Research Article

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On well-posedness of semilinear Rayleigh-Stokes problem with fractional derivative on $\mathbb{R}^N$

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Abstract: We are devoted to the study of a semilinear time fractional Rayleigh-Stokes problem on $\mathbb{R}^N$, which is derived from a non-Newtonian fluid for a generalized second grade fluid with Riemann-Liouville fractional derivative. We show that a solution operator involving the Laplacian operator is very effective to discuss the proposed problem. In this paper, we are concerned with the global/local well-posedness of the problem, the approaches rely on the Gagliardo-Nirenberg inequalities, operator theory, standard fixed point technique and harmonic analysis methods. We also present several results on the continuation, a blow-up alternative with a blow-up rate and the integrability in Lebesgue spaces.

Keywords: Fractional derivative, Rayleigh-Stokes problem, well-posedness, integrability

MSC: 26A33; 34A12; 35R11

1 Introduction

Fractional calculus has proved a powerful tool to describe the viscoelasticity of fluids and anomalous diffusion phenomena, such as the constitutive relationship of the fluid models [21], basic random walk models [25], and so on. Besides in recent years, mainly due to the nonlocal characteristic of the fractional derivative, there are some excellent works on stochastic processes driven by fractional Brownian motion [13] and on physical phenomena like inverse problems for heat equation [19] and memory effect [2]. It is worth mentioning some solid works about time-fractional derivatives [1, 4, 7, 12, 14, 17, 37–39] and space-fractional derivatives [15, 16, 33] and the references therein, most of conclusions in these works commuted with fractional models are quite different from the situation of integer derivative, for instance, decay and asymptotical behaviors, blow-up analysis, well-posedness analysis, stability and Liouville property etc..

It is known that many significant complex media are non-Newtonian and exhibit time-dependent behavior of thixotropy and rheopecty [23, 28]. The behavior in non-Newtonian fluid often follows the power law [24]. Time-dependent non-Newtonian properties are more closely linked to fractional viscoelasticity than previously thought. Especially in a generalized second grade fluid, it is also more difficult to construct a simple mathematical model for describing many different behavior of non-Newtonian fluids. From this physics point

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of view, in the second grade fluid, the employed constitutive relationship has the following form:

$$\sigma = -pI + \mu \epsilon_1 + \varrho_1 \epsilon_2 + \varrho_2 \epsilon_1^2,$$

where σ is the Cauchy stress tensor, p is the hydrostatic pressure, I is the identity tensor. $\mu \geq 0$, ϱ_1 and ϱ_2 are normal stress moduli. ϵ_1 and ϵ_2 are the kinematical tensors defined by

$$\epsilon_1 = \nabla V + (\nabla V)^T, \quad \epsilon_2 = \frac{d\epsilon_1}{dt} + \epsilon_1(\nabla V) + (\nabla V)^T \epsilon_1,$$

where V is the velocity, ∇ is the gradient operator and the superscript T denotes a transpose operation. When we consider the time-dependent of time derivative in the kinematical tensor ϵ_1 , generally the constitutive relationship of viscoelastic second grade fluids has the form as follows:

$$\epsilon_2 = \partial_t^\alpha \epsilon_1 + \epsilon_1(\nabla V) + (\nabla V)^T \epsilon_1,$$

where ∂_t^α is the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ defined by

$$\partial_t^\alpha u(t, x) = \frac{1}{\Gamma(1-\alpha)} \partial_t \int_0^t (t-s)^{-\alpha} u(s, x) ds, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N,$$

provided the right-hand side is pointwise defined, where $\Gamma(\cdot)$ stands for the Euler's Gamma function. The form of the model was selected for its ability to portray accurately the temperature distribution in a generalized second grade fluid that subject to a linear flow on a heated flat plate and within a heated edge, see [32]. In this paper, we concern with the following semilinear time-fractional Rayleigh-Stokes problem

$$\begin{cases} \partial_t u - \partial_t^\alpha \Delta u - \Delta u = f(u), & t > 0, \\ u(0, x) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where Δ is the Laplacian operator, f is a semilinear function and φ is a given initial condition in $L^p(\mathbb{R}^N)$.

Such type of problems play a central role in describing the viscoelasticity of non-Newtonian fluids behavior and characteristic. Many researchers showed a strong interest in this issue and they also obtained some satisfactory results. Here is a short description on the closely related works and comparison to our results. The Rayleigh-Stokes problem for a generalized second grade fluid subject to a flow on a heated flat plate and within a heated edge was introduced by Shen et al. [32] where they considered the exact solutions of the velocity and temperature fields. As for a viscous Newtonian fluid, they revealed that the solutions of the Stokes' first problem appear in the limiting for these exact solutions. The results in [32] were generalized later by Xue and Nie [34] in a porous half-space with a heated flat plate, they also obtained an exact solution of the velocity field and temperature fields, from which some classical results can be recovered. Both methods of two above papers are based on the Fourier sine transform and the fractional Laplace transform. For the smooth and nonsmooth initial data on a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 1, 2, 3$), Bazhlekova et al. [5] considered the solutions of the homogeneous problem on $C([0, T]; L^2(\Omega)) \cap C((0, T]; H_0^1(\Omega) \cap H^2(\Omega))$ for the initial value $\varphi \in L^2(\Omega)$, moreover, some operator theory and spectrum technique are used to also establish the related Sobolev regularity of the solutions. In the meantime, Bazhlekova [6] obtained a well-posedness result under the abstract analysis framework of a subordination identity relating to the solution operator associated with a bounded C_0 -semigroup and a two parameters probability density function. Zhou and Wang [40] also established the existence results on $C([0, T]; L^2(\Omega))$ of nonlinear problem in operator theory on a bounded domain Ω with smooth boundary. As for some numerical solution of Rayleigh-Stokes problem with fractional derivatives, several scholars have considered and developed it, for example Zaky [36], Chen et al. [10, 11], Yang and Jiang [35] etc.. Additionally, most of fluid flows and transport processes use more distribution parameters to establish equations, the inverse problem of parameter identification has been proposed to deal with this matter, see Nguyen et al. [26, 27].

In view of the results in above works, it is natural to consider whether the well-posedness result on $\mathbb{R}^N$ can be extended to the mixed norm $L^p(L^q)$ spaces and whether it is possible to obtain the integrability of

solutions. Unfortunately, it turns out that these extensions cannot be established by applying the technique of subordination principle in [6]. Moreover, some useful $L^p - L^q$ inequality estimates about solution operator generated by the Eq. (1.1) are not easy to obtain, in which it is not immediately suitable to build an integral equation when we consider that the time-fractional derivative depends on all the past states, and also we cannot apply the approach of classical solution operators to derive the relevant estimates, while it is readily to achieve at the classical solution operators of type heat operator, like fractional diffusion equations [22] and fractional Navier-Stokes equations [30]. To overcome the difficulty from the effect of time-fractional derivative, we propose a different technique to estimate the solution operator by means of the Gagliardo-Nirenberg inequality and generalized Gagliardo-Nirenberg inequality. For the proof, by using an admissible triplet concept that depends on the time-fractional derivative order $\alpha \in (0, 1)$ and the exponents p, q in $L^p(L^q)$ spaces and their dimensions, we shall use the standard fixed point argument to establish main well-posedness results. We also consider a contain special space to make the local existence, that is due to the decay exponent just depends on the order of time-fractional derivative deriving from the solution operator. We find that the local solution will blow up in $L^r(\mathbb{R}^N)$, and then the rate of the blow up solution may depend on the exponent of nonlinearity and $r \geq 2$ in $L^r(\mathbb{R}^N)$. Also, based on the standard harmonic analysis methods, such as Marcinkiewicz interpolation theorem and doubly weighted Hardy-Littlewood-Sobolev inequality, some new conclusions likely the integrability of global mild solutions in Lebesgue spaces $L^\mu(0, \infty; L^r(\mathbb{R}^N))$ are investigated.

