \leq p_+ < \infty, \quad (1.11)$$

and, for any $x, y \in \Omega$,

$$|p(x) - p(y)| \leq \omega(|x - y|), \quad (1.12)$$

where $\omega : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing continuous function with $\omega(0) = 0$ and

$$\limsup_{r \rightarrow 0^+} \omega(r) \log(1/r) < \infty. \quad (1.13)$$

By (1.13), it is easy to see that there exists a positive constant c_0 such that, for any $r \in (0, 1)$, $\omega(r) \log(1/r) \leq c_0$, which is equivalent to

$$r^{-\omega(r)} \leq e^{c_0}. \quad (1.14)$$

For any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $1 < p_- \leq p_+ < \infty$, let

$$\sigma \in (1, p_-) \text{ when } p_- \in (1, 2], \text{ or } \sigma := 2 \text{ when } p_- \in (2, \infty), \quad (1.15)$$

and $w \in A_\sigma(\mathbb{R}^n)$. Then, by Lemma 2.2(iii) below, we know that there exists some $q \in (1, \infty)$ such that

$$w \in RH_q(\mathbb{R}^n). \quad (1.16)$$

The following theorem is the main result of this article.

Theorem 1.7. *Let Ω be a bounded domain of $\mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (1.11), (1.12), and (1.13), $R \in (0, \infty)$, $M \in (0, \infty)$, σ be as in (1.15), $w \in A_\sigma(\mathbb{R}^n)$ with $[w]_{A_\sigma(\mathbb{R}^n)} \leq M$, and q be as in (1.16). Then there exists some $\delta := \delta(n, \Lambda, \sigma, M) \in (0, \infty)$ small enough such that, if Ω is a (δ, R) -Reifenberg flat domain and $\|A\|_{\text{BMO}_R(\Omega, w)} < \delta$, then there exists a unique weak solution $u \in W_0^{1, p_-}(\Omega, w)$ of (1.1) satisfying*

$$\|\nabla u\|_{L^{p(\cdot)}(\Omega, w)} \leq C \left\| \frac{\vec{F}}{w} \right\|_{L^{p(\cdot)}(\Omega, w)}, \quad (1.17)$$

where C is a positive constant depending only on $n, \Lambda, \delta, \sigma, q, M, R, w(\Omega), \text{diam}(\Omega), p_-$, and p_+ .

Remark 1.8. (i) In particular, if $p(\cdot) \equiv p \in (1, \infty)$ is a constant exponent, then Theorem 1.7 in this case coincides with [18, Theorem 2.7], namely, (1.17) in this case is just (1.6). However, when dealing with the variable Lebesgue space, the method used in [18] is not feasible. The approach used in [18] is called the approximation method of Caffarelli and Peral [16]. The main idea is to locally consider the equation (1.1) as the perturbation of an equation whose regularity of the solution is well known. This method depends on the Vitali covering lemma and the L^p boundedness of Hardy-Littlewood maximal function. We prove Theorem 1.7 by using the so-called “maximal function free technique”, which is introduced by Acerbi and Mingione [2, 3] to overcome the scaling deficiency for a class of p [or $p(\cdot)$] Laplacian systems, and developed by Byun et al. in [14]. This technique is a powerful tool to deal with the variable Sobolev type estimate and was also used in [8, 12]. Note that there have been some known results on the estimate like (1.17) with $w(\cdot) \equiv 1$ for some PDEs of divergence form, which were mainly based on the theory of Calderón–Zygmund operators (see [24, 25, 60]). But, in [14], it was pointed out that the tools from harmonic analysis, such as Calderón–Zygmund operators and the maximal functions, might not be feasible for the estimate (1.17) with $w(\cdot) \equiv 1$. This motivates us to use the maximal function free technique. Besides this, we establish the weighted L^p Caccioppoli estimate for any given $p \in (1, \infty)$ and the weighted gradient approximation estimate of u (see Sections 3 and 4 below). Moreover, to obtain (1.17), we need to establish a higher integrability result for ∇u (see Lemma 4.8 below). To achieve this, we take the advantage of the weighted estimate of level sets of ∇u (see Lemma 4.7 below) and borrow some ideas used in the proof of [18, Theorem 2.7] (see also [15, Theorem 1.5]), which are essentially influenced by Caffarelli and Peral [16].

- (ii) We make a comment on the weighted L^p Caccioppoli estimate for any given $p \in (1, \infty)$ in Section 3 below. The proof of Theorem 1.7 strongly depends on the weighted gradient approximation estimate of u established in Lemmas 4.3 and 4.5 below. When $p_- \in (2, \infty)$, the conclusions of Lemmas 4.3 and 4.5 for the integration index $p = 2$ therein are enough for (1.17) being true in this case, which has been established in [18, Propositions 4.4 and 5.5]. When $p(\cdot) = p \in (1, 2)$ is a constant exponent, (1.17) in this case was showed in [18, Theorem 2.7] via a dual argument of the case $p \in (2, \infty)$. In general, when $p_- \in (1, 2]$ and $p(\cdot)$ is not a constant, such a dual argument does not work anymore. Thus, we need to extend the results of [18, Propositions 4.4 and 5.5] to $p \in (1, 2)$. However, to the best of our knowledge, this is unknown. To achieve this, we need to establish the weighted L^p Caccioppoli estimate for any given $p \in (1, 2)$ [see Lemmas 3.3 and 3.5 below, where we prove a more general result for any given $p \in (1, \infty)$]. To this end, we apply the method used in [4, 38, 39, 50], which depends on an integration by parts combined with a covering and iteration argument.
- (iii) Via constructing a counterexample, Cao et al. [18, pp. 2232–2233] showed the necessity of the assumption $\|A\|_{\text{BMO}_R(\Omega, w)} < \delta$ such that (1.6) holds true, and constructed a specific example of such a matrix A . Indeed, let $n \geq 3$, $\Omega := B_1(\vec{0}_n)$, $\alpha := \frac{1}{n+1}$, and, for any $x \in \mathbb{R}^n$, $w_{(\alpha)}(x) := |x|^{2(\alpha+1)}$. Noticing that $2(\alpha+1) < n$, by [58, p. 218], we know that $w_{(\alpha)} \in A_2(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, define

$$A_{(\alpha)}(x) := w_{(\alpha)}(x)I_n \quad \text{and} \quad u_{(\alpha)}(x) := \frac{x_1}{|x|^{2\alpha}},$$

where I_n denotes the $n \times n$ identity matrix and $u_{(\alpha)}(x) := \infty$ when $x = \vec{0}_n$. Then it is easy to see that, for any given $\delta \in (0, 1)$, $\|A_{(\alpha)}\|_{\text{BMO}_R(\Omega, w)} > \delta$, and $u_{(\alpha)} \in W^{1,2}(\Omega, w)$ is a weak solution of $\text{div}[A_{(\alpha)} \nabla u] = 0$ in Ω . However, when $p \in (1, \frac{2\alpha+n+2}{2\alpha})$, we find that there exists a positive constant C , depending only on n and α , such that

$$0 < C \int_{\Omega} |x|^{2\alpha+2-2\alpha p} dx \leq \int_{\Omega} |\nabla u_{(\alpha)}(x)|^p w(x) dx \leq C^{-1} \int_{\Omega} |x|^{2\alpha+2-2\alpha p} dx < \infty,$$

which contradicts (1.6) with $\vec{F} := \vec{0}_n$ therein. This also shows that the assumption $\|A\|_{\text{BMO}_R(\Omega, w)} < \delta$ is necessary for (1.17). Moreover, for any given $\alpha \in (-1, 1)$ and any $x \in \mathbb{R}^n$, let $w_{(\alpha)}(x) := |x|^\alpha$ and $A_{(\alpha)}(x) := w_{(\alpha)}(x)I_n$. Then [18, Lemma 2.11] shows that there exists a positive constant C , depending only on n , such that, for any $\alpha \in (-1, 1)$,

$$\|A_{(\alpha)}\|_{\text{BMO}_R(B_1(\vec{0}_n), w)} \leq C|\alpha|.$$

- (iv) Let Ω be a bounded domain of $\mathbb{R}^n$. If $p(\cdot) \in \mathcal{P}(\Omega)$ satisfies (1.11), (1.12), and (1.13), then it is easy to see that $p(\cdot)$ is globally log-Hölder continuous in Ω . Recall that a function $p(\cdot) \in \mathcal{P}(\Omega)$ is said to be globally log-Hölder continuous in Ω , denoted by $p(\cdot) \in C^{\log}(\Omega)$, if there exist constants $C, \tilde{C} \in (0, \infty)$ and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \Omega$,

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x - y|)} \quad \text{and} \quad |p(x) - p_\infty| \leq \frac{\tilde{C}}{\log(e + |x|)}. \quad (1.18)$$

Conversely, if $p(\cdot) \in C^{\log}(\Omega)$, then, by taking $\omega(t) := \frac{C}{\log(e+1/t)}$ for any $t \in (0, \infty)$, and $\omega(0) := 0$, we know that $p(\cdot)$ satisfies (1.11), (1.12), and (1.13). For the study of PDEs in the setting of variable spaces, it is necessary to develop some tools for these spaces, such as, mollifications, the Riesz potential, Calderón–Zygmund operators, and the Hardy–Littlewood maximal function. The log-Hölder continuity condition makes these tools work in the variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$. Moreover, this regularity condition is, in some sense, optimal and can not be improved (see, for instance, [24, Chapter 4]).

The remainder of this article is organized as follows.

In Section 2, we recall some preliminary results on the Vitali covering lemma, Muckenhoupt weights, the truncated Fefferman–Stein sharp maximal function, and the weighted Sobolev and Poincaré inequalities; in Section 3, we establish the local weighted interior and boundary Caccioppoli type estimates of the weak solutions of degenerate elliptic equations for any given $p \in (1, \infty)$; in Section 4, based on the results obtained in Section 3, we establish the weighted interior and boundary gradient approximation estimates of the weak solutions of degenerate elliptic equations; Section 5 is devoted to the proof of Theorem 1.7.

2 Preliminaries

In this section, we recall several auxiliary and necessary well-known conclusions which are needed for the proof of Theorem 1.7.

The following is the well-known *Vitali covering lemma* (see, for instance, [57, p. 9]).

Lemma 2.1 ([57]). *Let E be a measurable subset of $\mathbb{R}^n$. Suppose that there exists a class $\{B_\alpha\}_{\alpha \in I}$ of balls of uniform bounded radii such that $E \subset \bigcup_{\alpha \in I} B_\alpha$, where I denotes an index set. Then there exists a subclass $\{B_{\alpha_i}\}_{i=1}^\infty$ of pairwise disjoint balls of $\{B_\alpha\}_{\alpha \in I}$ such that*

$$|E| \leq 5^n \sum_{i=1}^{\infty} |B_{\alpha_i}|.$$

Let $q \in (1, \infty]$. A nonnegative locally integrable function w is said to belong to the *reverse Hölder class* $RH_q(\mathbb{R}^n)$ if there exists a positive constant C such that, for any ball B of $\mathbb{R}^n$,

$$\left\{ \frac{1}{|B|} \int_B [w(x)]^q dx \right\}^{1/q} \leq C \frac{1}{|B|} \int_B w(x) dx,$$

where we replace $\{|B|^{-1} \int_B [w(x)]^q dx\}^{1/q}$ by $\|w\|_{L^\infty(B)}$ when $q = \infty$.

We recall some properties of Muckenhoupt weights and reverse Hölder classes in the following lemma (see, for instance, [34, Chapter 9]).

Lemma 2.2 ([34]). (i) *For any $1 \leq p \leq q < \infty$, $A_p(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$.*
(ii) *For any $1 < p \leq q \leq \infty$, $RH_q(\mathbb{R}^n) \subset RH_p(\mathbb{R}^n)$.*
(iii) $A_\infty(\mathbb{R}^n) = \bigcup_{p \in [1, \infty)} A_p(\mathbb{R}^n) = \bigcup_{r \in (1, \infty]} RH_r(\mathbb{R}^n)$.
(iv) *Let $p \in [1, \infty)$, $r \in (1, \infty]$, and $w \in A_p(\mathbb{R}^n) \cap RH_r(\mathbb{R}^n)$. Then there exists a positive constant $C_1 := C_{([w]_{A_p(\mathbb{R}^n)})}$ such that, for any ball B of $\mathbb{R}^n$ and any subset $E \subset B$,*

$$C_1^{-1} \left[\frac{|B|}{|E|} \right]^{\frac{r-1}{r}} \leq \frac{w(B)}{w(E)} \leq C_1 \left[\frac{|B|}{|E|} \right]^p.$$

Let $w \in A_\infty(\mathbb{R}^n)$. The *weighted central Hardy–Littlewood maximal function* $\mathcal{M}_w$ is defined by setting, for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}_w(f)(x) := \sup_{r \in (0, \infty)} \frac{1}{w(B_r(x))} \int_{B_r(x)} |f(y)| w(y) dy. \quad (2.1)$$

In particular, if $w \equiv 1$, we denote $\mathcal{M}_w(f)$ simply by $\mathcal{M}(f)$ which is just the classical central Hardy–Littlewood maximal function.

Remark 2.3. (i) *Let $w \in A_\infty(\mathbb{R}^n)$. By Lemma 2.2, we know that w is a doubling measure on $\mathbb{R}^n$. Then it is well known (see, for instance, [58, p. 13]) that, for any given $p \in (1, \infty)$, $\mathcal{M}_w$ is strong type (p, p) and weak type $(1, 1)$, namely, there exist positive constants $C := C_{(w, p)}$ and $\tilde{C} := C_{(w)}$ such that, for any $f \in L^p(\mathbb{R}^n, w)$,*

$$\|\mathcal{M}_w(f)\|_{L^p(\mathbb{R}^n, w)} \leq C \|f\|_{L^p(\mathbb{R}^n, w)}$$

and, for any $f \in L^1(\mathbb{R}^n, w)$ and $\lambda \in (0, \infty)$,

$$w(\{x \in \mathbb{R}^n : \mathcal{M}_w(f)(x) > \lambda\}) \leq \frac{\tilde{C}}{\lambda} \|f\|_{L^1(\mathbb{R}^n, w)}.$$

(ii) *Let $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. It is well known that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n, w)$, namely, there exists a positive constant $C := C_{(p, [w]_{A_p(\mathbb{R}^n)})}$ such that, for any $f \in L^p(\mathbb{R}^n, w)$,*

$$\|\mathcal{M}(f)\|_{L^p(\mathbb{R}^n, w)} \leq C \|f\|_{L^p(\mathbb{R}^n, w)}.$$

Let $\rho \in (0, \infty)$. Recall that the truncated *central Fefferman–Stein sharp maximal function* $\mathcal{M}^{\rho, \sharp}(f)$ is defined by setting, for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}^{\rho, \sharp}(f)(x) := \sup_{r \in (0, \rho]} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - \langle f \rangle_{B_r(x)}| dy,$$

where $\langle f \rangle_{B_r(x)}$ is as in (1.10). If $\rho = \infty$, $\mathcal{M}^{\rho, \sharp}$ is just the usual central sharp maximal function and we denote it simply by $\mathcal{M}^{\sharp}$. Let $p \in (1, \infty)$. It is well known (see, for instance, [35, Chapter 7]) that there exists a positive constant $C := C_{(p)}$ such that, for any $f \in L^p(\mathbb{R}^n)$,

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C \left\| \mathcal{M}^{\sharp}(f) \right\|_{L^p(\mathbb{R}^n)}. \quad (2.2)$$

The following lemma is a truncated weighted version of (2.2), which was established in [50, Corollary 2.7].

Lemma 2.4 ([50]). *Let $M \in (0, \infty)$, $\rho \in (0, \infty)$, $x_0 \in \mathbb{R}^n$, $p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^n)$ with $[w]_{A_p(\mathbb{R}^n)} \leq M$. Then there exist constants $\kappa := \kappa_{(n, p, M)} \in (\sqrt{n}, \infty)$ and $C := C_{(n, p, M)} \in (0, \infty)$ such that, for any $f \in L^p(\mathbb{R}^n)$ or $f \in L^p(\mathbb{R}^n, w)$ with $\text{supp } f \subset B_\rho(x_0)$,*

$$\int_{B_\rho(x_0)} |f(x)|^p w(x) dx \leq C \int_{B_{\kappa\rho}(x_0)} \left[\mathcal{M}^{\kappa\rho, \sharp}(f)(x) \right]^p w(x) dx. \quad (2.3)$$

The following lemma shows that the Reifenberg flat domain satisfies the measure density condition.

Lemma 2.5. *Let $p \in [1, \infty)$, $w \in A_p(\mathbb{R}^n)$, $\delta \in (0, 1)$, and $R \in (0, \infty)$. Assume that $\Omega \subset \mathbb{R}^n$ is a (δ, R) -Reifenberg flat domain. Then*

$$\sup_{r \in (0, 2R)} \sup_{x \in \Omega} \frac{w(B_r(x))}{w(\Omega \cap B_r(x))} \leq C_1 \left(\frac{4}{1-\delta} \right)^{pn},$$

where C_1 is the positive constant same as in Lemma 2.2(iv).

Proof. Let $p \in [1, \infty)$, $w \in A_p(\mathbb{R}^n)$, $\delta \in (0, 1)$, and $R \in (0, \infty)$. Fix any $r \in (0, 2R)$ and $x \in \Omega$. If $\text{dist}(x, \partial\Omega) \geq r/2$, then $B_{r/2}(x) \subset \Omega$. By this and Lemma 2.2, we have

$$\frac{w(B_r(x))}{w(\Omega \cap B_r(x))} \leq \frac{w(B_r(x))}{w(B_{r/2}(x))} \leq C_1 2^{pn} \leq C_1 \left(\frac{4}{1-\delta} \right)^{pn}. \quad (2.4)$$

If $\text{dist}(x, \partial\Omega) < r/2$, then there exists some $y \in \partial\Omega$ such that $\text{dist}(x, \partial\Omega) = \text{dist}(x, y) < r/2$. By Remark 1.4(ii), we know that there exists a Z -coordinate system $\{\vec{e}_{Z,1}, \dots, \vec{e}_{Z,n}\}$ such that, in this system, $y = \vec{0}_Z$ and

$$\left[B_{r/2}(y) \cap \left\{ z \in \mathbb{R}^n : z_n > \frac{\delta r}{2} \right\} \right] \subset [B_{r/2}(y) \cap \Omega] \subset \left[B_{r/2}(y) \cap \left\{ z \in \mathbb{R}^n : z_n > -\frac{\delta r}{2} \right\} \right].$$

Thus, we have

$$[B_r(x) \cap \Omega] \supset [B_{r/2}(y) \cap \Omega] \supset \left[B_{r/2}(y) \cap \left\{ z \in \mathbb{R}^n : z_n > \frac{\delta r}{2} \right\} \right]. \quad (2.5)$$

Moreover, it is easy to see that $B_{r/2}(y) \cap \{z \in \mathbb{R}^n : z_n > \delta r/2\} \supset B_{r(1-\delta)/4}(y_0)$, where $y_0 := \frac{(1+\delta)r}{4} \vec{e}_{Z,n}$. From this, (2.5), and Lemma 2.2, it follows that

$$\frac{w(B_r(x))}{w(B_r(x) \cap \Omega)} \leq \frac{w(B_r(x))}{w(B_{r/2}(y) \cap \{z \in \mathbb{R}^n : z_n > \delta r/2\})} \leq \frac{w(B_r(x))}{w(B_{r(1-\delta)/4}(y_0))} \leq C_1 \left(\frac{4}{1-\delta} \right)^{pn}.$$

This, together with (2.4), then finishes the proof of Lemma 2.5. $\square$

Let Ω be a measurable subset of $\mathbb{R}^n$, $w \in A_\infty(\mathbb{R}^n)$, and $p \in (0, \infty)$. It is well known that, for any $f \in L^p(\Omega, w)$,

$$\|f\|_{L^p(\Omega, w)}^p = \int_0^\infty p\lambda^{p-1} w(\{x \in \Omega : |f(x)| > \lambda\}) d\lambda. \quad (2.6)$$

From this, it is easy to deduce the following lemma, which is just [18, Lemma 3.1].

Lemma 2.6 ([18]). *Let $w \in A_\infty(\mathbb{R}^n)$ and $p \in [1, \infty)$. Assume that Ω is a bounded domain of $\mathbb{R}^n$ and $g \in L^p(\Omega, w)$. Then, for any given $\theta \in (0, \infty)$ and $\gamma \in (1, \infty)$, there exists a positive constant $C := C_{(\theta, \gamma, p)}$ such that*

$$C^{-1}S \leq \|g\|_{L^p(\Omega, w)}^p \leq C[w(\Omega) + S],$$

where

$$S := \sum_{j=1}^{\infty} \gamma^{pj} w\left(\left\{x \in \Omega : |g(x)| > \theta \gamma^j\right\}\right).$$

The following technical lemma was established in [37, Lemma 4.3].

Lemma 2.7 ([37]). *Let $0 < a < b < \infty$ and ϕ be a nonnegative bounded function on $[a, b]$. Assume that there exist positive constants $\theta \in (0, 1)$, $\tau \in (0, \infty)$, and $\kappa_1, \kappa_2 \in [0, \infty)$ such that, for any $a \leq \alpha \leq \beta \leq b$,*

$$\phi(\alpha) \leq \theta \phi(\beta) + \frac{\kappa_1}{(\beta - \alpha)^\tau} + \kappa_2.$$

Then there exists a positive constant $C := C_{(\theta, \tau)}$ such that, for any $a \leq \alpha \leq \beta \leq b$,

$$\phi(\alpha) \leq C \left[\frac{\kappa_1}{(\beta - \alpha)^\tau} + \kappa_2 \right].$$

To prove Theorem 1.7, we also need the following weighted imbedding inequality, which is just [30, Theorems (1.2) and (1.3)].

Lemma 2.8 ([30]). *Let Ω be a bounded domain of $\mathbb{R}^n$, $p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^n)$. Then there exist positive constants C and δ such that, for any given $k \in [1, \frac{n}{n-1} + \delta]$ and any $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_{L^{kp}(\Omega, w)} \leq C \|\nabla u\|_{L^p(\Omega, w)},$$

where the positive constant C depends only on $n, p, [w]_{A_p(\mathbb{R}^n)}$, and $\text{diam}(\Omega)$.

