Research Article

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On non-resistive limit of 1D MHD equations with no vacuum at infinity

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Abstract: In this paper, the Cauchy problem for the one-dimensional compressible isentropic magnetohydrodynamic (MHD) equations with no vacuum at infinity is considered, but the initial vacuum can be permitted inside the region. By deriving a priori ν (resistivity coefficient)-independent estimates, we establish the non-resistive limit of the global strong solutions with large initial data. Moreover, as a by-product, the global well-posedness of strong solutions for the compressible resistive MHD equations is also established.

Keywords: 1D compressible MHD equations; Cauchy problem; global strong solutions; non-resistive limit

MSC: 35D35; 35Q35; 76N10; 76W05

1 Introduction and Main Results

Compressible magnetohydrodynamics (MHD) is used to describe the macroscopic behavior of the electrically conducting fluid in a magnetic field. The application of MHD has a very wide of physical objects from liquid metals to cosmic plasmas. The system of the resistive MHD equations has the form:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho) + \frac{1}{2}b^2)_x = (\mu u_x)_x, \\ b_t + (ub)_x = \nu b_{xx}, \end{cases} \quad (1.1)$$

in $\mathbb{R} \times [0, \infty)$. Here, ρ , u , $P(\rho)$ and b denote the density, velocity, pressure and magnetic field, respectively. $\mu > 0$ is the viscosity coefficient, the constant $\nu > 0$ is the resistivity coefficient acting as the magnetic diffusion coefficient of the magnetic field. In this paper, we consider the isentropic compressible MHD equations in which the equation of the state has the form

$$P(\rho) = R\rho^\gamma, \quad \gamma > 1.$$

For simplicity, we set $R = 1$. We focus on the initial condition:

$$(\rho, u, b)|_{t=0} = (\rho_0(x), u_0(x), b_0(x)) \rightarrow (\bar{\rho}, 0, \bar{b}) \text{ as } |x| \rightarrow +\infty, \quad (1.2)$$

where $\bar{\rho}$ and $\bar{b}$ are both non-zero constants. Note that we can always normalize $\bar{\rho}$ such that $\bar{\rho} \geq 1$.

However, it is well known that the resistivity coefficient ν is inversely proportional to the electrical conductivity, therefore it is more reasonable to ignore the magnetic diffusion which means $\nu = 0$, when the

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conducting fluid considered is of highly conductivity, for example the ideal conductors. So instead of equations (1.1), when there is no resistivity, the system reduces to the so called compressible, isentropic, viscous and non-resistive MHD equations:

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u})_x = 0, \\ (\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + P(\tilde{\rho}) + \frac{1}{2}\tilde{b}^2)_x = (\mu\tilde{u}_x)_x, \\ \tilde{b}_t + (\tilde{u}\tilde{b})_x = 0, \end{cases} \quad (1.3)$$

in $\mathbb{R} \times [0, +\infty)$, with the following initial condition:

$$(\tilde{\rho}, \tilde{u}, \tilde{b})|_{t=0} = (\tilde{\rho}_0, \tilde{u}_0, \tilde{b}_0) \rightarrow (\tilde{\rho}, 0, \tilde{b}) \text{ as } |x| \rightarrow +\infty. \quad (1.4)$$

Because of the tight interaction between the dynamic motion and the magnetic field, the presence of strong nonlinearities, rich phenomena and mathematical challenges, many physicists and mathematicians are attracted to study in this field. Before stating our main theorems, we briefly recall some previous known results on compressible MHD equations. Firstly, we begin with the MHD equations with magnetic diffusion. For one-dimensional case, Vol'pert-Hudjaev [22] proved the local existence and uniqueness of strong solutions to the Cauchy problem and Kawashima-Okada [13] obtained the global smooth solutions with small initial data. For large initial data and the density containing vacuum, Ye-Li [27] proved the global existence of strong solutions to the 1D Cauchy problem with vacuum at infinity. When considering the full MHD equations and the heat conductivity depends on the temperature θ , Chen-Wang [3] studied the free boundary value problem and established the existence, uniqueness and Lipschitz dependence of strong solutions. Recently, Fan-Huang-Li [6] obtained the global strong solutions to the initial boundary value problem to the planner MHD equations with temperature-dependent heat conductivity. Later, with the effect of self-gravitation as well as the influence of radiation on the dynamics at high temperature regimes taken into account, Zhang-Xie [29] obtained the global strong solutions to the initial boundary value problem for the nonlinear planner MHD equations. For multi-dimensional MHD equations, Lv-Shi-Xu [19] considered the 2-D isentropic MHD equations and proved the global existence of classical solutions provided that the initial energy is small, where the decay rates of the solutions were also obtained. Vol'pert-Hudjaev [22] and Fan-Yu [5] obtained the local classical solution to the 3-D compressible MHD equations with the initial density is strictly positive or could contain vacuum, respectively. Hu-Wang [9] derived the global weak solutions to the 3-D compressible MHD equations with large initial data. Recently, Li-Xu-Zhang [14] established the global existence of classical solution of 3-D MHD equations with small energy but possibly large oscillations. Later, the result was improved by Hong-Hou-Peng-Zhu [8] just provided $((\gamma-1)^{\frac{1}{\gamma}} + \nu^{-\frac{1}{\gamma}})E_0$ is suitably small. When the resistivity is zero, then the magnetic equation is reduced from the heat-type equation to the hyperbolic-type equation, the problem becomes more challenging, hence the results are few. Kawashima [12] obtained the classical solutions to 3-D MHD equations when the initial data are of small perturbations in H^3 -norm and away from vacuum. Xu-Zhang [23] proved a blow-up criterion of strong solutions for 3-D isentropic MHD equations with vacuum. Fan-Hu [4] established the global strong solutions to the initial boundary value problem of 1-D heat-conducting MHD equations. With more general heat-conductivity, Zhang-Zhao [31] established the global strong solutions and also obtained the non-resistivity limits of the solutions in L^2 -norm. Li-Jiang [17] obtained the global solutions to the Cauchy problem of 1D heat-conductive MHD equations of viscous non-resistive gas under the frame work of Lagrangian coordinates. Jiang-Zhang [10] obtained the non-resistive limit of the strong solution and the "magnetic boundary layer" estimates to the initial boundary value problem of 1-D isentropic MHD equations as the resistivity $\nu \rightarrow 0$. Yu [28] obtained the global existence of strong solutions to the initial boundary value problem of 1-D isentropic MHD equations. For the Cauchy problem with large initial data and vacuum, Li-Wang-Ye [16] established the global well-posedness of strong solutions to the 1D isentropic MHD equations with vacuum at infinity, that is $\bar{\rho} = 0$. However, for the Cauchy problem with no vacuum at infinity, the global well-posedness of strong solutions and the non-resistive limits when the resistivity coefficient $\nu \rightarrow 0$ are still unknown. The goal of this paper is trying to answer these problems.

Now we give some comments on the analysis of this paper. The non-resistive limit of global strong solutions to 1D MHD equations (1.1)-(1.2) can be obtained by global uniform a priori estimates which are independent of resistivity ν . Thus, to obtain the a priori ν (resistivity coefficient)-independent estimates, some of

the main new difficulties will be encountered due to the initial density and initial magnetic which approach non-zero constants at infinity.

It turns out that the key issue in this paper is to derive upper bound for the density, magnetic field and the time-dependent higher norm estimates which are independent of resistivity ν . This is achieved by modifying upper bound estimate for the density developed in [25] and [27] in the theory of Cauchy problem with vacuum and initial magnetic field approaching zero at infinity. However, in comparison with the Cauchy problem with vacuum at infinity in [25] and [16], some new difficulties will be encountered. The first difficulty lies in no integrability for the density itself just from the elementary energy estimate (see Lemma 3.1), which is required when deriving the upper bound of the density. To overcome this difficulty, we use the technique of mathematical frequency decomposition to divide the momentum ξ into two parts:

$$\xi = \int_{-\infty}^x \rho u dy = \int_{-\infty}^x (\sqrt{\rho} - \sqrt{\bar{\rho}}) \sqrt{\rho} u dy + \sqrt{\bar{\rho}} \int_{-\infty}^x \sqrt{\rho} u dy = \xi_1 + \xi_2.$$

It is crucial to obtain the upper bound of ξ_1 and ξ_2 . For getting $\|\xi_1\|_{L^\infty}$, we use the technique of mathematical frequency decomposition to get the estimate of $\|\sqrt{\rho} - \sqrt{\bar{\rho}}\|_{L^2}$ by the elementary energy estimates, and then using Hölder's inequality, we can obtain the upper bound of ξ_1 (see (3.30)). For obtaining $\|\xi_2\|_{L^\infty}$, due to the Sobolev embedding theory, we need some $L^{\tilde{p}}$ integrability of ξ_2 . However, we can not obtain it just from $\|\xi_{2x}\|_{L^2}$ directly, because the Poincaré type inequality is no longer valid in the whole space $\mathbb{R}$. To overcome this difficulty, we use the Caffarelli-Kohn-Nirenberg weighted inequality and the Gagliardo-Nirenberg inequality to obtain the upper bound of ξ_2 (see (3.31)). It is worth noting that the non-resistive MHD equation (1.3) looks similar to the compressible model for gas and liquid two-phase fluids. Hence, in some previous results, it is technically assumed that $\tilde{\rho}, \tilde{b} \geq 0$ and $0 \leq \frac{\tilde{b}}{\tilde{\rho}} < \infty$ in (1.3), which implies the magnetic field $\tilde{b}$ is bounded provided the density $\tilde{\rho}$ is bounded. However, this is not physical and realistic in magnetohydrodynamics. Moreover, compared with that for the Cauchy problem with vacuum at infinity in [16], since the magnetic field $\tilde{b} \rightarrow \bar{b}$, as $|x| \rightarrow +\infty$, the method of getting the upper bound of magnetic field b in [16] can not be used here anymore. So another difficulty is how to get the uniform (independent of ν) upper bound of the magnetic field b without the assumption similar as that in two-phase fluids. To overcome this difficulty, we will make full use of the structure of the momentum equation and effective viscous flux (see Lemma 3.4 and Lemma 3.5).

Notations: We denote the material derivative of u and effective viscous flux by

$$\dot{u} \triangleq u_t + uu_x \text{ and } F \triangleq \mu u_x - \left(P(\rho) - P(\bar{\rho}) + \frac{b^2 - \bar{b}^2}{2} \right),$$

and define potential energy by

$$\Phi(\rho) = \rho \int_{\bar{\rho}}^{\rho} \frac{P(s) - P(\bar{\rho})}{s^2} ds = \frac{1}{\gamma - 1} \left(\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1} (\rho - \bar{\rho}) \right).$$

Sometimes we write $\int_{\mathbb{R}} f(x) dx$ as $\int f(x)$ for simplicity.

Now, we state the main result of this paper.

Theorem 1.1. Suppose that the initial data $(\rho_0, u_0, b_0)(x)$ satisfies

$$\begin{aligned} \rho_0 - \bar{\rho} &\in H^1(\mathbb{R}), \quad b_0 - \bar{b} \in H^1(\mathbb{R}), \quad u_0 \in H^2(\mathbb{R}), \\ \left(\frac{1}{2} \rho_0 u_0^2 + \Phi(\rho_0) + \frac{(b_0 - \bar{b})^2}{2} \right) |x|^\alpha &\in L^1(\mathbb{R}) \end{aligned} \quad (1.5)$$

for some $\alpha \in (1, 2]$, and the compatibility condition

$$\left(\mu u_{0x} - P(\rho_0) - \frac{1}{2} b_0^2 \right)_x = \sqrt{\rho_0} g(x), \quad x \in \mathbb{R} \quad (1.6)$$

with some $g \in L^2(\mathbb{R})$. Then for each fixed $\nu > 0$, there exist a positive constant C and a unique global strong solution (ρ, u, b) to the Cauchy problem (1.1)-(1.2) such that

$$0 \leq \rho(x, t) \leq C, \quad \forall (x, t) \in \mathbb{R} \times [0, T], \quad (1.7)$$

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left\| \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \tilde{b})^2}{2} \right) (1 + |x|^a) \right\|_{L^1} (t) \\ & + \int_0^T \left(\mu \left\| u_x (1 + |x|^{\frac{a}{2}}) \right\|_{L^2}^2 + \nu \left\| b_x (1 + |x|^{\frac{a}{2}}) \right\|_{L^2}^2 \right) dt \leq C, \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|u\|_{L^2} + \|u_{xx}\|_{L^2} + \|u_x\|_{L^2} + \|b_x\|_{L^2} + \|\rho_x\|_{L^2} + \|\sqrt{\rho} \dot{u}\|_{L^2}) (t) \\ & + \int_0^T (\|u_{xt}\|_{L^2}^2 + \nu \|b_{xx}\|_{L^2}^2) dt \leq C. \end{aligned} \quad (1.9)$$

Moreover, as $\nu \rightarrow 0$, we have

$$\begin{cases} (\rho, u, b) \rightarrow (\tilde{\rho}, \tilde{u}, \tilde{b}) \text{ strongly in } L^\infty(0, T; L^2), \\ \nu b_x \rightarrow 0, \quad u_x \rightarrow \tilde{u}_x \text{ strongly in } L^2(0, T; L^2), \end{cases} \quad (1.10)$$

and

$$\sup_{0 \leq t \leq T} (\|\rho - \tilde{\rho}\|_{L^2}^2 + \|u - \tilde{u}\|_{L^2}^2 + \|b - \tilde{b}\|_{L^2}^2) + \int_0^T \mu \|(u - \tilde{u})_x\|_{L^2}^2 dt \leq C\nu, \quad (1.11)$$

where C is a positive constant independent of ν .

Remark 1.1. In Theorem 1.1, we do not need the artificial assumption similarly as that in two-phase fluids. Moreover, if ignoring the magnetic field, then MHD system reduces to the compressible Navier-Stokes equations. So, Theorem 1.1 can be seen as an extension of that in [25].