This paper is organized as follows. In Section 2, we give some concepts about fractional calculus in Banach space and we introduce several useful analytic properties of Laplacian operator. By a rigorous analysis of solution operator $S(t)$, we establish two crucial estimates that will be used throughout this paper. After introducing a definition of mild solution in Section 3, the first subsection shows the global and local well-posedness of the semilinear problem (1.1). Further, we obtain continuation and blow-up alternative of local mild solution of problem. In the last subsection, we show several integrability results of the global mild solution in Lebesgue space.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space and let $\mathcal{L}(X, Y)$ stand for the space of all linear bounded operators maps Banach space X into Banach space Y , we remark that $C_b(\mathbb{R}_+, X)$ stands for the space of bounded continuous functions which is defined on $\mathbb{R}_+$ and takes values in X , equipped with the norm $\sup_{t \in \mathbb{R}_+} \|\cdot\|_X$ and $C(J, X)$ stands for the space of continuous functions which is defined on an interval $J \subseteq \mathbb{R}_+$ and takes values in X . If A is a linear closed operator, the symbols $\rho(A)$ and $\sigma(A)$ are called the resolvent set and the spectral set of A , respectively, identity $R(\lambda; A) = (\lambda I - A)^{-1}$ is the resolvent operator of A . We will denote by $D(A^\alpha)$, $\alpha \in (0, 1)$, the fractional power spaces associated with the linear closed operator A .

An operator A is called the sectorial operator, if it follows the next concept.

Definition 2.1. Let A be a densely defined linear closed operator on Banach space X , then A is called a sectorial operator if there exist $C > 0$ and $\theta \in (0, \pi/2)$ such that

$$\Sigma'_\theta = \{z \in \mathbb{C} : z \neq 0, \theta \leq |\arg z| \leq \pi\} \cup \{0\} \subset \rho(A),$$

and $\|R(z; A)\| \leq M/|z|$ for $z \in \Sigma'_\theta$, $z \neq 0$.

Additionally, from [8, Theorem 2.3.2], it is not difficult to check that the Laplacian operator Δ with maximal domain $D(\Delta) = \{u \in X : \Delta u \in X\}$ generates a bounded analytic semigroup of the spectral angle less than or equal to $\pi/2$ on $X := L^p(\mathbb{R}^N)$ with $1 \leq p < +\infty$. Moreover, the spectrum is given by $\sigma(-\Delta) = [0, +\infty)$ for $1 < p < +\infty$.

For $\delta > 0$ and $\theta \in (0, \pi/2)$ we introduce the contour $\Gamma_{\delta, \theta}$ defined by

$$\Gamma_{\delta, \theta} = \{re^{-i\theta} : r \geq \delta\} \cup \{\delta e^{i\psi} : |\psi| \leq \theta\} \cup \{re^{i\theta} : r \geq \delta\},$$

where the circular arc is oriented counterclockwise, and the two rays are oriented with an increasing imaginary part. In the sequel, let $A = -\Delta$, then A is a densely defined linear closed operator on Banach space $L^p(\mathbb{R}^N)$ with $1 < p < +\infty$, we define a linear operator $S(t)$ by means of Dunford integral as follows

$$S(t) = \frac{1}{2\pi i} \int_{\Gamma_{\delta, \theta}} e^{zt} H(z) dz, \quad (2.1)$$

where

$$H(z) = \frac{g(z)}{z} R(g(z), -A), \quad g(z) = \frac{z}{1 + z^\alpha}.$$

Remark 2.1. It is worth noting that the author [6] applied the technique of the subordination principle to study the solution operator in (2.1) of problem (1.1), that is,

$$S(t) = \int_0^\infty \phi(t, \tau) T(\tau) d\tau,$$

where $T(t)$ is a bounded C_0 -semigroup and function $\phi(t, \tau)$ is a probability density function with respect to both variables t and τ , that is $\int_0^\infty \phi(t, \tau) d\tau = \int_0^\infty \phi(t, \tau) dt = 1$. Nevertheless, we also find that it is hard to get the estimate of $AS(t)$ on a Banach space unless this estimate $\|AT(t)\|_{\mathcal{L}(X)} \leq M$ may be valid for all $t \geq 0$, constant $M > 0$ under the bounded analytic semigroup of $T(t)$. From this point of view, we can not apply the subordination principle to estimate the operator defined as in (2.1) in the current paper.

Recall that for any $\theta > 0$

$$\Sigma_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}.$$

It should be noticed that estimates of the operator $S(t)$ are standard in the theory of analytic semigroups as follows.

Lemma 2.1. [3, Lemma 4.1.1] Given $\theta \in (0, \pi/2)$, let $\mathcal{C}$ be an arbitrary piecewise smooth simple curve in $\Sigma_{\theta+\pi/2}$ running from $\infty e^{-i(\theta+\pi/2)}$ to $\infty e^{i(\theta+\pi/2)}$, and let X be a Banach space. Suppose that the map $f : \Sigma_{\theta+\pi/2} \times X \times \mathbb{R}^+ \rightarrow X$ has the following properties:

- (i) $f(\cdot, x, t) : \Sigma_{\theta+\pi/2} \rightarrow X$ is holomorphic for $(x, t) \in X \times \mathbb{R}^+$.
- (ii) $f(z, \cdot, \cdot) \in C(X \times \mathbb{R}^+, X)$ for $z \in \Sigma_{\theta+\pi/2}$.
- (iii) There are constants $\varrho \in \mathbb{R}$ and $M > 0$ such that

$$\|f(z, x, t)\| \leq M|z|^{\varrho-1} e^{t\operatorname{Re}(z)}, \quad (z, x, t) \in \Sigma_{\theta+\pi/2} \times X \times \mathbb{R}^+.$$

Then

$$(x, t) \mapsto \int_{\mathcal{C}} f(z, x, t) dz \in C(X \times \mathbb{R}^+, X),$$

and

$$\left\| \int_{\mathcal{C}} f(z, x, t) dz \right\| \leq M t^{-\varrho}, \quad (x, t) \in X \times \mathbb{R}^+.$$

Lemma 2.2. Let $\alpha \in (0, 1)$. The operator $S(t)$ defined in (2.1) is well defined, $S(t)x \in C([0, \infty); X)$ and $S(t)x \rightarrow x$ as $t \rightarrow 0$ for any $x \in X$. Furthermore, there exists a constant $M > 0$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq M, \quad \text{for } t \geq 0,$$

and moreover there exists $C > 0$ such that $AS(t)x \in C((0, \infty); X)$, we have

$$\|AS(t)\|_{\mathcal{L}(X)} \leq C t^{\alpha-1}, \quad \text{for } t > 0. \quad (2.2)$$