Remark 2.9. *Let $B := B(x_0, R)$ be a ball of $\mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $R \in (0, \infty)$. By the classical Sobolev imbedding theorem, we know that there exists a positive constant $C := C_{(n, p)}$ such that, for any $u \in W_0^{1,p}(B)$,*

$$\left[\frac{1}{|B|} \int_B |u(x)|^{kp} dx \right]^{1/kp} \leq CR \left[\frac{1}{|B|} \int_B |\nabla u(x)|^p dx \right]^{1/p},$$

where

$$k := \frac{n}{n-p} \text{ when } p \in (1, n), \text{ and } k := 2 \text{ when } p \in [n, \infty). \quad (2.7)$$

The following weighted Poincaré inequality was established in [30, Theorem (1.5)].

Lemma 2.10 ([30]). *Let $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. Then there exist positive constants δ and $C := C_{(n, p, [w]_{A_p(\mathbb{R}^n)})}$ such that, for any given $k \in [1, \frac{n}{n-1} + \delta]$, any ball B_R of $\mathbb{R}^n$ with $R \in (0, \infty)$, and any*

$$u \in W^{1,p}(B_R, w),$$

$$\left[\frac{1}{w(B_R)} \int_{B_R} |u(x) - A_{B_R}|^{kp} w(x) dx \right]^{1/kp} \leq CR \left[\frac{1}{w(B_R)} \int_{B_R} |\nabla u(x)|^p w(x) dx \right]^{1/p},$$

where either

$$A_{B_R} := \frac{1}{w(B_R)} \int_{B_R} u(x)w(x) dx \quad \text{or} \quad A_{B_R} := \frac{1}{|B_R|} \int_{B_R} u(x) dx.$$

3 Weighted Caccioppoli type estimates

In this section, we establish the weighted L^p Caccioppoli estimate for any given $p \in (1, \infty)$, which plays a key role in the proof of weighted gradient approximation estimates in Section 4.

Let $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. Then, by the reverse Hölder property and the self-improving property of Muckenhoupt weights (see, for instance, [35, Theorem 9.2.2 and Corollary 9.2.6]), we know that there exists some $q \in (1, \infty)$ such that

$$w \in A_{p^*}(\mathbb{R}^n) \cap RH_q(\mathbb{R}^n) \quad \text{with} \quad p^* := 1 + \frac{p-1}{q} \in (1, p). \quad (3.1)$$

Lemma 3.1. *Let B be a ball of $\mathbb{R}^n$, $M \in (0, \infty)$, $p \in [1, \infty)$, and $w \in A_p(\mathbb{R}^n)$ with $[w]_{A_p(\mathbb{R}^n)} \leq M$.*

(i) *Let $q_0 := \frac{p}{p^*} \in (1, \infty)$ and $p^* \in (1, p)$ be as in (3.1). Then there exists a positive constant $C := C_{(n,p,M)}$ such that, for any $u \in L^p(B, w)$,*

$$\left[\frac{1}{|B|} \int_B |u(x)|^{q_0} dx \right]^{1/q_0} \leq C \left[\frac{1}{w(B)} \int_B |u(x)|^p w(x) dx \right]^{1/p}.$$

(ii) *Let $\tilde{q} \in [1, \infty)$, $q \in (1, \infty)$ be as in (3.1), and $q_1 := \frac{\tilde{q}(q-1)}{q}$. Then there exists a positive constant $C := C_{(n,\tilde{q},q,M)}$ such that, for any $u \in L^{\tilde{q}}(B)$,*

$$\left[\frac{1}{w(B)} \int_B |u(x)|^{q_1} w(x) dx \right]^{1/q_1} \leq C \left[\frac{1}{|B|} \int_B |u(x)|^{\tilde{q}} dx \right]^{1/\tilde{q}}.$$

By the Hölder inequality, the proof of Lemma 3.1 is quite direct and we omit the details.

Definition 3.2. *Let $R \in (0, \infty)$, $B_R := B_R(\vec{0}_n)$, $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^n)$, A satisfy (1.2), and $|\vec{F}|/w \in L^p(B_R, w)$. A function $u \in W^{1,p}(B_R, w)$ is said to be a weak solution of*

$$\operatorname{div}(A \nabla u) = \operatorname{div} \vec{F} \quad \text{in } B_R \quad (3.2)$$

if, for any $\varphi \in C_c^\infty(B_R)$,

$$\int_{B_R} [A(x) \nabla u(x)] \cdot \nabla \varphi(x) dx = \int_{B_R} \vec{F}(x) \cdot \nabla \varphi(x) dx. \quad (3.3)$$

The following lemma can be viewed as a local interior weighted L^p Caccioppoli estimate.

Lemma 3.3. *Let $R \in (0, \infty)$, $M \in (0, \infty)$, $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^n)$ with $[w]_{A_p(\mathbb{R}^n)} \leq M$, $q \in (1, \infty)$ be as in (3.1), and $\kappa \in (\sqrt{n}, \infty)$ as in Lemma 2.4. Assume that A and $\vec{F}$ are as in Definition 3.2, and $u \in W^{1,p}(B_R, w)$ is a weak solution of (3.2). Then there exists a positive constant $h := h_{(n,\Lambda,p,q,M,R)} \in (1, \infty)$ such that, for any given*

$r \in (0, \frac{R}{2h\kappa})$ and $\theta \in (0, 1)$, there exist positive constants $\delta := \delta_{(n, \Lambda, p, q, M, R, h, \kappa)}$ and $C := C_{(n, \Lambda, p, q, M, R, h, \kappa, \theta)}$ such that, if $\|A\|_{\text{BMO}_R(B_R, w)} \leq \delta$, then

$$\int_{B_{\theta r}} |\nabla u(x)|^p w(x) dx \leq C \int_{B_r} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^p + |u(x)|^p \right] w(x) dx. \quad (3.4)$$

We prove Lemma 3.3 at the end of this section. Before that, we first present its local boundary version. To this end, let Ω be a bounded domain of $\mathbb{R}^n$. Recall that, for any given $R \in (0, \infty)$, $\Omega_R := \Omega \cap \{x \in \mathbb{R}^n : |x| < R\}$.

Definition 3.4. Let $R \in (0, \infty)$, $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^n)$, A satisfy (1.2), and $|\vec{F}|/w \in L^p(\Omega_R, w)$. A function $u \in W^{1,p}(\Omega_R, w)$ is said to be a weak solution of

$$\begin{cases} \operatorname{div}(A \nabla u) = \operatorname{div} \vec{F} & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega \cap B_R \end{cases} \quad (3.5)$$

if, for any $\varphi \in C_c^\infty(\Omega_R)$,

$$\int_{\Omega_R} [A(x) \nabla u(x)] \cdot \nabla \varphi(x) dx = \int_{\Omega_R} \vec{F}(x) \cdot \nabla \varphi(x) dx$$

and the zero extension of u to B_R is in $W^{1,p}(B_R, w)$, namely, there exists a $U \in W^{1,p}(B_R, w)$ such that $U(x) = u(x)$ for any $x \in \Omega_R$, and $U(x) = 0$ for any $x \in B_R \setminus \Omega$.

For the weak solution u of (3.5), we have the following Lemma 3.5, which is a local boundary version of Lemma 3.3. The proof of Lemma 3.5 is similar to that of Lemma 3.3 (see also the proof of [4, Theorem 6.1]). To limit the length of this paper, we omit the details here.

Lemma 3.5. Let $R \in (0, \infty)$, $M \in (0, \infty)$, $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^n)$ with $[w]_{A_p(\mathbb{R}^n)} \leq M$, $q \in (1, \infty)$ be as in (3.1), and $\kappa \in (\sqrt{n}, \infty)$ as in Lemma 2.4. Assume that A and $\vec{F}$ are as in Definition 3.4 and $u \in W^{1,p}(B_R, w)$ is a weak solution of (3.5). Then there exists a positive constant $h := h_{(n, \Lambda, p, q, M, R)} \in (1, \infty)$ such that, for any given $r \in (0, \frac{R}{2h\kappa})$ and $\theta \in (0, 1)$, there exist positive constants $\delta := \delta_{(n, \Lambda, p, q, M, R, h, \kappa)}$ and $C := C_{(n, \Lambda, p, q, M, R, h, \kappa, \theta)}$ such that, if Ω is a (δ, R) -Reifenberg flat domain and $\|A\|_{\text{BMO}_R(\Omega_R, w)} \leq \delta$, then

$$\int_{\Omega_{\theta r}} |\nabla u(x)|^p w(x) dx \leq C \int_{\Omega_r} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^p + |u(x)|^p \right] w(x) dx.$$

In the remainder of this section, we prove Lemma 3.3 and always let $R \in (0, \infty)$, $p \in (1, \infty)$, u be a weak solution of (3.2), $h \in (1, \infty)$, and $\kappa \in (\sqrt{n}, \infty)$ as in Lemma 2.4. For any $r \in (0, \frac{R}{2h\kappa})$ and $\theta \in (0, 1)$, choose a function $\zeta \in C_c^\infty(B_r)$ satisfying that $0 \leq \zeta(x) \leq 1$ for any $x \in B_r$, $\zeta(x) = 1$ for any $x \in B_{\theta r}$, and there exists a positive constant $c := c_{(n, \theta)}$ such that, for any $x \in B_r$,

$$|\nabla \zeta(x)| \leq \frac{c}{r} \quad \text{and} \quad |\nabla^2 \zeta(x)| \leq \frac{c}{r^2}. \quad (3.6)$$

For any $x \in B_R$, define

$$u_\star(x) := u(x)\zeta(x). \quad (3.7)$$

Let $p_0 \in (1, \infty)$. For any $z \in B_{\kappa r}$ and $\tau \in (0, h\kappa r)$, consider the following equation

$$\begin{cases} \operatorname{div}(\langle A \rangle_{B_\tau(z)} \nabla v) = 0 & \text{in } B_\tau(z), \\ v - u_\star \in W_0^{1,p_0}(B_\tau(z)). \end{cases} \quad (3.8)$$

To show Lemma 3.3, we need the following estimate on $v - u_\star$ on $B_\tau(z)$.

Lemma 3.6. Let $p \in (1, \infty)$, $M \in (0, \infty)$, $w \in A_p(\mathbb{R}^n)$, $[w]_{A_p(\mathbb{R}^n)} \leq M$, p^* and q be as in (3.1), and z , τ , u_* as in (3.8). Let

$$q^* := q \left[\left(\frac{1}{p} + \frac{p-1}{q+p-1} \right) (q-1) + 1 \right]^{-1} \in (1, p). \quad (3.9)$$

For any given $p_0 \in (1, q^*)$, assume that $v \in W^{1, p_0}(B_\tau(z))$ is a weak solution of (3.8). Then there exist positive constants $\gamma \in (1, \frac{p}{p_0})$ and $C := C(n, R, A, p_0, p, q, M)$ such that

$$\begin{aligned} & \left[\frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} |\nabla v(x) - \nabla u_*(x)|^{p_0} dx \right]^{1/p_0} \\ & \leq C \|A\|_{\text{BMO}_R(B_R, w)} \left[\frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} |\nabla u_*(x)|^{\gamma p_0} w(x) dx \right]^{\frac{1}{\gamma p_0}} \\ & \quad + Ch\kappa \|A\|_{\text{BMO}_R(B_R, w)} \left[\frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} |\nabla u(x)|^{\gamma p_0} \mathbf{1}_{B_r}(x) w(x) dx \right]^{\frac{1}{\gamma p_0}} \\ & \quad + Ch\kappa \left[\frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} \left| \frac{u(x)}{r} \right|^{p_0} \mathbf{1}_{B_r}(x) w(x) dx \right]^{\frac{1}{p_0}} \\ & \quad + Ch\kappa \left\{ \frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^{\gamma p_0} + \left| \frac{u(x)}{r} \right|^{\gamma p_0} \right] \mathbf{1}_{B_r}(x) w(x) dx \right\}^{\frac{1}{\gamma p_0}}. \end{aligned} \quad (3.10)$$

Proof. Let z , τ , u_* be as in (3.8) and $w := v - u_*$. Then (3.8) is equivalent to

$$\begin{cases} \operatorname{div}(\langle A \rangle_{B_\tau(z)} \nabla w) = -\operatorname{div}(\langle A \rangle_{B_\tau(z)} \nabla u_*) & \text{in } B_\tau(z), \\ w \in W_0^{1, p_0}(B_\tau(z)). \end{cases} \quad (3.11)$$

Let $\mathcal{D}'(B_\tau(z))$ be the dual space of $C_c^\infty(B_\tau(z))$. Then, by (3.2), (3.7), and $B_\tau(z) \subset B_R$, we easily conclude that

$$\operatorname{div}(A \nabla u_*) = \operatorname{div}(\zeta \vec{F} + u A \nabla \zeta) + (A \nabla u - \vec{F}) \cdot \nabla \zeta \quad \text{in } \mathcal{D}'(B_\tau(z)).$$

From this and (3.11), we deduce that

$$\begin{aligned} \operatorname{div}(\langle A \rangle_{B_\tau(z)} \nabla w) &= -\operatorname{div}(\langle A \rangle_{B_\tau(z)} \nabla u_*) \\ &= \operatorname{div} \left[(A - \langle A \rangle_{B_\tau(z)}) \nabla u_* - (\zeta \vec{F} + u A \nabla \zeta) \right] \\ &\quad - [\langle A \rangle_{B_\tau(z)} \nabla u] \cdot \nabla \zeta - [(A - \langle A \rangle_{B_\tau(z)}) \nabla u] \cdot \nabla \zeta + \vec{F} \cdot \nabla \zeta \\ &=: \operatorname{div} \vec{G} + H. \end{aligned}$$

Now, we consider the following two equations

$$\begin{cases} \operatorname{div}(\langle A \rangle_{B_\tau(z)} \nabla \tilde{u}) = \operatorname{div} \vec{G} & \text{in } B_\tau(z), \\ \tilde{u} \in W_0^{1, p_0}(B_\tau(z)) \end{cases} \quad (3.12)$$

and

$$\begin{cases} \operatorname{div}(\langle A \rangle_{B_\tau(z)} \nabla \tilde{v}) = H & \text{in } B_\tau(z), \\ \tilde{v} \in W_0^{1, p_0}(B_\tau(z)). \end{cases} \quad (3.13)$$

Then $w = \tilde{u} + \tilde{v}$. By (1.2), we know that, for any $\xi \in \mathbb{R}^n$,

$$\Lambda^{-1} \frac{w(B_\tau(z))}{|B_\tau(z)|} |\xi|^2 \leq \langle A \rangle_{B_\tau(z)} \leq \Lambda \frac{w(B_\tau(z))}{|B_\tau(z)|} |\xi|^2.$$

Let

$$A_{\tau,z} := \langle A \rangle_{B_\tau(z)} \frac{|B_\tau(z)|}{w(B_\tau(z))}, \quad \tilde{u}_{\tau,z} := \tilde{u} \frac{w(B_\tau(z))}{|B_\tau(z)|}, \quad \text{and} \quad \tilde{v}_{\tau,z} := \tilde{v} \frac{w(B_\tau(z))}{|B_\tau(z)|}.$$

Then, from (3.12) and (3.13), we deduce that

$$\begin{cases} \operatorname{div}(A_{\tau,z} \nabla \tilde{u}_{\tau,z}) = \operatorname{div} \vec{G} & \text{in } B_\tau(z), \\ \tilde{u}_{\tau,z} \in W_0^{1,p_0}(B_\tau(z)) \end{cases} \quad \text{and} \quad \begin{cases} \operatorname{div}(A_{\tau,z} \nabla \tilde{v}_{\tau,z}) = H & \text{in } B_\tau(z), \\ \tilde{v}_{\tau,z} \in W_0^{1,p_0}(B_\tau(z)), \end{cases}$$

where $A_{\tau,z}$ satisfies the classical uniform elliptic condition, namely, (1.2) with $w \equiv 1$ therein. It is well known that

$$\|\nabla \tilde{u}_{\tau,z}\|_{L^{p_0}(B_\tau(z))} \lesssim \|\vec{G}\|_{L^{p_0}(B_\tau(z))}. \quad (3.14)$$

To estimate $\|\nabla \tilde{v}_{\tau,z}\|_{L^{p_0}(B_\tau(z))}$, by using $\tilde{u}_{\tau,z}$ as a test function of (3.13) and vice versa, and the assumption that A is symmetric, we obtain

$$\begin{aligned} \int_{B_\tau(z)} H(x) \tilde{u}_{\tau,z}(x) dx &= \int_{B_\tau(z)} [A_{\tau,z}(x) \nabla \tilde{v}_{\tau,z}(x)] \cdot \nabla \tilde{u}_{\tau,z}(x) dx \\ &= \int_{B_\tau(z)} [A_{\tau,z}(x) \nabla \tilde{u}_{\tau,z}(x)] \cdot \nabla \tilde{v}_{\tau,z}(x) dx = \int_{B_\tau(z)} \vec{G}(x) \cdot \nabla \tilde{v}_{\tau,z}(x) dx. \end{aligned}$$

From this and Remark 2.9, it follows that

$$\begin{aligned} \|\nabla \tilde{v}_{\tau,z}\|_{L^{p_0}(B_\tau(z))} &= \sup_{\|\vec{G}\|_{L^{p'_0}(B_\tau(z))} \leq 1} \left| \int_{B_\tau(z)} \vec{G}(x) \cdot \nabla \tilde{v}_{\tau,z}(x) dx \right| \\ &= \sup_{\|\vec{G}\|_{L^{p'_0}(B_\tau(z))} \leq 1} \left| \int_{B_\tau(z)} H(x) \tilde{u}_{\tau,z}(x) dx \right| \\ &\leq \sup_{\|\vec{G}\|_{L^{p'_0}(B_\tau(z))} \leq 1} \|H\|_{W^{-1,p_0}(B_\tau(z))} \|\tilde{u}_{\tau,z}\|_{W_0^{1,p'_0}(B_\tau(z))} \\ &\lesssim \sup_{\|\vec{G}\|_{L^{p'_0}(B_\tau(z))} \leq 1} \|H\|_{W^{-1,p_0}(B_\tau(z))} \|\nabla \tilde{u}_{\tau,z}\|_{L^{p'_0}(B_\tau(z))} \\ &\lesssim \|H\|_{W^{-1,p_0}(B_\tau(z))}, \end{aligned} \quad (3.15)$$

where, for any given $p \in (1, \infty)$ and subset U of $\mathbb{R}^n$, $W^{-1,p'}(U)$ denotes the *dual space* of $W_0^{1,p}(U)$, which is defined as the completion of all $v \in L^{p'}(U)$ with respect to the norm

$$\|v\|_{W^{-1,p'}(U)} := \sup_{u \in W_0^{1,p}(U), \|u\|_{W_0^{1,p}(U)} \leq 1} \left| \int_U u(x) v(x) dx \right|. \quad (3.16)$$

Combining (3.14) and (3.15), we find that

$$\begin{aligned} \|\nabla w\|_{L^{p_0}(B_\tau(z))} &\lesssim \frac{|B_\tau(z)|}{w(B_\tau(z))} \|(A - \langle A \rangle_{B_\tau(z)}) \nabla u^*\|_{L^{p_0}(B_\tau(z))} + \frac{|B_\tau(z)|}{w(B_\tau(z))} \|\zeta \vec{F} + u A \nabla \zeta\|_{L^{p_0}(B_\tau(z))} \\ &\quad + \frac{|B_\tau(z)|}{w(B_\tau(z))} \|[(A - \langle A \rangle_{B_\tau(z)}) \nabla u] \cdot \nabla \zeta\|_{W^{-1,p_0}(B_\tau(z))} \end{aligned}$$

$$\begin{aligned}
& + \frac{|B_\tau(z)|}{w(B_\tau(z))} \left\| \langle A \rangle_{B_\tau(z)} \nabla u \cdot \nabla \zeta \right\|_{W^{-1}, p_0(B_\tau(z))} \\
& + \frac{|B_\tau(z)|}{w(B_\tau(z))} \left\| \vec{F} \cdot \nabla \zeta \right\|_{W^{-1}, p_0(B_\tau(z))} \\
& =: \sum_{i=1}^5 I_i.
\end{aligned} \tag{3.17}$$

For I_1 , by $p_0 \in (1, q_*)$ with q_* as in (3.9), we easily conclude that

$$0 < p_0 \frac{p-1}{q+p-1} + \frac{p_0-1}{q-1} + \frac{p_0}{p} < 1.$$

Thus, there exist positive constants $\alpha, \beta, \gamma \in (1, \infty)$ such that

$$\frac{1}{\alpha} \in \left(p_0 \frac{p-1}{q+p-1}, 1 \right), \quad \frac{1}{\beta} \in \left(\frac{p_0-1}{q-1}, 1 \right), \quad \frac{1}{\gamma} \in \left(\frac{p_0}{p}, 1 \right), \quad \text{and} \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1. \tag{3.18}$$

Moreover, we have

$$(\alpha p_0)' \in (p_*, \infty) \quad \text{and} \quad q \in (\beta(p_0-1)+1, \infty), \tag{3.19}$$

where $p_* = 1 + \frac{p-1}{q}$ is as in (3.1) such that $w \in A_{p_*}(\mathbb{R}^n) \cap RH_q(\mathbb{R}^n)$. From this, the Hölder inequality, respectively, with exponents α, β , and γ , and Remark 1.6, we deduce that

$$\begin{aligned}
I_1^{p_0} & \leq \left[\frac{|B_\tau(z)|}{w(B_\tau(z))} \right]^{p_0} \left[\int_{B_\tau(z)} \left| \frac{A - \langle A \rangle_{B_\tau(z)}}{w} \right|^{\alpha p_0} w(x) dx \right]^{\frac{1}{\alpha}} \\
& \quad \times \left\{ \int_{B_\tau(z)} [w(x)]^{\beta(p_0-1)+1} dx \right\}^{\frac{1}{\beta}} \left[\int_{B_\tau(z)} |\nabla u_*(x)|^{\gamma p_0} w(x) dx \right]^{\frac{1}{\gamma}} \\
& \leq \|A\|_{\text{BMO}_R^{\alpha p_0}(B_R, w)}^{p_0} \frac{|B_\tau(z)|^{p_0 + \frac{1}{\beta}}}{[w(B_\tau(z))]^{p_0 - \frac{1}{\alpha}}} \left\{ \frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} [w(x)]^{q_0} dx \right\}^{\frac{1}{\beta}} \\
& \quad \times \left[\int_{B_\tau(z)} |\nabla u_*(x)|^{\gamma p_0} w(x) dx \right]^{\frac{1}{\gamma}} \\
& \lesssim \|A\|_{\text{BMO}_R^{\alpha p_0}(B_R, w)}^{p_0} \frac{|B_\tau(z)|^{p_0 + \frac{1}{\beta}}}{[w(B_\tau(z))]^{p_0 - \frac{1}{\alpha}}} \left[\frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} w(x) dx \right]^{\frac{q_0}{\beta}} \\
& \quad \times \left[\int_{B_\tau(z)} |\nabla u_*(x)|^{\gamma p_0} w(x) dx \right]^{\frac{1}{\gamma}} \\
& \lesssim \|A\|_{\text{BMO}_R(B_R, w)}^{p_0} |B_\tau(z)| \left[\frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} |\nabla u_*(x)|^{\gamma p_0} w(x) dx \right]^{\frac{1}{\gamma}},
\end{aligned} \tag{3.20}$$

where $q_0 := \beta(p_0-1)+1$ and we used the fact that $w \in A_{p_*}(\mathbb{R}^n) \subset A_{(\alpha p_0)'}(\mathbb{R}^n)$, and $RH_q(\mathbb{R}^n) \subset RH_{q_0}(\mathbb{R}^n)$ [see Lemma 2.2(ii)].