Remark 1.2. In Theorem 1.1, we give that the global strong solution of resistive MHD equation (1.1)-(1.2) converges to that of non-resistive MHD equation (1.3)-(1.4) in L^2 -norm as $\nu \rightarrow 0$, moreover, the convergence rates are also justified.

The rest of the paper is organized as follows. In Section 2, we recall some preliminary lemmas which will be used later. Section 3 is devoted to establishing global ν -independent estimates for (1.1) and (1.2), which will be used to justify the non-resistive limit. Section 4 is devoted to proving Theorem 1.1.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

Lemma 2.1 (Gagliardo-Nirenberg inequality [7, 20]). For any $f \in W^{1,m}(\mathbb{R}) \cap L^r(\mathbb{R})$, there exists some generic constant $C > 0$ which may depend on q, r such that

$$\|f\|_{L^q} \leq C \|f\|_{L^r}^{1-\theta} \|\nabla f\|_{L^m}^\theta, \quad (2.1)$$

where $\theta = (\frac{1}{r} - \frac{1}{q})(\frac{1}{r} - \frac{1}{m} + 1)^{-1}$, if $m = 1$, then $q \in [r, \infty)$, if $m > 1$, then $q \in [r, \infty]$.

The following Caffarelli-Kohn-Nirenberg weighted inequality is the key to deal with the Cauchy problem in this paper.

Lemma 2.2 (Caffarelli-Kohn-Nirenberg weighted inequality [2]). $\forall h \in C_0^\infty(\mathbb{R})$, it holds that

$$\| |x|^\kappa h \|_r \leq C \| |x|^\alpha |\partial_x h| \|_p^\theta \| |x|^\beta h \|_q^{1-\theta} \quad (2.2)$$

where $1 \leq p, q < \infty$, $0 < r < \infty$, $0 \leq \theta \leq 1$, $\frac{1}{p} + \alpha > 0$, $\frac{1}{q} + \beta > 0$, $\frac{1}{r} + \kappa > 0$ and satisfy

$$\frac{1}{r} + \kappa = \theta \left(\frac{1}{p} + \alpha - 1 \right) + (1 - \theta) \left(\frac{1}{q} + \beta \right), \quad (2.3)$$

and

$$\kappa = \theta \sigma + (1 - \theta) \beta$$

with $0 \leq \alpha - \sigma$ if $\theta > 0$ and $0 \leq \alpha - \sigma \leq 1$ if $\theta > 0$ and $\frac{1}{p} + \alpha - 1 = \frac{1}{r} + \kappa$.

Proof. The proof can be found in [2]. Here, we omit the details. $\square$

Remark 2.1. The lemma 2.2 is also valid for any $h \in D_{\alpha, \beta}^{p, q}(\mathbb{R})$, where $D_{\alpha, \beta}^{p, q}(\mathbb{R})$ is the completion of the space of smooth compactly supported functions with the norm $\left(\int_{\mathbb{R}} (|x|^\alpha |\partial_x h|)^p dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} (|x|^\beta |h|)^q dx \right)^{\frac{1}{q}}$.

By direct calculation, the potential energy $\Phi(\rho)$ has the following properties:

Lemma 2.3. Observing the function of the potential energy $\Phi(\rho)$, we will easily find the following properties for positive constants c_1, c_2, C_1, C_2 :

- (1) if $0 \leq \rho \leq 2\bar{\rho}$, $c_1(\rho - \bar{\rho})^2 \leq \Phi(\rho) \leq c_2(\rho - \bar{\rho})^2$;
- (2) if $\rho > 2\bar{\rho}$, $\rho^\gamma - \bar{\rho}^\gamma \leq C_1(\rho - \bar{\rho})^\gamma \leq C_2\Phi(\rho)$.

3 Global v -independent estimates for (1.1) and (1.2)

The main purpose of this section is to derive the global v -independent a priori estimates of the solutions (ρ, u, b) to the system (1.1) and (1.2), which is used to justify the non-resistive limit. To do this, before going any further, we first let the initial density have lower bound $\delta > 0$ and get the global v -independent a priori estimates of the smooth solutions (ρ, u, b) , which is uniform of δ . Then taking $\delta \rightarrow 0^+$, we will get what we want.

Due to the initial density approaches no vacuum at infinity $\lim_{|x| \rightarrow +\infty} \rho_0(x) = \bar{\rho} > 0$, then there exists a large number $M > 0$ such that if $|x| \geq M$, $\rho_0(x) \geq \frac{\bar{\rho}}{2}$. For any $0 < \delta < \frac{\bar{\rho}}{2}$, we define

$$\rho_0^\delta(x) = \begin{cases} \rho_0(x) + \delta, & \text{if } |x| \leq M, \\ \rho_0(x) + \delta s(x), & \text{if } M \leq |x| \leq M + 1, \\ \rho_0(x), & \text{if } |x| \geq M + 1, \end{cases} \quad (3.1)$$

where $s(x) = s(|x|)$ is a smooth and decreasing function satisfying $s(x) = 1$ if $|x| \leq M$ and $s(x) = 0$ if $|x| \geq M + 1$. Clearly, we have

$$(\rho_0^\delta - \bar{\rho}, P(\rho_0^\delta) - P(\bar{\rho})) \rightarrow (\rho_0 - \bar{\rho}, P(\rho_0) - P(\bar{\rho})) \text{ in } H^1(\mathbb{R}),$$

and

$$\Phi(\rho_0^\delta)(1 + |x|^\alpha) \rightarrow \Phi(\rho_0)(1 + |x|^\alpha) \text{ in } L^1(\mathbb{R}), \text{ as } \delta \rightarrow 0^+.$$

To approximate the initial velocity, we define u_0^δ as

$$u_0^\delta = \begin{cases} \tilde{u}_0^\delta, & |x| < M+1, \\ u_0, & |x| \geq M+1, \end{cases} \quad (3.2)$$

where $\tilde{u}_0^\delta$ is the unique solution to the following elliptic equation:

$$\begin{cases} (\mu \tilde{u}_{0x}^\delta)_x = P(\rho_0^\delta)_x + \frac{1}{2}(b_0^2)_x + \sqrt{\rho_0}g(x), & \text{in } \Omega_M := \{x \mid |x| < M+1\}, \\ \tilde{u}_0^\delta|_{|x|=M+1} = u_0. \end{cases} \quad (3.3)$$

Since $(\rho_0^\delta - \bar{\rho}, P(\rho_0^\delta) - P(\bar{\rho})) \in H^1(\mathbb{R})$, $b_0 \in H^1(\mathbb{R})$ and $\sqrt{\rho_0}g \in L^2(\mathbb{R})$, by the elliptic theory, (1.6) and (3.3), we have

$$\begin{aligned} \|\tilde{u}_0^\delta\|_{H^2(\Omega_M)} &\leq C \left(\|P(\rho_0^\delta)_x\|_{L^2} + \|(b_0^2)_x\|_{L^2} + \|\sqrt{\rho_0}g\|_{L^2} + 1 \right) \\ &\leq C \left(\|P(\rho_0^\delta)_x\|_{L^2} + \|b_0\|_{L^\infty} \|b_{0x}\|_{L^2} + \|\sqrt{\rho_0}\|_{L^\infty} \|g\|_{L^2} + 1 \right) \leq C. \end{aligned} \quad (3.4)$$

From the compatibility conditions (1.6) and (3.3), it follows that

$$\begin{cases} (\mu(\tilde{u}_0^\delta - u_0)_x)_x = (P(\rho_0^\delta) - P(\rho_0))_x, & \text{in } \Omega_M, \\ (\tilde{u}_0^\delta - u_0)|_{|x|=M+1} = 0, \end{cases} \quad (3.5)$$

which yields

$$\tilde{u}_0^\delta - u_0 \in H_0^1(\Omega_M) \cap H^2(\Omega_M),$$

and

$$\begin{aligned} \|\tilde{u}_0^\delta - u_0\|_{H^2(\Omega_M)} &\leq C \| (P(\rho_0^\delta) - P(\rho_0))_x \|_{L^2} \\ &\leq C\delta \rightarrow 0, \quad \text{as } \delta \rightarrow 0^+. \end{aligned} \quad (3.6)$$

Furthermore

$$\sqrt{\rho_0^\delta} u_0^\delta (1 + |x|^{\frac{\gamma}{2}}) \rightarrow \sqrt{\rho_0} u_0 (1 + |x|^{\frac{\gamma}{2}}) \text{ in } L^2(\mathbb{R}), \text{ as } \delta \rightarrow 0^+.$$

For any $T \in (0, \infty)$, let $(\rho^\delta, u^\delta, b^\delta)$ with positive ρ^δ be the smooth solution to (1.1)-(1.2). Without confusion, we still denote the solution by (ρ, u, b) instead of $(\rho^\delta, u^\delta, b^\delta)$ to simplify the presentation. Throughout this section, we denote C to be a generic constant which is uniform of δ for any $\delta \in (0, 1)$ and may depend on $\rho_0, u_0, b_0, \gamma, \mu, T$ and some other known constants but independent of v .

First of all, we can prove the following elementary energy estimates.

Lemma 3.1. *Let (ρ, u, b) be a smooth solution of (1.1) and (1.2). Then for any $T > 0$, it holds that*

$$\sup_{0 \leq t \leq T} \left\| \frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right\|_{L^1(\mathbb{R})} + \int_0^T \left(\mu \|u_x\|_{L^2}^2 + v \|b_x\|_{L^2}^2 \right) dt \leq C. \quad (3.7)$$

Proof. By the definition of the potential energy Φ , and using (1.1)₁, we deduce

$$\Phi_t + (u\Phi)_x + (\rho^\gamma - \bar{\rho}^\gamma)u_x = 0. \quad (3.8)$$

Adding the equation (1.1)₂ multiplied by u , (1.1)₃ multiplied by $b - \bar{b}$ into (3.8), and then integrating the resulting equation over $\mathbb{R} \times [0, T]$ with respect to the variables x and t , we have

$$\begin{aligned} &\int \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right) dx + \int_0^T \left(\mu \|u_x\|_{L^2}^2 + v \|b_x\|_{L^2}^2 \right) dt \\ &\leq \int \left(\frac{1}{2} \rho_0 u^2 + \Phi(\rho_0) + \frac{(b_0 - \bar{b})^2}{2} \right) dx \\ &\leq C \|\rho_0\|_{L^\infty} \|u_0\|_{L^2}^2 + C (\|\rho_0 - \bar{\rho}\|_{L^2}^2 + \|b_0 - \bar{b}\|_{L^2}^2) \\ &\leq C (\|\rho_0 - \bar{\rho}\|_{H^1}^2 + \bar{\rho}) (\|u_0\|_{L^2}^2 + 1) + C \|b_0 - \bar{b}\|_{L^2}^2 \\ &\leq C. \end{aligned}$$

Then the proof of Lemma 3.1 is completed. $\square$

To obtain the upper bound of the density ρ , we need the following weighted energy estimates.

Lemma 3.2. *Let (ρ, u, b) be a smooth solution of (1.1) and (1.2). Then for any $T > 0$ and some index $1 < \alpha \leq 2$, it holds that*

$$\sup_{0 \leq t \leq T} \left\| \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right) |x|^\alpha \right\|_{L^1(\mathbb{R})} + \int_0^T \left(\mu \|u_x |x|^{\frac{\alpha}{2}}\|_{L^2}^2 + \nu \|b_x |x|^{\frac{\alpha}{2}}\|_{L^2}^2 \right) dt \leq C. \quad (3.9)$$

Proof. Multiplying the equation (1.1)₂ by $u|x|^\alpha$ and integrating the resulting equation over $\mathbb{R}$ with respect to x , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho u^2 |x|^\alpha + \mu \int u_x^2 |x|^\alpha \\ &= \frac{1}{2} \int \rho u^3 \alpha |x|^{\alpha-2} x - \mu \int \alpha |x|^{\alpha-2} x u u_x - \int \left(P(\rho) + \frac{b^2}{2} \right)_x u |x|^\alpha. \end{aligned} \quad (3.10)$$

It follows from the integration by parts that

$$\begin{aligned} & - \int \left(P(\rho) + \frac{b^2}{2} \right)_x u |x|^\alpha \\ &= - \int \left((P(\rho) - P(\bar{\rho}))_x + b b_x \right) u |x|^\alpha \\ &= - \int \left((P(\rho) - P(\bar{\rho}))_x + (b - \bar{b})(b - \bar{b})_x + \bar{b}(b - \bar{b})_x \right) u |x|^\alpha \\ &= \int \left(P(\rho) - P(\bar{\rho}) + \frac{(b - \bar{b})^2}{2} + \bar{b}(b - \bar{b}) \right) \left(u_x |x|^\alpha + u \alpha |x|^{\alpha-2} x \right). \end{aligned} \quad (3.11)$$

Then, combining (3.10) and (3.11) gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho u^2 |x|^\alpha + \mu \int u_x^2 |x|^\alpha = \frac{1}{2} \int \rho u^3 \alpha |x|^{\alpha-2} x - \mu \int \alpha |x|^{\alpha-2} x u u_x \\ &+ \int \left(P(\rho) - P(\bar{\rho}) + \frac{(b - \bar{b})^2}{2} + \bar{b}(b - \bar{b}) \right) \left(u_x |x|^\alpha + u \alpha |x|^{\alpha-2} x \right). \end{aligned} \quad (3.12)$$