Proof. Let $t > 0$, $\theta \in (0, \pi/2)$, $\delta > 0$. We choose $\delta = 1/t$, since operator $-A$ generates a bounded analytic semigroup of the spectral angle is less than or equal to $\pi/2$, i.e., for any $\theta \in (0, \pi/2)$

$$\|(z + A)^{-1}\|_{\mathcal{L}(X)} \leq M/|z|, \quad z \in \Sigma_{\theta+\pi/2}, \quad (2.3)$$

where $\Sigma_{\theta+\pi/2} \subset \rho(-A)$. As a similarly approach in [5, Lemma 2.1], we conclude that $g(z) \in \Sigma_{\theta+\pi/2}$ and $|g(z)| \leq M|z|^{1-\alpha}$ for any $z \in \Sigma_{\theta+\pi/2}$, and thus from Lemma 2.2 we get $\|S(t)\|_{\mathcal{L}(X)} \leq M$ and $S(t)x \in C((0, \infty); X)$ which shows the well-defined part for $t > 0$. Additionally, for the Laplace transform of $S(t)$, by virtue of Fubini's theorem and Cauchy's integral formula, we obtain for $\lambda > 0$,

$$\begin{aligned} \widehat{S}(\lambda) &= \int_0^\infty e^{-\lambda t} S(t) dt = \frac{1}{2\pi i} \int_0^\infty \int_{\Gamma_{\delta, \theta}} e^{-\lambda t} e^{\mu t} H(\mu) d\mu dt \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\delta, \theta}} (\lambda - \mu)^{-1} H(\mu) d\mu \\ &= H(\lambda). \end{aligned}$$

Consequently, in order to prove the limit point at $t = 0$ in $S(t)x$ for $x \in X$, the similar technique as in [31, Corollary 2.2] deduces that $S(z) \rightarrow I$ strongly as $z \rightarrow 0$, $z \in \Sigma_{\theta+\pi/2}$. This means that $S(t)x \in C([0, \infty); X)$. Moreover, by using the identity

$$AH(z) = (-H(z) + z^{-1}I)g(z),$$

we see from (2.3) that

$$\|AH(z)\|_{\mathcal{L}(X)} \leq M(M+1)|z|^{-\alpha}, \quad \text{for any } z \in \Sigma_{\theta+\pi/2}.$$

Thus, the estimate (2.2) of $AS(t)$ for $t > 0$ is given by

$$\begin{aligned} \|AS(t)\|_{\mathcal{L}(X)} &\leq \int_{\Gamma_{1/t, \pi-\theta_1}} e^{\operatorname{Re}(z)t} \|AH(z)\|_{\mathcal{L}(X)} |dz| \\ &\leq M(M+1) \left(2 \int_{1/t}^\infty r^{-\alpha} e^{-rt \cos(\theta_1)} dr + \int_{-\pi+\theta_1}^{\pi-\theta_1} t^{\alpha-1} e^{\cos(\psi)} d\psi \right) \\ &\leq M_\alpha t^{\alpha-1}, \end{aligned}$$

where $\theta_1 = \pi/2 - \theta$, M_α is a positive constant and it may depend on M , α and θ . Consequently, it follows that $AS(t)x \in X$ for $x \in X$, $t > 0$. Moreover, $AS(t)x \in C((0, \infty); X)$ according to Lemma 2.1 for $x \in X$. The proof is completed. $\square$

The inequality in (2.2) enables us to get another estimate about $S(t)x$ in fractional power spaces. To do this, we need the following inequality.

Lemma 2.3. [29] Let $\omega \in (0, 1)$. Then there exists a constant $C > 0$ such that

$$\|A^\omega x\| \leq C \|x\|^{1-\omega} \|Ax\|^\omega, \quad x \in D(A).$$

Lemma 2.4. Let $\alpha \in (0, 1)$ and $\omega \in (0, 1)$. Then there exists a constant $C > 0$ such that $S(t)x \in D(A^\omega)$ for all $x \in X$ and for every $t > 0$, moreover,

$$\|A^\omega S(t)x\| \leq Ct^{-(1-\alpha)\omega} \|x\|, \quad \text{for } t > 0.$$

and $A^\omega S(t)x \in C((0, \infty); X)$ for $x \in X$.

Proof. This conclusion is an immediate result of Lemma 2.2 and Lemma 2.3. So, we omit it. $\square$

Now, we introduce the concept of admissible triplet.

Definition 2.2. We call (p, q, μ) as an admissible triplet with respect to $\alpha \in (0, 1)$ if

$$\frac{1}{\mu} = \frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{q} \right), \quad \text{for } N \geq 1, \quad 1 \leq p \leq q \leq +\infty.$$

Remark 2.2. For the definition of admissible triplet, it is inspired by [20] where they concerned with space-time estimates to a fractional integro-differential equation, in the meantime, one finds that this concept matches the $L^p - L^q$ estimates of operator $S(t)$ appropriately, see below lemma 2.5. Furthermore, it is worth noting that $\mu = \mu(p, q)$ is completely determined by p and q .

Thenceforth, we give some useful $L^p - L^q$ estimates about the linear operator $S(t)$.

Lemma 2.5. The operator $S(t)$ has the following properties:

(i) Let $N \geq 1$, if $v \in L^p(\mathbb{R}^N)$ for $1 < p < +\infty$, $p \leq q \leq +\infty$, then there exists a constant $C > 0$ such that for $t > 0$

$$\|S(t)v\|_{L^q(\mathbb{R}^N)} \leq Ct^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{q} \right)} \|v\|_{L^p(\mathbb{R}^N)}.$$

(ii) Let $N > 1$, if $v \in L^p(\mathbb{R}^N)$ for $1 < p < N$, $p \leq q < +\infty$, then there exists a constant $C > 0$ such that for $t > 0$

$$\|A^{\frac{1}{2}}S(t)v\|_{L^q(\mathbb{R}^N)} \leq Ct^{-\frac{1-\alpha}{2} \left(1+N \left(\frac{1}{p} - \frac{1}{q} \right) \right)} \|v\|_{L^p(\mathbb{R}^N)}.$$

Proof. Using the classical Gagliardo-Nirenberg inequality, we know that there exists a constant $C > 0$ such that

$$\|S(t)v\|_{L^q(\mathbb{R}^N)} \leq C \|AS(t)v\|_{L^p(\mathbb{R}^N)}^\theta \|S(t)v\|_{L^p(\mathbb{R}^N)}^{1-\theta},$$

where $\frac{1}{q} = \theta \left(\frac{1}{p} - \frac{2}{N} \right) + (1-\theta)\frac{1}{p}$ for any $\theta \in [0, 1]$ with respect to each $N \geq 1$. Thus, from (2.2) in Lemma 2.2, we have

$$\|S(t)v\|_{L^q(\mathbb{R}^N)} \leq Ct^{(\alpha-1)\theta} \|v\|_{L^p(\mathbb{R}^N)}^\theta \|S(t)v\|_{L^p(\mathbb{R}^N)}^{1-\theta},$$

it follows that

$$\|S(t)v\|_{L^q(\mathbb{R}^N)} \leq Ct^{(\alpha-1)\theta} \|v\|_{L^p(\mathbb{R}^N)},$$

taking the exponents of p, q into above inequality, we immediately obtain the $L^p - L^q$ estimate of operator $S(t)$. Hence, we have showed (i).