For I_2 , let β and γ be as in (3.18). Then we have $\beta > \gamma'$. By (1.2), the Hölder inequality, respectively, with exponents γ and γ' , and $w \in RH_q(\mathbb{R}^n) \subset RH_{\beta(p_0-1)+1}(\mathbb{R}^n) \subset RH_{\gamma'(p_0-1)+1}(\mathbb{R}^n)$, we conclude that

$$I_2^{p_0} = \left[\frac{|B_\tau(z)|}{w(B_\tau(z))} \right]^{p_0} \int_{B_\tau(z)} \left| \frac{\zeta \vec{F} + u A \nabla \zeta}{w} \right|^{p_0} [w(x)]^{p_0-1} w(x) dx$$

$$\begin{aligned}
&\lesssim \left[\frac{|B_\tau(z)|}{w(B_\tau(z))} \right]^{p_0} \left\{ \int_{B_\tau(z)} [w(x)]^{\gamma'(p_0-1)+1} dx \right\}^{\frac{1}{\gamma}} \left\{ \int_{B_\tau(z)} \left[\left| \frac{\tilde{F}(x)}{w(x)} \right|^{\gamma p_0} + \left| \frac{u(x)}{r} \right|^{\gamma p_0} \right] w(x) dx \right\}^{\frac{1}{\gamma}} \\
&\sim \left[\frac{|B_\tau(z)|}{w(B_\tau(z))} \right]^{p_0} \left\{ \frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} [w(x)]^{q_1} dx \right\}^{\frac{1}{q_1} \frac{q_1}{\gamma}} |B_\tau(z)|^{\frac{1}{\gamma}} \\
&\quad \times \left\{ \int_{B_\tau(z)} \left[\left| \frac{\tilde{F}(x)}{w(x)} \right|^{\gamma p_0} + \left| \frac{u(x)}{r} \right|^{\gamma p_0} \right] w(x) dx \right\}^{\frac{1}{\gamma}} \\
&\lesssim \left[\frac{|B_\tau(z)|}{w(B_\tau(z))} \right]^{p_0} \left[\frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} w(x) dx \right]^{\frac{q_1}{\gamma}} |B_\tau(z)|^{\frac{1}{\gamma}} \\
&\quad \times \left\{ \int_{B_\tau(z)} \left[\left| \frac{\tilde{F}(x)}{w(x)} \right|^{\gamma p_0} + \left| \frac{u(x)}{r} \right|^{\gamma p_0} \right] w(x) dx \right\}^{\frac{1}{\gamma}} \\
&\lesssim |B_\tau(z)| \left\{ \frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} \left[\left| \frac{\tilde{F}(x)}{w(x)} \right|^{\gamma p_0} + \left| \frac{u(x)}{r} \right|^{\gamma p_0} \right] w(x) dx \right\}^{\frac{1}{\gamma}}, \tag{3.21}
\end{aligned}$$

where $q_1 := \gamma'(p_0 - 1) + 1$.

Next, we estimate I_3 , I_4 , and I_5 , respectively. For I_3 , from (3.16) and Remark 2.9, it follows that

$$\begin{aligned}
I_3 &\lesssim \frac{|B_\tau(z)|}{w(B_\tau(z))} \sup \left\{ \left| \int_{B_\tau(z)} \{ [A(x) - \langle A \rangle_{B_\tau(z)}] \nabla u(x) \} \cdot \nabla \zeta(x) \phi(x) dx \right| \right. \\
&\quad \left. : \phi \in C_c^\infty(B_\tau(z)), \|\nabla \phi\|_{L^{p'_0}(B_\tau(z))} \leq 1 \right\}.
\end{aligned}$$

Let $\gamma \in (1, \frac{p}{p_0})$ be as in (3.18) and $\theta := k \frac{q-1}{q}$, where $k \in (1, \infty)$ is as in (2.7). By (3.18) and (3.19), we know that

$$\frac{1}{p'_0} + \frac{1}{\beta p_0} + \frac{1}{\gamma p_0} = \frac{1}{(\alpha p_0)'} < \frac{1}{p^*}. \tag{3.22}$$

Moreover, from $\frac{1}{\beta} \in (\frac{p_0-1}{q-1}, 1)$, we deduce that

$$\frac{1}{p'_0} + \frac{1}{\beta p_0} > \frac{1}{p'_0} + \frac{1}{p_0} \frac{p_0-1}{q-1} = \frac{1}{p'_0} + \frac{1}{q-1} \frac{1}{p'_0} = \frac{q}{(q-1)p'_0} > \frac{q}{(q-1)kp'_0} = \frac{1}{\theta p'_0}.$$

By this and (3.22), we find that there exists some $\tilde{\theta} \in (1, \infty)$ such that

$$\frac{1}{\tilde{\theta}} + \frac{1}{\gamma p_0} + \frac{1}{\theta p'_0} = 1 \quad \text{and} \quad \tilde{\theta}' \in (p^*, \infty). \tag{3.23}$$

For any $\phi \in C_c^\infty(B_\tau(z))$ with $\|\nabla \phi\|_{L^{p'_0}(B_\tau(z))} \leq 1$, from the Hölder inequality, (3.6), Lemma 3.1(ii), Remarks 1.6 and 2.9, $p'_0 > n$, and $\tau \in (0, h\kappa r)$, it follows that

$$\begin{aligned}
I_3 &\lesssim \frac{1}{r} \frac{|B_\tau(z)|}{w(B_\tau(z))} \left\| \frac{A - \langle A \rangle_{B_\tau(z)}}{w} \right\|_{L^{\tilde{\theta}}(B_\tau(z), w)} \|\nabla u \mathbf{1}_{B_\tau}\|_{L^{\gamma p_0}(B_\tau(z), w)} \|\phi\|_{L^{\theta p'_0}(B_\tau(z), w)} \\
&\lesssim \frac{1}{r} \frac{|B_\tau(z)|}{w(B_\tau(z))} \|A\|_{\text{BMO}_R^{\tilde{\theta}}(B_R, w)} [w(B_\tau(z))]^{\frac{1}{\tilde{\theta}}} \|\nabla u \mathbf{1}_{B_\tau}\|_{L^{\gamma p_0}(B_\tau(z), w)}
\end{aligned}$$

$$\begin{aligned}
& \times [w(B_\tau(z))]^{\frac{1}{\theta p_0}} \left[\frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} |\phi(x)|^{\theta p'_0} w(x) dx \right]^{\frac{1}{\theta p_0}} \\
& \lesssim \frac{1}{r} \frac{|B_\tau(z)|}{[w(B_\tau(z))]^{1/\gamma p_0}} \|A\|_{\text{BMO}_R^{\theta}(B_R, w)} \|\nabla u \mathbf{1}_{B_r}\|_{L^{\gamma p_0}(B_\tau(z), w)} \left[\frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} |\phi(x)|^{k p'_0} dx \right]^{\frac{1}{k p_0}} \\
& \lesssim \frac{\tau}{r} \frac{|B_\tau(z)|}{[w(B_\tau(z))]^{1/\gamma p_0}} \|A\|_{\text{BMO}_R^{\theta}(B_R, w)} \|\nabla u \mathbf{1}_{B_r}\|_{L^{\gamma p_0}(B_\tau(z), w)} \left[\frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} |\nabla \phi(x)|^{p'_0} dx \right]^{\frac{1}{p_0}} \\
& \lesssim h\kappa \|A\|_{\text{BMO}_R(B_R, w)} |B_\tau(z)|^{\frac{1}{p_0}} \left[\frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} |\nabla u(x)|^{\gamma p_0} \mathbf{1}_{B_r}(x) w(x) dx \right]^{\frac{1}{\gamma p_0}}. \quad (3.24)
\end{aligned}$$

For I_4 , by (3.16) and Remark 2.9, we conclude that

$$\begin{aligned}
I_4 & \lesssim \frac{|B_\tau(z)|}{w(B_\tau(z))} \\
& \times \sup \left\{ \left| \int_{B_\tau(z)} [\langle A \rangle_{B_\tau(z)} \nabla u(x)] \cdot \nabla \zeta(x) \phi(x) dx \right| : \phi \in C_c^\infty(B_\tau(z)), \|\nabla \phi\|_{L^{p'_0}(B_\tau(z))} \leq 1 \right\}. \quad (3.25)
\end{aligned}$$

For any $\phi \in C_c^\infty(B_\tau(z))$, from integration by parts, we deduce that

$$\begin{aligned}
\left| \int_{B_\tau(z)} [\langle A \rangle_{B_\tau(z)} \nabla u(x)] \cdot \nabla \zeta(x) \phi(x) dx \right| & = \left| \int_{B_\tau(z)} u(x) \operatorname{div} [\langle A \rangle_{B_\tau(z)} \nabla \zeta(x) \phi(x)] dx \right| \\
& \leq |\langle A \rangle_{B_\tau(z)}| \int_{B_\tau(z)} |u(x)| |\nabla^2 \zeta(x)| |\phi(x)| \mathbf{1}_{B_r}(x) dx \\
& \quad + |\langle A \rangle_{B_\tau(z)}| \int_{B_\tau(z)} |u(x)| |\nabla \zeta(x)| |\nabla \phi(x)| \mathbf{1}_{B_r}(x) dx \\
& =: K_1 + K_2. \quad (3.26)
\end{aligned}$$

For K_1 , by (1.2), (3.6), the Hölder inequality, Remark 2.9, and $\tau \in (0, h\kappa r)$, we conclude that

$$\begin{aligned}
K_1 & \lesssim \frac{1}{r} \frac{w(B_\tau(z))}{|B_\tau(z)|} \left\| \frac{u}{r} \mathbf{1}_{B_r} \right\|_{L^{p_0}(B_\tau(z))} \|\phi\|_{L^{p'_0}(B_\tau(z))} \\
& \lesssim h\kappa \frac{w(B_\tau(z))}{|B_\tau(z)|} \left\| \frac{u}{r} \mathbf{1}_{B_r} \right\|_{L^{p_0}(B_\tau(z))} \|\nabla \phi\|_{L^{p'_0}(B_\tau(z))}. \quad (3.27)
\end{aligned}$$

For K_2 , from the Hölder inequality and an argument similar to that used in the estimation of (3.27), it follows that

$$K_2 \lesssim \frac{w(B_\tau(z))}{|B_\tau(z)|} \left\| \frac{u}{r} \mathbf{1}_{B_r} \right\|_{L^{p_0}(B_\tau(z))} \|\nabla \phi\|_{L^{p'_0}(B_\tau(z))}.$$

Combining this, (3.27), (3.26), and (3.25), we obtain

$$I_4 \lesssim h\kappa \left\| \frac{u}{r} \mathbf{1}_{B_r} \right\|_{L^{p_0}(B_\tau(z))}. \quad (3.28)$$

Next, we estimate I_5 . By (3.16) and Remark 2.9, we know that

$$I_5 \lesssim \frac{|B_\tau(z)|}{w(B_\tau(z))} \sup \left\{ \left| \int_{B_\tau(z)} \vec{F}(x) \cdot \nabla \zeta(x) \phi(x) dx \right| : \phi \in C_c^\infty(B_\tau(z)), \|\nabla \phi\|_{L^{p'_0}(B_\tau(z))} \leq 1 \right\}.$$

For any $\phi \in C_c^\infty(B_\tau(z))$ with $\|\nabla \phi\|_{L^{p'_0}(B_\tau(z))} \leq 1$, by the Hölder inequality and $\tau \in (0, h\kappa r)$, we have

$$\begin{aligned} \left| \int_{B_\tau(z)} \vec{F}(x) \cdot \nabla \zeta(x) \phi(x) dx \right| &\lesssim \frac{1}{r} \|\vec{F}\|_{L^{p_0}(B_\tau(z))} \|\phi\|_{L^{p'_0}(B_\tau(z))} \\ &\lesssim \frac{\tau}{r} \|\vec{F}\|_{L^{p_0}(B_\tau(z))} \|\nabla \phi\|_{L^{p'_0}(B_\tau(z))} \\ &\lesssim h\kappa \left[\int_{B_\tau(z)} \left| \frac{\vec{F}(x)}{w(x)} \right|^{p_0} [w(x)]^{p_0-1} w(x) dx \right]^{\frac{1}{p_0}}. \end{aligned}$$

From this and an argument similar to that used in the estimation of (3.21), we deduce that

$$I_5 \lesssim h\kappa |B_\tau(z)|^{\frac{1}{p_0}} \left\{ \frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\gamma p_0} w(x) dx \right\}^{\frac{1}{\gamma p_0}},$$

where $\gamma \in (1, \frac{p}{p_0})$ is as in (3.18). By this, (3.28), (3.24), (3.21), (3.20), and (3.17), we obtain (3.10). This finishes the proof of Lemma 3.6. $\square$

The following lemma gives a $C^{1,\alpha}$ estimate for the weak solution of (3.8), which was established in [4, Lemma 5.5].

Lemma 3.7 ([4]). *Let $p \in (1, \infty)$, u be a weak solution of (3.2), $p_0 \in (1, p)$, and z, τ, u_* be as in (3.8). Assume that $v \in W^{1,p_0}(B_\tau(z))$ is a weak solution of (3.8). Then there exist positive constants $C := C_{(n, \Lambda, p_0, M, R)}$ and $\alpha := \alpha_{(n, \Lambda)} \in (0, 1)$ such that, for any $\rho \in (0, \frac{\tau}{2})$,*

$$\frac{1}{|B_\rho(z)|} \int_{B_\rho(z)} \left| |\nabla v(x)| - \langle |\nabla v| \rangle_{B_\rho(z)} \right| dx \leq C \left(\frac{\rho}{\tau} \right)^\alpha \left[\frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} |\nabla u_*(x)|^{p_0} dx \right]^{1/p_0}.$$

We are now ready to prove Lemma 3.3 by using Lemmas 3.6 and 3.7.

Proof of Lemma 3.3. Let $R \in (0, \infty)$, $M \in (0, \infty)$, $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^n)$ with $[w]_{A_p(\mathbb{R}^n)} \leq M$, $q \in (1, \infty)$ be as in (3.1), and $\kappa \in (\sqrt{n}, \infty)$ as in Lemma 2.4. Assume that A satisfies $\|A\|_{\text{BMO}_R(B_R, w)} \leq \delta$, $\vec{F}$ is as in Definition 3.2, $u \in W^{1,p}(B_R, w)$ is a weak solution of (3.2), u_* is as in (3.7), v is a weak solution of (3.8), $r \in (0, \frac{R}{2h\kappa})$, and $z \in B_{\kappa r}$, where $h \in (1, \infty)$ and $\delta \in (0, \infty)$ are fixed later.

Take $p_0 \in (1, p)$ sufficiently small such that $w \in A_{p/p_0}(\mathbb{R}^n)$. From Lemmas 3.6 and 3.7, and the Hölder inequality, we deduce that there exists a positive constant $c_1 := C_{(n, \Lambda, p, q, M, R)}$ such that, for any $\rho \in (0, \kappa r)$ and $\tau = \rho h$,

$$\begin{aligned} &\frac{1}{|B_\rho(z)|} \int_{B_\rho(z)} \left| |\nabla u_*(x)| - \langle |\nabla u_*| \rangle_{B_\rho(z)} \right| dx \\ &\leq 2 \frac{1}{|B_\rho(z)|} \int_{B_\rho(z)} \left| |\nabla u_*(x)| - \langle |\nabla v| \rangle_{B_\rho(z)} \right| dx \\ &\leq 2 \frac{1}{|B_\rho(z)|} \int_{B_\rho(z)} |\nabla u_*(x) - \nabla v(x)| dx + 2 \frac{1}{|B_\rho(z)|} \int_{B_\rho(z)} \left| |\nabla v(x)| - \langle |\nabla v| \rangle_{B_\rho(z)} \right| dx \\ &\leq 2 \left[\frac{1}{|B_\rho(z)|} \int_{B_\rho(z)} |\nabla u_*(x) - \nabla v(x)|^{p_0} dx \right]^{\frac{1}{p_0}} + 2 \frac{1}{|B_\rho(z)|} \int_{B_\rho(z)} \left| |\nabla v(x)| - \langle |\nabla v| \rangle_{B_\rho(z)} \right| dx \end{aligned}$$

$$\begin{aligned}
&\leq c_1 h^n \|A\|_{\text{BMO}_R(B_R, w)} \left[\frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} |\nabla u_\star(x)|^{\gamma p_0} w(x) dx \right]^{\frac{1}{\gamma p_0}} \\
&\quad + c_1 h^{n+1} \kappa \|A\|_{\text{BMO}_R(B_R, w)} \left[\frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} |\nabla u(x)|^{\gamma p_0} \mathbf{1}_{B_r}(x) w(x) dx \right]^{\frac{1}{\gamma p_0}} \\
&\quad + c_1 h^{n+1} \kappa \left[\frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} \left| \frac{u(x)}{r} \right|^{p_0} \mathbf{1}_{B_r}(x) dx \right]^{\frac{1}{p_0}} \\
&\quad + c_1 h^{n+1} \kappa \left\{ \frac{1}{w(B_\tau(z))} \int_{B_\tau(z)} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^{\gamma p_0} + \left| \frac{u(x)}{r} \right|^{\gamma p_0} \right] \mathbf{1}_{B_r}(x) w(x) dx \right\}^{\frac{1}{\gamma p_0}} \\
&\quad + c_1 h^{-\alpha} \left[\frac{1}{|B_\tau(z)|} \int_{B_\tau(z)} |\nabla u_\star(x)|^{p_0} dx \right]^{1/p_0},
\end{aligned}$$

where $\gamma \in (1, \frac{p}{p_0})$ is as in Lemma 3.6, and $\alpha \in (0, 1)$ as in Lemma 3.7. Thus, we further know that, for any $z \in B_{Kr}$,

$$\begin{aligned}
\mathcal{M}^{Kr, \sharp}(|\nabla u_\star|)(z) &\leq c_1 h^n \|A\|_{\text{BMO}_R(B_R, w)} [\mathcal{M}_w(|\nabla u_\star|^{\gamma p_0})(z)]^{\frac{1}{\gamma p_0}} \\
&\quad + c_1 h^{n+1} \kappa \|A\|_{\text{BMO}_R(B_R, w)} [\mathcal{M}_w(|\nabla u|^{\gamma p_0} \mathbf{1}_{B_r})(z)]^{\frac{1}{\gamma p_0}} \\
&\quad + c_1 h^{n+1} \kappa \left[\mathcal{M} \left(\left| \frac{u}{r} \mathbf{1}_{B_r} \right|^{p_0} \right)(x) \right]^{\frac{1}{p_0}} \\
&\quad + c_1 h^{n+1} \kappa \left[\mathcal{M}_w \left(\left| \frac{u}{r} \mathbf{1}_{B_r} \right|^{\gamma p_0} + \left| \frac{\vec{F}}{w} \mathbf{1}_{B_r} \right|^{\gamma p_0} \right)(z) \right]^{\frac{1}{\gamma p_0}} \\
&\quad + c_1 h^{-\alpha} [\mathcal{M}(|\nabla u_\star|^{p_0})(z)]^{\frac{1}{p_0}}.
\end{aligned} \tag{3.29}$$

Observing that $\text{supp } |\nabla u_\star| \subset B_r$, by Lemma 2.4, we conclude that there exists a positive constant $c_2 := C_{(n, p, M)}$ such that

$$\int_{B_r} |\nabla u_\star(x)|^p w(x) dx \leq c_2 \int_{B_{Kr}} [\mathcal{M}^{Kr, \sharp}(|\nabla u_\star|)(z)]^p w(x) dx.$$

From this, (3.29), Remark 2.3, and $w \in A_{p/p_0}(\mathbb{R}^n)$, it follows that there exists a positive constant $c_3 := C_{(n, \Lambda, p, q, M, R)}$ such that

$$\begin{aligned}
\int_{B_r} |\nabla u_\star(x)|^p w(x) dx &\leq c_3 h^{np} \|A\|_{\text{BMO}_R(B_R, w)}^p \int_{B_r} |\nabla u_\star(x)|^p w(x) dx \\
&\quad + c_3 (h^{n+1} \kappa)^p \|A\|_{\text{BMO}_R(B_R, w)}^p \int_{B_r} |\nabla u(x)|^p w(x) dx \\
&\quad + c_3 (h^{n+1} \kappa)^p \int_{B_r} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^p + \left| \frac{u(x)}{r} \right|^p \right] w(x) dx + c_3 h^{-\alpha p} \int_{B_r} |\nabla u_\star(x)|^p w(x) dx.
\end{aligned}$$

Fix $h \in (1, \infty)$ large enough such that $0 < c_3 h^{-\alpha p} < \frac{1}{2}$. Then we obtain

$$\int_{B_r} |\nabla u_\star(x)|^p w(x) dx$$

$$\begin{aligned} &\leq c_4 \|A\|_{\text{BMO}_R(B_R, w)}^p \int_{B_r} |\nabla u_*(x)|^p w(x) dx \\ &\quad + c_4 \|A\|_{\text{BMO}_R(B_R, w)}^p \int_{B_r} |\nabla u(x)|^p w(x) dx + c_4 \int_{B_r} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^p + \left| \frac{u(x)}{r} \right|^p \right] w(x) dx, \end{aligned}$$

where $c_4 := C_{(n, \Lambda, p, q, M, R, h, \kappa)}$. Let $\delta_0 \in (0, \infty)$ be such that

$$c_4 \delta_0^p = \frac{1}{2} \quad \text{and} \quad \|A\|_{\text{BMO}_R(B_R, w)} \leq \delta < \delta_0. \quad (3.30)$$