To deal with the last term on the right-hand side of (3.12), first, multiplying (3.8) by $|x|^\alpha$ and integrating over $\mathbb{R}$ with respect to x , yields

$$\frac{d}{dt} \int \Phi(\rho) |x|^\alpha - \int u \Phi(\rho) \alpha |x|^{\alpha-2} x + \int (\rho^\gamma - \bar{\rho}^\gamma) |x|^\alpha u_x = 0, \quad (3.13)$$

and then multiplying the equation (1.1)₃ by $(b - \bar{b})|x|^\alpha$ and integrating over $\mathbb{R}$ with respect to x , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (b - \bar{b})^2 |x|^\alpha + \frac{1}{2} \int \left((b - \bar{b})^2 \right)_x u |x|^\alpha + \int (b - \bar{b})^2 u_x |x|^\alpha + \bar{b} \int (b - \bar{b}) u_x |x|^\alpha \\ &= \nu \int b_{xx} (b - \bar{b}) |x|^\alpha, \end{aligned}$$

which together with the integration by parts implies that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (b - \bar{b})^2 |x|^\alpha + \nu \int b_x^2 |x|^\alpha = - \int \frac{(b - \bar{b})^2}{2} (u_x |x|^\alpha - u \alpha |x|^{\alpha-2} x) \\ & - \bar{b} \int (b - \bar{b}) u_x |x|^\alpha - \nu \int b_x (b - \bar{b}) \alpha |x|^{\alpha-2} x. \end{aligned} \quad (3.14)$$

Now, putting (3.13) and (3.14) into (3.12), we have

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right) |x|^\alpha dx + \mu \int u_x^2 |x|^\alpha + \nu \int b_x^2 |x|^\alpha \\ &= \int \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + (b - \bar{b})^2 \right) u \alpha |x|^{\alpha-2} x + \int (\rho^\gamma - \bar{\rho}^\gamma) u \alpha |x|^{\alpha-2} x \\ &\quad - \mu \int \alpha |x|^{\alpha-2} x u u_x + \bar{b} \int (b - \bar{b}) u \alpha |x|^{\alpha-2} x - \nu \int (b - \bar{b}) b_x \alpha |x|^{\alpha-2} x \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (3.15)$$

Next, we estimate the terms I_1 - I_5 as follows:

$$\begin{aligned} I_1 &\leq \int \left(\left| \frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right| \right) |u| |x|^{\alpha-1} \\ &\leq \int \left(\left| \frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right| |x|^\alpha \right)^{\frac{\alpha-1}{\alpha}} \left(\left| \frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right| \right)^{\frac{1}{\alpha}} |u| \\ &\leq C \|u\|_{L^\infty} \left\| \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right) |x|^\alpha \right\|_{L^1}^{1-\frac{1}{\alpha}} \left\| \frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right\|_{L^1}^{\frac{1}{\alpha}} \\ &\leq C(1 + \|u_x\|_{L^2}) \left(\left\| \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right) |x|^\alpha \right\|_{L^1} + 1 \right), \end{aligned} \quad (3.16)$$

where $\alpha > 1$ and we have used Hölder's inequality, Lemma 3.1 and the following facts:

$$\begin{aligned} \bar{\rho} \int u^2 &= \int (\bar{\rho} - \rho + \rho) u^2 \\ &\leq \int ((\bar{\rho} - \rho)|_{\{0 \leq \rho \leq 2\bar{\rho}\}} + (\bar{\rho} - \rho)|_{\{\rho > 2\bar{\rho}\}}) u^2 + \int \rho u^2 \\ &\leq C \left(\|(\rho - \bar{\rho})|_{\{0 \leq \rho \leq 2\bar{\rho}\}}\|_{L^2} \|u\|_{L^4}^2 + \|(\rho - \bar{\rho})|_{\{\rho > 2\bar{\rho}\}}\|_{L^\gamma} \|u\|_{L^{\frac{2\gamma}{\gamma-1}}}^2 + 1 \right) \\ &\leq C \left[\|\Phi(\rho)\|_{L^1}^{\frac{1}{2}} \left(\|u\|_{L^2}^{\frac{3}{4}} \|u_x\|_{L^2}^{\frac{1}{4}} \right)^2 + \|\Phi(\rho)\|_{L^1}^{\frac{1}{\gamma}} \left(\|u\|_{L^2}^{1-\frac{1}{2\gamma}} \|u_x\|_{L^2}^{\frac{1}{2\gamma}} \right)^2 + 1 \right] \\ &\leq \varepsilon \|u\|_{L^2}^2 + C \left(1 + \|u_x\|_{L^2}^2 \right), \end{aligned}$$

where we have used the properties of the potential energy Φ in Lemma 2.3. This together with the Gagliardo-Nirenberg inequality implies that

$$\|u\|_{L^2}^2 \leq C(1 + \|u_x\|_{L^2}^2), \quad (3.17)$$

and

$$\|u\|_{L^\infty} \leq C \|u\|_{L^2}^{\frac{1}{2}} \|u_x\|_{L^2}^{\frac{1}{2}} \leq C(1 + \|u_x\|_{L^2}). \quad (3.18)$$

Applying the property of $\Phi(\rho)$, Hölder's inequality, the Caffarelli-Kohn-Nirenberg weighted inequality (2.3), Lemma 3.1, Young's inequality and (3.18), we obtain

$$\begin{aligned} I_2 &= \int ((\rho^\gamma - \bar{\rho}^\gamma)|_{\{0 \leq \rho \leq 2\bar{\rho}\}} + (\rho^\gamma - \bar{\rho}^\gamma)|_{\{\rho > 2\bar{\rho}\}}) u \alpha |x|^{\alpha-2} x \\ &\leq C \int (|\rho - \bar{\rho}|_{\{0 \leq \rho \leq 2\bar{\rho}\}} + (\rho - \bar{\rho})^\gamma|_{\{\rho > 2\bar{\rho}\}}) |u| |x|^{\alpha-1} \\ &\leq C \int \left(\Phi^{\frac{1}{2}}(\rho)|_{\{0 \leq \rho \leq 2\bar{\rho}\}} + \Phi(\rho)|_{\{\rho > 2\bar{\rho}\}} \right) |u| |x|^{\alpha-1} \\ &\leq C \int \left(\Phi^{\frac{1}{2}}(\rho)|_{\{0 \leq \rho \leq 2\bar{\rho}\}} |x|^{\frac{\alpha}{2}} \right) |u| |x|^{\frac{\alpha}{2}-1} + C \int (\Phi(\rho)|_{\{\rho > 2\bar{\rho}\}} |x|^\alpha)^{\frac{\alpha-1}{\alpha}} \Phi^{\frac{1}{\alpha}}(\rho) |u| \\ &\leq C \left(\|\Phi(\rho)|x|^\alpha\|_{L^1}^{\frac{1}{2}} \left\| |x|^{\frac{\alpha}{2}-1} u \right\|_{L^2} + \|\Phi(\rho)|x|^\alpha\|_{L^1}^{1-\frac{1}{\alpha}} \|\Phi(\rho)\|_{L^1}^{\frac{1}{\alpha}} \|u\|_{L^\infty} \right) \\ &\leq C(1 + \|u_x\|_{L^2}^2)(1 + \|\Phi(\rho)|x|^\alpha\|_{L^1}), \end{aligned} \quad (3.19)$$

where we have used the fact: for $1 < \alpha \leq 2$,

$$\begin{aligned} \| |x|^{\frac{\alpha}{2}-1} u \|_{L^2} &\leq C \|u\|_{L^2}^{\frac{\alpha}{2}} \|u_x\|_{L^2}^{1-\frac{\alpha}{2}} \\ &\leq C (\|u\|_{L^2} + \|u_x\|_{L^2}) \\ &\leq C(1 + \|u_x\|_{L^2}). \end{aligned} \quad (3.20)$$

Similarly, using the Caffarelli-Kohn-Nirenberg weighted inequality (2.3), we have

$$\begin{aligned} I_3 &\leq C \| |x|^{\frac{\alpha}{2}} u_x \|_{L^2} \| |x|^{\frac{\alpha}{2}-1} u \|_{L^2} \\ &\leq \varepsilon \| |x|^{\frac{\alpha}{2}} u_x \|_{L^2}^2 + C \left(1 + \|u_x\|_{L^2}^2 \right). \end{aligned} \quad (3.21)$$

For the term I_4 , it follows from the Hölder's inequality and the Caffarelli-Kohn-Nirenberg weighted inequality that

$$\begin{aligned} I_4 &\leq C \left\| (b - \bar{b}) |x|^{\frac{\alpha}{2}} \right\|_{L^2} \left\| |x|^{\frac{\alpha}{2}-1} u \right\|_{L^2} \\ &\leq C \left(\left\| (b - \bar{b}) |x|^{\frac{\alpha}{2}} \right\|_{L^2}^2 + \|u_x\|_{L^2}^2 + 1 \right). \end{aligned} \quad (3.22)$$

Similarly, using the Caffarelli-Kohn-Nirenberg weighted inequality (2.3), one has

$$\begin{aligned} I_5 &\leq C v \| |x|^{\frac{\alpha}{2}} b_x \|_{L^2} \| (b - \bar{b}) |x|^{\frac{\alpha}{2}-1} \|_{L^2} \\ &\leq C v \| |x|^{\frac{\alpha}{2}} b_x \|_{L^2} \left(\|b - \bar{b}\|_{L^2}^{\frac{\alpha}{2}} \|b_x\|_{L^2}^{1-\frac{\alpha}{2}} \right) \\ &\leq \varepsilon v \| |x|^{\frac{\alpha}{2}} b_x \|_{L^2}^2 + C v \left(1 + \|b_x\|_{L^2}^2 \right), \end{aligned} \quad (3.23)$$

where $1 < \alpha \leq 2$.

Then, substituting (3.16), (3.19), (3.21), (3.22) and (3.23) into (3.15), choosing $\varepsilon > 0$ small enough, one deduces

$$\begin{aligned} \frac{d}{dt} \int \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right) |x|^\alpha + \mu \int |x|^\alpha u_x^2 + v \int |x|^\alpha b_x^2 \\ \leq C(1 + \|u_x\|_{L^2}^2) \left\| \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right) |x|^\alpha \right\|_{L^1} + C(1 + \|u_x\|_{L^2}^2 + v \|b_x\|_{L^2}^2). \end{aligned}$$

This together with Gronwall's inequality, (1.5) and Lemma 3.1 gives

$$\int \left(\frac{1}{2} \rho u^2 + \Phi(\rho) + \frac{(b - \bar{b})^2}{2} \right) |x|^\alpha + \mu \int_0^T \int |x|^\alpha u_x^2 + v \int_0^T \int |x|^\alpha b_x^2 \leq C(T).$$

Then, the proof of Lemma 3.2 is completed. $\square$

The upper bound of the density ρ can be shown in the similar manner as that in [25]. However, for completeness of the paper, we give the details here.

Lemma 3.3. *Let (ρ, u, b) be a smooth solution of (1.1) and (1.2). Then for any $T > 0$, it holds that*

$$0 < \delta e^{-C(T)} \leq \rho(x, t) \leq C, \text{ for all } (x, t) \in \mathbb{R} \times [0, T],$$

and

$$\sup_{0 \leq t \leq T} \|\rho - \bar{\rho}\|_{L^2(\mathbb{R})} \leq C.$$

Proof. Let $\xi = \int_{-\infty}^x \rho u dy$, then the momentum equation (1.1)₂ can be rewritten as

$$\xi_{tx} + \left(\rho u^2 + P(\rho) - P(\bar{\rho}) + \frac{b^2 - \bar{b}^2}{2} \right)_x = (\mu u_x)_x.$$

Integrating the above equality with respect to x over $(-\infty, x)$ yields that

$$\xi_t + \rho u^2 + P(\rho) - P(\bar{\rho}) + \frac{b^2 - \bar{b}^2}{2} = \mu u_x, \quad (3.24)$$

which together with (1.1)₁ gives

$$\xi_t + \rho u^2 + P(\rho) - P(\bar{\rho}) + \frac{b^2 - \bar{b}^2}{2} + \mu \frac{\rho_t + u \rho_x}{\rho} = 0. \quad (3.25)$$

Next, we define the particle trajectory $X(x, t)$ as follows:

$$\begin{cases} \frac{dX(x, t)}{dt} = u(X(x, t), t), \\ X(x, 0) = x, \end{cases} \quad (3.26)$$

which implies

$$\frac{d\xi}{dt} = \xi_t + u \xi_x = \xi_t + \rho u^2.$$

Then combining the above equality and (3.25), we infer that

$$\frac{d}{dt} \left(\xi + \mu \ln \rho \right) (X(x, t), t) + \left(P(\rho) + \frac{b^2}{2} \right) (X(x, t), t) - \left(P(\bar{\rho}) + \frac{\bar{b}^2}{2} \right) = 0, \quad (3.27)$$

which together with $\left(P(\rho) + \frac{b^2}{2} \right) (X(x, t), t) \geq 0$ yields

$$\frac{d}{dt} \left(\xi + \mu \ln \rho \right) (X(x, t), t) \leq \left(P(\bar{\rho}) + \frac{\bar{b}^2}{2} \right) (X(x, t), t) \leq C. \quad (3.28)$$

Thus, integrating it over $[0, T]$ with respect to t , we have

$$(\xi + \mu \ln \rho) (X(x, t), t) \leq (\xi + \mu \ln \rho)(x, 0) + C.$$

By direct calculation, we obtain

$$\begin{aligned} \ln \rho &\leq \frac{1}{\mu} (\xi_0 + \mu \ln \rho_0 + C - \xi) \\ &\leq \frac{1}{\mu} \left(\xi_0 + \mu \ln \rho_0 + C - \int_{-\infty}^x \rho u dy \right) \\ &\leq \frac{1}{\mu} \left(\xi_0 + \mu \ln \rho_0 + C - \int_{-\infty}^x \sqrt{\rho} u (\sqrt{\rho} - \sqrt{\bar{\rho}}) dy - \sqrt{\bar{\rho}} \int_{-\infty}^x \sqrt{\rho} u dy \right) \\ &\leq \frac{1}{\mu} (\xi_0 + \mu \ln \rho_0 + C - \xi_1 - \xi_2). \end{aligned} \quad (3.29)$$