On the other hand, similarly, by virtue of the Gagliardo-Nirenberg inequality of fractional version, (see e.g. [18, Corollary 2.3.]), in view of Lemma 2.4, there exists a constant $C > 0$ such that

$$\|A^{\frac{1}{2}}S(t)v\|_{L^q(\mathbb{R}^N)} \leq C \|AS(t)v\|_{L^p(\mathbb{R}^N)}^\theta \|A^{\frac{1}{2}}S(t)v\|_{L^p(\mathbb{R}^N)}^{1-\theta},$$

where $\frac{1}{q} = \theta \left(\frac{1}{p} - \frac{1}{N} \right) + (1-\theta)\frac{1}{p}$ for any $\theta \in (0, 1)$. Thus, the $L^p - L^q$ estimate of $A^{\frac{1}{2}}S(t)v$ follows. The proof of (ii) is completed. $\square$

In the sequel, we set a function W_f with respect to $f \in L^1(0, T; X)$ with any $T > 0$ (or $T = +\infty$), given by

$$W_f(t) = \int_0^t S(t-s)f(s)ds,$$

in which we will prove some properties of this function.

Lemma 2.6. Let $0 < T < +\infty$. If $f \in L^1(0, T; X)$, then $W_f(\cdot) \in C([0, T]; X)$. If $f \in L^p(0, T; X)$ with $p > \frac{1}{1-(1-\alpha)\omega}$ for some $0 < \omega \leq 1$, then $A^\omega W_f(\cdot) \in C([0, T]; X)$. Furthermore, let $1 < p < +\infty$, if $r \in [p, +\infty]$ satisfies $\frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right) < 1$ and there is $\xi \in [0, 1)$ such that $\sup_{t \in [0, T]} t^\xi \|f(t)\|_{L^p(\mathbb{R}^N)} < +\infty$, then $W_f(\cdot) \in C((0, T]; L^r(\mathbb{R}^N))$ and $W_f(\cdot) \in C([0, T]; L^r(\mathbb{R}^N))$ provided with $\xi < 1 - \frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right)$.

Proof. Observe that for any $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$W_f(t_2) - W_f(t_1) = \int_{t_1}^{t_2} S(t_2 - s)f(s)ds + \int_0^{t_1} (S(t_2 - s) - S(t_1 - s))f(s)ds.$$

Since $f \in L^1(0, T; X)$ and from Lemma 2.2, it follows that

$$\left\| \int_{t_1}^{t_2} S(t_2 - s)f(s)ds \right\|_X \leq M \int_{t_1}^{t_2} \|f(s)\|_X ds \rightarrow 0, \quad \text{as } t_2 \rightarrow t_1.$$

Additionally, we have

$$\|(S(t_2 - s) - S(t_1 - s))f(s)\|_X \leq 2M\|f(s)\|_X, \quad s \in [0, t],$$

which is integrable in $L^1(0, t; X)$. By virtue of $S(t)f(\cdot) \in C([0, T]; X)$, we thus conclude that $W_f(\cdot) \in C([0, T]; X)$ by Lebesgue's dominated convergence theorem. Next, for $\omega \in (0, 1]$, we know $(1 - \alpha)\omega \in (0, 1)$. By Lemma 2.4 we obtain

$$\begin{aligned} \left\| \int_{t_1}^{t_2} A^\omega S(t_2 - s)f(s)ds \right\|_X &\leq C \int_{t_1}^{t_2} (t_2 - s)^{-(1-\alpha)\omega} \|f(s)\|_X ds \\ &\leq C \left(\frac{p-1}{-(1-\alpha)\omega p + p-1} \right)^{1-\frac{1}{p}} (t_2 - t_1)^{-(1-\alpha)\omega + 1-\frac{1}{p}} \|f\|_{L^p(0, T; X)}, \end{aligned}$$

which tends to zero as $t_2 \rightarrow t_1$. On the other hand, we have

$$\|A^\omega(S(t_2 - s) - S(t_1 - s))f(s)\|_X \leq 2C(t_1 - s)^{-(1-\alpha)\omega} \|f(s)\|_X, \quad s \in [0, t],$$

which is integrable in $L^1(0, T; X)$. By virtue of $A^\omega S(t)f(\cdot) \in C((0, T]; X)$, we thus conclude that $A^\omega W_f(\cdot) \in C([0, T]; X)$ by Lebesgue's dominated convergence theorem.

By Lemma 2.5, for $t_1, t_2 \in (0, T]$ with $t_1 < t_2$, we see from $\sup_{t \in [0, T]} t^\xi \|f(t)\|_{L^p(\mathbb{R}^N)} < +\infty$ for $\xi \in [0, 1]$ that

$$\begin{aligned} \left\| \int_{t_1}^{t_2} S(t_2 - s)f(s)ds \right\|_{L^r(\mathbb{R}^N)} &\leq M \int_{t_1}^{t_2} (t_2 - s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right)} \|f(s)\|_{L^p(\mathbb{R}^N)} ds \\ &\leq M t_2^{1-\xi - \frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right)} \int_{t_1/t_2}^1 (1-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right)} s^{-\xi} ds, \end{aligned}$$

which tends to zero as $t_2 \rightarrow t_1$ by the properties of incomplete Beta function. Moreover,

$$\begin{aligned} \|(S(t_2 - s) - S(t_1 - s))f(s)\|_{L^r(\mathbb{R}^N)} &\leq 2M(t_1 - s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right)} \|f(s)\|_{L^p(\mathbb{R}^N)} \\ &\leq 2M(t_1 - s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right)} s^{-\xi} \sup_{s \in [0, T]} s^\xi \|f(s)\|_{L^p(\mathbb{R}^N)}, \end{aligned}$$

which is integrable in $L^1(0, t_1)$. Therefore, it follows from the similar method that $W_f(\cdot) \in C((0, T]; L^r(\mathbb{R}^N))$.

In addition, if $\xi < 1 - \frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right)$, then it is easy to check that there exists a constant $C > 0$ such that

$$\|W_f(t)\|_{L^r(\mathbb{R}^N)} \leq C t^{1-\xi - \frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right)}.$$

This implies that $W_f(t)$ tends to zero as $t \rightarrow 0$ in $L^r(\mathbb{R}^N)$. Thus, $W_f(\cdot) \in C([0, T]; L^r(\mathbb{R}^N))$. The proof is completed. $\square$

3 Well-posedness

Let u be a solution of problem (1.1), taking the Laplace transform into (1.1) yields

$$\hat{u}(z) = H(z)\varphi + H(z)\hat{f}(u)(z),$$

by means of the inverse Laplace transform, we thus derive an integral representation of problem (1.1) by

$$u(t) = S(t)\varphi + \int_0^t S(t-s)f(u(s))ds, \quad (3.1)$$

following this, we regard $S(t)$ defined in (2.1) as the solution operator of problem (1.1). Next, we introduce the definitions of global/local mild solutions to the problem (1.1).

Definition 3.1. Let $p > 1$.