By this and $u_*(x) = u(x)$ for any $x \in B_{\theta r}$, we obtain

$$\begin{aligned} &\int_{B_{\theta r}} |\nabla u(x)|^p w(x) dx \\ &\leq \int_{B_r} |\nabla u_*(x)|^p w(x) dx \\ &\leq 2c_4 \|A\|_{\text{BMO}_R(\Omega, w)}^p \int_{B_r} |\nabla u(x)|^p w(x) dx + 2c_4 \int_{B_r} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^p + \left| \frac{u(x)}{r} \right|^p \right] w(x) dx. \end{aligned} \quad (3.31)$$

To complete the proof of this lemma, we use the covering and iteration method to absorb the first term on the right hand side. Indeed, by the proof of (3.31), we know that (3.31) holds true not only for balls B_r centered at the origin $\tilde{O}_n$, but also for balls $B_r(y)$ with $y \in B_{2\kappa}$ when $h \in (3, \infty)$. Let $\theta r < l_1 < l_2 < r$. We find that there exists a sequence $\{B_i := B_{(l_2-l_1)/2}(\tilde{z}_i)\}_{i \in I}$ of balls, with $\tilde{z}_i \in B_{l_1}(z)$ for any $i \in I$, and I a finite index set, such that

$$B_{l_1}(z) \subset \bigcup_{i \in I} B_i \quad \text{and} \quad \sum_{i \in I} \mathbf{1}_{2B_i}(x) \leq N_{(n)} \quad \text{for any } x \in \mathbb{R}^n, \quad (3.32)$$

where $N_{(n)}$ is a positive constant depending only on n . By (3.31), we know that, for any $i \in I$,

$$\begin{aligned} \int_{B_i} |\nabla u(x)|^p w(x) dx &\leq 2c_4 \|A\|_{\text{BMO}_R(B_R, w)}^p \int_{2B_i} |\nabla u(x)|^p w(x) dx \\ &\quad + 2c_4 (1 - \theta)^p \int_{2B_i} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^p + \left| \frac{u(x)}{l_2 - l_1} \right|^p \right] w(x) dx. \end{aligned}$$

From this and (3.32), via summing up over $i \in I$, we deduce that

$$\begin{aligned} \int_{B_{l_1}} |\nabla u(x)|^p w(x) dx &\leq 2c_4 N_{(n)} \|A\|_{\text{BMO}_R(B_R, w)}^p \int_{B_{l_2}} |\nabla u(x)|^p w(x) dx \\ &\quad + 2c_4 (1 - \theta)^p N_{(n)} \int_{B_{l_2}} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^p + \left| \frac{u(x)}{l_2 - l_1} \right|^p \right] w(x) dx, \end{aligned}$$

where we used $2B_i \subset B_{l_2}$ for any $i \in I$. Let $\delta_1 \in (0, \infty)$ be such that

$$2c_4 N_{(n)} \delta_1^p = \frac{1}{2}.$$

Fixing $\delta \in (0, \infty)$ small enough such that $\|A\|_{\text{BMO}_R(B_R, w)} \leq \delta < \min\{\delta_0, \delta_1\}$, where δ_0 is as in (3.30), we find that, for any $\theta r < l_1 < l_2 < r$,

$$\int_{B_{l_1}} |\nabla u(x)|^p w(x) dx \leq \frac{1}{2} \int_{B_{l_2}} |\nabla u(x)|^p w(x) dx + c_5 \int_{B_{l_2}} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^p + \left| \frac{u(x)}{l_2 - l_1} \right|^p \right] w(x) dx,$$

where $c_5 := 2c_4 (1 - \theta)^p N_{(n)}$. Using Lemma 2.7 with $\phi(t) := \int_{B_t} |\nabla u(x)|^p w(x) dx$ for any $t \in [\theta r, r]$ therein, we conclude that (3.4) holds true. This finishes the proof of Lemma 3.3. $\square$

4 Weighted gradient approximation estimates

In this section, we establish the weighted interior and boundary gradient approximation estimates of the weak solution u of degenerate elliptic equations, which play a key role in the proof of Theorem 1.7. Moreover, we obtain the higher integrability result for ∇u (see Lemma 4.8 below).

Let $w \in A_2(\mathbb{R}^n)$, A satisfy (1.2), and $\vec{F} := (f_1, \dots, f_n) : B_4 \rightarrow \mathbb{R}^n$ be a given vector field satisfying $|\vec{F}|/w \in L^2(B_4, w)$. Recall that $B_4 := \{x \in \mathbb{R}^n : |x| < 4\}$. Consider the following degenerate elliptic equation

$$\operatorname{div}(A \nabla u) = \operatorname{div} \vec{F} \quad \text{in } B_4 \quad (4.1)$$

and the corresponding homogeneous elliptic equation

$$\operatorname{div}(\mathbb{A}_0 \nabla v) = 0 \quad \text{in } B_4, \quad (4.2)$$

where $\mathbb{A}_0$ is an $n \times n$ matrix of constant coefficients satisfying the classical uniform elliptic condition, namely, (1.2) with $w(x) \equiv 1$ therein.

By [18, Lemma 4.1] and its proof (see also [37, Theorem 4.1]), we have the following well-known result on the regularity of the weak solution of (4.2).

Lemma 4.1 ([18]). *Let $\mathbb{A}_0$ be an $n \times n$ matrix of constant coefficients satisfying the classical uniform elliptic condition, namely, (1.2) with $w(\cdot) \equiv 1$ therein. Let $\theta \in (0, 1)$, $r \in (0, 4]$, $p \in (1, \infty)$, and $v \in W^{1,p}(B_4)$ be a weak solution of (4.2). Then there exists a positive constant $C := C_{(n,p,\Lambda,\theta,r)}$ such that*

$$\|v\|_{L^\infty(B_{\theta r})} \leq C \left[\frac{1}{|B_r|} \int_{B_r} |v(x)|^p dx \right]^{1/p}.$$

Remark 4.2. *Let $p \in (1, \infty)$ and $v \in W^{1,p}(B_4)$ be a weak solution of (4.2). Since $\mathbb{A}_0$ is a matrix of constant coefficients, by the classical regularity theory of elliptic equations, we know that $v \in C^\infty(B_4)$. Moreover, it is easy to see that, for any $j \in \mathbb{N}$, $\frac{\partial v}{\partial x_j}$ is also a weak solution of (4.2). Thus, for any given $\theta \in (0, 1)$ and $r \in (0, 4]$, there exists a positive constant $C := C_{(n,p,\Lambda,\theta,r)}$ such that*

$$\|\nabla v\|_{L^\infty(B_{\theta r})} \leq C \left[\frac{1}{|B_r|} \int_{B_r} |\nabla v|^p dx \right]^{1/p}.$$

For a weak solution u of (4.1), we have the following lemma, which shows that the gradient ∇u can be approximated via the gradient of a weak solution v of the associated homogeneous equation (4.2).

Lemma 4.3. *Let $p \in (1, \infty)$, $M \in (0, \infty)$, $w \in A_p(\mathbb{R}^n)$ with $[w]_{A_p(\mathbb{R}^n)} \leq M$, $q \in (1, \infty)$ be as in (3.1), A satisfy (1.2), and $|\vec{F}|/w \in L^p(B_4, w)$. Then, for any given $\varepsilon \in (0, \infty)$, there exists a positive constant $\delta_1 := \delta_{(\varepsilon,n,\Lambda,p,q,M)}$ such that, for any given $\delta \in (0, \delta_1)$, if $u \in W^{1,p}(B_4, w)$ is a weak solution of (4.1) satisfying*

$$\frac{1}{w(B_4)} \int_{B_4} |\nabla u(x)|^p w(x) dx \leq 1,$$

and

$$\|A\|_{\operatorname{BMO}_4^{p'}(B_4, w)} + \left[\frac{1}{w(B_4)} \int_{B_4} \left| \frac{\vec{F}(x)}{w(x)} \right|^p w(x) dx \right]^{1/p} \leq \delta, \quad (4.3)$$

then there exist an $n \times n$ matrix $\mathbb{A}_0$ of constant coefficients and a weak solution $v \in W^{1,q_0}(B_4)$ of (4.2), with $q_0 := \frac{p}{p_*} \in (1, \infty)$ and $p_* \in [1, p)$ as in (3.1), satisfying

$$|\langle A \rangle_{B_4} - \mathbb{A}_0| \leq \varepsilon \frac{w(B_4)}{|B_4|}, \quad \frac{1}{w(B_{7/2})} \int_{B_{7/2}} |\hat{u}(x) - v(x)|^p w(x) dx \leq \varepsilon, \quad (4.4)$$

and

$$\frac{1}{|B_3|} \int_{B_3} |v(x)|^p dx \leq C_2, \quad (4.5)$$

where

$$\hat{u}(\cdot) := u(\cdot) - \langle u \rangle_{B_4, w}, \quad \langle u \rangle_{B_4, w} := \frac{1}{w(B_4)} \int_{B_4} u(x)w(x) dx, \quad (4.6)$$

and $C_2 := C_{(n, \Lambda, p, q, M)}$ is a positive constant. Moreover, it holds true that

$$\frac{1}{w(B_2)} \int_{B_2} |\nabla u(x) - \nabla v(x)|^p w(x) dx \leq \varepsilon \quad \text{and} \quad \|\nabla v\|_{L^\infty(B_2)} \leq C_2. \quad (4.7)$$

Remark 4.4. In particular, if $p = 2$, then Lemma 4.3 in this case is just [18, Proposition 4.4].

Now, we show Lemma 4.3.

Proof of Lemma 4.3. Let $p \in (1, \infty)$, $M \in (0, \infty)$, $w \in A_p(\mathbb{R}^n) \cap RH_q(\mathbb{R}^n)$ with $[w]_{A_p(\mathbb{R}^n)} \leq M$, $q \in (1, \infty)$ be as in (3.1), A satisfy (1.2), $|\vec{F}|/w \in L^p(\Omega, w)$, and $u \in W^{1,p}(B_4, w)$ be a weak solution of (4.1). For any given $\tau \in (0, \infty)$, let $A_\tau := A/\tau$, $w_\tau := w/\tau$, and $\vec{F}_\tau := \vec{F}/\tau$. Then we find that u is also a weak solution of $\operatorname{div}(A_\tau \nabla u) = \operatorname{div} \vec{F}_\tau$. If the conclusion of this lemma holds true for A/τ , $\vec{F}_\tau$, and $w_\tau := w/\tau$, it also holds true for A , $\vec{F}$, and w . Thus, we can prove this lemma under the additional assumption that

$$\langle w \rangle_{B_4} = \frac{1}{|B_4|} \int_{B_4} w(x) dx = 1. \quad (4.8)$$

We first prove (4.4) by using a contradiction argument. Indeed, if (4.4) is not true, then there exists some $\varepsilon_0 \in (0, \infty)$ such that, for any $k \in \mathbb{N}$, if $w_k \in A_p(\mathbb{R}^n)$ with $[w_k]_{A_p(\mathbb{R}^n)} \leq M$ and $\langle w \rangle_{B_4} = 1$, A_k is a matrix satisfying (1.2) with w therein replaced by w_k , $|\vec{F}_k|/w_k \in L^p(B_4, w_k)$, and $u_k \in W^{1,p}(B_4, w_k)$ is a weak solution of $\operatorname{div}(A_k \nabla u) = \operatorname{div} \vec{F}_k$ in B_4 , which satisfy that

$$\|A_k\|_{\operatorname{BMO}_4^{p'}(B_4, w)} + \left[\frac{1}{w(B_4)} \int_{B_4} \left| \frac{\vec{F}_k(x)}{w(x)} \right|^p w(x) dx \right]^{1/p} \leq \frac{1}{k}, \quad (4.9)$$

$$\langle w_k \rangle_{B_4} = \frac{1}{|B_4|} \int_{B_4} w_k(x) dx = 1, \quad (4.10)$$

and

$$\frac{1}{w_k(B_4)} \int_{B_4} |\nabla u_k(x)|^p w_k(x) dx \leq 1, \quad (4.11)$$

then, for any constant matrix $\mathbb{A}_0$ satisfying $|\langle A_k \rangle_{B_4} - \mathbb{A}_0| \leq \varepsilon_0$, and any weak solution $v \in W^{1, q_0}(B_4)$ of (4.2) with $q_0 = \frac{p}{p^*}$ and $p^* \in (1, p)$ as in (3.1), we have

$$\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\hat{u}_k(x) - v(x)|^p w_k(x) dx \geq \varepsilon_0. \quad (4.12)$$

Now, we show that there exist a constant matrix $\mathbb{A}_0$ and a weak solution $v \in W^{1, q_0}(B_4)$ of (4.2) which contradicts (4.12). Indeed, by the assumption that, for any $k \in \mathbb{N}$, A_k satisfies (1.2) and $\langle w_k \rangle_{B_4} = 1$, we find that, for any $\xi \in \mathbb{R}^n$,

$$\Lambda^{-1} |\xi|^2 \leq \langle \langle A_k \rangle_{B_4} \xi, \xi \rangle \leq \Lambda |\xi|^2,$$

namely, $\{\langle A_k \rangle_{B_4}\}_{k \in \mathbb{N}}$ is a bounded sequence of constant matrices in $\mathbb{R}^{n \times n}$. Thus, there exists a subsequence of $\{\langle A_k \rangle_{B_4}\}_{k \in \mathbb{N}}$ (without loss of generality, we may use the same notation as the original sequence) and a constant matrix $\mathbb{A}_0$ such that

$$\lim_{k \rightarrow \infty} \langle A_k \rangle_{B_4} = \mathbb{A}_0. \quad (4.13)$$

On the other hand, from Lemma 2.10 and (4.11), it follows that there exists a positive constant $c_1 := C_{(n,p,M)}$ such that

$$\frac{1}{w_k(B_4)} \int_{B_4} |\widehat{u}_k(x)|^p w_k(x) dx \leq c_1 \frac{1}{w_k(B_4)} \int_{B_4} |\nabla u_k(x)|^p w_k(x) dx \leq c_1,$$

where $\widehat{u}_k$ is as in (4.6) with u therein replaced by u_k . This, together with $\nabla \widehat{u}_k = \nabla u_k$, $w_k(B_4) = |B_4|$, and (4.11), implies that there exists a positive constant $C_{(n,p,M)}$ such that, for any $k \in \mathbb{N}$, $\|\widehat{u}_k\|_{W^{1,p}(B_4, w_k)} \leq C_{(n,p,M)}$. Moreover, by this and Lemma 3.1(i), we know that, for any $k \in \mathbb{N}$,

$$\|\widehat{u}_k\|_{W^{1,q_0}(B_4)} \leq C_{(n,p,p^*,M)} \|\widehat{u}_k\|_{W^{1,p}(B_4, w_k)} \leq C_{(n,p,p^*,M)},$$

where $q_0 = \frac{p}{p^*}$ and $p^* \in (1, p)$ is as in (3.1). Thus, $\{\widehat{u}_k\}_{k \in \mathbb{N}}$ is a bounded set of $W^{1,q_0}(B_4)$. From this, the fact that $L^{q_0}(B_4)$ is reflexive, and the Eberlein–Šmul'yan theorem (see, for instance, [61, p. 141]), we deduce that there exists a subsequence of $\{\widehat{u}_k\}_{k \in \mathbb{N}}$ (without loss of generality, we may use the same notation as the original sequence) and some $v_0 \in W^{1,q_0}(B_4)$ such that

$$\begin{cases} \widehat{u}_k \rightharpoonup v_0, \\ \nabla \widehat{u}_k \rightharpoonup \nabla v_0 \end{cases} \text{ weakly in } L^{q_0}(B_4), \quad \text{and} \quad \|v_0\|_{W^{1,q_0}(B_4)} \leq C_{(n,p,p_0,M)}.$$

Moreover, by the fact that $W^{1,q_0}(B_4)$ is compactly imbedded into $L^{q_0}(B_4)$, we know that

$$\lim_{k \rightarrow \infty} \|\widehat{u}_k - v_0\|_{L^{q_0}(B_4)} = 0. \quad (4.14)$$

We claim that v_0 is a weak solution of $\operatorname{div}(\mathbb{A}_0 \nabla v) = 0$ in B_4 , where $\mathbb{A}_0$ is as in (4.13). Indeed, fix any $\varphi \in C_c^\infty(B_4)$. Then, from the assumption that u_k a weak solution of $\operatorname{div}(A_k \nabla u) = \operatorname{div} \vec{F}_k$ in B_4 , we deduce that

$$\int_{B_4} [A_k(x) \nabla u_k(x)] \cdot \nabla \varphi(x) dx = \int_{B_4} \vec{F}_k(x) \cdot \nabla \varphi(x) dx. \quad (4.15)$$

By (4.13) and $\nabla u_k = \nabla \widehat{u}_k \rightharpoonup \nabla v_0$ weakly in $L^{q_0}(B_4)$, we conclude that

$$\lim_{k \rightarrow \infty} \int_{B_4} [\langle A_k \rangle_{B_4} \nabla u_k(x)] \cdot \nabla \varphi(x) dx = \int_{B_4} [\mathbb{A}_0 \nabla v_0(x)] \cdot \nabla \varphi(x) dx. \quad (4.16)$$

Moreover, from (4.9), (4.11), and the Hölder inequality, we deduce that

$$\begin{aligned} & \left| \frac{1}{|B_4|} \int_{B_4} \{ [A_k(x) - \langle A_k \rangle_{B_4}] \nabla u_k \} \cdot \nabla \varphi(x) dx \right| \\ & \leq \frac{1}{w_k(B_4)} \int_{B_4} |A_k(x) - \langle A_k \rangle_{B_4}| [w_k(x)]^{-1/p} |\nabla u_k| [w_k(x)]^{1/p} |\nabla \varphi(x)| dx \\ & \leq \|\nabla \varphi\|_{L^\infty(B_4)} \left\{ \frac{1}{w_k(B_4)} \int_{B_4} |A_k(x) - \langle A_k \rangle_{B_4}|^{p'} [w_k(x)]^{1-p'} dx \right\}^{\frac{1}{p'}} \\ & \quad \times \left[\frac{1}{w_k(B_4)} \int_{B_4} |\nabla u_k(x)|^p w_k(x) dx \right]^{\frac{1}{p}} \end{aligned}$$

$$\leq \|\nabla \varphi\|_{L^\infty(B_4)} \|A_k\|_{\text{BMO}_4'(B_4, w)} < \frac{\|\nabla \varphi\|_{L^\infty(B_4)}}{k}. \quad (4.17)$$

This, combined with (4.16), implies that

$$\lim_{k \rightarrow \infty} \int_{B_4} [A_k(x) \nabla u_k(x)] \cdot \nabla \varphi(x) \, dx = \int_{B_4} [\mathbb{A}_0 \nabla v_0(x)] \cdot \nabla \varphi(x) \, dx. \quad (4.18)$$

By an argument similar to that used in the estimation of (4.17), we know that

$$\lim_{k \rightarrow \infty} \int_{B_4} \vec{F}_k(x) \cdot \nabla \varphi(x) \, dx = 0.$$

From this, (4.18), and (4.15), it follows that v_0 is a weak solution of $\text{div}(\mathbb{A}_0 \nabla v) = 0$ in B_4 . For any $k \in \mathbb{N}$, let

$$v_k := v_0 + \langle \widehat{u}_k - v_0 \rangle_{B_{7/2}} := v_0 + \frac{1}{|B_{7/2}|} \int_{B_{7/2}} [\widehat{u}_k(x) - v_0(x)] \, dx.$$

Then, for any $k \in \mathbb{N}$, v_k is also a weak solution of $\text{div}(\mathbb{A}_0 \nabla v) = 0$ in B_4 .

Next, we prove that

$$\lim_{k \rightarrow \infty} \frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\widehat{u}_k(x) - v_k(x)|^p w_k(x) \, dx = 0. \quad (4.19)$$

If this is true, then it contradicts (4.12), which further shows that (4.4) holds true. To prove (4.19), fix $\eta \in (1, \frac{n}{n-1}]$. Then, by Lemma 2.10, we find that there exists a positive constant $c_2 := C_{(n, p, M)}$ such that, for any $k \in \mathbb{N}$,

$$\begin{aligned} & \left[\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\widehat{u}_k(x) - v_0(x) - \langle \widehat{u}_k - v_0 \rangle_{B_{7/2}}|^{p\eta} w_k(x) \, dx \right]^{1/p\eta} \\ & \leq c_2 \left[\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\nabla \widehat{u}_k(x) - \nabla v_0(x)|^p w_k(x) \, dx \right]^{1/p}. \end{aligned} \quad (4.20)$$

Let $\tau \in (0, p)$ be a small number which is determined later, and

$$\theta := \frac{\frac{1}{p} - \frac{1}{p\eta}}{\frac{1}{\tau} - \frac{1}{p\eta}} \in (0, 1).$$

From this, the Hölder inequality, respectively, with exponents $\tau/p\theta$ and $(\tau/p\theta)'$, and (4.20), we deduce that

$$\begin{aligned} & \left[\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\widehat{u}_k(x) - v_0(x) - \langle \widehat{u}_k - v_0 \rangle_{B_{7/2}}|^p w_k(x) \, dx \right]^{1/p} \\ & \leq \left[\frac{1}{|B_{7/2}|} \int_{B_{7/2}} |\widehat{u}_k(x) - v_0(x) - \langle \widehat{u}_k - v_0 \rangle_{B_{7/2}}|^\tau w_k(x) \, dx \right]^{\theta/\tau} \\ & \quad \times \left[\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\widehat{u}_k(x) - v_0(x) - \langle \widehat{u}_k - v_0 \rangle_{B_{7/2}}|^{p\eta} w_k(x) \, dx \right]^{\frac{1-\theta}{p\eta}} \\ & \leq c_2 \left[\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\widehat{u}_k(x) - v_0(x) - \langle \widehat{u}_k - v_0 \rangle_{B_{7/2}}|^\tau w_k(x) \, dx \right]^{\theta/\tau} \end{aligned}$$

$$\times \left[\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\nabla \hat{u}_k(x) - \nabla v_0(x)|^p w_k(x) dx \right]^{\frac{1-\theta}{p}}. \quad (4.21)$$

Moreover, by Lemma 3.1(ii), we know that there exists a positive constant $c_3 := C_{(n, q_0, q, M)}$ such that

$$\begin{aligned} & \left[\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\hat{u}_k(x) - v_0(x) - \langle \hat{u}_k - v_0 \rangle_{B_{7/2}}|^\tau w_k(x) dx \right]^{1/\tau} \\ & \leq c_3 \left[\frac{1}{|B_{7/2}|} \int_{B_{7/2}} |\hat{u}_k(x) - v_0(x) - \langle \hat{u}_k - v_0 \rangle_{B_{7/2}}|^{q_0} dx \right]^{1/q_0} \\ & \leq 2c_3 \left[\frac{1}{|B_{7/2}|} \int_{B_{7/2}} |\hat{u}_k(x) - v_0(x)|^{q_0} dx \right]^{1/q_0} \leq C_{(n, q_0, q, M)} \|\hat{u}_k - v_0\|_{L^{q_0}(B_4)}, \end{aligned} \quad (4.22)$$

where we fixed some $\tau \in (0, \frac{q_0(q-1)}{q}]$. On the other hand, from Lemma 2.2(iv), (4.11), $\nabla \hat{u}_k = \nabla u_k$, and Remark 4.2, we deduce that

$$\begin{aligned} & \left[\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\nabla \hat{u}_k(x) - \nabla v_0(x)|^p w_k(x) dx \right]^{\frac{1}{p}} \\ & \leq \frac{w_k(B_4)}{w_k(B_{7/2})} \left[\frac{1}{w_k(B_4)} \int_{B_4} |\nabla \hat{u}_k(x)|^p w_k(x) dx \right]^{\frac{1}{p}} + \|\nabla v_0\|_{L^\infty(B_{7/2})} \\ & \leq C_{(n, p, M)} + C_{(n, p, \Lambda)} \left[\frac{1}{|B_4|} \int_{B_4} |v_0(x)|^p dx \right]^{1/p} \leq C_{(n, p, M, \Lambda)}. \end{aligned}$$

Combining this, (4.22), and (4.21), we conclude that

$$\left[\frac{1}{w_k(B_{7/2})} \int_{B_{7/2}} |\hat{u}_k(x) - v_0(x) - \langle \hat{u}_k - v_0 \rangle_{B_{7/2}}|^p w_k(x) dx \right]^{1/p} \leq C_{(n, q_0, q, p, M, \Lambda)} \|\hat{u}_k - v_0\|_{L^{q_0}(B_4)}.$$

This, together with (4.14), implies that (4.19) holds true. Thus, we complete the proof of (4.4).