Firstly, it follows from Lemma 2.3 and Lemma 3.1 that $\|\xi_1\|_{L^\infty}$ can be estimated as

$$\begin{aligned} |\xi_1| &= \left| \int_{-\infty}^x \sqrt{\rho} u (\sqrt{\rho} - \sqrt{\bar{\rho}}) dy \right| \\ &\leq C \|\sqrt{\rho} u\|_{L^2} \left(\|(\sqrt{\rho} - \sqrt{\bar{\rho}})|_{\{0 \leq \rho \leq 2\bar{\rho}\}}\|_{L^2} + \|(\sqrt{\rho} - \sqrt{\bar{\rho}})|_{\{\rho > 2\bar{\rho}\}}\|_{L^2} \right) \\ &\leq C \|\sqrt{\rho} u\|_{L^2} \left(\|(\rho - \bar{\rho})|_{\{0 \leq \rho \leq 2\bar{\rho}\}}\|_{L^2} + \|\sqrt{\rho - \bar{\rho}}|_{\{\rho > 2\bar{\rho}\}}\|_{L^2} \right) \\ &\leq C \|\sqrt{\rho} u\|_{L^2} \left(\|(\rho - \bar{\rho})|_{\{0 \leq \rho \leq 2\bar{\rho}\}}\|_{L^2} + \|(\rho - \bar{\rho})|_{\{\rho > 2\bar{\rho}\}}\|_{L^1}^{\frac{1}{2}} \right) \\ &\leq C \|\sqrt{\rho} u\|_{L^2} \left(\|(\rho - \bar{\rho})|_{\{0 \leq \rho \leq 2\bar{\rho}\}}\|_{L^2} + \|(\rho - \bar{\rho})^\gamma|_{\{\rho > 2\bar{\rho}\}}\|_{L^1}^{\frac{1}{2}} \right) \\ &\leq C \|\sqrt{\rho} u\|_{L^2} \left(\|\Phi(\rho)\|_{L^1}^{\frac{1}{2}} + \|\Phi(\rho)\|_{L^1}^{\frac{1}{2}} \right) \leq C. \end{aligned} \quad (3.30)$$

Secondly, using the Gagliardo-Nirenberg inequality, the Caffarelli-Kohn-Nirenberg weighted inequality, Lemma 3.1 and Lemma 3.2, we can bound $\|\xi_2\|_{L^\infty}$ as

$$\begin{aligned}
 \|\xi_2\|_{L^\infty} &\leq C\|\xi_2\|_{L^{\frac{\tilde{p}}{\tilde{p}+2}}}^{\frac{\tilde{p}}{\tilde{p}+2}}\|\xi_{2x}\|_{L^2}^{\frac{2}{\tilde{p}+2}} \\
 &\leq C(\|\xi_{2x}\|_{L^2}^\eta\|x\|_{L^q}^\kappa\|\xi_2\|_{L^2}^{1-\eta})^{\frac{\tilde{p}}{\tilde{p}+2}}\|\xi_{2x}\|_{L^2}^{\frac{2}{\tilde{p}+2}} \\
 &\leq C(\|\xi_{2x}\|_{L^2}^\eta\|x\|_{L^2}^{\frac{\alpha}{2}}\|\xi_{2x}\|_{L^2}^{1-\eta})^{\frac{\tilde{p}}{\tilde{p}+2}}\|\xi_{2x}\|_{L^2}^{\frac{2}{\tilde{p}+2}} \\
 &\leq C\|\xi_{2x}\|_{L^2}^{1-(1-\eta)\frac{\tilde{p}}{\tilde{p}+2}}\|x\|_{L^2}^{\frac{\alpha}{2}}\|\xi_{2x}\|_{L^2}^{(1-\eta)\frac{\tilde{p}}{\tilde{p}+2}} \\
 &= C\|\xi_{2x}\|_{L^2}^{1-\frac{1}{\alpha}}\|x\|_{L^2}^{\frac{\alpha}{2}}\|\xi_{2x}\|_{L^2}^{\frac{1}{\alpha}} \\
 &\leq C(\|\sqrt{\rho}u\|_{L^2}^{1-\frac{1}{\alpha}}\|x\|_{L^2}^{\frac{\alpha}{2}}\|\sqrt{\rho}u\|_{L^2}^{\frac{1}{\alpha}}) \leq C.
 \end{aligned} \tag{3.31}$$

Here the indexes $1 \leq \tilde{p} < \infty$, $q > 1$, $\eta \in (0, 1)$ and satisfy

$$\begin{aligned}
 \frac{1}{\tilde{p}} &= \left(\frac{1}{2} - 1\right)\eta + \left(\frac{1}{q} + \kappa\right)(1 - \eta), \\
 \frac{1}{q} + \kappa &= \frac{1}{2} + \frac{\alpha}{2} - 1 > 0,
 \end{aligned}$$

which gives

$$\alpha > 1, \eta = \frac{\tilde{p}(\alpha - 1) - 2}{\alpha\tilde{p}} > 0 \Rightarrow \tilde{p} > \frac{2}{\alpha - 1}. \tag{3.32}$$

Similarly as that for (3.30) and (3.31), we can obtain

$$|\xi_0| \leq C. \tag{3.33}$$

Then, substituting (3.30), (3.31) and (3.33) into (3.28), we have

$$\ln \rho \leq C,$$

which gives

$$\rho \leq C.$$

On the other hand, if $\rho_0 \geq \delta > 0$, then combining $\rho \leq C$ and integrating (3.27) over $[0, T]$, we have

$$\begin{aligned}
 &\left| \mu \left(\ln \rho(X(x, t), t) - \ln \rho(x, 0) \right) \right| \\
 &= \left| -\xi(X(x, t), t) + \xi(x, 0) - \int_0^T \left(\left(P(\rho) + \frac{b^2}{2} \right) - \left(P(\bar{\rho}) + \frac{\bar{b}^2}{2} \right) \right) (X(x, t), t) dt \right| \\
 &\leq C(T),
 \end{aligned}$$

so

$$|\ln \rho(X(x, t), t) - \ln \rho(x, 0)| \leq C(T) \Rightarrow \ln \rho(X(x, t), t) - \ln \rho(x, 0) \geq -C(T),$$

which implies

$$\ln \rho \geq \ln \rho_0 - C(T) \Rightarrow \rho \geq \rho_0 e^{-C(T)} \geq \delta e^{-C(T)} > 0.$$

It should be noted that we have used the $\|b\|_{L^\infty(0, T; L^\infty)}$ in advance to obtain the lower bound of the density, which will be proved in the later Lemma 3.5. However, this makes sense, because we will only use the upper bound of the density to prove the upper bound of the magnetic field b in the Lemma 3.5. Hence, we put the proof ahead of time just for the sake of convenience.

Furthermore, it follows from Lemma 2.3 and Lemma 3.1 that

$$\begin{aligned}
 \|\rho - \bar{\rho}\|_{L^2} &= \left\| (\rho - \bar{\rho})^2|_{\{\rho \leq 2\bar{\rho}\}} \right\|_{L^1}^{\frac{1}{2}} + \|(\rho - \bar{\rho})|_{\{\rho \geq 2\bar{\rho}\}}\|_{L^2} \\
 &\leq \|\Phi(\rho)\|_{L^1}^{\frac{1}{2}} + \|(\rho - \bar{\rho})|_{\{\rho \geq 2\bar{\rho}\}}\|_{L^2} \\
 &\leq C + \|(\rho - \bar{\rho})|_{\{\rho \geq 2\bar{\rho}\}}\|_{L^2}.
 \end{aligned} \tag{3.34}$$

To deal with the last term on the right-hand side of (3.34), we discuss it in the following two cases:

Case 1: if $1 < \gamma \leq 2$, by Lemma 2.3, Lemma 3.1 and the fact $\rho \leq C$, it holds

$$\|(\rho - \bar{\rho})|_{\{\rho \geq 2\bar{\rho}\}}\|_{L^2(\mathbb{R})} \leq \|\rho - \bar{\rho}\|_{L^\gamma(\mathbb{R})}^{\frac{\gamma}{2}} \|\rho - \bar{\rho}\|_{L^\infty(\mathbb{R})}^{1-\frac{\gamma}{2}} \leq \|\Phi(\rho)\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \|\rho - \bar{\rho}\|_{L^\infty(\mathbb{R})}^{1-\frac{\gamma}{2}} \leq C(T). \quad (3.35)$$

Case 2: if $\gamma > 2$, using the fact that $\rho - \bar{\rho} > \bar{\rho} \Rightarrow \frac{1}{(\rho - \bar{\rho})^{\gamma-2}} < \frac{1}{\bar{\rho}^{\gamma-2}}$ and Lemma 3.1, we have

$$\begin{aligned} & \|(\rho - \bar{\rho})|_{\{\rho \geq 2\bar{\rho}\}}\|_{L^2(\mathbb{R})} \\ & \leq \|(\rho - \bar{\rho})^2|_{\{\rho \geq 2\bar{\rho}\}}\|_{L^1}^{\frac{1}{2}} \\ & \leq \left\| (\rho - \bar{\rho})^\gamma |_{\{\rho \geq 2\bar{\rho}\}} \frac{1}{(\rho - \bar{\rho})^{\gamma-2}} \right\|_{L^1}^{\frac{1}{2}} \\ & \leq \left\| (\rho - \bar{\rho})^\gamma |_{\{\rho \geq 2\bar{\rho}\}} \frac{1}{\bar{\rho}^{\gamma-2}} \right\|_{L^1}^{\frac{1}{2}} \\ & \leq \frac{1}{\bar{\rho}^{\gamma-2}} \|\Phi(\rho)\|_{L^1}^{\frac{1}{2}} \leq C. \end{aligned} \quad (3.36)$$

Thus, combining (3.35), (3.36) and (3.34), we can obtain $\|\rho - \bar{\rho}\|_{L^2} \leq C$. This completes the proof of Lemma 3.3. $\square$

Lemma 3.4. Let (ρ, u, b) be a smooth solution of (1.1)-(1.2). Then for any $T > 0$, it holds that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\mu \|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 + \|b - \bar{b}\|_{L^4}^4 \right) + \int_0^T \|b - \bar{b}\|_{L^6}^6 + \mu \nu \|(b - \bar{b})b_x\|_{L^2}^2 dt \\ & + \int_0^T \nu^2 \|b_{xx}\|_{L^2}^2 dt + \int_0^T \|\sqrt{\rho} \dot{u}\|_{L^2}^2 dt \leq C, \end{aligned}$$

and

$$\sup_{0 \leq t \leq T} (\|u\|_{L^2} + \|u\|_{L^\infty}) \leq C.$$

Proof. The proof of Lemma 3.4 will be divided into four steps.

Step 1. Multiplying the equation (1.1)₂ by $\dot{u}$ and integrating the resulting equation over $\mathbb{R}$ with respect to x yields

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \int u_x^2 + \int \rho \dot{u}^2 \\ & = -\mu \int u_x (u u_x)_x - \int P(\rho)_x (u_t + u u_x) - \int \left(\frac{b^2}{2} \right)_x (u_t + u u_x) \\ & =: J_1 + J_2 + J_3. \end{aligned} \quad (3.37)$$

Firstly, by integration by parts, we find

$$\begin{aligned} J_1 & = -\mu \int u_x^3 - \mu \int u \left(\frac{u_x^2}{2} \right)_x \\ & = -\mu \int u_x^3 + \mu \int \frac{u_x^2}{2} \cdot u_x \\ & = -\frac{\mu}{2} \int u_x^3. \end{aligned} \quad (3.38)$$

Similarly, we have

$$\begin{aligned} J_2 & = - \int (P(\rho) - P(\bar{\rho}))_x (u_t + u u_x) \\ & = \frac{d}{dt} \int (P(\rho) - P(\bar{\rho})) u_x - \int [(P(\rho) - P(\bar{\rho}))_t + (P(\rho) - P(\bar{\rho}))_x u] u_x \\ & = \frac{d}{dt} \int (P(\rho) - P(\bar{\rho})) u_x + \gamma \int \rho^\gamma u_x^2, \end{aligned} \quad (3.39)$$

and

$$\begin{aligned}
 J_3 &= - \int b(b - \bar{b})_x (u_t + uu_x) \\
 &= - \int \left(\frac{(b - \bar{b})^2}{2} \right)_x (u_t + uu_x) - \bar{b} \int (b - \bar{b})_x (u_t + uu_x) \\
 &= \frac{d}{dt} \int \left(\frac{(b - \bar{b})^2}{2} + \bar{b}(b - \bar{b}) \right) u_x - \int \left(\left(\frac{(b - \bar{b})^2}{2} \right)_t + \left(\frac{(b - \bar{b})^2}{2} \right)_x u \right) u_x \\
 &\quad - \bar{b} \int ((b - \bar{b})_t + u(b - \bar{b})_x) u_x \\
 &= \frac{d}{dt} \int \left(\frac{(b - \bar{b})^2}{2} + \bar{b}(b - \bar{b}) \right) u_x - \int ((b - \bar{b}) + \bar{b}) (vb_{xx} - bu_x) u_x,
 \end{aligned} \tag{3.40}$$

where we have used the following facts:

$$\begin{aligned}
 (P(\rho) - P(\bar{\rho}))_t + u(P(\rho) - P(\bar{\rho}))_x + \gamma \rho^\gamma u_x &= 0, \\
 \left(\frac{(b - \bar{b})^2}{2} \right)_t + u \left(\frac{(b - \bar{b})^2}{2} \right)_x + b(b - \bar{b})u_x &= vb_{xx}(b - \bar{b}),
 \end{aligned}$$

and

$$(b - \bar{b})_t + u(b - \bar{b})_x + bu_x = vb_{xx}.$$

Substituting (3.38)-(3.40) into (3.37), using Hölder's and Cauchy-Schwarz's inequalities, we have

$$\begin{aligned}
 &\frac{d}{dt} \int \frac{\mu}{2} u_x^2 - (P(\rho) - P(\bar{\rho})) u_x - \frac{(b - \bar{b})^2}{2} u_x - \bar{b}(b - \bar{b})u_x dx + \int \rho \dot{u}^2 \\
 &= -\frac{\mu}{2} \int u_x^3 + \gamma \int \rho^\gamma u_x^2 - v \int b_{xx}(b - \bar{b})u_x + \int (b - \bar{b})^2 u_x^2 \\
 &\quad + 2\bar{b} \int (b - \bar{b})u_x^2 - \bar{b} \int vb_{xx}u_x + \bar{b}^2 \int u_x^2 \\
 &\leq C(\|u_x\|_{L^3}^3 + \|u_x\|_{L^2}^2 + v\|b_{xx}\|_{L^2}\|b - \bar{b}\|_{L^6}\|u_x\|_{L^3} + \|u_x\|_{L^3}^2\|b - \bar{b}\|_{L^6}^2 \\
 &\quad + \|u_x\|_{L^3}^2\|b - \bar{b}\|_{L^3} + v\|b_{xx}\|_{L^2}\|u_x\|_{L^2}) \\
 &\leq \varepsilon v^2\|b_{xx}\|_{L^2}^2 + C\left(\|u_x\|_{L^3}^3 + \|b - \bar{b}\|_{L^6}^6 + (\|b - \bar{b}\|_{L^2}^{\frac{1}{2}}\|b - \bar{b}\|_{L^6}^{\frac{1}{2}})^3 + \|u_x\|_{L^2}^2\right) \\
 &\leq \varepsilon v^2\|b_{xx}\|_{L^2}^2 + C\left(\|u_x\|_{L^3}^3 + \|b - \bar{b}\|_{L^6}^6 + \|u_x\|_{L^2}^2 + 1\right).
 \end{aligned} \tag{3.41}$$