- (i) A continuous function $u : [0, +\infty) \rightarrow L^p(\mathbb{R}^N)$ satisfying (3.1) for $t \in [0, +\infty)$ is called a global mild solution to problem (1.1) in $L^p(\mathbb{R}^N)$.
- (ii) If there exists $0 < T < +\infty$ such that a continuous function $u : [0, T] \rightarrow L^p(\mathbb{R}^N)$ satisfies (3.1) for $t \in [0, T]$, we say that u is a local mild solution to problem (1.1) in $L^p(\mathbb{R}^N)$.

In the sequel, the well-posedness of problem (1.1) will be considered, in order to achieve this goal, the following general hypothesis of the semilinear function introduced by [9] will be also considered. Let r', r be the conjugate indices.

(Hf) We suppose that $f(0) = 0$ and $f : L^r(\mathbb{R}^N) \rightarrow L^{r'}(\mathbb{R}^N)$, for some

$$r \in \left[2, \frac{2N}{N-2} \right), \text{ if } N \geq 2, \quad (r \in [2, \infty], \text{ if } N = 1).$$

Additionally, we suppose that there exist constants $\sigma \geq 0$ and $K > 0$ such that

$$\|f(u) - f(v)\|_{L^{r'}(\mathbb{R}^N)} \leq K(\|u\|_{L^r(\mathbb{R}^N)}^\sigma + \|v\|_{L^r(\mathbb{R}^N)}^\sigma)\|u - v\|_{L^r(\mathbb{R}^N)},$$

for all $u, v \in L^r(\mathbb{R}^N)$.

We first consider the case $T = +\infty$, i.e., the global well-posedness of the problem for mild solutions. For any $\alpha \in (0, 1)$, let (p, r, μ) be an admissible triplet such that $1 < r' \leq p < r \leq +\infty$, consider the Banach space X_{pr} of continuous functions $v : [0, \infty) \rightarrow L^p(\mathbb{R}^N)$ equipped with its natural norm

$$\|v\|_{X_{pr}} = \sup_{t \geq 0} \|v(t)\|_{L^p(\mathbb{R}^N)} + \sup_{t \geq 0} t^{\frac{1}{\mu}} \|v(t)\|_{L^{r'}(\mathbb{R}^N)}.$$

Theorem 3.1. Let $1 \leq N < \frac{2}{(1-\alpha)(1-2/r)}$ and $\frac{1}{\mu} = \frac{1}{\sigma} \left(1 - \frac{N(1-\alpha)}{2} \left(1 - \frac{2}{r} \right) \right)$. If (Hf) holds and there exists $\lambda > 0$ for all $\varphi \in L^p(\mathbb{R}^N)$ such that $\|\varphi\|_{L^p(\mathbb{R}^N)} \leq \lambda$. Then problem (1.1) admits a unique global mild solution $u \in X_{pr}$. If u and $\tilde{u}$ are solutions starting at φ and ψ , both values on $L^p(\mathbb{R}^N)$ respectively, then there exists a constant $C > 0$ such that

$$\|u - \tilde{u}\|_{X_{pr}} \leq C\|\varphi - \psi\|_{L^p(\mathbb{R}^N)}.$$

Proof. Let $\varepsilon > 0$ and set

$$\Omega_\varepsilon = \{u \in X_{pr} : \|u\|_{X_{pr}} \leq 2\varepsilon\}.$$

It is easy to see that Ω_ε is a closed ball of X_{pr} with center 0 and radius 2ε . Define the operator Φ in Ω_ε as

$$\Phi(u)(t) = S(t)\varphi + \int_0^t S(t-s)f(u(s))ds. \quad (3.2)$$

The proof of the existence of unique global solution to problem (1.1) is based on the contraction mapping technique. From this point, we shall need some estimates which comes from this argument, we recall Lemma 2.2 and Lemma 2.5, it follows that there exists a constant $C > 0$ such that

$$\|S(t)\varphi\|_{L^p(\mathbb{R}^N)} \leq C\|\varphi\|_{L^p(\mathbb{R}^N)} \quad \text{and} \quad t^{\frac{1}{\mu}}\|S(t)\varphi\|_{L^r(\mathbb{R}^N)} \leq C\|\varphi\|_{L^p(\mathbb{R}^N)}.$$

Let us define $\lambda := \varepsilon/(2C)$ and observe that $\varphi \in L^p(\mathbb{R}^N)$ with $\|\varphi\|_{L^p(\mathbb{R}^N)} \leq \lambda$, we thus get $S(t)\varphi \in X_{pr}$. Moreover, for any $u \in \Omega_\varepsilon$, from the hypothesis (Hf) we deduce that

$$\|f(u(t))\|_{L^r(\mathbb{R}^N)} \leq K\|u(t)\|_{L^r(\mathbb{R}^N)}^{\sigma+1}, \quad \text{for } t \geq 0.$$

Form the choice of μ that also determined by (p, r) , we get

$$\frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{p} \right) + \frac{N(1-\alpha)}{2} \left(\frac{1}{p} - \frac{1}{r} \right) (\sigma+1) = 1,$$

which implies that $\frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{p} \right) < 1$ and $\sigma+1 < \mu$, combined with Lemma 2.5, we can easily derive the following estimates

$$\begin{aligned} \|W_{f(u)}(t)\|_{L^p(\mathbb{R}^N)} &\leq \int_0^t \|S(t-s)f(u(s))\|_{L^p(\mathbb{R}^N)} ds \\ &\leq C \int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{p} \right)} \|f(u(s))\|_{L^r(\mathbb{R}^N)} ds \\ &\leq CK \int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{p} \right)} s^{-\vartheta} \left(\sup_{s \geq 0} s^{\frac{1}{\mu}} \|u(s)\|_{L^r(\mathbb{R}^N)} \right)^{\sigma+1} ds \\ &\leq CKB(\vartheta, 1-\vartheta) \|u\|_{X_{pr}}^{\sigma+1}, \end{aligned}$$

where $B(\cdot, \cdot)$ stands for Beta function, $\vartheta = (\sigma+1)/\mu \in (0, 1)$, and $1 - \frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{p} \right) = \vartheta$.

Consequently, one derives

$$\|W_{f(u)}(t)\|_{L^p(\mathbb{R}^N)} \leq CKB(\vartheta, 1-\vartheta) (2\varepsilon)^{\sigma+1}.$$

On the other hand, the choice of μ implies that $\frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{r} \right) < 1$, as the same way in above arguments, we have

$$\begin{aligned} \|W_{f(u)}(t)\|_{L^r(\mathbb{R}^N)} &\leq \int_0^t \|S(t-s)f(u(s))\|_{L^r(\mathbb{R}^N)} ds \\ &\leq C \int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{r} \right)} \|f(u(s))\|_{L^r(\mathbb{R}^N)} ds \\ &\leq CK \int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{r} \right)} s^{-\vartheta} \left(\sup_{s \geq 0} s^{\frac{1}{\mu}} \|u(s)\|_{L^r(\mathbb{R}^N)} \right)^{\sigma+1} ds \\ &\leq CKB \left(1 - \frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{r} \right), 1-\vartheta \right) \|u\|_{X_{pr}}^{\sigma+1} t^{-\frac{1}{\mu}} \\ &= CKB \left(\frac{\sigma}{\mu}, 1-\vartheta \right) \|u\|_{X_{pr}}^{\sigma+1} t^{-\frac{1}{\mu}}. \end{aligned}$$