Now, we prove (4.5). Let $\mathbb{A}_0$ be a matrix of constant coefficients and $v \in W^{1, q_0}(B_4)$ a weak solution of (4.2), with $q_0 := \frac{p}{p^*} \in (1, \infty)$ and $p^* \in [1, p]$ as in (3.1), satisfying (4.4). By Lemma 3.3, Remark 4.2, Lemma 3.1, and (4.4), we find that there exists a positive constant $c_4 := C_{(n, \Lambda, p, q, M)}$ such that

$$\begin{aligned} & \frac{1}{|B_3|} \int_{B_3} |\nabla v(x)|^p dx \\ & \leq c_4 \frac{1}{|B_{16/5}|} \int_{B_{16/5}} |v(x)|^p dx \leq c_4 \|v\|_{L^\infty(B_{16/5})} \\ & \leq c_4 \left[\frac{1}{|B_{7/2}|} \int_{B_{7/2}} |v(x)|^{q_0} dx \right]^{1/q_0} \leq c_4 \left[\frac{1}{w(B_{7/2})} \int_{B_{7/2}} |v(x)|^p w(x) dx \right]^{1/p} \end{aligned}$$

$$\begin{aligned} &\leq c_4 \left\{ \left[\frac{1}{w(B_{7/2})} \int_{B_{7/2}} |\widehat{u}(x) - v(x)|^p w(x) dx \right]^{1/p} + \left[\frac{1}{w(B_{7/2})} \int_{B_{7/2}} |\widehat{u}(x)|^p w(x) dx \right]^{1/p} \right\} \\ &\leq c_4 \left\{ \varepsilon + \left[\frac{w(B_4)}{w(B_{7/2})} \frac{1}{w(B_4)} \int_{B_4} |\nabla u(x)|^p w(x) dx \right]^{1/p} \right\} \leq C_{(n, \Lambda, p, q, M)}. \end{aligned}$$

This shows (4.5).

To complete the proof of this lemma, we still need to prove (4.7). Indeed, from (4.5) and Remark 4.2, we deduce that

$$\|\nabla v\|_{L^\infty(B_{5/2})} \leq C_{(n, p, \Lambda)} \left[\frac{1}{|B_3|} \int_{B_3} |\nabla v(x)|^p dx \right]^{1/p} \leq C_{(n, \Lambda, p, q, M)} =: c_5.$$

On the other hand, by the assumption that v and u solve, respectively, (4.2) and (4.1), we conclude that $\widehat{u} - v$ is a weak solution of

$$\operatorname{div}[A \nabla(\widehat{u} - v)] = \operatorname{div}[\vec{F} - (A - \mathbb{A}_0) \nabla v] \quad \text{in } B_4,$$

where $\widehat{u} = u - \langle u \rangle_{B_4, w}$. From this and Lemma 3.3, it follows that there exists a positive constant $c_6 := C_{(n, \Lambda, p, q, M)}$ such that

$$\begin{aligned} &\int_{B_2} |\nabla u(x) - \nabla v(x)|^p w(x) dx \\ &= \int_{B_2} |\nabla \widehat{u}(x) - \nabla v(x)|^p w(x) dx \\ &\leq c_6 \int_{B_{5/2}} \left| \frac{\vec{F}(x)}{w(x)} \right|^p w(x) dx + c_6 \int_{B_{5/2}} \left| \frac{[A(x) - \mathbb{A}_0] \nabla v}{w(x)} \right|^p w(x) dx + c_6 \int_{B_{5/2}} |\widehat{u}(x) - v(x)|^p w(x) dx \\ &=: \text{I} + \text{II} + \text{III}. \end{aligned} \tag{4.23}$$

For I, by (4.3), we find that

$$\text{I} \leq c_6 \delta^p w(B_4). \tag{4.24}$$

For II, from (4.4) and $\|A\|_{\text{BMO}_4(B_4, w)} \leq \delta$, we deduce that

$$\begin{aligned} \text{II} &\leq 2^{p-1} c_6 \|\nabla v\|_{L^\infty(B_{5/2})}^p \int_{B_4} |A(x) - \langle A \rangle_{B_4}|^p [w(x)]^{1-p} dx \\ &\quad + 2^{p-1} c_6 \|\nabla v\|_{L^\infty(B_{5/2})}^p |\langle A \rangle_{B_4} - \mathbb{A}_0|^p \int_{B_4} [w(x)]^{1-p} dx \\ &\leq 2^{p-1} c_6 (c_5 \delta)^p w(B_4) + 2^{p-1} c_6 (c_5 \varepsilon)^p \left[\frac{w(B_4)}{|B_4|} \right]^p \int_{B_4} [w(x)]^{1-p} dx. \end{aligned} \tag{4.25}$$

For III, by (4.4), we further have

$$\text{III} \leq c_6 \varepsilon w(B_{7/2}).$$

From this, (4.25), (4.24), (4.23), and $w \in A_p(\mathbb{R}^n) \subset A_p(\mathbb{R}^n)$ for any given $p \in (1, 2]$, we deduce that

$$\frac{1}{w(B_2)} \int_{B_2} |\nabla u(x) - \nabla v(x)|^p w(x) dx \leq c_7 \delta^p + c_7 \varepsilon^p \left[\frac{w(B_4)}{|B_4|} \right]^{p-1} \frac{1}{|B_4|} \int_{B_4} [w(x)]^{1-p} dx + c_7 \varepsilon$$

$$\leq c_7 \delta^p + c_7 \varepsilon \leq c_8 \varepsilon,$$

where δ is chosen small enough such that $\delta^p \in (0, \varepsilon)$, and $c_8 := C_{(n, \Lambda, p, q, M)}$ same as above. Since $\varepsilon \in (0, \infty)$ is arbitrary, we obtain (4.7). This finishes the proof of Lemma 4.3. $\square$

Let $w \in A_2(\mathbb{R}^n)$, A satisfy (1.2), and $\vec{F} := (f_1, \dots, f_n) : \Omega_4 \rightarrow \mathbb{R}^n$ be a given vector field satisfying $|\vec{F}|/w \in L^2(\Omega_4, w)$. Consider the following homogeneous elliptic equation

$$\begin{cases} \operatorname{div}(\mathbb{A}_0 \nabla v) = 0 & \text{in } B_4^+, \\ v = 0 & \text{on } (\partial B_4^+) \cap \{x \in \mathbb{R}^n : x_n = 0\}, \end{cases} \quad (4.26)$$

where $\mathbb{A}_0$ is an $n \times n$ matrix of constant coefficients as in (4.2). Let $q \in (1, \infty)$. A function $v \in W^{1, q}(B_4^+)$ is said to be a weak solution of (4.26) if, for any $\varphi \in C_c^\infty(B_4^+)$,

$$\int_{B_4^+} [\mathbb{A}_0 \nabla v(x)] \cdot \nabla \varphi(x) dx = 0$$

and the zero extension of v to B_4 is in $W^{1, q}(B_4)$, namely, there exists a function $V \in W^{1, q}(B_4)$ such that $V(x) = v(x)$ for any $x \in B_4^+$, and $V(x) = 0$ for any $x \in B_4 \setminus B_4^+$.

By Lemma 3.5 and an argument similar to that used in the proof of Lemma 4.3, we have the following approximation estimate at the boundary for the weak solution of (3.5) and we omit the details here.

Lemma 4.5. *Let $p \in (1, \infty)$, $M \in (0, \infty)$, $w \in A_p(\mathbb{R}^n)$ with $[w]_{A_p(\mathbb{R}^n)} \leq M$, $q \in (1, \infty)$ be as in (3.1), A satisfy (1.2), and $|\vec{F}|/w \in L^p(B_4, w)$. Then, for any given $\varepsilon \in (0, \infty)$, there exists a positive constant $\delta_2 := \delta_{(\varepsilon, n, \Lambda, p, q, M)}$ such that, for any given $\delta \in (0, \delta_2)$, if Ω is a (δ, R) -Reifenberg flat domain, $u \in W^{1, p}(\Omega_4, w)$ a weak solution of (3.5) satisfying*

$$\frac{1}{w(B_4)} \int_{\Omega_4} |\nabla u(x)|^p w(x) dx \leq 1,$$

and

$$\|A\|_{\operatorname{BMO}_4^{p'}(\Omega_4, w)} + \left[\frac{1}{w(B_4)} \int_{\Omega_4} \left| \frac{\vec{F}(x)}{w(x)} \right|^p w(x) dx \right]^{1/p} \leq \delta,$$

then there exist an $n \times n$ matrix $\mathbb{A}_0$ of constant coefficients and a weak solution $v \in W^{1, q_0}(B_4^+)$ of (4.26), with $q_0 := \frac{p}{p^*} \in (1, \infty)$ and $p^* \in (1, p)$ as in (3.1), satisfying

$$|\langle A \rangle_{B_4} - \mathbb{A}_0| \leq \varepsilon \frac{w(B_4)}{|B_4|}, \quad \frac{1}{w(\Omega_2)} \int_{\Omega_2} |\nabla u(x) - \nabla V(x)|^p w(x) dx \leq \varepsilon, \quad \text{and} \quad \|\nabla v\|_{L^\infty(B_2^+)} \leq C_3,$$

where V is the zero extension of v to B_4 and $C_3 := C_{(n, \Lambda, p, q, M)}$ is a positive constant.

Remark 4.6. *In particular, if $p = 2$, then Lemma 4.5 in this case is just [18, Proposition 5.5].*

By Lemma 4.5 and an argument similar to that used in the proof of [18, Lemma 5.10], we have the following conclusion and we omit the details here. Recall that $\Omega_1 := \Omega \cap \{x \in \mathbb{R}^n : |x| < 1\}$ and $\Omega_6 := \Omega \cap \{x \in \mathbb{R}^n : |x| < 6\}$.

Lemma 4.7. *Let $R \in (0, \infty)$, $p \in (1, \infty)$, $M \in (0, \infty)$, $w \in A_p(\mathbb{R}^n)$ with $[w]_{A_p(\mathbb{R}^n)} \leq M$, $q \in (1, \infty)$ be as in (3.1), and $|\vec{F}|/w \in L^p(\Omega, w)$. Then, for any given $\varepsilon \in (0, \infty)$, there exist some $\delta_3 := \delta_{(\varepsilon, n, \Lambda, p, q, M)} \in (0, 1/4)$ and a constant $\gamma := \gamma_{(n, \Lambda, p, q, M)} \in (1, \infty)$ such that, for any given $\delta \in (0, \delta_3)$ and $R \in (0, \infty)$, if Ω is a (δ, R) -Reifenberg flat domain, $\|A\|_{\operatorname{BMO}_R(\Omega, w)} < \delta$, $u \in W_0^{1, p}(\Omega, w)$ is a weak solution of (1.1), and, for any $y \in \Omega_1$,*

$$w(\{x \in \Omega_1 : \mathcal{M}_w(\mathbf{1}_{\Omega_6} |\nabla u|^p)(x) > \gamma^p\}) < \varepsilon w(B_{1/2}(y)),$$

where $\mathcal{M}_w$ is the weighted Hardy-Littlewood maximal function as in (2.1), then, for any $j \in \mathbb{N}$ and $\eta := \varepsilon(\frac{10}{1-4\delta})^{pn}M^2$,

$$\begin{aligned} & w\left(\left\{x \in \Omega_1 : \mathcal{M}_w\left(\mathbf{1}_{\Omega_6}|\nabla u|^p\right)(x) > \gamma^{pj}\right\}\right) \\ & \leq \sum_{i=1}^j \eta^i w\left[\left\{x \in \Omega_1 : \mathcal{M}_w\left(\mathbf{1}_{\Omega_6}\left|\frac{\vec{F}}{w}\right|^p\right)(x) > \delta^p \gamma^{p(j-i)}\right\}\right] \\ & \quad + \eta^j w\left[\left\{x \in \Omega_1 : \mathcal{M}_w\left(\mathbf{1}_{\Omega_6}|\nabla u|^p\right)(x) > 1\right\}\right]. \end{aligned}$$

To prove Theorem 1.7, we also need the following lemma which establishes the higher integrability result for the gradient of the weak solution of (1.1).

Lemma 4.8. *Let R, p, M, w, q , and $\vec{F}$ be as in Lemma 4.7. Then there exists a positive constant $\delta_3 := \delta_{(n, \Lambda, p, q, M)} \in (0, 1/4)$, which is as in Lemma 4.7, such that, for any given $\delta \in (0, \delta_3)$, if Ω is a (δ, R) -Reifenberg flat domain, A is a matrix satisfying (1.2) and $\|A\|_{\text{BMO}_R(\Omega, w)} < \delta$, $r \in (p, \infty)$, and $u \in W_0^{1,p}(\Omega, w)$ is a weak solution of (1.1), then*

$$\begin{aligned} & \left[\frac{1}{w(\Omega_1)} \int_{\Omega_1} |\nabla u(x)|^r w(x) dx\right]^{1/r} \\ & \leq C_4 \left\{ \left[\frac{1}{w(\Omega_6)} \int_{\Omega_6} |\nabla u(x)|^p w(x) dx\right]^{1/p} + \left[\frac{1}{w(\Omega_6)} \int_{\Omega_6} \left|\frac{\vec{F}(x)}{w(x)}\right|^r w(x) dx\right]^{1/r} \right\}, \end{aligned}$$

where $C_4 := C_{(n, \Lambda, p, q, r, M)}$ is a positive constant.

Proof. We prove this lemma by borrowing some ideas from the proof of [18, Theorem 2.7]. Let $R \in (0, \infty)$, $p \in (1, \infty)$, $r \in (p, \infty)$, $M \in (0, \infty)$, $w \in A_p(\mathbb{R}^n)$, $q \in (1, \infty)$ be as in (3.1), $[w]_{A_p(\mathbb{R}^n)} \leq M$, and $|\vec{F}|/w \in L^p(\Omega, w)$. Assume that $u \in W_0^{1,p}(\Omega, w)$ is a weak solution of (1.1). For any given $\varepsilon \in (0, \infty)$, let $0 < \delta < \delta_3 < 1/4$ and $\gamma \in (1, \infty)$ be as in Lemma 4.7. We claim that there exists some $N \in (0, \infty)$ large enough such that, for $u_N := u/N$ and any $y \in \Omega_1$,

$$w\left(\left\{x \in \Omega_1 : \mathcal{M}_w\left(\mathbf{1}_{\Omega_6}|\nabla u_N|^p\right)(x) > \gamma^p\right\}\right) < \varepsilon w(B_{1/2}(y)). \quad (4.27)$$

Indeed, by Remark 2.3(i), we find that there exists a positive constant $c_0 := C_{(M)}$ such that

$$w\left(\left\{x \in \Omega_1 : \mathcal{M}_w\left(\mathbf{1}_{\Omega_6}|\nabla u_N|^p\right)(x) > \gamma^p\right\}\right) \leq \frac{c_0}{\gamma^p N^p} \int_{\Omega_6} |\nabla u|^p w(x) dx. \quad (4.28)$$

We fix N large enough such that

$$\frac{c_0}{\gamma^p N^p} \int_{\Omega_6} |\nabla u(x)|^p w(x) dx = \frac{\varepsilon w(\Omega_1)}{C_1 4^{pn}}. \quad (4.29)$$

Moreover, from Lemma 2.2(iv), we deduce that, for any $y \in \Omega_1$,

$$w(\Omega_1) \leq w(B_1) \leq w(B_2) < C_1 4^{pn} w(B_{1/2}(y)), \quad (4.30)$$

where $C_1 \in (0, \infty)$ is as in Lemma 2.2(iv). This, combined with (4.28) and (4.30), implies that (4.27) holds true.

Let

$$S := \sum_{j=1}^{\infty} \gamma^{rj} w\left(\left\{x \in \Omega_1 : \mathcal{M}_w\left(\mathbf{1}_{\Omega_6}|\nabla u_N|^p\right)(x) > \gamma^{pj}\right\}\right).$$

Then, by Lemma 2.6, we conclude that there exists a positive constant $c_1 := C_{(\gamma, p, r)}$ such that

$$\|\mathcal{M}_w(\mathbf{1}_{\Omega_6} |\nabla u_N|^p)\|_{L^{r/p}(\Omega_1, w)}^{r/p} \leq c_1 [w(\Omega_1) + S]. \quad (4.31)$$

On the other hand, from (4.27) and Lemmas 4.7 and 2.6, it follows that

$$\begin{aligned} S &\leq \sum_{j=1}^{\infty} \gamma^{rj} \sum_{i=1}^j \varepsilon_1^i w \left(\left\{ x \in \Omega_1 : \mathcal{M}_w \left(\mathbf{1}_{\Omega_6} \left| \frac{\vec{F}_N}{w} \right|^p \right) (x) > \delta^p \gamma^{p(j-i)} \right\} \right) \\ &\quad + \sum_{j=1}^{\infty} \gamma^{rj} \varepsilon_1^j w [\{x \in \Omega_1 : \mathcal{M}_w(\mathbf{1}_{\Omega_6} |\nabla u_N|^p)(x) > 1\}] \\ &= \sum_{i=1}^{\infty} (\gamma^r \varepsilon_1)^i \left\{ \sum_{j=i}^{\infty} \gamma^{r(j-i)} w \left(\left\{ x \in \Omega_1 : \mathcal{M}_w \left(\mathbf{1}_{\Omega_6} \left| \frac{\vec{F}_N}{w} \right|^p \right) (x) > \delta^p \gamma^{p(j-i)} \right\} \right) \right\} \\ &\quad + \sum_{j=1}^{\infty} (\gamma^r \varepsilon_1)^j w [\{x \in \Omega_1 : \mathcal{M}_w(\mathbf{1}_{\Omega_6} |\nabla u_N|^p)(x) > 1\}] \\ &\leq c_2 \sum_{i=1}^{\infty} (\gamma^r \varepsilon_1)^i \int_{\Omega_1} \left[\mathcal{M}_w \left(\mathbf{1}_{\Omega_6} \left| \frac{\vec{F}_N}{w} \right|^p \right) (x) \right]^{r/p} w(x) dx + c_2 \sum_{j=1}^{\infty} (\gamma^r \varepsilon_1)^j \int_{\Omega_6} |\nabla u_N(x)|^p w(x) dx, \end{aligned} \quad (4.32)$$

where $c_2 := C_{(\gamma, \delta, p, r)}$ is a positive constant, $\varepsilon_1 := \varepsilon(\frac{10}{1-4\delta})^{pn} M^2$, and $\vec{F}_N := \vec{F}/N$. Choosing ε small enough such that $\gamma^q \varepsilon_1 \in (0, 1/2)$, and using (4.32) and Remark 2.3(i), we find that

$$\begin{aligned} S &\leq c_2 \int_{\Omega_1} \left[\mathcal{M}_w \left(\mathbf{1}_{\Omega_6} \left| \frac{\vec{F}_N}{w} \right|^p \right) (x) \right]^{r/p} w(x) dx + c_2 \int_{\Omega_6} |\nabla u_N(x)|^p w(x) dx \\ &\leq c_3 \int_{\Omega_6} \left| \frac{\vec{F}_N(x)}{w(x)} \right|^r w(x) dx + c_3 \int_{\Omega_6} |\nabla u_N(x)|^p w(x) dx, \end{aligned}$$

where $c_3 := C_{(\gamma, \delta, p, r, M)}$ is a positive constant. Combining this, (4.31), and, for almost every $x \in \Omega_1$, $|\nabla u_N(x)|^p \leq \mathcal{M}_w(\mathbf{1}_{\Omega_6} |\nabla u_N|^p)(x)$, we obtain

$$\begin{aligned} \int_{\Omega_1} |\nabla u_N(x)|^r w(x) dx &\leq \|\mathcal{M}_w(\mathbf{1}_{\Omega_6} |\nabla u_N|^p)\|_{L^{r/p}(\Omega_1, w)}^{r/p} \\ &\leq c_1 w(\Omega_1) + c_1 c_3 \int_{\Omega_6} \left| \frac{\vec{F}_N(x)}{w(x)} \right|^r w(x) dx + c_1 c_3 \int_{\Omega_6} |\nabla u_N(x)|^p w(x) dx. \end{aligned}$$

Multiplying the both sides of the above formula by N^r , we conclude that

$$\int_{\Omega_1} |\nabla u(x)|^r w(x) dx \leq c_1 N^r w(\Omega_1) + c_1 c_3 \int_{\Omega_6} \left| \frac{\vec{F}(x)}{w(x)} \right|^r w(x) dx + c_1 c_3 N^{r-p} \int_{\Omega_6} |\nabla u_N(x)|^p w(x) dx. \quad (4.33)$$

Moreover, from (4.29), we deduce that

$$N = c_4 \left[\frac{1}{w(\Omega_1)} \int_{\Omega_6} |\nabla u(x)|^p w(x) dx \right]^{1/p},$$

where $c_4 := (c_0 C_1 4^{pn} / \varepsilon)^{1/p} \gamma^{-1}$. By this and (4.33), we find that

$$\int_{\Omega_1} |\nabla u(x)|^r w(x) dx \leq c_5 \int_{\Omega_6} \left| \frac{\vec{F}(x)}{w(x)} \right|^r w(x) dx + \frac{c_5}{[w(\Omega_1)]^{\frac{r}{p}-1}} \left[\int_{\Omega_6} |\nabla u(x)|^p w(x) dx \right]^{r/p}, \quad (4.34)$$

where $c_5 := \max\{c_1 c_3, c_1 c_4^r + c_1 c_3 c_4^{r-p}\}$. Moreover, from Lemmas 2.2(iv) and 2.5, we deduce that

$$w(\Omega_6) \leq w(B_6) \leq C_1 6^{pn} w(B_1) \leq C_1^2 \left(\frac{24}{1-\delta} \right)^{pn} w(\Omega_1),$$

where C_1 is as in Lemma 2.2(iv). This, together with (4.34), implies that

$$\begin{aligned} & \left[\frac{1}{w(\Omega_1)} \int_{\Omega_1} |\nabla u(x)|^r w(x) dx \right]^{1/r} \\ & \leq C \left[\frac{1}{w(\Omega_6)} \int_{\Omega_6} \left| \frac{\vec{F}(x)}{w(x)} \right|^r w(x) dx \right]^{1/r} + C \left[\frac{1}{w(\Omega_1)} \int_{\Omega_6} |\nabla u(x)|^p w(x) dx \right]^{1/p}, \end{aligned}$$

where $C := C_{(n, \Lambda, p, q, r, M)}$ is a positive constant. This finishes the proof of Lemma 4.8. $\square$

5 Proof of Theorem 1.7

In this section, we show Theorem 1.7. We begin with establishing the following lemma, which plays a key role in the proof of Theorem 1.7.