Now, we estimate the term $\|u_x\|_{L^3}^3$ on the right-hand side of (3.41). Employing the effective viscous flux, momentum equation, Lemma 3.1 and Lemma 3.3, we have

$$\begin{aligned}
 \|F\|_{L^3} &\leq \|F\|_{L^2}^{\frac{5}{6}} \|F_x\|_{L^2}^{\frac{1}{6}} \\
 &\leq C \left\| \left[\mu u_x - (P(\rho) - P(\bar{\rho})) - \left(\frac{(b - \bar{b})^2}{2} + \bar{b}(b - \bar{b}) \right) \right] \right\|_{L^2}^{\frac{5}{6}} \|\rho \dot{u}\|_{L^2}^{\frac{1}{6}} \\
 &\leq C \left(\|u_x\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2} + \|b - \bar{b}\|_{L^4}^2 + \|b - \bar{b}\|_{L^2} \right)^{\frac{5}{6}} \|\rho \dot{u}\|_{L^2}^{\frac{1}{6}} \\
 &\leq C \left(\|u_x\|_{L^2} + \|b - \bar{b}\|_{L^2}^{\frac{1}{2}} \|b - \bar{b}\|_{L^6}^{\frac{3}{2}} + 1 \right)^{\frac{5}{6}} \|\rho \dot{u}\|_{L^2}^{\frac{1}{6}} \\
 &\leq C \left(\|u_x\|_{L^2} + \|b - \bar{b}\|_{L^6}^{\frac{3}{2}} + 1 \right)^{\frac{5}{6}} \|\sqrt{\rho} \dot{u}\|_{L^2}^{\frac{1}{6}}.
 \end{aligned} \tag{3.42}$$

Moreover, we can obtain

$$\begin{aligned}
 \|u_x\|_{L^3} &= \left\| \frac{F + P(\rho) - P(\bar{\rho}) + \frac{b^2 - \bar{b}^2}{2}}{\mu} \right\|_{L^3} \\
 &\leq C \left(\|F\|_{L^3} + \|\rho - \bar{\rho}\|_{L^3} + \left\| \frac{(b - \bar{b})^2}{2} + \bar{b}(b - \bar{b}) \right\|_{L^3} \right) \\
 &\leq C \left(\|F\|_{L^3} + \|\rho - \bar{\rho}\|_{L^2}^{\frac{2}{3}} \|\rho - \bar{\rho}\|_{L^\infty}^{\frac{1}{3}} + \|b - \bar{b}\|_{L^6}^2 + \|b - \bar{b}\|_{L^3} \right) \\
 &\leq C \left(\left(\|u_x\|_{L^2} + \|b - \bar{b}\|_{L^6}^{\frac{3}{2}} + 1 \right)^{\frac{5}{6}} \|\sqrt{\rho} \dot{u}\|_{L^2}^{\frac{1}{6}} + \|b - \bar{b}\|_{L^6}^2 + 1 \right),
 \end{aligned} \tag{3.43}$$

which implies

$$\begin{aligned}
 \|u_x\|_{L^3}^3 &\leq C \left(\left(\left(\|u_x\|_{L^2} + \|b - \bar{b}\|_{L^6}^{\frac{3}{2}} + 1 \right)^{\frac{5}{6}} \|\sqrt{\rho} \dot{u}\|_{L^2}^{\frac{1}{6}} \right)^3 + \|b - \bar{b}\|_{L^6}^6 + 1 \right) \\
 &\leq \varepsilon \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \left(\|u_x\|_{L^2}^4 + \|b - \bar{b}\|_{L^6}^6 + 1 \right).
 \end{aligned} \tag{3.44}$$

Then, substituting (3.44) into (3.41) and choosing ε sufficiently small, we have

$$\begin{aligned}
 \frac{d}{dt} \int \frac{\mu}{2} u_x^2 - (P(\rho) - P(\bar{\rho})) u_x - \frac{(b - \bar{b})^2}{2} u_x - \bar{b}(b - \bar{b}) u_x + \int \rho \dot{u}^2 \\
 \leq \varepsilon v^2 \|b_{xx}\|_{L^2}^2 + \varepsilon \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \left(\|u_x\|_{L^2}^4 + \|b - \bar{b}\|_{L^6}^6 + 1 \right).
 \end{aligned} \tag{3.45}$$

Step 2. To control the term $\|b - \bar{b}\|_{L^6}^6$ on the right-hand side of (3.45), we rewrite the magnetic field equation as

$$(b - \bar{b})_t + u(b - \bar{b})_x + (b - \bar{b})u_x + \bar{b}u_x = \nu b_{xx}.$$

Then multiplying the above equation by $(b - \bar{b})^3$ and integrating it over $\mathbb{R}$ with respect to x , we have

$$\begin{aligned}
 \frac{1}{4} \frac{d}{dt} \int (b - \bar{b})^4 + 3\nu \int (b - \bar{b})^2 b_x^2 &= - \int \left(\frac{3}{4} (b - \bar{b})^4 + \bar{b}(b - \bar{b})^3 \right) u_x \\
 &= - \int \left(\frac{3}{4} (b - \bar{b})^4 + \bar{b}(b - \bar{b})^3 \right) \frac{F + P(\rho) - P(\bar{\rho}) + \frac{b^2 - \bar{b}^2}{2}}{\mu} \\
 &= - \int \left(\frac{3}{4} (b - \bar{b})^4 + \bar{b}(b - \bar{b})^3 \right) \frac{F + P(\rho) - P(\bar{\rho}) + \frac{(b - \bar{b})^2}{2} + \bar{b}(b - \bar{b})}{\mu} \\
 &= - \frac{3}{4\mu} \int (b - \bar{b})^4 (F + P(\rho) - P(\bar{\rho})) - \frac{\bar{b}}{\mu} \int (b - \bar{b})^3 (F + P(\rho) - P(\bar{\rho})) \\
 &\quad - \frac{3}{8\mu} \int (b - \bar{b})^6 - \frac{5\bar{b}}{4\mu} \int (b - \bar{b})^5 - \frac{\bar{b}^2}{\mu} \int (b - \bar{b})^4,
 \end{aligned}$$

which gives

$$\begin{aligned}
 \frac{1}{4} \frac{d}{dt} \int (b - \bar{b})^4 + \frac{3}{8\mu} \int (b - \bar{b})^6 + \frac{\bar{b}^2}{\mu} \int (b - \bar{b})^4 + 3\nu \int (b - \bar{b})^2 b_x^2 \\
 = - \frac{3}{4\mu} \int (b - \bar{b})^4 (F + P(\rho) - P(\bar{\rho})) - \frac{\bar{b}}{\mu} \int (b - \bar{b})^3 (F + P(\rho) - P(\bar{\rho})) - \frac{5\bar{b}}{4\mu} \int (b - \bar{b})^5 \\
 =: K_1 + K_2 + K_3.
 \end{aligned} \tag{3.46}$$

Next, we estimate $K_1 - K_3$ term by term. Using (3.42), Lemma 3.3 and Young's inequality, we have

$$\begin{aligned}
 K_1 &\leq C \|b - \bar{b}\|_{L^6}^4 (\|F\|_{L^3} + \|P(\rho) - P(\bar{\rho})\|_{L^3}) \\
 &\leq C \|b - \bar{b}\|_{L^6}^4 \left(\left(\|u_x\|_{L^2} + \|b - \bar{b}\|_{L^6}^{\frac{3}{2}} + 1 \right)^{\frac{5}{6}} \|\sqrt{\rho} \dot{u}\|_{L^2}^{\frac{1}{6}} + \|\rho - \bar{\rho}\|_{L^3} \right) \\
 &\leq \varepsilon \|b - \bar{b}\|_{L^6}^6 + \varepsilon \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C(1 + \|u_x\|_{L^2}^4).
 \end{aligned} \tag{3.47}$$

Similarly, we can get

$$\begin{aligned}
 K_2 &\leq C \|b - \bar{b}\|_{L^6}^3 (\|F\|_{L^2} + \|P(\rho) - P(\bar{\rho})\|_{L^2}) \\
 &\leq C \|b - \bar{b}\|_{L^6}^3 \left(\|u_x\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2} + \|b - \bar{b}\|_{L^4}^2 + \|b - \bar{b}\|_{L^2} \right) \\
 &\leq C \|b - \bar{b}\|_{L^6}^3 \left(\|u_x\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2} + \|b - \bar{b}\|_{L^2}^{\frac{1}{2}} \|b - \bar{b}\|_{L^6}^{\frac{3}{2}} + \|b - \bar{b}\|_{L^2} \right) \\
 &\leq \varepsilon \|b - \bar{b}\|_{L^6}^6 + C \left(\|u_x\|_{L^2}^4 + 1 \right),
 \end{aligned} \tag{3.48}$$

and

$$K_3 \leq C \|b - \bar{b}\|_{L^5}^5 \leq C \|b - \bar{b}\|_{L^2}^{\frac{1}{2}} \|b - \bar{b}\|_{L^6}^{\frac{9}{2}} \leq \varepsilon \|b - \bar{b}\|_{L^6}^6 + C. \tag{3.49}$$

Then, substituting (3.47)-(3.49) into (3.46) and choosing $\varepsilon > 0$ small enough yields

$$\begin{aligned}
 \frac{1}{4} \frac{d}{dt} \int (b - \bar{b})^4 + \frac{3}{8\mu} \int (b - \bar{b})^6 + \frac{\bar{b}^2}{\mu} \int (b - \bar{b})^4 + 3\nu \int (b - \bar{b})^2 b_x^2 \\
 \leq \varepsilon \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \left(1 + \|u_x\|_{L^2}^4 \right).
 \end{aligned} \tag{3.50}$$

Step 3. To control the term $\nu^2 \|b_{xx}\|_{L^2}^2$ on the right hand-side of (3.45), we multiply the equation (1.1)₃ by νb_{xx} and integrate it over $\mathbb{R}$ to get

$$\begin{aligned}
 \frac{\nu}{2} \frac{d}{dt} \int b_x^2 + \nu^2 \int b_{xx}^2 &= \nu \int b_{xx} u b_x + \nu \int b_{xx} (b - \bar{b}) u_x + \bar{b} \nu \int b_{xx} u_x \\
 &= -\frac{\nu}{2} \int b_x^2 u_x + \nu \int b_{xx} (b - \bar{b}) u_x + \bar{b} \nu \int b_{xx} u_x \\
 &=: H_1 + H_2 + H_3.
 \end{aligned} \tag{3.51}$$

For the term H_1 , by the effective viscous flux, it shows that

$$\begin{aligned}
 H_1 &= -\frac{\nu}{2} \int b_x^2 \frac{F + P(\rho) - P(\bar{\rho}) + \frac{b^2 - \bar{b}^2}{2}}{\mu} \\
 &\leq -\frac{\nu}{2\mu} \int b_x^2 F + \frac{\nu}{2\mu} \int b_x^2 \left(P(\bar{\rho}) + \frac{\bar{b}^2}{2} \right) \\
 &\leq C (1 + \|F\|_{L^\infty}) \nu \|b_x\|_{L^2}^2 \\
 &\leq C \left(1 + \|\sqrt{\rho} \dot{u}\|_{L^2} + \|u_x\|_{L^2} + \|b - \bar{b}\|_{L^6}^{\frac{3}{2}} \right) \nu \|b_x\|_{L^2}^2 \\
 &\leq \varepsilon \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|b - \bar{b}\|_{L^6}^6 + C \left(1 + \|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right) \nu \|b_x\|_{L^2}^2,
 \end{aligned} \tag{3.52}$$

where we have used the following inequality:

$$\begin{aligned}
 \|F\|_{L^\infty} &\leq C \|F\|_{L^2}^{\frac{1}{2}} \|F_x\|_{L^2}^{\frac{1}{2}} \\
 &\leq C \left(\|u_x\|_{L^2} + \|\rho - \bar{\rho}\|_{L^2} + \|b - \bar{b}\|_{L^4} + \|b - \bar{b}\|_{L^2} \right)^{\frac{1}{2}} \|\sqrt{\rho} \dot{u}\|_{L^2}^{\frac{1}{2}} \\
 &\leq C \left(1 + \|\sqrt{\rho} \dot{u}\|_{L^2} + \|u_x\|_{L^2} + \|b - \bar{b}\|_{L^6}^{\frac{3}{2}} \right).
 \end{aligned} \tag{3.53}$$