It follows that

$$t^{\frac{1}{\mu}} \|W_{f(u)}(t)\|_{L^r(\mathbb{R}^N)} \leq CKB \left(\frac{\sigma}{\mu}, 1-\vartheta \right) (2\varepsilon)^{\sigma+1}.$$

Noting that for $\vartheta = (\sigma + 1)/\mu > 0$

$$B(\vartheta, 1 - \vartheta) = \int_0^1 z^{\vartheta-1} (1-z)^{-\vartheta} dz \leq \int_0^1 z^{\frac{\sigma}{\mu}-1} (1-z)^{-\vartheta} dz = B(\sigma/\mu, 1 - \vartheta),$$

for choosing $\varepsilon \leq \left(\frac{1}{2^{\sigma+3} CKB(\sigma/\mu, 1-\vartheta)} \right)^{1/\sigma}$, we thus get $\|W_{f(u)}\|_{X_{pr}} \leq \varepsilon$ for $u \in \Omega_\varepsilon$. Hence, by the same choice of ε , it yields

$$\begin{aligned} \|S(t)\varphi\|_{X_{pr}} + \|W_{f(u)}\|_{X_{pr}} &\leq 2C\|\varphi\|_{L^p(\mathbb{R}^N)} + 2CKB(\sigma/\mu, 1 - \vartheta)(2\varepsilon)^{\sigma+1} \\ &\leq 2C\lambda + \varepsilon/2 \leq 2\varepsilon, \quad \text{for } u \in \Omega_\varepsilon. \end{aligned}$$

In addition, we now concern with continuity properties of (3.2). By virtue of Lemma 2.2 and similarly to Lemma 2.6, we know that $\Phi(u) \in C([0, \infty); L^p(\mathbb{R}^N)) \cap C((0, \infty); L^r(\mathbb{R}^N))$. Consequently, operator Φ maps Ω_ε into itself.

From the assumption of f and Lemma 2.5, for any $u, v \in \Omega_\varepsilon$, we further obtain that

$$\begin{aligned} &\|\Phi(u)(t) - \Phi(v)(t)\|_{L^p(\mathbb{R}^N)} \\ &\leq \int_0^t \|S(t-s)(f(u(s)) - f(v(s)))\|_{L^p(\mathbb{R}^N)} ds \\ &\leq C \int_0^t (t-s)^{-\vartheta} \|f(u(s)) - f(v(s))\|_{L^{r'}(\mathbb{R}^N)} ds \\ &\leq CK \int_0^t (t-s)^{-\vartheta} (\|u(s)\|_{L^r(\mathbb{R}^N)}^\sigma + \|v(s)\|_{L^{r'}(\mathbb{R}^N)}^\sigma) \|u(s) - v(s)\|_{L^r} ds \\ &\leq 2CKB(\vartheta, 1 - \vartheta)(2\varepsilon)^\sigma \|u - v\|_{X_{pr}}, \end{aligned}$$

with a similar argument, we get

$$t^{\frac{1}{\mu}} \|\Phi(u)(t) - \Phi(v)(t)\|_{L^r(\mathbb{R}^N)} \leq 2CKB(\sigma/\mu, 1 - \vartheta)(2\varepsilon)^\sigma \|u - v\|_{X_{pr}}.$$

For the choice of ε , we have

$$\|\Phi(u) - \Phi(v)\|_{X_{pr}} \leq \frac{1}{2} \|u - v\|_{X_{pr}}, \quad (3.3)$$

which shows that Φ is a strict contraction on Ω_ε . Thus Φ has a fixed point u , and it is the unique solution of problem (1.1).

Next, it just remains to prove the continuous dependence upon the initial data. Let $\tilde{u}$ be another mild solution of problem (1.1) associated with initial data $\psi \in L^p(\mathbb{R}^N)$. We perform as in (3.3) to get

$$\|u - \tilde{u}\|_{X_{pr}} \leq C\|\varphi - \psi\|_{L^p(\mathbb{R}^N)} + \frac{1}{2} \|u - \tilde{u}\|_{X_{pr}},$$

and then the continuous dependence follows. The proof is completed. $\square$

Remark 3.1. It is notice that if the solution takes value in $L^p(\mathbb{R}^N)$ of $p = 2$, the choice of μ in $\frac{1}{\mu} = \frac{1}{\sigma} \left(1 - \frac{N(1-\alpha)}{2} \left(1 - \frac{2}{r} \right) \right)$ for $1 \leq N < \frac{2}{(1-\alpha)(1-2/r)}$ combined with the admissible triplet $(2, r, \mu)$ can reduce to $r = \frac{2(\sigma+2)N(1-\alpha)}{(\sigma+2)N(1-\alpha)-4}$ provided with $N \geq \frac{4}{(\sigma+2)(1-\alpha)}$, and we find the explicit value $\mu = \sigma + 2$. Furthermore, let us take the semilinear function $f(u) = \varrho|u|^\sigma u$ to the current problem for $\varrho \in \mathbb{R}$, if $N \geq 1$, $\sigma = \frac{4}{N(1-\alpha)}$ and $r = \sigma + 2$, or if $\sqrt{\frac{5-\alpha}{1-\alpha}} - 1 \leq N < \frac{1+\sqrt{4(1-\alpha)+1}}{1-\alpha}$, $\sigma = N$ and $r = \sigma + 2$, then from Theorem 3.1, we know that there exists a unique global solution.

Corollary 3.1. Let $\sqrt{\frac{5-\alpha}{1-\alpha}} - 1 \leq N < \frac{1+\sqrt{4(1-\alpha)+1}}{1-\alpha}$. For the problem

$$\partial_t u - \partial_t^\alpha \Delta u - \Delta u = \varrho|u|^N u, \quad \varrho \in \mathbb{R},$$

associated with initial value condition $u(0, x) = \varphi(x)$ for $x \in \mathbb{R}^N$, if there exists $\lambda > 0$ such that $\|\varphi\|_{L^2(\mathbb{R}^N)} \leq \lambda$. Then there admits a unique global mild solution on space X of continuous functions $v : [0, \infty) \rightarrow L^2(\mathbb{R}^N)$ equipped with its norm

$$\|v\|_X = \sup_{t \geq 0} \|v(t)\|_{L^2(\mathbb{R}^N)} + \sup_{t \geq 0} t^{\frac{1}{N+2}} \|v(t)\|_{L^{N+2}(\mathbb{R}^N)}.$$

In particular, from the restriction on μ in Theorem 3.1 and the admissible triplet $(2, N+2, \mu)$, the dimension N in Corollary 3.1 is $N = \sqrt{\frac{4}{1-\alpha}}$ for some suitable $\alpha \in (0, 1)$.