Lemma 5.1. *Let Ω be a bounded domain of $\mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (1.11), (1.12), and (1.13), σ be as in (1.15), q as in (1.16), $M \in (0, \infty)$, $w \in A_{\sigma}(\mathbb{R}^n)$ with $[w]_{A_{\sigma}(\mathbb{R}^n)} \leq M$, A satisfy (1.2), and $\|\vec{F}/w\|_{L^{p(\cdot)}(\Omega, w)} \leq 1$. Then, for any given $\varepsilon \in (0, \infty)$, there exists a positive constant $\delta_4 := \delta_{(\varepsilon, n, \Lambda, p, q, M)} \in (0, 1/5]$ such that, for any given $\delta \in (0, \delta_4)$ and $R \in (0, \infty)$, if Ω is a (δ, R) -Reifenberg flat domain, $\|A\|_{\text{BMO}_R(\Omega, w)} < \delta$, and $u \in W_0^{1, \sigma}(\Omega, w)$ is a weak solution of (1.1) satisfying $\|\nabla u\|_{L^{p(\cdot)}(\Omega, w)} < \infty$, then there exists a positive constant $C_5 := C_{(n, \sigma, q, M)}$ such that, for any $x_0 \in \overline{\Omega}$ (the closure of Ω), $R_0 \in (0, \frac{R}{\varepsilon})$, $1 \leq \alpha < \beta \leq 2$, and $\lambda \in (K_1 \lambda_0, \infty)$,*

$$w(E(K_2 \lambda)) \leq C_5 \frac{\varepsilon}{\lambda} \left[\int_{D_1} |\nabla u(x)|^{\frac{\sigma p(x)}{p_0}} w(x) dx + \int_{D_2} \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0}} w(x) dx \right], \quad (5.1)$$

where $c^* := \max\{R[2 + w(\Omega)], 1\}$, K_1 and K_2 are positive constants, respectively, as in (5.6) and (5.33) below,

$$p_0^- := \inf_{x \in \Omega_{2R_0}(x_0)} p(x), \quad E(\lambda) := \left\{ x \in \Omega_{\alpha R_0}(x_0) : |\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} > \lambda \right\}, \quad 1 \leq \alpha < \beta \leq 2, \quad (5.2)$$

$$D_1 := \left\{ x \in \Omega_{\beta R_0}(x_0) : |\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} > \frac{\lambda}{4} \right\}, \quad \text{and} \quad D_2 := \left\{ x \in \Omega_{\beta R_0}(x_0) : \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} > \frac{\lambda}{4} \right\}.$$

Proof. Assume that p_0^- is as in (5.2), σ as in (1.15), and $u \in W_0^{1, \sigma}(\Omega, w)$ a weak solution of (1.1). For any given $\varepsilon \in (0, \infty)$, let

$$\delta_4 := \min \left\{ \delta_1, \delta_2, \frac{1}{5} \right\},$$

where δ_1 and δ_2 are, respectively, as in Lemmas 4.3 and 4.5. For any $x_0 \in \overline{\Omega}$, $\delta \in (0, \delta_4)$, and $R_0 \in (0, \frac{R}{\varepsilon})$, define

$$\lambda_0 := \frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} \left[|\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} + \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} \right] w(x) dx + \frac{1}{\delta}, \quad (5.3)$$

where $c^* := \max\{R[2 + w(\Omega)], 1\}$. By the assumption that $\|\nabla u\|_{L^{p(\cdot)}(\Omega, w)} < \infty$ and $\|\vec{F}/w\|_{L^{p(\cdot)}(\Omega, w)} \leq 1$, we know that λ_0 is well defined.

For any given $\delta \in (0, \delta_4)$, $1 \leq \alpha < \beta \leq 2$, and $\lambda \in (0, \infty)$, and for any $y \in E(\lambda)$ and $r \in (0, (\beta - \alpha)R_0)$, define

$$\Phi_y(r) := \frac{1}{w(\Omega_r(y))} \int_{\Omega_r(y)} \left[|\nabla u(x)|^{\frac{\sigma p(x)}{p_0}} + \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0}} \right] w(x) dx. \quad (5.4)$$

From $y \in E(\lambda) \subset B_{\alpha R_0}(x_0)$ and $r < (\beta - \alpha)R_0 \leq R_0$, we deduce that $B_r(y) \subset B_{2R_0}(x_0)$. By this, Lemmas 2.2 and 2.5, and $\delta \in (0, 1/5)$, we conclude that, for any given $\lambda \in (0, \infty)$ and $y \in E(\lambda)$, and for any $r \in [\frac{\beta - \alpha}{70}R_0, (\beta - \alpha)R_0]$,

$$\begin{aligned} \Phi_y(r) &\leq \frac{w(\Omega_{2R_0}(x_0))}{w(\Omega_r(y))} \frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} \left[|\nabla u(x)|^{\frac{\sigma p(x)}{p_0}} + \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0}} \right] w(x) dx \\ &\leq C_1 \left(\frac{2R_0}{r} \right)^{\sigma n} \frac{w(B_r(y))}{w(\Omega_r(y))} \frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} \left[|\nabla u(x)|^{\frac{\sigma p(x)}{p_0}} + \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0}} \right] w(x) dx \\ &\leq C_1^2 \left(\frac{2R_0}{r} \right)^{\sigma n} \left(\frac{4}{1 - \delta} \right)^{\sigma n} \lambda_0 \leq C_1^2 \left(\frac{700}{\beta - \alpha} \right)^{\sigma n} \lambda_0, \end{aligned} \quad (5.5)$$

where C_1 is as in Lemma 2.2. Let

$$K_1 := \max \left\{ C_1^2 \left(\frac{700}{\beta - \alpha} \right)^{\sigma n}, 1 \right\}. \quad (5.6)$$

Then, from (5.5), it follows that, for any given $\lambda \in (K_1 \lambda_0, \infty)$ and $y \in E(\lambda)$, and for any $r \in [\frac{\beta - \alpha}{70}R_0, (\beta - \alpha)R_0]$,

$$\Phi_y(r) \leq K_1 \lambda_0 < \lambda. \quad (5.7)$$

On the other hand, by (5.4), it is easy to see that, for any given $\lambda \in (0, \infty)$ and $y \in E(\lambda)$, $\Phi_y(\cdot)$ is continuous on $(0, [\beta - \alpha]R_0)$. Moreover, from the definition of $E(\lambda)$ in (5.2) and the Lebesgue differential theorem, we deduce that, for any given $\lambda \in (0, \infty)$ and almost every $y \in E(\lambda)$, $\lim_{r \rightarrow 0} \Phi_y(r) > \lambda$. This, combined with (5.7), implies that, for any given $\lambda \in (K_1 \lambda_0, \infty)$ and almost every $y \in E(\lambda)$, there exists some $r_y \in (0, \frac{\beta - \alpha}{70}R_0)$ such that $\Phi_y(r_y) = \lambda$ and, for any $r \in (r_y, [\beta - \alpha]R_0)$, $\Phi_y(r) < \lambda$. Then, by Lemma 2.1, we find that, for any given $\lambda \in (K_1 \lambda_0, \infty)$, there exists a family $\{B_{r_i}(y_i)\}_{i \in \mathbb{N}}$ of pairwise disjoint balls, with $\{y_i\}_{i \in \mathbb{N}} \subset E(\lambda)$ and $\{r_i\}_{i \in \mathbb{N}} \subset (0, \frac{\beta - \alpha}{70}R_0)$, satisfying, for any $i \in \mathbb{N}$,

$$\Phi_{y_i}(r_i) = \lambda, \quad \Phi_{y_i}(r) < \lambda \quad \text{for any } r \in (r_i, (\beta - \alpha)R_0), \quad (5.8)$$

and

$$E(\lambda) \subset \left[\bigcup_{i=1}^{\infty} \Omega_{5r_i}(y_i) \bigcup E_0 \right], \quad (5.9)$$

where E_0 is a measurable subset of $\mathbb{R}^n$ and $|E_0| = 0$. Let $K_2 \in (1, \infty)$ be a constant fixed later. Then, for any $\lambda \in (0, \infty)$, we have $E(K_2 \lambda) \subset E(\lambda)$. From this and (5.9), we deduce that, for any given $\lambda \in (K_1 \lambda_0, \infty)$,

$$\begin{aligned} w(E(K_2 \lambda)) &\leq \sum_{i=1}^{\infty} w \left(\left\{ x \in \Omega_{5r_i}(y_i) : |\nabla u(x)|^{\frac{\sigma p(x)}{p_0}} > K_2 \lambda \right\} \right) \\ &= \sum_{i \in \Lambda_1} w \left(\left\{ x \in \Omega_{5r_i}(y_i) : |\nabla u(x)|^{\sigma} > (K_2 \lambda)^{\frac{p_0}{p(x)}} \right\} \right) + \sum_{i \in \Lambda_2} \cdots, \end{aligned} \quad (5.10)$$

where

$$\Lambda_1 := \{i \in \mathbb{N} : B_{10r_i}(y_i) \subset \Omega\} \quad \text{and} \quad \Lambda_2 := \{i \in \mathbb{N} : B_{10r_i}(y_i) \not\subset \Omega\}.$$

Now, we consider two cases.

Case 1) $i \in \Lambda_1$. In this case, for any $i \in \Lambda_1$, let

$$p_i^- := \inf_{x \in B_{10r_i}(y_i)} p(x) \quad \text{and} \quad p_i^+ := \sup_{x \in B_{10r_i}(y_i)} p(x).$$

Then, for any $i \in \Lambda_1$, by $B_{10r_i}(y_i) \subset \Omega$, $10r_i \in (r_i, \frac{\beta-\alpha}{7}R_0)$, and (5.8), we know that, for any $\lambda \in (K_1\lambda_0, \infty)$,

$$\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} |\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx \leq \Phi_{y_i}(10r_i) < \lambda \quad (5.11)$$

and

$$\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx \leq \Phi_{y_i}(10r_i) < \lambda. \quad (5.12)$$

Next, we show that there exists a positive constant $c_6 := C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}$ such that, for any $i \in \Lambda_1$ and $\lambda \in (K_1\lambda_0, \infty)$,

$$\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} |\nabla u(x)|^\sigma w(x) dx \leq c_6 \lambda^{\frac{p_0^-}{p_i^-}} \quad (5.13)$$

and

$$\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} \left| \frac{\vec{F}(x)}{w(x)} \right|^\sigma w(x) dx \leq c_6 \lambda^{\frac{p_0^-}{p_i^+}} \delta^{\frac{p_-}{p_+}}, \quad (5.14)$$

where p_- and p_+ are as in (1.11).

To this end, we first prove that there exists a positive constant $C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}$ such that, for any $i \in \Lambda_1$,

$$\left[\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} |\nabla u(x)|^\sigma w(x) dx \right]^{p_i^+ - p_i^-} \leq C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}. \quad (5.15)$$

Indeed, fix any $y^* \in \Omega$. Then we have $B_{10r_i}(y_i) \subset \Omega \subset B_{\text{diam}(\Omega)}(y^*)$. From this and Lemma 2.2, we deduce that

$$w\left(B_{\text{diam}(\Omega)}(y^*)\right) \leq C_1 \left[\frac{\text{diam}(\Omega)}{10r_i} \right]^{\sigma n} w(B_{10r_i}(y_i)),$$

where $C_1 \in (0, \infty)$ is as in Lemma 2.2. This, together with (1.12), (1.14), and $20r_i < R_0 < \frac{R}{c^*} < 1$, further implies that, for any $i \in \Lambda_1$,

$$\begin{aligned} \left[\frac{1}{w(B_{10r_i}(y_i))} \right]^{p_i^+ - p_i^-} &\leq \left\{ C_1 \left[\frac{\text{diam}(\Omega)}{10r_i} \right]^{\sigma n} \frac{1}{w(B_{\text{diam}(\Omega)}(y^*))} \right\}^{p_i^+ - p_i^-} \\ &\leq \left\{ C_1 \left[\frac{\text{diam}(\Omega)}{10} \right]^{\sigma n} \frac{1}{w(\Omega)} \right\}^{\omega(\text{diam}(\Omega))} r_i^{-2n\omega(20r_i)} \\ &\leq C_{(n, \sigma, M, w(\Omega), \text{diam}(\Omega))}. \end{aligned} \quad (5.16)$$

On the other hand, by the assumption that $\|\vec{F}/w\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$, $\sigma < p_-$, and Remark 1.2(i), we find that

$$\int_{\Omega} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\sigma} w(x) dx \leq \int_{\Omega} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^{p(x)} + 1 \right] w(x) dx \leq 1 + w(\Omega).$$

Combining this and (1.6), we conclude that

$$\int_{\Omega} |\nabla u(x)|^{\sigma} w(x) dx \leq C_0^{\sigma} \int_{\Omega} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\sigma} w(x) dx \leq C_0^{\sigma} [1 + w(\Omega)],$$

where C_0 is a positive constant as in (1.6). From this, $0 < r_i < (\beta - \alpha)R_0 < \frac{R}{c^*}$, $c^* > R[1 + w(\Omega)]$, and (1.14), we deduce that, for any $i \in \Lambda_1$,

$$\begin{aligned} \left[\int_{B_{10r_i}(y_i)} |\nabla u(x)|^{\sigma} w(x) dx \right]^{p_i^+ - p_i^-} &\leq \left[\int_{\Omega} |\nabla u(x)|^{\sigma} w(x) dx \right]^{p_i^+ - p_i^-} \leq C_0^{\sigma \omega(20r_i)} [1 + w(\Omega)]^{\omega(20r_i)} \\ &\leq C_{(M, R)} r_i^{-\omega(20r_i)} \leq C_{(M, R)}. \end{aligned} \quad (5.17)$$

Combining this and (5.16), we obtain (5.15). Observe that $\sigma < p_- \leq \inf_{x \in \Omega_{2R_0}(x_0)} p(x) =: p_0^-$. By $y_i \in E(\lambda) \subset \Omega_{\alpha R_0}(x_0)$ and $r_i \in (0, \frac{\beta - \alpha}{70} R_0)$, we find that $B_{10r_i}(y_i) \subset \Omega_{2R_0}(x_0)$. Thus, $\sigma < p_0^- \leq p_i^-$. From this, (5.15), the Hölder inequality, and (5.11), it follows that

$$\begin{aligned} &\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} |\nabla u(x)|^{\sigma} w(x) dx \\ &= \left[\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} |\nabla u(x)|^{\sigma} w(x) dx \right]^{\frac{p_i^+ - p_i^-}{p_i^+}} \left[\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} |\nabla u(x)|^{\sigma} w(x) dx \right]^{\frac{p_i^-}{p_i^+}} \\ &\leq [C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}]^{1/p_i^+} \left[\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} |\nabla u(x)|^{\frac{\sigma p_i^-}{p_0^-}} w(x) dx \right]^{\frac{p_0^-}{p_i^+}} \\ &\leq [C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}]^{1/\sigma} \left[\frac{1}{w(B_{10r_i}(y_i))} \int_{B_{10r_i}(y_i)} |\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx + 1 \right]^{\frac{p_0^-}{p_i^+}} \\ &\leq [C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}]^{1/\sigma} (\lambda + 1)^{\frac{p_0^-}{p_i^+}} \leq [C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}]^{1/\sigma} \lambda^{\frac{p_0^-}{p_i^+}}, \end{aligned}$$

where we used $\lambda > K_1 \lambda_0 \geq \lambda_0 > 1$. This shows that (5.13) holds true. By (5.12) and an argument similar to that used in the estimation of (5.13), we obtain (5.14).

For any $i \in \Lambda_1$ and $x \in \mathbb{R}^n$, let

$$u_i(x) := \frac{2}{5} r_i^{-1} \left[C_3 \lambda^{\frac{p_0^-}{p_i^+}} \right]^{-1/\sigma} u \left(\frac{5r_i}{2} x + y_i \right), \quad \vec{F}_i(x) := \frac{2}{5} r_i^{-1} \left[C_3 \lambda^{\frac{p_0^-}{p_i^+}} \right]^{-1/\sigma} \vec{F} \left(\frac{5r_i}{2} x + y_i \right),$$

$$w_i(x) := w \left(\frac{5r_i}{2} x + y_i \right), \quad \text{and} \quad A_i(x) := A \left(\frac{5r_i}{2} x + y_i \right),$$

where C_3 is a positive constant as in Lemma 4.5. Then it is easy to see that $[w_i]_{A_{\sigma}(\mathbb{R}^n)} = [w]_{A_{\sigma}(\mathbb{R}^n)}$. From the assumption that u solves (1.1), it follows that u is a weak solution of $\text{div}(A \nabla u) = \text{div} \vec{F}$ in $B_{10r_i}(y_i)$, namely,

u_i is a weak solution of $\operatorname{div}(A_i \nabla u_i) = \operatorname{div} \vec{F}_i$ in B_4 . Moreover, by (5.13), (5.14), and changing of variables, we conclude that, for any $i \in \Lambda_1$,

$$\frac{1}{w_i(B_4)} \int_{B_4} |\nabla u_i(x)|^\sigma w_i(x) dx \leq 1 \quad \text{and} \quad \frac{1}{w_i(B_4)} \int_{B_4} \left| \frac{\vec{F}_i(x)}{w_i(x)} \right|^\sigma w_i(x) dx \leq \delta^{\frac{p_-}{p_+}}. \quad (5.18)$$

From $\|A\|_{\operatorname{BMO}_R(Q, w)} < \delta$, Remark 1.6, $10r_i < \frac{\beta-\alpha}{7c^*} R < R$, and $B_{10r_i}(y_i) \subset \Omega$, we deduce that $\|A\|_{\operatorname{BMO}_{10r_i}(B_{10r_i}(y_i), w)} < \delta$. This further implies that, for any $i \in \Lambda_1$,

$$\|A_i\|_{\operatorname{BMO}_4(B_4, w_i)} < \delta.$$

Combining this, Remark 1.6, (5.18), and Lemma 4.3, we find that there exists a matrix $\mathbb{A}_{i,0}$ of constant coefficients and a weak solution $v_i \in W^{1,q_0}(B_4)$ of $\operatorname{div}(\mathbb{A}_{i,0} \nabla v_i) = 0$ in B_4 , for some $q_0 \in (1, \sigma)$, such that

$$\frac{1}{w_i(B_2)} \int_{B_2} |\nabla u_i(x) - \nabla v_i(x)|^\sigma w_i(x) dx \leq \varepsilon \quad \text{and} \quad \|\nabla v_i\|_{L^\infty(B_2)} \leq C_2,$$

where C_2 is as in Lemma 4.3. By this and changing of variables, we know that, for any $i \in \Lambda_1$,

$$\frac{1}{w(B_{5r_i}(y_i))} \int_{B_{5r_i}(y_i)} |\nabla u(x) - \nabla v(x)|^\sigma w(x) dx \leq \varepsilon c_6 \lambda^{\frac{p_0^-}{p_i^+}} \quad \text{and} \quad \|\nabla v\|_{L^\infty(B_{5r_i}(y_i))}^\sigma \leq c_7 \lambda^{\frac{p_0^-}{p_i^+}},$$

where $c_7 := C_2^\sigma c_6$, $\varepsilon \in (0, \infty)$ is chosen in the beginning of the proof and, for any $x \in \mathbb{R}^n$,

$$v(x) = \frac{5r_i}{2} \left[C_3 \lambda^{\frac{p_0^-}{p_i^+}} \right]^{1/\sigma} v_i \left(\frac{2}{5r_i} [x - y_i] \right).$$

From $|\nabla u(x)|^\sigma \leq 2^{\sigma-1} |\nabla u(x) - \nabla v(x)|^\sigma + 2^{\sigma-1} |\nabla v(x)|^\sigma$ for any $x \in \mathbb{R}^n$, it follows that, for any $i \in \Lambda_1$,

$$\begin{aligned} & w \left(\left\{ x \in \Omega_{5r_i}(y_i) : |\nabla u(x)|^\sigma > (K_2 \lambda)^{\frac{p_0^-}{p(x)}} \right\} \right) \\ & \leq w \left(\left\{ x \in B_{5r_i}(y_i) : |\nabla u(x) - \nabla v(x)|^\sigma > c_6 \lambda^{\frac{p_0^-}{p_i^+}} \right\} \right) \\ & \quad + w \left(\left\{ x \in B_{5r_i}(y_i) : |\nabla v(x)|^\sigma > c_7 \lambda^{\frac{p_0^-}{p_i^+}} \right\} \right) \\ & \leq \frac{1}{c_6 \lambda^{p_0^-/p_i^+}} \int_{B_{5r_i}(y_i)} |\nabla u(x) - \nabla v(x)|^\sigma w(x) dx \\ & \leq \varepsilon w(B_{5r_i}(y_i)) \leq \varepsilon C_1 5^{2n} w(B_{r_i}(y_i)) = \varepsilon C_1 5^{2n} w(\Omega_{r_i}(y_i)), \end{aligned} \quad (5.19)$$

where $K_2 \geq [2^\sigma \max\{c_6, c_7, 1\}]^{p_+/p_-}$ is a positive constant determined in (5.33) below.