For the terms H_2 and H_3 , using Hölder's inequality, Cauchy's inequality and (3.44), we have

$$\begin{aligned}
 H_2 &\leq \nu \|b_{xx}\|_{L^2} \|b - \bar{b}\|_{L^6} \|u_x\|_{L^3} \\
 &\leq \varepsilon \nu^2 \|b_{xx}\|_{L^2}^2 + C \left(\|b - \bar{b}\|_{L^6}^6 + \|u_x\|_{L^3}^3 \right) \\
 &\leq \varepsilon \nu^2 \|b_{xx}\|_{L^2}^2 + \varepsilon \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \left(1 + \|u_x\|_{L^2}^4 + \|b - \bar{b}\|_{L^6}^6 \right),
 \end{aligned} \tag{3.54}$$

and

$$H_3 \leq C \nu \|b_{xx}\|_{L^2} \|u_x\|_{L^2} \leq \varepsilon \nu^2 \|b_{xx}\|_{L^2}^2 + C \|u_x\|_{L^2}^2. \tag{3.55}$$

Thus, substituting (3.52), (3.54) and (3.55) into (3.51) and choosing $\varepsilon > 0$ small enough, we have

$$\begin{aligned} \frac{\nu}{2} \frac{d}{dt} \int b_x^2 + \nu^2 \int b_{xx}^2 &\leq 2\varepsilon \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \left(1 + \|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right) \nu \|b_x\|_{L^2}^2 \\ &\quad + C \left(1 + \|u_x\|_{L^2}^4 + \|b - \bar{b}\|_{L^6}^6 \right). \end{aligned} \quad (3.56)$$

Step 4. Adding the equation (3.50) multiplied by $\frac{8\mu}{3}(2C+1)$ and (3.45) into (3.56), we have

$$\begin{aligned} \frac{d}{dt} \int \left(\frac{\mu}{2} u_x^2 + \frac{\nu}{2} b_x^2 + \frac{2\mu(2C+1)}{3} (b - \bar{b})^4 - \psi(\rho, u, b) \right) &+ \int \rho \dot{u}^2 + \nu^2 \int b_{xx}^2 \\ &+ \int \left((b - \bar{b})^6 + \frac{8\bar{b}^2(2C+1)}{3} (b - \bar{b})^4 + 8\mu\nu(2C+1)(b - \bar{b})^2 b_x^2 \right) \\ &\leq C + C \left(1 + \|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right) \left(\|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right), \end{aligned} \quad (3.57)$$

where $\psi(\rho, u, b) = (P(\rho) - P(\bar{\rho})) u_x + \frac{(b-\bar{b})^2}{2} u_x + \bar{b}(b - \bar{b})u_x$. Integrating (3.57) over $[0, T]$ with respect to t , we obtain

$$\begin{aligned} &\int \left(\mu u_x^2 + \nu b_x^2 + (b - \bar{b})^4 \right) + \int_0^T \int \left(\rho \dot{u}^2 + (b - \bar{b})^6 + \mu\nu(b - \bar{b})^2 b_x^2 + \nu^2 b_{xx}^2 \right) \\ &\leq C + C \int_0^T \left(1 + \|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right) \left(\|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right) \\ &\quad + \int \psi(\rho, u, b) - \int \psi(\rho_0, u_0, b_0). \end{aligned} \quad (3.58)$$

By Lemma 3.1, Lemma 3.3 and Cauchy-Schwarz's inequality, we can obtain

$$\begin{aligned} &\int \psi(\rho, u, b) - \int \psi(\rho_0, u_0, b_0) \\ &\leq C \left(\|P(\rho) - P(\bar{\rho})\|_{L^2} + \|b - \bar{b}\|_{L^4}^2 + \|b - \bar{b}\|_{L^2} \right) \|u_x\|_{L^2} + C \\ &\leq C \left(\|\rho - \bar{\rho}\|_{L^2} + \|b - \bar{b}\|_{L^4}^2 + 1 \right) \|u_x\|_{L^2} + C \\ &\leq \varepsilon \mu \|u_x\|_{L^2}^2 + C(1 + \|b - \bar{b}\|_{L^4}^4), \end{aligned}$$

which together with (3.58) gives (choosing $\varepsilon > 0$ small enough)

$$\begin{aligned} &\int \left(\mu u_x^2 + \nu b_x^2 + (b - \bar{b})^4 \right) + \int_0^T \int \left(\rho \dot{u}^2 + (b - \bar{b})^6 + \mu\nu(b - \bar{b})^2 b_x^2 + \nu^2 b_{xx}^2 \right) \\ &\leq C(1 + \|b - \bar{b}\|_{L^4}^4) + C \int_0^T \left(1 + \|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right) \left(\|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right). \end{aligned} \quad (3.59)$$

Now, integrating (3.50) over $[0, T]$ with respect to t , and then adding the resulting inequality multiplied by $4C$ into (3.59) show that

$$\begin{aligned} &\int \left(\mu u_x^2 + \nu b_x^2 + (b - \bar{b})^4 \right) + \int_0^T \int \left(\rho \dot{u}^2 + (b - \bar{b})^6 + \mu\nu(b - \bar{b})^2 b_x^2 + \nu^2 b_{xx}^2 \right) \\ &\leq C + C \int_0^T \left(1 + \|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right) \left(\|u_x\|_{L^2}^2 + \nu \|b_x\|_{L^2}^2 \right), \end{aligned}$$

which together with Gronwall's inequality and Lemma 3.1 yields

$$\begin{aligned} & \int \mu u_x^2 + \nu b_x^2 + (b - \bar{b})^4 dx + \int_0^T \int (b - \bar{b})^6 + \mu \nu (b - \bar{b})^2 b_x^2 dx dt \\ & + \int_0^T \int \nu^2 b_{xx}^2 dx dt + \int_0^T \int \rho \dot{u}^2 dx dt \leq C(T). \end{aligned}$$

Thus, it follows from (3.17) and (3.18) that

$$\|u\|_{L^2}^2 \leq C(1 + \|u_x\|_{L^2}^2) \leq C,$$

and

$$\|u\|_{L^\infty} \leq C \|u\|_{L^2}^{\frac{1}{2}} \|u_x\|_{L^2}^{\frac{1}{2}} \leq C(1 + \|u_x\|_{L^2}) \leq C.$$

Then, we complete the proof of Lemma 3.4. $\square$

To get the uniform upper bound of the magnetic field b , we need to re-estimate the $\|b_x\|_{L^2}$ independent of ν as follows.

Lemma 3.5. *Let (ρ, u, b) be a smooth solution of (1.1)-(1.2). Then for any $T > 0$, it holds that*

$$\sup_{0 \leq t \leq T} \left(\|b\|_{L^\infty(\mathbb{R})} + \|\rho_x\|_{L^2(\mathbb{R})}^2 + \|b_x\|_{L^2(\mathbb{R})}^2 \right) + \int_0^T \left(\mu \|u_{xx}\|_{L^2(\mathbb{R})}^2 + \nu \|b_{xx}\|_{L^2(\mathbb{R})}^2 \right) dt \leq C.$$

Proof. Differentiating the equality (1.1)₃ with respect to x , then multiplying the resulting equation by b_x and integrating by parts over $\mathbb{R}$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int b_x^2 + \nu \int b_{xx}^2 &= -2 \int b_x^2 u_x - \int b b_x u_{xx} - \int u b_x b_{xx} \\ &= -\frac{3}{2} \int b_x^2 u_x - \int b b_x u_{xx}. \end{aligned} \quad (3.60)$$

To control the second term on the right-hand side of (3.60), multiplying the momentum equation (1.1)₂ by u_{xx} gives

$$\mu \int u_{xx}^2 = \int \rho \dot{u} u_{xx} + \gamma \int \rho^{\gamma-1} \rho_x u_{xx} + \int b b_x u_{xx}. \quad (3.61)$$

Combining (3.60) and (3.61) yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int b_x^2 + \nu \int b_{xx}^2 + \mu \int u_{xx}^2 &= -\frac{3}{2} \int b_x^2 u_x + \int \rho \dot{u} u_{xx} + \gamma \int \rho^{\gamma-1} \rho_x u_{xx} \\ &= -\frac{3}{2} \int b_x^2 \frac{F + P(\rho) - P(\bar{\rho}) + \frac{b^2 - \bar{b}^2}{2}}{\mu} + \int \rho \dot{u} u_{xx} + \gamma \int \rho^{\gamma-1} \rho_x u_{xx} \\ &\leq -\frac{3}{2} \int b_x^2 \frac{F - P(\bar{\rho}) + \frac{-\bar{b}^2}{2}}{\mu} + \int \rho \dot{u} u_{xx} + \gamma \int \rho^{\gamma-1} \rho_x u_{xx} \\ &\leq C \left(\|F\|_{L^\infty} \|b_x\|_{L^2}^2 + \|b_x\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2} \|u_{xx}\|_{L^2} + \|\rho_x\|_{L^2} \|u_{xx}\|_{L^2} \right) \\ &\leq \varepsilon \|u_{xx}\|_{L^2}^2 + C(\|F\|_{L^\infty} + 1) \|b_x\|_{L^2}^2 + C \left(\|\rho_x\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right) \\ &\leq \varepsilon \|u_{xx}\|_{L^2}^2 + C \left(\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|b - \bar{b}\|_{L^6}^6 + 1 \right) \|b_x\|_{L^2}^2 + C \left(\|\rho_x\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right), \end{aligned} \quad (3.62)$$

where we have used (3.53) and Lemma 3.4.

Next, differentiating the density equation (1.1)₁ with respect to x , multiplying the resulting equation by ρ_x , and integrating it over $\mathbb{R}$ implies that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \rho_x^2 &= -2 \int \rho_x^2 u_x - \int u \rho_x \rho_{xx} - \int \rho \rho_x u_{xx} \\ &= -\frac{3}{2} \int \rho_x^2 u_x - \int \rho \rho_x u_{xx} \\ &= -\frac{3}{2} \int \rho_x^2 \frac{F + P(\rho) - P(\bar{\rho}) + \frac{b^2 - \bar{b}^2}{2}}{\mu} - \int \rho \rho_x u_{xx} \\ &\leq -\frac{3}{2\mu} \int \rho_x^2 F + \frac{3}{2\mu} \int \left(P(\bar{\rho}) + \frac{\bar{b}^2}{2} \right) \rho_x^2 + C \|\rho_x\|_{L^2} \|u_{xx}\|_{L^2} \\ &\leq \varepsilon \|u_{xx}\|_{L^2}^2 + C (\|F\|_{L^\infty} + 1) \|\rho_x\|_{L^2}^2 \\ &\leq \varepsilon \|u_{xx}\|_{L^2}^2 + C \left(\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|b - \bar{b}\|_{L^6}^6 + 1 \right) \|\rho_x\|_{L^2}^2, \end{aligned} \quad (3.63)$$

where we have used (3.53) and Lemma 3.4. Then adding the (3.63) into (3.62), we have

$$\frac{1}{2} \frac{d}{dt} \int (b_x^2 + \rho_x^2) + \nu \int b_{xx}^2 + \mu \int u_{xx}^2 \leq C \left(\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|b - \bar{b}\|_{L^6}^6 + 1 \right) \left(1 + \|\rho_x\|_{L^2}^2 + \|b_x\|_{L^2}^2 \right),$$

which together with Gronwall's inequality and Lemma 3.4 gives

$$\int \left(\rho_x^2 + b_x^2 \right) + \int_0^T \int \left(\nu b_{xx}^2 + \mu u_{xx}^2 \right) \leq C(T).$$

Thus, it follows from Lemma 3.1 and the Gagliardo-Nirenberg inequality (2.1) that

$$\|b\|_{L^\infty} \leq \|b - \bar{b}\|_{L^\infty} + \bar{b} \leq C \|b - \bar{b}\|_{L^2}^{\frac{1}{2}} \|b_x\|_{L^2}^{\frac{1}{2}} + \bar{b} \leq C(T).$$

Thus, the proof of Lemma 3.5 is finished. $\square$

With the help of the Lemma 3.5, we can get the estimates of the first order derivative with respect to t of the density and the magnetic field, respectively.

Lemma 3.6. *Let (ρ, u, b) be a smooth solution to (1.1)-(1.2). Then for any $T > 0$, it holds that*

$$\sup_{0 \leq t \leq T} \|\rho_t\|_{L^2(\mathbb{R})} + \|b_t\|_{L^2(0, T; L^2(\mathbb{R}))} \leq C.$$

Proof. By direct calculation, we have

$$\begin{aligned} \|\rho_t\|_{L^2} &\leq C (\|\rho u_x\|_{L^2} + \|u \rho_x\|_{L^2}) \\ &\leq C (\|\rho\|_{L^\infty} \|u_x\|_{L^2} + \|u\|_{L^\infty} \|\rho_x\|_{L^2}) \in L^\infty(0, T), \end{aligned}$$

and

$$\begin{aligned} \|b_t\|_{L^2} &\leq C (\|b u_x\|_{L^2} + \|u b_x\|_{L^2} + \|\nu b_{xx}\|_{L^2}) \\ &\leq C (\|b\|_{L^\infty} \|u_x\|_{L^2} + \|u\|_{L^\infty} \|b_x\|_{L^2} + \|\nu b_{xx}\|_{L^2}) \\ &\leq C (1 + \|\nu b_{xx}\|_{L^2}) \in L^2(0, T), \end{aligned}$$

where we have used Lemma 3.4 and Lemma 3.5. $\square$

The following estimates of the second order derivative of the velocity play an important role in the analysis of the non-resistive limit.