Now, let us turn to the case $T > 0$ and discuss the local well-posedness of the problem. Let $\alpha \in (0, 1)$ and (p, r, μ) be an admissible triplet such that $1 < p \leq r < +\infty$ and $p = r' < N$, and

$$\begin{cases} 2 \leq r \leq +\infty, & \text{if } 1 < N \leq \frac{2}{1-\alpha}, \\ 2 \leq r < \frac{2N(1-\alpha)}{N(1-\alpha)-2}, & \text{if } N > \frac{2}{1-\alpha}. \end{cases}$$

Consider the Banach space $Y_{pr}[T]$ of continuous functions $v : [0, T] \rightarrow L^p(\mathbb{R}^N)$ under this admissible triplet by satisfying

$$\begin{aligned} t^{\frac{1}{\mu}} v &\in C_b([0, T]; L^r(\mathbb{R}^N)), \quad \lim_{t \rightarrow 0} t^{\frac{1}{\mu}} v(t) = 0, \\ t^{\frac{1-\alpha}{2}} (-\Delta)^{\frac{1}{2}} v &\in C_b([0, T]; L^p(\mathbb{R}^N)), \quad \lim_{t \rightarrow 0} t^{\frac{1-\alpha}{2}} (-\Delta)^{\frac{1}{2}} v(t) = 0, \end{aligned}$$

equipped with its natural norm

$$\|v\|_{Y_{pr}[T]} = \sup_{t \in (0, T]} t^{\frac{1}{\mu}} \|v(t)\|_{L^r(\mathbb{R}^N)} + \sup_{t \in (0, T]} t^{\frac{1-\alpha}{2}} \|(-\Delta)^{\frac{1}{2}} v(t)\|_{L^p(\mathbb{R}^N)}.$$

Theorem 3.2. Let $\mu > \sigma + 1$ and (Hf) hold, then there exists $T_* > 0$ such that problem (1.1) has a unique local mild solution u in $Y_{pr}[T_*]$. Moreover,

$$t^{\frac{1}{\mu}} u \in C_b([0, T_*]; L^r(\mathbb{R}^N)), \quad t^{\frac{1-\alpha}{2}} (-\Delta)^{\frac{1}{2}} u \in C_b([0, T_*]; L^p(\mathbb{R}^N))$$

both values zero at $t = 0$ except for $r = 2$ in the first term, in which $u(0) = \varphi$. And if u and $\tilde{u}$ are solutions starting at φ and ψ , both values on $L^p(\mathbb{R}^N)$ respectively, there exists a constant $C > 0$ such that

$$\|u - \tilde{u}\|_{Y_{pr}[T_*]} \leq C \|\varphi - \psi\|_{L^p(\mathbb{R}^N)}.$$

Proof. Define the operator Φ in $B(R)$ a closed ball of $Y_{pr}[T]$ with center 0 and radius $R > 0$ as in (3.2) and we take $4C\|\varphi\|_{L^p(\mathbb{R}^N)} = R$. From (Hf), assumption $\mu > \sigma + 1$ and the proof in Theorem 3.1, it is easy to check that $\Phi(u)$ belongs to $L^p(\mathbb{R}^N)$ for $u \in B(R)$. Moreover, there exists a constant $C > 0$ such that

$$t^{\frac{1}{\mu}} \|S(t)\varphi\|_{L^r(\mathbb{R}^N)} \leq C \|\varphi\|_{L^p(\mathbb{R}^N)}, \quad \text{and} \quad t^{\frac{1-\alpha}{2}} \|(-\Delta)^{\frac{1}{2}} S(t)\varphi\|_{L^p(\mathbb{R}^N)} \leq C \|\varphi\|_{L^p(\mathbb{R}^N)}.$$

Moreover, the assumption $\mu > \sigma + 1$ derives that $\mu > 1$ for $\sigma \geq 0$, that is

$$\frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{r} \right) (\sigma + 1) < 1, \quad \text{and} \quad \frac{N(1-\alpha)}{2} \left(\frac{1}{r'} - \frac{1}{r} \right) < 1.$$

From Lemma 2.5 (i) we have

$$\begin{aligned} \|W_{f(u)}(t)\|_{L^r(\mathbb{R}^N)} &\leq C \int_0^t (t-s)^{-\frac{1}{\mu}} \|f(u(s))\|_{L^r(\mathbb{R}^N)} ds \\ &\leq CK \int_0^t (t-s)^{-\frac{1}{\mu}} s^{-\theta} \left(\sup_{s \in [0, T]} s^{\frac{1}{\mu}} \|u(s)\|_{L^r(\mathbb{R}^N)} \right)^{\sigma+1} ds \\ &\leq CKB \left(1 - \frac{1}{\mu}, 1 - \theta \right) t^{1-\frac{\sigma+2}{\mu}} \|u\|_{Y_{pr}[T]}^{\sigma+1}, \end{aligned}$$

for $\vartheta = (\sigma + 1)/\mu \in (0, 1)$, which implies that

$$t^{\frac{1}{\mu}} \|W_{f(u)}(t)\|_{L^r(\mathbb{R}^N)} \leq CKB \left(1 - \frac{1}{\mu}, 1 - \vartheta\right) T^{1-\vartheta} R^{\sigma+1},$$

On the other hand, from Lemma 2.5 (ii) we have

$$\begin{aligned} \|(-\Delta)^{\frac{1}{2}} W_{f(u)}(t)\|_{L^p(\mathbb{R}^N)} &\leq \int_0^t \|(-\Delta)^{\frac{1}{2}} S(t-s)f(u(s))\|_{L^p(\mathbb{R}^N)} ds \\ &\leq C \int_0^t (t-s)^{-\frac{1-\alpha}{2}} \|f(u(s))\|_{L^r(\mathbb{R}^N)} ds \\ &\leq CK \int_0^t (t-s)^{-\frac{1-\alpha}{2}} s^{-\vartheta} \left(\sup_{s \in [0, T]} s^{\frac{1}{\mu}} \|u(s)\|_{L^r(\mathbb{R}^N)}\right)^{\sigma+1} ds \\ &\leq CKB \left(\frac{1+\alpha}{2}, 1 - \vartheta\right) \|u\|_{Y_{pr}[T]}^{\sigma+1} t^{\frac{1+\alpha}{2}-\vartheta}, \end{aligned}$$

it follows that

$$t^{\frac{1-\alpha}{2}} \|(-\Delta)^{\frac{1}{2}} W_{f(u)}(t)\|_{L^p(\mathbb{R}^N)} \leq CKB \left(\frac{1+\alpha}{2}, 1 - \vartheta\right) T^{1-\vartheta} R^{\sigma+1}.$$

Let

$$M_\sigma = \max \left\{ B \left(1 - \frac{1}{\mu}, 1 - \vartheta\right), B \left(\frac{1+\alpha}{2}, 1 - \vartheta\right) \right\},$$

then there exists $T_* > 0$ small enough such that $4CKM_\sigma T_*^{1-(\sigma+1)/\mu} R^\sigma < 1$, we thus get $\|W_{f(u)}\|_{Y_{pr}[T_*]} \leq R/2$ for $u \in B(R)$, that is,

$$\|S(t)\varphi\|_{Y_{pr}[T_*]} + \|W_{f(u)}\|_{Y_{pr}[T_*]} \leq 2C\|\varphi\|_{L^p(\mathbb{R}^N)} + 2CKM_\sigma T_*^{1-\vartheta} R^{\sigma+1} \leq R. \quad (3.4)$$

Additionally, Lemma 2.2 and Lemma 2.4 ensure the continuity of $\Phi(u)$. Hence, the operator Φ maps $B(R)$ into itself. Proceeding as in the proof of Theorem 3.1, we can conclude that Φ has a fixed point u , which is the unique solution of problem (1.1) in $[0, T_*]$. To complete the proof, it just remains to prove $t^{\frac{1}{\mu}} u \in C_b([0, T_*]; L^r(\mathbb{R}^N))$ and $t^{\frac{1-\alpha}{2}} (-\Delta)^{\frac{1}{2}} u \in C_b([0, T_*]; L^p(\mathbb{R}^N))$ both vanishing at $t = 0$.