Case 2) $i \in \Lambda_2$, namely, $B_{10r_i}(y_i) \not\subset \Omega$. In this case, we then know that there exists some $y_0 \in \partial\Omega$ such that $\operatorname{dist}(y_i, \partial\Omega) = \operatorname{dist}(y_i, y_0) < 10r_i$. By $c^* \geq 1$ and $0 < \delta < \delta_4 \leq 1/5$, it is easy to see that $50r_i < \frac{5(\beta-\alpha)}{7} R_0 < \frac{5(\beta-\alpha)}{7} \frac{1}{c^*} R < (1-\delta)R$. From this and Remark 1.4(ii), it follows that there exists a Z -coordinate system $\{\vec{e}_{Z,1}, \dots, \vec{e}_{Z,n}\}$ such that $\vec{0}_Z \in \Omega$, $y_0 = -50\delta r_i \vec{e}_{Z,n}$, and, for any $i \in \Lambda_2$,

$$B_{50r_i}^+(\vec{0}_Z) \subset \Omega_{50r_i}(\vec{0}_Z) \subset \left[B_{50r_i}(\vec{0}_Z) \cap \left\{ z \in \mathbb{R}^n : z_n > -\frac{100\delta}{1-\delta} r_i \right\} \right]. \quad (5.20)$$

By this and $\delta \in (0, 1/5)$, we find that, in this Z -coordinate system, for any $i \in \Lambda_2$,

$$|y_i - \vec{0}_Z| \leq |y_i - y_0| + |y_0 - \vec{0}_Z| \leq 20r_i, \quad (5.21)$$

which further implies that $B_{5r_i}(y_i) \subset B_{25r_i}(\vec{0}_Z)$ and $\Omega_{5r_i}(y_i) \subset \Omega_{25r_i}(\vec{0}_Z)$. It is well known that there exists an orthogonal matrix T such that, for any $x := (x_1, \dots, x_n)$ in the original coordinate system, there exists a

unique $z := (z_1, \dots, z_n)$ in the Z -coordinate system such that $x = Tz$. From this and (5.21), we deduce that, for any $i \in \Lambda_2$,

$$w \left(\left\{ x \in \Omega_{5r_i}(y_i) : |\nabla u(x)|^\sigma > (K_2 \lambda)^{\frac{p_0^-}{p(x)}} \right\} \right) \leq w \left(\left\{ z \in \Omega_{25r_i}(\vec{0}_Z) : |\nabla u(Tz)|^\sigma > (K_2 \lambda)^{\frac{p_0^-}{p(x)}} \right\} \right).$$

In what follows, we denote $u(Tz)$, $\vec{F}(Tz)$, $w(Tz)$, and $A(Tz)$ simply, respectively, by $u(z)$, $\vec{F}(z)$, $w(z)$, and $A(z)$.

By (5.21), we find that $\Omega_{50r_i}(\vec{0}_Z) \subset \Omega_{70r_i}(y_i)$. From this, Lemmas 2.2 and 2.5, $70r_i < (\beta - \alpha)R_0$, and (5.8), it follows that, for any $\delta \in (0, 1/5)$ and $i \in \Lambda_2$,

$$\begin{aligned} & \frac{1}{w(\Omega_{50r_i}(\vec{0}_Z))} \int_{\Omega_{50r_i}(\vec{0}_Z)} \left[|\nabla u(z)|^{\frac{\sigma p(z)}{p_0^-}} + \frac{1}{\delta} \left| \frac{\vec{F}(z)}{w(z)} \right|^{\frac{\sigma p(z)}{p_0^-}} \right] w(z) dz \\ & \leq \frac{w(\Omega_{70r_i}(y_i))}{w(\Omega_{50r_i}(\vec{0}_Z))} \frac{1}{w(\Omega_{70r_i}(y_i))} \int_{\Omega_{70r_i}(y_i)} \left[|\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} + \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} \right] w(x) dx \\ & \leq C_1 \left(\frac{7}{5} \right)^{\sigma n} \frac{w(B_{50r_i}(y_i))}{w(\Omega_{50r_i}(\vec{0}_Z))} \Phi_{y_i}(70r_i) < C_1^2 \left(\frac{7}{5} \right)^{\sigma n} \left(\frac{4}{1-\delta} \right)^{\sigma n} \lambda < C_1^2 7^{\sigma n} \lambda, \end{aligned} \quad (5.22)$$

where $\lambda \in (K_1 \lambda_0, \infty)$. For any $i \in \Lambda$, let

$$p_i^- := \inf_{x \in \Omega_{50r_i}} p(x) \quad \text{and} \quad p_i^+ := \sup_{x \in \Omega_{50r_i}} p(x). \quad (5.23)$$

Next, we show that there exists a positive constant $c_8 := C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}$ such that, for any $i \in \Lambda_2$,

$$\frac{1}{w(\Omega_{50r_i}(\vec{0}_Z))} \int_{\Omega_{50r_i}(\vec{0}_Z)} |\nabla u(z)|^\sigma w(z) dz \leq c_8 \lambda^{\frac{p_0^-}{p_i^-}} \quad (5.24)$$

and

$$\frac{1}{w(\Omega_{50r_i}(\vec{0}_Z))} \int_{\Omega_{50r_i}(\vec{0}_Z)} \left| \frac{\vec{F}(z)}{w(z)} \right|^\sigma w(z) dz \leq c_8 \lambda^{\frac{p_0^-}{p_i^-}} \delta^{\frac{p_-}{p_i^+}}, \quad (5.25)$$

where p_0^- is as in (5.2) and p_-, p_+ as in (1.11). To achieve this, we first prove that there exists a positive constant $C_{(n, \sigma, M, w(\Omega), \text{diam}(\Omega))}$ such that, for any $i \in \Lambda_2$,

$$\left[\frac{1}{w(\Omega_{50r_i})} \int_{\Omega_{50r_i}(\vec{0}_Z)} |\nabla u(z)|^\sigma w(z) dz \right]^{p_i^+ - p_i^-} \leq C_{(n, \sigma, M, w(\Omega), \text{diam}(\Omega))}. \quad (5.26)$$

Indeed, by $B_{50r_i}^+(\vec{0}_Z) \subset \Omega_{50r_i}(\vec{0}_Z)$ [see (5.20)], we know that

$$B_{25r_i}(z_0) \subset B_{50r_i}^+(\vec{0}_Z) \subset \Omega_{50r_i}(\vec{0}_Z) \subset \Omega \subset B_{\text{diam}(\Omega)}(z_0),$$

where $z_0 := 25r_i \vec{e}_{Z,n}$. From this, Lemma 2.2, (1.14), and $100r_i < \frac{10}{7}R_0 < \frac{10R}{7c^*} < 1$, we deduce that, for any $i \in \Lambda_2$,

$$\begin{aligned} \left[\frac{1}{w(\Omega_{50r_i})} \right]^{p_i^+ - p_i^-} & \leq \left[\frac{1}{B_{25r_i}(z_0)} \right]^{p_i^+ - p_i^-} \leq \left\{ C_1 \left[\frac{\text{diam}(\Omega)}{25r_i} \right]^{\sigma n} \frac{1}{w(B_{\text{diam}(\Omega)}(z_0))} \right\}^{p_i^+ - p_i^-} \\ & \leq \left\{ C_1 \left[\frac{\text{diam}(\Omega)}{25} \right]^{\sigma n} \frac{1}{w(\Omega)} \right\}^{\omega(\text{diam}(\Omega))} r_i^{-2n\omega(100r_i)} \\ & \leq C_{(n, \sigma, M, w(\Omega), \text{diam}(\Omega))}. \end{aligned} \quad (5.27)$$

On the other hand, by an argument similar to that used in the estimation of (5.17), we know that, for any $i \in \Lambda_2$,

$$\left[\int_{\Omega_{50r_i}} |\nabla u(z)|^\sigma w(z) dz \right]^{p_i^+ - p_i^-} \leq C_{(M, R)}.$$

Combining this and (5.27), we obtain (5.26). By $y_i \in E(\lambda) \subset \Omega_{\alpha R_0}(x_0)$, $r_i \in (0, \frac{\beta - \alpha}{70} R_0)$, and (5.21), we find that $\Omega_{50r_i}(\vec{0}_Z) \subset \Omega_{70r_i}(y_i) \subset \Omega_{2R_0}(x_0)$. This, together with (5.23) and (5.2), implies that $\sigma < p_- \leq p_0^- \leq p_i^-$. From this, the Hölder inequality, (5.26), and (5.22), it follows that, for any $i \in \Lambda_2$,

$$\begin{aligned} & \frac{1}{w(\Omega_{50r_i}(\vec{0}_Z))} \int_{\Omega_{50r_i}(\vec{0}_Z)} |\nabla u(z)|^\sigma w(z) dz \\ &= \left[\frac{1}{w(\Omega_{50r_i}(\vec{0}_Z))} \int_{\Omega_{50r_i}(\vec{0}_Z)} |\nabla u(z)|^\sigma w(z) dz \right]^{\frac{p_i^+ - p_i^-}{p_i^+}} \left[\frac{1}{w(\Omega_{50r_i}(\vec{0}_Z))} \int_{\Omega_{50r_i}(\vec{0}_Z)} |\nabla u(z)|^\sigma w(z) dz \right]^{\frac{p_i^-}{p_i^+}} \\ &\leq [C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}]^{1/p_i^+} \left[\frac{1}{w(\Omega_{50r_i}(\vec{0}_Z))} \int_{\Omega_{50r_i}(\vec{0}_Z)} |\nabla u(z)|^{\frac{\sigma p_i^-}{p_0^-}} w(z) dz \right]^{\frac{p_0^-}{p_i^+}} \\ &\leq [C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}]^{1/\sigma} \left[\frac{1}{w(\Omega_{50r_i}(\vec{0}_Z))} \int_{\Omega_{50r_i}(\vec{0}_Z)} |\nabla u(z)|^{\frac{\sigma p(z)}{p_0^-}} w(z) dz + 1 \right]^{\frac{p_0^-}{p_i^+}} \\ &\leq [C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}]^{1/\sigma} (C_1^2 7^{\sigma n} \lambda + 1)^{\frac{p_0^-}{p_i^+}} \leq [C_{(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega))}]^{1/\sigma} \lambda^{\frac{p_0^-}{p_i^+}}. \end{aligned}$$

Thus, we obtain (5.24). Similarly, we can show that (5.25) holds true and we omit the details.

For any $i \in \Lambda_2$ and $z \in \mathbb{R}^n$, let

$$\begin{aligned} u_i(z) &:= \frac{2}{25} r_i^{-1} \left[C_5 \lambda^{\frac{p_0^-}{p_i^+}} \right]^{-1/\sigma} u\left(\frac{25r_i}{2} z\right), \quad \vec{F}_i(z) := \frac{2}{25} r_i^{-1} \left[C_5 \lambda^{\frac{p_0^-}{p_i^+}} \right]^{-1/\sigma} \vec{F}\left(\frac{25r_i}{2} z\right), \\ w_i(z) &:= w\left(\frac{25r_i}{2} z\right), \quad \text{and} \quad A_i(z) := A\left(\frac{25r_i}{2} z\right). \end{aligned}$$

By the assumption that u solves (1.1) and Remark 1.4(ii), we find that u_i is a weak solution of

$$\begin{cases} \operatorname{div}(A_i \nabla u_i) = \operatorname{div} \vec{F}_i & \text{in } \Omega_i \cap B_4, \\ u_i = 0 & \text{on } \partial \Omega_i \cap B_4, \end{cases} \quad (5.28)$$

where $\Omega_i := \{\frac{2}{25r_i} z : z \in \Omega\}$. Moreover, from (5.24), (5.25), and changing of variables, we deduce that, for any $i \in \Lambda_2$,

$$\frac{1}{w_i(\Omega_{i,4}(\vec{0}_Z))} \int_{\Omega_{i,4}(\vec{0}_Z)} |\nabla u_i(x)|^\sigma w_i(x) dx \leq 1 \quad (5.29)$$

and

$$\frac{1}{w_i(\Omega_{i,4}(\vec{0}_Z))} \int_{\Omega_{i,4}(\vec{0}_Z)} \left| \frac{\vec{F}_i(x)}{w_i(x)} \right|^\sigma w_i(x) dx \leq \delta^{\frac{p_-}{p_+}}, \quad (5.30)$$

here and thereafter, for any $r \in (0, \infty)$, $\Omega_{i,r}(\vec{0}_Z) := \Omega_i \cap B_r(\vec{0}_Z)$. By $\|A\|_{\text{BMO}_R(\Omega, w)} < \delta$ and $50r_i < R$, we conclude that, for any $i \in \Lambda_2$, $\|A\|_{\text{BMO}_{50r_i}(\Omega_{50r_i}(\vec{0}_Z), w)} < \delta$. This further implies that, for any $i \in \Lambda_2$,

$$\|A_i\|_{\text{BMO}_4(\Omega_{i,4}(\vec{0}_Z), w_i)} < \delta.$$

Combining this, (5.28), (5.29), (5.30), and Lemma 4.5, we further conclude that there exist a matrix $\mathbb{A}_0$ of constant coefficients and a weak solution $v_i \in W^{1,q_0}(B_4^+)$ of (4.26), for some $q_0 \in (1, \sigma)$, such that, for any $i \in \Lambda_2$,

$$\frac{1}{w_i(\Omega_{i,2}(\vec{0}_Z))} \int_{\Omega_{i,2}(\vec{0}_Z)} |\nabla u_i(z) - \nabla V_i(z)|^\sigma w(z) dz \leq \varepsilon \quad \text{and} \quad \|\nabla v_i\|_{L^\infty(B_2^+(\vec{0}_Z))} \leq C_3,$$

where C_3 is as in Lemma 4.5 and V_i the zero extension of v_i to B_4 . By changing of variables, we know that, for any $i \in \Lambda_2$,

$$\frac{1}{w(\Omega_{25r_i}(\vec{0}_Z))} \int_{\Omega_{25r_i}(\vec{0}_Z)} |\nabla u(z) - \nabla V(z)|^\sigma w(z) dz \leq \varepsilon c_8 \lambda^{\frac{p_0^-}{p_i^+}} \quad (5.31)$$

and

$$\|\nabla V\|_{L^\infty(B_{25r_i}^+(\vec{0}_Z))}^\sigma \leq c_9 \lambda^{\frac{p_0^-}{p_i^+}}, \quad (5.32)$$

where $c_9 := C_3^\sigma c_8$ and, for any $z \in \mathbb{R}^n$,

$$V(z) := \frac{25r_i}{2} \left[C_5 \lambda^{\frac{p_0^-}{p_i^+}} \right]^{1/\sigma} V_i \left(\frac{2}{25r_i} z \right).$$

Fix

$$K_2 := [2^\sigma \max\{c_6, c_7, c_8, c_9, 1\}]^{\frac{p_+}{p_-}}. \quad (5.33)$$

From this, $\Omega_{5r_i}(y_i) \subset \Omega_{25r_i}(\vec{0}_Z)$ for any $i \in \Lambda_2$, (5.31), and (5.32), it follows that

$$\begin{aligned} & w \left(\left\{ x \in \Omega_{5r_i}(y_i) : |\nabla u(x)|^\sigma > (K_2 \lambda)^{\frac{p_0^-}{p(x)}} \right\} \right) \\ & \leq w \left(\left\{ z \in \Omega_{25r_i}(\vec{0}_Z) : |\nabla u(z) - \nabla V(z)|^\sigma > c_8 \lambda^{\frac{p_0^-}{p_i^+}} \right\} \right) \\ & \quad + w \left(\left\{ z \in \Omega_{25r_i}(\vec{0}_Z) : |\nabla V(z)|^\sigma > c_9 \lambda^{\frac{p_0^-}{p_i^+}} \right\} \right) \\ & \leq \frac{1}{c_8 \lambda^{p_0^-/p_i^+}} \int_{\Omega_{25r_i}(\vec{0}_Z)} |\nabla u(z) - \nabla V(z)|^\sigma w(z) dz \leq \varepsilon w(\Omega_{25r_i}). \end{aligned} \quad (5.34)$$

By $B_{5r_i}(y_i) \subset B_{25r_i}(\vec{0}_Z)$ and Lemma 2.2, we find that, for any $i \in \Lambda_2$,

$$w(\Omega_{25r_i}(\vec{0}_Z)) \leq C_1 5^{\sigma n} w(B_{5r_i}(y_i)) \leq C_1^2 25^{\sigma n} w(B_{r_i}(y_i)) \leq C_1^3 25^{\sigma n} \left(\frac{4}{1-\delta} \right)^{\sigma n} w(\Omega_{r_i}(y_i)).$$

This, together with (5.34) and $\delta \in (0, 1/5)$, implies that, for any $i \in \Lambda_2$,

$$\begin{aligned} w \left(\left\{ x \in \Omega_{5r_i}(y_i) : |\nabla u(x)|^\sigma > (K_2 \lambda)^{\frac{p_0^-}{p(x)}} \right\} \right) & \leq \varepsilon C_1^3 25^{\sigma n} \left(\frac{4}{1-\delta} \right)^{\sigma n} w(\Omega_{r_i}(y_i)) \\ & \leq \varepsilon C_1^3 125^{\sigma n} w(\Omega_{r_i}(y_i)). \end{aligned}$$

From this, (5.19), and (5.10), we deduce that, for any $\lambda \in (K_1\lambda_0, \infty)$,

$$w(E(K_2\lambda)) \leq c_{10}\varepsilon \sum_{i=1}^{\infty} w(\Omega_{r_i}(y_i)), \quad (5.35)$$

where

$$c_{10} := \max \left\{ C_1 5^{\sigma n}, C_1^3 125^{\sigma n} \right\}.$$

On the other hand, by (5.8), we know that, for any $i \in \mathbb{N}$ and $\lambda \in (K_1\lambda_0, \infty)$,

$$\begin{aligned} w(\Omega_{r_i}(y_i)) &= \frac{1}{\lambda} \left[\int_{\Omega_{r_i}(y_i)} |\nabla u(x)|^{\frac{\sigma p(x)}{p_0}} w(x) dx + \frac{1}{\delta} \int_{\Omega_{r_i}(y_i)} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0}} w(x) dx \right] \\ &\leq \frac{1}{\lambda} \left[\int_{\left\{ x \in \Omega_{r_i}(y_i) : |\nabla u(x)|^{\sigma p(x)/p_0^-} > \lambda/4 \right\}} |\nabla u(x)|^{\frac{\sigma p(x)}{p_0}} w(x) dx + \frac{\lambda}{4} w(\Omega_{r_i}(y_i)) \right. \\ &\quad \left. + \frac{1}{\delta} \int_{\left\{ x \in \Omega_{r_i}(y_i) : \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\sigma p(x)/p_0^-} > \lambda/4 \right\}} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0}} w(x) dx + \frac{\lambda}{4} w(\Omega_{r_i}(y_i)) \right], \end{aligned}$$

which further implies that, for any $i \in \mathbb{N}$ and $\lambda \in (K_1\lambda_0, \infty)$,

$$\begin{aligned} w(\Omega_{r_i}(y_i)) &\leq \frac{2}{\lambda} \left[\int_{\left\{ x \in \Omega_{r_i}(y_i) : |\nabla u(x)|^{\sigma p(x)/p_0^-} > \lambda/4 \right\}} |\nabla u(x)|^{\frac{\sigma p(x)}{p_0}} w(x) dx \right. \\ &\quad \left. + \frac{1}{\delta} \int_{\left\{ x \in \Omega_{r_i}(y_i) : \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\sigma p(x)/p_0^-} > \lambda/4 \right\}} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0}} w(x) dx \right]. \end{aligned}$$

From this, $B_{r_i}(y_i) \subset B_{\beta R_0}(x_0)$ for any $i \in \mathbb{N}$, and $\{\Omega_{r_i}(y_i)\}_{i=1}^{\infty}$ are pairwise disjoint, it follows that, for any $\lambda \in (K_1\lambda_0, \infty)$,

$$\sum_{i=1}^{\infty} w(\Omega_{r_i}(y_i)) \leq \frac{2}{\lambda} \left[\int_{D_1} |\nabla u(x)|^{\frac{\sigma p(x)}{p_0}} w(x) dx + \frac{1}{\delta} \int_{D_2} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0}} w(x) dx \right].$$

Combining this and (5.35), we obtain (5.1), which completes the proof of Lemma 5.1. $\square$

We now prove Theorem 1.7.

Proof of Theorem 1.7. Let $R \in (0, \infty)$, Ω be a (δ, R) -Reifenberg domain of $\mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (1.11), (1.12), and (1.13), $M \in (0, \infty)$, σ be as in (1.15), $w \in A_{\sigma}(\mathbb{R}^n)$ with $[w]_{A_{\sigma}(\mathbb{R}^n)} \leq M$, and $q \in (1, \infty)$ as in (1.16), where δ is a positive constant determined later. Suppose that A is a matrix satisfying (1.2) and $\|A\|_{\text{BMO}_R(\Omega, w)} < \delta$, and $\vec{F} : \Omega \rightarrow \mathbb{R}^n$ is a given vector field satisfying $|\vec{F}|/w \in L^{p(\cdot)}(\Omega, w)$. We now prove this theorem by two steps.