Lemma 3.7. *Let (ρ, u, b) be a smooth solution to (1.1)-(1.2). Then for any $T > 0$, it holds that*

$$\sup_{0 \leq t \leq T} \|\sqrt{\rho} \dot{u}\|_{L^2(\mathbb{R})} + \int_0^T \mu \|u_{xt}\|_{L^2(\mathbb{R})}^2 \leq C,$$

and

$$\sup_{0 \leq t \leq T} (\|u_{xx}\|_{L^2(\mathbb{R})} + \|\sqrt{\rho}u_t\|_{L^2(\mathbb{R})}) \leq C.$$

Proof. Differentiating the momentum equation (1.1)₂ with respect to t , we have

$$\rho_t \dot{u} + \rho \dot{u}_t + \left(P(\rho) + \frac{b^2}{2} \right)_{xt} = \mu u_{xxt}.$$

Multiplying the above equation by $\dot{u}$ and integrating the resulting equation over $\mathbb{R}$ yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho \dot{u}^2 + \mu \int u_{xt}^2 \\ &= -\frac{1}{2} \int \rho_t \dot{u}^2 - \int \left(P(\rho) + \frac{b^2}{2} \right)_{xt} \dot{u} - \mu \int u_{xt} (u_x^2 + uu_{xx}) \\ &=: L_1 + L_2 + L_3. \end{aligned} \quad (3.64)$$

Now, we estimate the terms $L_1 - L_3$ as

$$\begin{aligned} L_1 &= \frac{1}{2} \int (\rho u)_x \dot{u}^2 = \int -\rho u \dot{u} \dot{u}_x \\ &\leq C \|\sqrt{\rho} \dot{u}\|_{L^2} \|u\|_{L^\infty} \|\dot{u}_x\|_{L^2} \\ &\leq C \|\sqrt{\rho} \dot{u}\|_{L^2} \left(\|u_{xt}\|_{L^2} + \|u_x\|_{L^4}^2 + \|u\|_{L^\infty} \|u_{xx}\|_{L^2} \right) \\ &\leq C \|\sqrt{\rho} \dot{u}\|_{L^2} \left(\|u_{xt}\|_{L^2} + \|u_x\|_{L^2}^{\frac{3}{2}} \|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^\infty} \|u_{xx}\|_{L^2} \right) \\ &\leq \varepsilon \|u_{xt}\|_{L^2}^2 + C \left(\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + 1 \right), \end{aligned} \quad (3.65)$$

and similarly, we also have

$$\begin{aligned} L_2 &= \int \left(P(\rho) + \frac{b^2}{2} \right)_t \dot{u}_x \\ &\leq C \|\rho^{\gamma-1} \rho_t + b b_t\|_{L^2} \|\dot{u}_x\|_{L^2} \\ &\leq C (\|\rho_t\|_{L^2} + \|b_t\|_{L^2}) \|u_{xt} + u_x^2 + uu_{xx}\|_{L^2} \\ &\leq C (1 + \|b_t\|_{L^2}) \left(\|u_{xt}\|_{L^2} + \|u_x\|_{L^2}^{\frac{3}{2}} \|u_{xx}\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^\infty} \|u_{xx}\|_{L^2} \right) \\ &\leq \varepsilon \|u_{xt}\|_{L^2}^2 + C \left(\|u_{xx}\|_{L^2}^2 + \|b_t\|_{L^2}^2 + 1 \right), \end{aligned} \quad (3.66)$$

and

$$\begin{aligned} L_3 &\leq C \|u_{xt}\|_{L^2} \left(\|u_x\|_{L^4}^2 + \|u\|_{L^\infty} \|u_{xx}\|_{L^2} \right) \\ &\leq \varepsilon \|u_{xt}\|_{L^2}^2 + C \left(1 + \|u_{xx}\|_{L^2}^2 \right), \end{aligned} \quad (3.67)$$

where we have used the Gagliardo-Nirenberg inequality, Lemma 3.4 and Lemma 3.5. Thus, substituting (3.65)-(3.67) into (3.64) and choosing $\varepsilon > 0$ small enough, we have

$$\frac{1}{2} \frac{d}{dt} \int \rho \dot{u}^2 + \mu \int u_{xt}^2 \leq C \left(1 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \|b_t\|_{L^2}^2 \right),$$

which combining Gronwall's inequality, Lemma 3.4 and Lemma 3.5 gives

$$\frac{1}{2} \int \rho \dot{u}^2 + \int_0^T \int \mu u_{xt}^2 \leq C(T).$$

This together with the momentum equation yields that

$$\begin{aligned} \mu \|u_{xx}\|_{L^2} &\leq C \left(\|\rho \dot{u}\|_{L^2} + \left\| \left(P(\rho) + \frac{b^2}{2} \right)_x \right\|_{L^2} \right) \\ &\leq C (\|\sqrt{\rho} \dot{u}\|_{L^2} + \|\rho_x\|_{L^2} + \|b_x\|_{L^2}) \leq C, \end{aligned}$$

and

$$\begin{aligned}\|\sqrt{\rho}u_t\|_{L^2} &\leq C\|\sqrt{\rho}\dot{u} - \sqrt{\rho}uu_x\|_{L^2} \\ &\leq C\|\sqrt{\rho}\dot{u}\|_{L^2} + C\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^\infty}\|u_x\|_{L^2} \leq C.\end{aligned}$$

We complete the proof of Lemma 3.7. $\square$

In conclusion, we have established a global ν -independent priori estimates for the smooth solution $(\rho^\delta, u^\delta, b^\delta)$ to (1.1)-(1.2) with the initial density away from vacuum ($\rho_0^\delta \geq \delta > 0$), which is uniform of δ .

4 Proof of the Theorem 1.1

Before giving the proof of the Theorem 1.1, we first give the global existence of strong solutions to 1D non-resistive MHD equations (1.3)-(1.4) when the initial density and initial magnetic approach non-zero constants at infinity.

Proposition 4.1. Suppose that the initial data $(\tilde{\rho}_0, \tilde{u}_0, \tilde{b}_0)(x)$ satisfies

$$\begin{aligned}\tilde{\rho}_0 - \bar{\rho} &\in H^1(\mathbb{R}), \quad \tilde{b}_0 - \bar{b} \in H^1(\mathbb{R}), \quad \tilde{u}_0 \in H^2(\mathbb{R}), \\ \left(\frac{1}{2}\tilde{\rho}_0\tilde{u}_0^2 + \Phi(\tilde{\rho}_0) + \frac{(\tilde{b}_0 - \bar{b})^2}{2}\right)|x|^\alpha &\in L^1(\mathbb{R})\end{aligned}\quad (4.1)$$

for some $1 < \alpha \leq 2$, and the compatibility condition

$$\left(\mu\tilde{u}_{0x} - P(\tilde{\rho}_0) - \frac{1}{2}\tilde{b}_0^2\right)_x = \sqrt{\tilde{\rho}_0}\tilde{g}(x), \quad x \in \mathbb{R} \quad (4.2)$$

with some $\tilde{g} \in L^2(\mathbb{R})$. Then for any $T > 0$, there exists a unique global strong solution $(\tilde{\rho}, \tilde{u}, \tilde{b})$ to the Cauchy problem (1.3) and (1.4) such that

$$0 \leq \tilde{\rho}(x, t) \leq C, \quad \forall (x, t) \in \mathbb{R} \times [0, T], \quad (4.3)$$

$$\sup_{0 \leq t \leq T} \left\| \left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \Phi(\tilde{\rho}) + \frac{(\tilde{b} - \bar{b})^2}{2} \right) (1 + |x|^\alpha) \right\|_{L^1} + \int_0^T \mu \left\| \tilde{u}_x (1 + |x|^{\frac{\alpha}{2}}) \right\|_{L^2}^2 dt \leq C, \quad (4.4)$$

and

$$\sup_{0 \leq t \leq T} \left(\|\tilde{u}\|_{H^2} + \|\tilde{b}_x\|_{L^2} + \|\tilde{\rho}_x\|_{L^2} + \|\sqrt{\tilde{\rho}}\tilde{u}\|_{L^2} \right) + \int_0^T \mu \|\tilde{u}_{xt}\|_{L^2}^2 dt \leq C. \quad (4.5)$$

Proof. For the details, the readers can refer to [1]. $\square$

The existence and uniqueness of global strong solution in Theorem 1.1:

Before proving Theorem 1.1, by the standard fixed point Theorem and the global uniform of δ priori estimates derived in section 3, we have the following auxiliary theorem.

Theorem 4.2. Under the same assumption in Theorem 1.1, then for any $\delta \in (0, 1)$, there exists a unique global solution $(\rho^\delta, u^\delta, b^\delta)$ to (1.1)-(1.2) with the initial data replaced by $(\rho_0^\delta, u_0^\delta, b_0)$ such that for any $T > 0$,

$$0 < \delta e^{-CT} \leq \rho^\delta(x, t) \leq C, \quad \forall (x, t) \in \mathbb{R} \times [0, T],$$

$$\begin{aligned}\sup_{0 \leq t \leq T} \left\| \left(\frac{1}{2}(\sqrt{\rho^\delta}u^\delta)^2 + \Phi(\rho^\delta) + \frac{(b^\delta - \bar{b})^2}{2} \right) (1 + |x|^\alpha) \right\|_{L^1} (t) \\ + \int_0^T \left(\mu \left\| u_x^\delta (1 + |x|^{\frac{\alpha}{2}}) \right\|_{L^2}^2 + \nu \left\| b_x^\delta (1 + |x|^{\frac{\alpha}{2}}) \right\|_{L^2}^2 \right) dt \leq C,\end{aligned}$$

and

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\|u^\delta\|_{L^2} + \|u_{xx}^\delta\|_{L^2} + \|u_x^\delta\|_{L^2} + \|b_x^\delta\|_{L^2} + \|\rho_x^\delta\|_{L^2} + \|\sqrt{\rho^\delta} \dot{u}^\delta\|_{L^2} \right) (t) \\ & + \int_0^T \left(\|u_{xt}^\delta\|_{L^2}^2 + \nu \|b_{xx}^\delta\|_{L^2}^2 \right) dt \leq C, \end{aligned}$$

where C is independent of ν and δ .

From Theorem 4.2, we know that there exists a unique global strong solution $(\rho^\delta, u^\delta, b^\delta)$, such that Lemma 3.1- Lemma 3.7 are valid when we replace (ρ, u, b) by $(\rho^\delta, u^\delta, b^\delta)$. With the uniform estimates for δ , we let $\delta \rightarrow 0^+$ (take subsequence if necessary) to get a solution to (1.1)-(1.2) still denoted by (ρ, u, b) which satisfies Lemma 3.1- Lemma 3.7 by the lower semicontinuity of the norms. By a standard continuity argument c.f. [27], we can obtain the global existence of the solution as in Theorem 1.1. The uniqueness of the solution can be proved by the standard L^2 energy method, here we omit the details for brevity. Hence, the first part of the proof of Theorem 1.1 is completed.

The non-resistive limit in Theorem 1.1: To justify the non-resistive limit as $\nu \rightarrow 0$, we consider the difference of these two solutions $(\rho - \tilde{\rho}, u - \tilde{u}, b - \tilde{b})$ which satisfy the following equations:

$$\begin{cases} (\rho - \tilde{\rho})_t + (\rho - \tilde{\rho})u_x + \tilde{\rho}(u - \tilde{u})_x + (\rho - \tilde{\rho})_x u + \tilde{\rho}_x(u - \tilde{u}) = 0, \\ \rho(u - \tilde{u})_t + \rho u(u - \tilde{u})_x - \mu(u - \tilde{u})_{xx} = -(\rho - \tilde{\rho})(\tilde{u}_t + \tilde{u}\tilde{u}_x) - \rho(u - \tilde{u})\tilde{u}_x \\ \quad - (P(\rho) - P(\tilde{\rho}))_x - \left(\frac{b^2 - \tilde{b}^2}{2}\right)_x, \\ (b - \tilde{b})_t + u_x(b - \tilde{b}) + \tilde{b}(u - \tilde{u})_x + u(b - \tilde{b})_x + (u - \tilde{u})\tilde{b}_x = \nu b_{xx}. \end{cases} \quad (4.6)$$

Firstly, multiplying the equation (4.6)₁ by $2(\rho - \tilde{\rho})$, integrating the resultant over $\mathbb{R}$, and integrating by parts, it follows from Hölder's inequality, the Gagliardo-Nirenberg inequality, the Cauchy inequality, Lemma 3.4 and Proposition 4.1 that

$$\begin{aligned} & \frac{d}{dt} \int (\rho - \tilde{\rho})^2 = - \int (\rho - \tilde{\rho})^2 u_x - 2 \int \tilde{\rho}(\rho - \tilde{\rho})(u - \tilde{u})_x - 2 \int \tilde{\rho}_x(\rho - \tilde{\rho})(u - \tilde{u}) \\ & \leq C(\|u_x\|_{L^\infty} \|\rho - \tilde{\rho}\|_{L^2}^2 + \|\tilde{\rho}\|_{L^\infty} \|\rho - \tilde{\rho}\|_{L^2} \|(u - \tilde{u})_x\|_{L^2} + \|\tilde{\rho}_x\|_{L^2} \|\rho - \tilde{\rho}\|_{L^2} \|u - \tilde{u}\|_{L^\infty}) \\ & \leq C \left[\|u_x\|_{L^2}^{\frac{1}{2}} \|u_{xx}\|_{L^2}^{\frac{1}{2}} \|\rho - \tilde{\rho}\|_{L^2}^2 + \left(\|(u - \tilde{u})_x\|_{L^2} + \|u - \tilde{u}\|_{L^2}^{\frac{1}{2}} \|(u - \tilde{u})_x\|_{L^2}^{\frac{1}{2}} \right) \|\rho - \tilde{\rho}\|_{L^2} \right] \\ & \leq \varepsilon \|(u - \tilde{u})_x\|_{L^2}^2 + C(1 + \|u_{xx}\|_{L^2}) \left(\|\rho - \tilde{\rho}\|_{L^2}^2 + 1 \right). \end{aligned} \quad (4.7)$$

Secondly, we multiply (4.6)₂ by $2(u - \tilde{u})$, integrate the resultant over $\mathbb{R}$, integrate by parts, and use Hölder's inequality, the Gagliardo-Nirenberg inequality, Young's inequality, the fact that $\|u - \tilde{u}\|_{L^2}^2 \leq C(1 + \|(u - \tilde{u})_x\|_{L^2}^2)$ (see (3.17)) and Proposition 4.1 to get

$$\begin{aligned} & \frac{d}{dt} \int \rho(u - \tilde{u})^2 + 2\mu \int (u - \tilde{u})_x^2 = - \int 2(\rho - \tilde{\rho})(u - \tilde{u})(\tilde{u}_t + \tilde{u}\tilde{u}_x) \\ & \quad - \int 2\rho(u - \tilde{u})^2 \tilde{u}_x - \int 2(P(\rho) - P(\tilde{\rho}))_x(u - \tilde{u}) - \int (b^2 - \tilde{b}^2)_x(u - \tilde{u}) \\ & \leq C \left(\|\dot{u}\|_{L^2} \|\rho - \tilde{\rho}\|_{L^2} \|u - \tilde{u}\|_{L^\infty} + \|\tilde{u}_x\|_{L^\infty} \|\sqrt{\rho}(u - \tilde{u})\|_{L^2}^2 \right. \\ & \quad \left. + (\|\rho - \tilde{\rho}\|_{L^2} + \|b - \tilde{b}\|_{L^2}) \|(u - \tilde{u})_x\|_{L^2} \right) \\ & \leq C \left[\|\dot{u}\|_{L^2} \|u - \tilde{u}\|_{L^2}^{\frac{1}{2}} \|(u - \tilde{u})_x\|_{L^2}^{\frac{1}{2}} \|\rho - \tilde{\rho}\|_{L^2} + \|\tilde{u}_x\|_{L^2}^{\frac{1}{2}} \|\tilde{u}_{xx}\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho}(u - \tilde{u})\|_{L^2}^2 \right. \\ & \quad \left. + (\|\rho - \tilde{\rho}\|_{L^2} + \|b - \tilde{b}\|_{L^2}) \|(u - \tilde{u})_x\|_{L^2} \right] \\ & \leq \varepsilon \|(u - \tilde{u})_x\|_{L^2}^2 + C \left(1 + \|\rho - \tilde{\rho}\|_{L^2}^2 + \|b - \tilde{b}\|_{L^2}^2 + \|\sqrt{\rho}(u - \tilde{u})\|_{L^2}^2 \right) \\ & \quad \left(1 + \|\dot{u}\|_{L^2}^2 + \|\tilde{u}_{xx}\|_{L^2} \right). \end{aligned} \quad (4.8)$$

Thirdly, multiplying the equation (4.6)₃ by $2(b - \tilde{b})$, integrating the resultant over $\mathbb{R}$, integrating by parts and using Proposition 4.1, the third equation of (1.3), the fact that $\|\tilde{b}\|_{L^\infty} \leq C\|\tilde{b}\|_{L^2}^{\frac{1}{2}}\|\tilde{b}_x\|_{L^2}^{\frac{1}{2}}$, we deduce

$$\begin{aligned} & \frac{d}{dt} \int (b - \tilde{b})^2 \\ &= - \int (b - \tilde{b})^2 u_x - 2 \int \tilde{b}(b - \tilde{b})(u - \tilde{u})_x - 2 \int (b - \tilde{b})(u - \tilde{u})\tilde{b}_x + 2\nu \int (b - \tilde{b})b_{xx} \\ &\leq C \left(\|u_x\|_{L^\infty} \|b - \tilde{b}\|_{L^2}^2 + \|\tilde{b}\|_{L^\infty} \|b - \tilde{b}\|_{L^2} \|(u - \tilde{u})_x\|_{L^2} \right. \\ &\quad \left. + \|b - \tilde{b}\|_{L^2} \|\tilde{b}_x\|_{L^2} \|u - \tilde{u}\|_{L^\infty} + \nu \|b - \tilde{b}\|_{L^2} \|b_{xx}\|_{L^2} \right) \\ &\leq \varepsilon \|(u - \tilde{u})_x\|_{L^2}^2 + C(1 + \|u_{xx}\|_{L^2}) \|b - \tilde{b}\|_{L^2}^2 + \nu^2 \|b_{xx}\|_{L^2}^2. \end{aligned} \quad (4.9)$$

Then combining (4.7)-(4.9) and choosing $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned} & \frac{d}{dt} \int \left((\rho - \tilde{\rho})^2 + \rho(u - \tilde{u})^2 + (b - \tilde{b})^2 \right) + 2\mu \int (u - \tilde{u})_x^2 \\ &\leq C \left(1 + \|u_{xx}\|_{L^2} + \|\tilde{u}_{xx}\|_{L^2} + \|\dot{u}\|_{L^2}^2 \right) \left(\|\rho - \tilde{\rho}\|_{L^2}^2 + \|b - \tilde{b}\|_{L^2}^2 + \|\sqrt{\rho}(u - \tilde{u})\|_{L^2}^2 + 1 \right) \\ &\quad + \nu^2 \|b_{xx}\|_{L^2}^2, \end{aligned}$$

which together with Gronwall's inequality, Lemma 3.5 and Proposition 4.1 yields

$$\begin{aligned} & \int \left((\rho - \tilde{\rho})^2 + \rho(u - \tilde{u})^2 + (b - \tilde{b})^2 \right) + 2\mu \int (u - \tilde{u})_x^2 \\ &\leq C\nu^2 \int_0^T \|b_{xx}\|_{L^2}^2 dt \exp \left\{ C \int_0^T \left(1 + \|u_{xx}\|_{L^2} + \|\tilde{u}_{xx}\|_{L^2} + \|\dot{u}\|_{L^2}^2 \right) dt \right\} \\ &\leq C\nu, \end{aligned} \quad (4.10)$$

where we have used Lemmas 3.5 and 3.7, Proposition 4.1 and the following fact:

$$\begin{aligned} \|\dot{u}\|_{L^2} &\leq C\|\sqrt{\rho}\dot{u}\|_{L^2} \leq C \left(\|(\sqrt{\rho} - \sqrt{\tilde{\rho}})\dot{u}\|_{L^2} + \|\sqrt{\tilde{\rho}}\dot{u}\|_{L^2} \right) \\ &\leq C \left(\|\tilde{\rho} - \rho\|_{L^2} \|\dot{u}\|_{L^\infty} + 1 \right) \\ &\leq \varepsilon \|\dot{u}\|_{L^2} + C(\|\tilde{u}_x\|_{L^2} + 1), \end{aligned}$$

which gives

$$\begin{aligned} \|\dot{u}\|_{L^2} &\leq C(\|\tilde{u}_t + \tilde{u}\tilde{u}_x\|_{L^2} + 1) \\ &\leq C \left(\|\tilde{u}_{xt}\|_{L^2} + \|\tilde{u}_x\|_{L^4}^2 + \|\tilde{u}\tilde{u}_{xx}\|_{L^2} + 1 \right) \\ &\leq C(\|\tilde{u}_{xt}\|_{L^2} + \|\tilde{u}_{xx}\|_{L^2} + 1) \in L^2(0, T). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \int \tilde{\rho}(u - \tilde{u})^2 &= \int (\tilde{\rho} - \rho)(u - \tilde{u})^2 + \int \rho(u - \tilde{u})^2 \\ &\leq C\|\rho - \tilde{\rho}\|_{L^2} \|u - \tilde{u}\|_{L^4}^2 + C\nu \\ &\leq C\|\rho - \tilde{\rho}\|_{L^2} \|u - \tilde{u}\|_{L^2}^{\frac{3}{2}} \|(u - \tilde{u})_x\|_{L^2}^{\frac{1}{2}} + C\nu \\ &\leq \varepsilon \|u - \tilde{u}\|_{L^2}^2 + C\|\rho - \tilde{\rho}\|_{L^2}^4 \|(u - \tilde{u})_x\|_{L^2}^2 + C\nu \\ &\leq \varepsilon \|u - \tilde{u}\|_{L^2}^2 + C\nu^2 + C\nu, \end{aligned}$$

which together with (4.10) yields that

$$\sup_{0 \leq t \leq T} \|u - \tilde{u}\|_{L^2}^2 \leq C\nu. \quad (4.11)$$

Then, we obtain the desired results of Theorem 1.1. $\square$

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References

- [1] X.L. Ai, Z.L. Li and Y.L. Ye, Global strong solutions to Cauchy problem of 1D non-resistive MHD equations with no vacuum at infinity, *Acta Appl. Math.* 175 (2021), Paper No. 7.
- [2] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequality with weights, *Compositio Mathematica* 53 (1984), 259-275.
- [3] G.Q. Chen and D.H. Wang, Global solutions of nonlinear magnetohydrodynamics with large initial data, *J. Differential Equations* 182 (2002), 344-376.
- [4] J.S. Fan and Y.X. Hu, Global strong solutions to the 1-D compressible magnetohydrodynamic equations with zero resistivity, *J. Math. Phys.* 56 (2015), no. 2, 023101, 13 pp.
- [5] J.S. Fan and W.H. Yu, Strong solution to the compressible magnetohydrodynamic equations with vacuum, *Nonlinear Anal. Real World Appl.* 10 (2009), no. 1, 392-409.
- [6] J.S. Fan, S.X. Huang and F.C. Li, Global strong solutions to the planar compressible magnetohydrodynamic equations with large initial data and vacuum, *Kinet. Relat. Models.* 10 (2017), no. 4, 1035-1053.
- [7] E. Gagliardo, Ulteriori proprietà di alcune classi di funzioni in più variabili, *Ricerche Mat.* 8 (1959), 24-51.
- [8] G.Y. Hong, X.F. Hou, H.Y. Peng and C.J. Zhu, Global existence for a class of large solutions to three-dimensional compressible magnetohydrodynamic equations with vacuum, *SIAM J. Math. Anal.* 49 (2017), no. 4, 2409-2441.
- [9] X.P. Hu and D.H. Wang, Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows, *Arch. Ration. Mech. Anal.* 197 (2010), no. 1, 203-238.
- [10] S. Jiang and J.W. Zhang, On the non-resistive limit and the magnetic boundary-layer for one-dimensional compressible magnetohydrodynamics, *Nonlinearity* 30 (2017), no. 9, 3587-3612.
- [11] Q.S. Jiu, M.J. Li and Y.L. Ye, Global classical solution of the Cauchy problem to 1D compressible Navier-Stokes equations with large initial data, *J. Differential Equations* 257 (2014), no. 2, 311-350.
- [12] S. Kawashima, Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics, PhD thesis, Kyoto University, (1983).
- [13] S. Kawashima and M. Okada, Smooth global solutions for the one-dimensional equations in magnetohydrodynamics, *Proc. Japan Acad. Ser. A Math. Sci.* 58 (1982), 384-387.
- [14] H.L. Li, X.Y. Xu and J.W. Zhang, Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum, *SIAM J. Math. Anal.* 45 (2013), no. 3, 1356-1387.
- [15] J. Li and Z.L. Liang, On local classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum, *J. Math. Pures Appl.* 102 (2014), no. 4, 640-671.
- [16] Z.L. Li, H.Q. Wang and Y.L. Ye, Global strong solutions to the Cauchy problem of 1D compressible MHD equations with no resistivity, *Commun. Math. Sci.* 18 (2020), no. 3, 851-873.
- [17] Y. Li and L.Y. Jiang, Global weak solutions for the Cauchy problem to one-dimensional heat-conductive MHD equations of viscous non-resistive gas, *Acta Appl. Math.* 163 (2019), 185-206.
- [18] Z. Luo, Local existence of classical solutions to the two-dimensional viscous compressible flows with vacuum, *Commun. Math. Sci.* 10 (2012), no. 2, 527-554.
- [19] B.Q. Lv, X.D. Shi and X.Y. Xu, Global existence and large-time asymptotic behavior of strong solutions to the compressible magnetohydrodynamic equations with vacuum, *Indiana Univ. Math. J.* 65 (2016), no. 3, 925-975.
- [20] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa*, 13 (1959), 115-162.
- [21] V.A. Vaigant and A.V. Kazhikhov, On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid, (*Russian*) *Sibirsk. Mat. Zh.*, 36 (1995), no. 6, 1283-1316, ii; translation in *Siberian Math. J.* 36 (1995), no. 6, 1108-1141.
- [22] A.I. Volpert and S.I. Hudjaev, On the Cauchy problem for composite systems of nonlinear differential equations, *Math. USSR-Sb.* 16 (1972), 517-544.

- [23] X.Y. Xu and J.W. Zhang, A blow-up criterion for 3D non-resistive compressible magnetohydrodynamic equations with initial vacuum, *Nonlinear Anal. Real World Appl.* 12 (2011), 3442-3451.
- [24] L. Yao and C.J. Zhu, Existence and uniqueness of global weak solution to a two-phase flow model with vacuum, *Math. Ann.* 349 (2011), no. 4, 903-928.
- [25] Y.L. Ye, Global classical solution to 1D compressible Navier-Stokes equations with no vacuum at infinity, *Math. Methods Appl. Sci.* 39 (2016), no. 4, 776-795.
- [26] Y.L. Ye, Q.S. Jiu and C.S. Dou, Local well-posedness to the Cauchy problem of the 3-D compressible Navier-Stokes equations with density-dependent viscosity, *Acta Math. Sci. Ser. B (Engl. Ed.)* 34 (2014), no. 3, 851-871.
- [27] Y.L. Ye and Z.L. Li, Global strong solution to the Cauchy problem of 1D compressible MHD equations with large initial data and vacuum, *Z. Angew. Math. Phys.* 70 (2019), no. 1, Art. 38, 20 pp.
- [28] H.B. Yu, Global classical large solutions with vacuum to 1D compressible MHD with zero resistivity, *Acta Appl. Math.* 128 (2013), 193-209.
- [29] J.W. Zhang and F. Xie, Global solution for a one-dimensional model problem in thermally radiative magnetohydrodynamics, *J. Differential Equations* 245 (2008), no. 7, 1853-1882.
- [30] J.W. Zhang and J.N. Zhao, Some decay estimates of solutions for the 3-D compressible isentropic magnetohydrodynamics, *Commun. Math. Sci.* 8 (2010), no. 4, 835-850.
- [31] J.W. Zhang and X.K. Zhao, On the global solvability and the non-resistive limit of the one-dimensional compressible heat-conductive MHD equations, *J. Math. Phys.* 58 (2017), no. 3, 031504, 17 pp.