Step 1) Assume that $u \in W_0^{1,p}(\Omega, w)$ is a weak solution of (1.1) and satisfies

$$\int_{\Omega} |\nabla u(x)|^{p(x)} w(x) dx < \infty. \quad (5.36)$$

Under this prior assumption, we now prove (1.17). To this end, by an argument of scaling, we only need to show that there exists a positive constant $C_6 := C(n, \delta, \sigma, q, M, R, w(\Omega), \text{diam}(\Omega), p_-, p_+, \Lambda)$ such that

$$\|\nabla u\|_{L^{p(\cdot)}(\Omega, w)} \leq C_6 \quad (5.37)$$

under the assumption that

$$\left\| \frac{\vec{F}}{w} \right\|_{L^{p(\cdot)}(\Omega, w)} \leq 1, \quad (5.38)$$

where p_- and p_+ are as in (1.7). For any given $\varepsilon \in (0, \infty)$, let δ_3 and δ_4 be the positive constants, respectively, as in Lemmas 4.8 and 5.1 and fix some $\delta \in (0, \min\{\delta_3, \delta_4\})$. Let $1 \leq \alpha < \beta \leq 2$ and $R_0 \in (0, \frac{R}{c^*})$, where c^* is a positive constant sufficiently large, which is determined later. By (2.6), we know that, for any given $x_0 \in \overline{\Omega}$,

$$\begin{aligned} \int_{\Omega_{aR_0}(x_0)} |\nabla u(x)|^{p(x)} w(x) dx &= \int_{\Omega_{aR_0}(x_0)} \left[|\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} \right]^{\frac{p_0^-}{\sigma}} w(x) dx \\ &= \frac{p_0^-}{\sigma} \int_0^\infty \lambda^{\frac{p_0^-}{\sigma}-1} w(E(\lambda)) d\lambda = \frac{p_0^-}{\sigma} K_2^{\frac{p_0^-}{\sigma}} \int_0^\infty \lambda^{\frac{p_0^-}{\sigma}-1} w(E(K_2\lambda)) d\lambda \\ &\leq (K_1 K_2 \lambda_0)^{\frac{p_0^-}{\sigma}} w(\Omega_{aR_0}(x_0)) + \frac{p_0^-}{\sigma} K_2^{\frac{p_0^-}{\sigma}} \int_{K_1 \lambda_0}^\infty \lambda^{\frac{p_0^-}{\sigma}-1} w(E(K_2\lambda)) d\lambda \\ &=: \text{I} + \text{II}, \end{aligned}$$

where p_0^- , $E(\lambda)$, λ_0 , K_1 , and K_2 are as in Lemma 5.1.

For I, from (5.3) and (5.6), we deduce that, for any given $x_0 \in \overline{\Omega}$,

$$\begin{aligned} \text{I} &\leq c_{11} \frac{w(\Omega_{2R_0}(x_0))}{(\beta - \alpha)^{np_0^-}} \left\{ \frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} \left[|\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} + \frac{1}{\delta} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} \right] w(x) dx + \frac{1}{\delta} \right\}^{\frac{p_0^-}{\sigma}} \\ &\leq c_{11} 2^{\frac{p_0^-}{\sigma}-1} \frac{w(\Omega_{2R_0}(x_0))}{(\beta - \alpha)^{np_+}} \left[\frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} |\nabla u(x)|^{\frac{\sigma p_0^+}{p_0^-}} w(x) dx + 1 \right]^{\frac{p_0^-}{\sigma}} \\ &\quad + c_{11} 2^{\frac{p_0^-}{\sigma}-1} \frac{w(\Omega_{2R_0}(x_0))}{(\beta - \alpha)^{np_+}} \delta^{-\frac{p_0^-}{\sigma}} \left[\frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx + 1 \right]^{\frac{p_0^-}{\sigma}}, \end{aligned} \quad (5.39)$$

where $c_{11} := [(C_1^2 700^{\sigma n} + 1)K_2]^{p_0^-/\sigma}$, C_1 is as in Lemma 2.2, and

$$p_0^+ := \sup_{x \in \Omega_{2R_0}(x_0)} p(x).$$

By $1 < \sigma < p_- \leq p_0^-$, the Hölder inequality, (5.38), and Remark 1.2(i), we find that, for any given $x_0 \in \overline{\Omega}$,

$$\begin{aligned} &\left[\frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx + 1 \right]^{\frac{p_0^-}{\sigma}} \\ &\leq 2^{\frac{p_0^-}{\sigma}-1} \left\{ \left[\frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx \right]^{\frac{p_0^-}{\sigma}} + 1 \right\} \end{aligned}$$

$$\leq 2^{p_+} \left[\frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} \left| \frac{\vec{F}(x)}{w(x)} \right|^{p(x)} w(x) dx + 1 \right] \leq 2^{p_+} \left[\frac{1}{w(\Omega_{2R_0}(x_0))} + 1 \right]. \quad (5.40)$$

On the other hand, from $\lim_{r \rightarrow 0^+} \omega(r) = 0$, it follows that there exists some $c^* \in (0, \infty)$ large enough such that, for any $R_0 \in (0, \frac{R}{c^*})$, $\omega(4R_0) \leq (p_- - \sigma)p_-/\sigma$. Thus, we obtain

$$\sigma \frac{p_0^+}{p_0^-} = \sigma + \sigma \frac{p_0^+ - p_0^-}{p_0^-} \leq \sigma + \sigma \frac{\omega(4R_0)}{p_-} < p_-.$$

By this, Lemma 4.8, (5.38), and Remark 1.2(i), we conclude that there exists a positive constant $c_{12} := C_{(n, \sigma, q, M, p_0^+, p_0^-, \Lambda)}$ such that, for any given $x_0 \in \bar{\Omega}$,

$$\begin{aligned} & \frac{1}{w(\Omega_{2R_0}(x_0))} \int_{\Omega_{2R_0}(x_0)} |\nabla u(x)|^{\sigma \frac{p_0^+}{p_0^-}} w(x) dx \\ & \leq c_{12} \left[\frac{1}{w(\Omega_{12R_0}(x_0))} \int_{\Omega_{12R_0}(x_0)} |\nabla u(x)|^\sigma w(x) dx \right]^{\frac{p_0^+}{p_0^-}} + \frac{c_{12}}{w(\Omega_{12R_0}(x_0))} \int_{\Omega_{12R_0}(x_0)} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\sigma \frac{p_0^+}{p_0^-}} w(x) dx \\ & \leq c_{12} C_0^{\sigma \frac{p_0^+}{p_0^-}} \left\{ \frac{1}{w(\Omega_{12R_0}(x_0))} \int_{\Omega} \left| \frac{\vec{F}(x)}{w(x)} \right|^\sigma w(x) dx \right\}^{\frac{p_0^+}{p_0^-}} \\ & \quad + \frac{c_{12}}{w(\Omega_{12R_0}(x_0))} \int_{\Omega_{12R_0}(x_0)} \left[\left| \frac{\vec{F}(x)}{w(x)} \right|^{p(x)} + 1 \right] w(x) dx \\ & \leq c_{12} C_0^{\sigma \frac{p_0^+}{p_0^-}} \left[\frac{1 + w(\Omega)}{w(\Omega_{12R_0}(x_0))} \right]^{\frac{p_0^+}{p_0^-}} + c_{12} \left[\frac{1}{w(\Omega_{12R_0}(x_0))} + 1 \right] \leq c_{13} \left[\frac{1 + w(\Omega)}{w(\Omega_{12R_0}(x_0))} \right]^{\frac{p_0^+}{p_0^-}}, \end{aligned}$$

where C_0 is a positive constant as in (1.6) and $c_{13} := C_{(n, \sigma, q, M, p_-, p_+, \Lambda, R, \text{diam}(\Omega))}$. From this, $p_0^+ \leq p_+$, (5.40), and (5.39), we deduce that, for any given $x_0 \in \bar{\Omega}$,

$$\begin{aligned} \text{I} & \leq c_{11} 2^{p_+} \frac{w(\Omega_{2R_0}(x_0))}{(\beta - \alpha)^{np_+}} \left\{ c_{13} \left[\frac{1 + w(\Omega)}{w(\Omega_{12R_0}(x_0))} \right]^{\frac{p_0^+}{p_0^-}} + 1 \right\}^{\frac{p_0^-}{\sigma}} \\ & \quad + c_{11} 4^{p_+} \frac{w(\Omega_{2R_0}(x_0))}{(\beta - \alpha)^{np_+}} \delta^{-\frac{p_0^-}{\sigma}} \left[\frac{1}{w(\Omega_{2R_0}(x_0))} + 1 \right] \\ & \leq c_{11} 2^{p_+} (c_{13} + 1)^{\frac{p_+}{\sigma}} \frac{w(\Omega_{2R_0}(x_0))}{(\beta - \alpha)^{np_+}} \left[\frac{1 + w(\Omega)}{w(\Omega_{12R_0}(x_0))} \right]^{\frac{p_+}{\sigma}} + c_{11} 4^{p_+} \delta^{-\frac{p_0^-}{\sigma}} \frac{1}{(\beta - \alpha)^{np_+}} [1 + w(\Omega_{2R_0}(x_0))] \\ & \leq c_{14} \frac{1}{(\beta - \alpha)^{np_+}} \frac{[1 + w(\Omega)]^{1 + \frac{p_+}{\sigma}}}{[w(\Omega_{R_0}(x_0))]^{\frac{p_+}{\sigma}}}, \end{aligned} \quad (5.41)$$

where $c_{14} := C_{(n, \delta, \sigma, q, M, R, w(\Omega), \text{diam}(\Omega), p_-, p_+, \Lambda)}$ is a positive constant.

For II, by Lemma 5.1, we know that

$$\begin{aligned} \text{II} & = \frac{p_0^-}{\sigma} K_2^{\frac{p_0^-}{\sigma}} \int_{K_1 \lambda_0}^{\infty} \lambda^{\frac{p_0^-}{\sigma} - 1} w(E(K_2 \lambda)) d\lambda \\ & \leq C_5 \varepsilon \frac{p_0^-}{\sigma} K_2^{\frac{p_0^-}{\sigma}} \int_0^{\infty} \lambda^{\frac{p_0^-}{\sigma} - 2} \left[\int_{D_1} |\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx + \frac{1}{\delta} \int_{D_2} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx \right] d\lambda \end{aligned}$$

$$\begin{aligned}
&\leq C_5 4^{\frac{p_0^-}{\sigma}} \varepsilon K_2^{\frac{p_0^-}{\sigma}} \int_0^\infty \left(\frac{\lambda}{4}\right)^{\frac{p_0^-}{\sigma}-2} \int_{D_1} |\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx d\left(\frac{\lambda}{4}\right) \\
&\quad + C_5 4^{\frac{p_0^-}{\sigma}} \varepsilon K_2^{\frac{p_0^-}{\sigma}} \delta^{-\frac{p_0^-}{\sigma}} \int_0^\infty \left(\frac{\lambda \delta}{4}\right)^{\frac{p_0^-}{\sigma}-2} \int_{D_2} \left| \frac{\vec{F}(x)}{w(x)} \right|^{\frac{\sigma p(x)}{p_0^-}} w(x) dx d\left(\frac{\lambda \delta}{4}\right) \\
&=: \Pi_1 + \Pi_2,
\end{aligned}$$

where C_5 is as in (5.1), λ_0 as in (5.3), and D_1, D_2 are as in (5.2). Notice that, for any given subset U of $\mathbb{R}^n$, any $0 < r < p < \infty$ and $f \in L^p(U, w)$,

$$\int_U |f(x)|^p w(x) dx = \int_0^\infty (p-r)t^{p-r-1} \int_{\{x \in U: |f(x)| > t\}} |f(x)|^r w(x) dx dt. \quad (5.42)$$

To estimate Π_1 , using (5.42) with $U := \Omega_{\beta R_0}(x_0)$, $p := \frac{p_0^-}{\sigma}$, $r := 1$, $t := \frac{\lambda}{4}$, and $f(x) := |\nabla u(x)|^{\frac{\sigma p(x)}{p_0^-}}$ therein, we obtain

$$\Pi_1 \leq c_{15} \varepsilon \int_{\Omega_{\beta R_0}(x_0)} |\nabla u(x)|^{p(x)} w(x) dx, \quad (5.43)$$

where

$$c_{15} := C_5 4^{\frac{p_+}{\sigma}} K_2^{\frac{p_+}{\sigma}} \left(\frac{p_-}{\sigma} - 1\right)^{-1} := C(n, \sigma, M, R, w(\Omega), \text{diam}(\Omega), p_-, p_+).$$

Similarly, from (5.38) and Remark 1.2(i), we deduce that, for any given $x_0 \in \overline{\Omega}$,

$$\Pi_2 \leq c_{15} \varepsilon \delta^{-\frac{p_+}{\sigma}} \int_{\Omega_{\beta R_0}(x_0)} \left| \frac{\vec{F}(x)}{w(x)} \right|^{p(x)} w(x) dx \leq c_{15} \varepsilon \delta^{-\frac{p_+}{\sigma}}.$$

Combining this and (5.43), we find that, for any given $x_0 \in \overline{\Omega}$,

$$\Pi \leq c_{15} \varepsilon \int_{\Omega_{\beta R_0}(x_0)} |\nabla u(x)|^{p(x)} w(x) dx + c_{15} \varepsilon \delta^{-\frac{p_+}{\sigma}}.$$

This, together with (5.41), implies that, for any given $x_0 \in \overline{\Omega}$,

$$\begin{aligned}
&\int_{\Omega_{\alpha R_0}(x_0)} |\nabla u(x)|^{p(x)} w(x) dx \\
&\leq c_{14} \frac{1}{(\beta - \alpha)^{np_+}} \frac{[1 + w(\Omega)]^{1+\frac{p_+}{\sigma}}}{[w(\Omega_{R_0}(x_0))]^{\frac{p_+}{\sigma}}} + c_{15} \varepsilon \int_{\Omega_{\beta R_0}(x_0)} |\nabla u(x)|^{p(x)} w(x) dx + c_{15} \varepsilon \delta^{-\frac{p_+}{\sigma}}.
\end{aligned}$$

By fixing $\varepsilon \in (0, \infty)$ small enough such that $0 < c_{15} \varepsilon < 1/2$, we further know that, for any given $x_0 \in \overline{\Omega}$,

$$\int_{\Omega_{\alpha R_0}(x_0)} |\nabla u(x)|^{p(x)} w(x) dx \leq \frac{1}{2} \int_{\Omega_{\beta R_0}(x_0)} |\nabla u(x)|^{p(x)} w(x) dx + \frac{c_{14}}{(\beta - \alpha)^{np_+}} \frac{[1 + w(\Omega)]^{1+\frac{p_+}{\sigma}}}{[w(\Omega_{R_0}(x_0))]^{\frac{p_+}{\sigma}}} + c_{16},$$

where $c_{16} := c_{15} \varepsilon \delta^{-\frac{p_+}{\sigma}}$. Applying Lemma 2.7 with $\phi(\cdot) := \int_{\Omega_{R_0}(x_0)} |\nabla u(x)|^{p(x)} w(x) dx$, $\alpha = 1$, and $\beta = 2$ therein, we conclude that, for any given $x_0 \in \overline{\Omega}$,

$$\int_{\Omega_{R_0}(x_0)} |\nabla u(x)|^{p(x)} w(x) dx \leq C_{(p_+)} \left\{ c_{14} \frac{[1 + w(\Omega)]^{1+\frac{p_+}{\sigma}}}{[w(\Omega_{R_0}(x_0))]^{\frac{p_+}{\sigma}}} + c_{16} \right\}$$

$$\leq C_7 \frac{[1 + w(\Omega)]^{1 + \frac{p_+}{\sigma}}}{[w(\Omega_{R_0}(x_0))]^{\frac{p_+}{\sigma}}} \leq C_7 \frac{[1 + w(\Omega)]^{1 + \frac{p_+}{\sigma}}}{[w(B_{R_0}(x_0))]^{\frac{p_+}{\sigma}}},$$

where $C_7 := C_{(n, \delta, \sigma, q, M, R, w(\Omega), \text{diam}(\Omega), p_-, p_+, \Lambda)}$ and we used $[1 + w(\Omega)]^{1 + \frac{p_+}{\sigma}} > [w(\Omega_{R_0}(x_0))]^{\frac{p_+}{\sigma}}$. From this and the fact that $\overline{\Omega}$ is compact, we deduce that there exist finitely many points $\{x_0^k\}_{k=1}^N \subset \overline{\Omega}$ such that $\overline{\Omega} \subset \cup_{k=1}^N \Omega_{R_0}(x_0^k)$ and

$$\int_{\Omega_{R_0}(x_0^k)} |\nabla u(x)|^{p(x)} w(x) dx \leq C_7 \frac{[1 + w(\Omega)]^{1 + \frac{p_+}{\sigma}}}{[w(B_{R_0}(x_0^k))]^{\frac{p_+}{\sigma}}}.$$

Thus, we have

$$\int_{\Omega} |\nabla u(x)|^{p(x)} w(x) dx \leq \sum_{k=1}^N \int_{\Omega_{R_0}(x_0^k)} |\nabla u(x)|^{p(x)} w(x) dx \leq NC_7,$$

which further implies (5.37).

Step 2) In this step, we show the existence and the uniqueness of weak solution $u \in W_0^{1, p^-}(\Omega, w)$ of (1.1), and that the prior assumption (5.36) can be removed. Indeed, by $|\vec{F}|/w \in L^{p(\cdot)}(\Omega, w)$ and Remark 1.2(ii), we conclude that there exists a sequence $\{\vec{\phi}_k\}_{k \in \mathbb{N}}$ with $\vec{\phi}_k := (\phi_{k,1}, \dots, \phi_{k,n})$ and $\phi_{k,i} \in C_c^\infty(\Omega)$ for any $k \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, such that

$$\lim_{k \rightarrow \infty} \left\| \vec{\phi}_k - \frac{\vec{F}}{w} \right\|_{L^{p(\cdot)}(\Omega, w)} = 0. \quad (5.44)$$

Obviously, for any $k \in \mathbb{N}$, $|\vec{\phi}_k| \in W_0^{1, p^+}(\Omega, w)$. From this and [18, Theorem 2.7], we deduce that, for any $k \in \mathbb{N}$ and $w\vec{\phi}_k$, there exists a unique $u_k \in W_0^{1, p^+}(\Omega, w)$ such that u_k is a weak solution of

$$\begin{cases} \operatorname{div}(A \nabla u_k) = \operatorname{div}(w \vec{\phi}_k) & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial \Omega. \end{cases} \quad (5.45)$$

Moreover, by Remark 1.2(iii), we find that, for any $k \in \mathbb{N}$, $\|\nabla u_k\|_{L^{p(\cdot)}(\Omega, w)} < \infty$. Thus, for any $k \in \mathbb{N}$, $\int_{\Omega} |\nabla u_k(x)|^{p(x)} w(x) dx < \infty$. From this, *Step 1)*, and (5.44), it follows that, for any $k \in \mathbb{N}$ large enough,

$$\|\nabla u_k\|_{L^{p(\cdot)}(\Omega, w)} \leq C_6 \left\| \vec{\phi}_k \right\|_{L^{p(\cdot)}(\Omega, w)} \leq 2C_6 \left\| \frac{\vec{F}}{w} \right\|_{L^{p(\cdot)}(\Omega, w)},$$

where C_6 is a positive constant as in (5.37). By this, Lemma 2.8, and Remark 1.2(iii), we know that there exist positive constants $C_{(n, \sigma, M, \text{diam}(\Omega))}$ and $C_{(p(\cdot), n, \sigma, M, \text{diam}(\Omega))}$ such that, for any $k, l \in \mathbb{N}$,

$$\begin{aligned} \|u_k - u_l\|_{L^{p^-}(\Omega, w)} &\leq C_{(n, \sigma, M, \text{diam}(\Omega))} \|\nabla u_k - \nabla u_l\|_{L^{p^-}(\Omega, w)} \\ &\leq C_{(p(\cdot), n, \sigma, M, \text{diam}(\Omega))} \|\nabla u_k - \nabla u_l\|_{L^{p(\cdot)}(\Omega, w)} \\ &\leq C_{(p(\cdot), n, \sigma, M, \text{diam}(\Omega))} C_6 \|\phi_k - \phi_l\|_{L^{p(\cdot)}(\Omega, w)}. \end{aligned}$$

This, combined with (5.44), further shows that $\{u_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $W_0^{1, p^-}(\Omega, w)$. Thus, there exists a unique $u \in W_0^{1, p^-}(\Omega, w)$ such that

$$\lim_{k \rightarrow \infty} \|u_k - u\|_{W^{1, p^-}(\Omega, w)} = 0. \quad (5.46)$$

From this, we deduce that, for almost every $x \in \Omega$, $\lim_{k \rightarrow \infty} |\nabla u_k(x)| = |\nabla u(x)|$. By the Fatou lemma for $L^{p(\cdot)}(\Omega, w)$ (see, for instance, [21, Theorem 2.61] or [24, Lemma 3.2.8]), we conclude that

$$\|\nabla u\|_{L^{p(\cdot)}(\Omega, w)} \leq \liminf_{k \rightarrow \infty} \|\nabla u_k\|_{L^{p(\cdot)}(\Omega, w)} \leq 2C_6 \left\| \frac{\vec{F}}{w} \right\|_{L^{p(\cdot)}(\Omega, w)}. \quad (5.47)$$

Now, we prove that u is a weak solution of (1.1). Indeed, from the fact that u_k is a weak solution of (5.45), it follows that, for any $k \in \mathbb{N}$ and $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega} [A(x) \nabla u_k(x)] \cdot \nabla \varphi(x) \, dx = \int_{\Omega} w(x) \vec{\phi}_k(x) \cdot \nabla \varphi(x) \, dx.$$

By this, (5.46), the Hölder inequality, (1.2), $w \in A_{p^-}(\mathbb{R}^n)$, and taking $k \rightarrow \infty$, we find that, for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega} [A(x) \nabla u(x)] \cdot \nabla \varphi(x) \, dx = \int_{\Omega} \vec{F}(x) \cdot \nabla \varphi(x) \, dx.$$

This shows that $u \in W_0^{1,p^-}(\Omega, w)$ is a weak solution of (1.1).

To complete the proof of Theorem 1.7, we need to prove that the above weak solution u is unique. Indeed, if $u, v \in W_0^{1,p^-}(\Omega, w)$ are two weak solutions of (1.1), then, by the linearity of (1.1), we know that $u - v$ is a weak solution of

$$\begin{cases} \operatorname{div}(A \nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

From (5.47), we deduce that $\|\nabla(u - v)\|_{L^{p(\cdot)}(\Omega, w)} = 0$. By this, Remark 1.2(iii), and Lemma 2.8, we conclude that $\|u - v\|_{L^{p^-}(\Omega, w)} = 0$. Thus, for almost every $x \in \Omega$, $u(x) = v(x)$. This shows the uniqueness of the weak solution u of (1.1) and hence finishes the proof of Theorem 1.7. $\square$

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