For this purpose, we first claim that for $1 < p \leq r$ and $\varphi \in L^p(\mathbb{R}^N)$, then

$$\lim_{t \rightarrow 0} t^{\frac{1}{\mu}} \|S(t)\varphi\|_{L^r(\mathbb{R}^N)} = 0.$$

To see this, note that by Lemma 2.5 (i) the maps $t^{\frac{1}{\mu}} S(t) : L^p(\mathbb{R}^N) \rightarrow L^r(\mathbb{R}^N)$, $t \in (0, T]$, are uniformly bounded and converge strongly to zero on the dense subset $L^r(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$. Hence, we get the desired argument. Similarly, Lemma 2.5 (ii) shows that the maps $t^{\frac{1-\alpha}{2}} (-\Delta)^{\frac{1}{2}} S(t)$ are uniformly bounded from $L^p(\mathbb{R}^N)$ to itself and converge to zero strongly as $t \rightarrow 0$. This means that $t^{\frac{1-\alpha}{2}} (-\Delta)^{\frac{1}{2}} \|S(t)\varphi\|_{L^p(\mathbb{R}^N)} \rightarrow 0$ as $t \rightarrow 0$.

Now, for $t \in (0, T_0]$ with T_0 enough small, let $u, v \in B(R)$, we get

$$\begin{aligned} \sup_{0 < t \leq T_0} t^{\frac{1}{\mu}} \|W_{f(u)}(t)\|_{L^r(\mathbb{R}^N)} &\leq \sup_{0 < t \leq T_0} t^{\frac{1}{\mu}} \int_0^t \|S(t-s)f(u(s))\|_{L^r(\mathbb{R}^N)} ds \\ &\leq CKM_\sigma T_0^{1-\vartheta} \|u\|_{Y_{pr}[T_0]}^{\sigma+1}. \end{aligned}$$

and

$$\begin{aligned} \sup_{0 < t \leq T_0} t^{\frac{1-\alpha}{2}} \|(-\Delta)^{\frac{1}{2}} W_{f(u)}(t)\|_{L^p(\mathbb{R}^N)} &\leq \sup_{0 < t \leq T_0} t^{\frac{1-\alpha}{2}} \int_0^t \|(-\Delta)^{\frac{1}{2}} S(t-s)f(u(s))\|_{L^r(\mathbb{R}^N)} ds \\ &\leq CKM_\sigma T_0^{1-\vartheta} \|u\|_{Y_{pr}[T_0]}^{\sigma+1}. \end{aligned}$$

For the choice of T_0 , we get

$$t^{\frac{1}{\mu}} \|W_{f(u)}(t)\|_{L^r(\mathbb{R}^N)} + t^{\frac{1-\alpha}{2}} \|(-\Delta)^{\frac{1}{2}} W_{f(u)}(t)\|_{L^p(\mathbb{R}^N)} \rightarrow 0, \quad \text{as } t \rightarrow 0.$$

Consequently, it follows that $\lim_{t \rightarrow 0} t^{\frac{1}{\mu}} u(t) = 0$ in $L^r(\mathbb{R}^N)$ and $\lim_{t \rightarrow 0} t^{\frac{1-\alpha}{2}} (-\Delta)^{\frac{1}{2}} u(t) = 0$ in $L^p(\mathbb{R}^N)$. The proof is completed. $\square$

Remark 3.2. It is notice that if $p = r$ in Theorem 3.1, it may not exist a global continuous and bounded solution in X_{pr} because a singular integral term is not globally essentially bounded for $t \in [0, +\infty)$ at the showing estimate $W_{f(u)}(\cdot)$ in $L^p(\mathbb{R}^N)$, that is

$$\int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r} - \frac{1}{p}\right)} \|u(s)\|_{L^r(\mathbb{R}^N)}^{\sigma+1} ds \leq \|u\|_{L^p(\mathbb{R}^N)}^{\sigma+1} \int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r} - \frac{1}{p}\right)} ds,$$

for the assumption (Hf) and admissible triplet $(p, p, +\infty)$, where X_{pr} reduces to $L^p(\mathbb{R}^N)$. Obviously, the above right-hand side inequality of integral term is not integrable for all $t \in [0, +\infty)$. Thus, the case $p = r$ is not valid on this argument. However, let $p = r$, it is easy to show that the local solution will belong to $L^\infty(0, T; L^2(\mathbb{R}^N))$ in Theorem 3.2 for some $T > 0$.

In the sequel, we establish the continuation and a blow-up alternative.

Theorem 3.3. Let the assumptions of Theorem 3.2 hold and u be the mild solution. Then u can be extended to a maximal interval $[0, T_{\max})$. If $T_{\max} < +\infty$, then $\|u(t)\|_{L^r(\mathbb{R}^N)} \rightarrow +\infty$ as $t \rightarrow T_{\max}$. Furthermore, there exists a constant $C > 0$ such that

$$\|u(t)\|_{L^2(\mathbb{R}^N)} \geq C(T_{\max} - t)^{-\frac{1}{\sigma}}.$$

Proof. Since the assumptions of Theorem 3.2 hold, let $u \in Y_{pr}[T^*]$ be the solution, we proceed in a similar way to the proof of continuation as in Theorem 3.2, we next just point out the differences of the proof. In fact, define $\Phi : \widetilde{Y_{pr}}[T] \rightarrow \widetilde{Y_{pr}}[T]$ by (3.2), where $\widetilde{Y_{pr}}[T]$ is the space of the functions $v \in Y_{pr}[T^*]$ equipped with the same norm form such that $u \equiv v$ on $t \in [0, T^*]$ and for $t \in [T^*, T]$

$$\sup_{t \in [T^*, T]} t^{\frac{1}{\mu}} \|v(t) - u(T)\|_{L^r(\mathbb{R}^N)} + \sup_{t \in [T^*, T]} t^{\frac{1-\alpha}{2}} \|(-\Delta)^{\frac{1}{2}} (v(t) - u(T))\|_{L^p(\mathbb{R}^N)} \leq R.$$

Given $v \in \widetilde{Y_{pr}}[T]$, the continuity of $\Phi(v) : (0, T] \rightarrow \widetilde{Y_{pr}}[T]$ follows as in Theorem 3.2. Clearly, one finds that $\Phi(v)(t) = u(t)$, for every $t \in [0, T^*]$. For $t \in [T^*, T]$, we have

$$\begin{aligned} \Phi(v)(t) - u(T^*) &= S(t)\varphi - S(T^*)\varphi + \int_{T^*}^t S(t-s)f(v(s))ds \\ &\quad + \int_{T^*}^t (S(t-s) - S(T^*-s))f(u(s))ds. \end{aligned} \quad (3.5)$$

We note that Lemma 2.5 implies that the first term of the right hand side of (3.5) is in $\widetilde{Y_{pr}}[T]$, and then it goes to zero as $t \rightarrow T^*$. For this reason, we can choose T_a so close to T^* such that $\|S(\cdot)\varphi - S(T^*)\varphi\|_{\widetilde{Y_{pr}}[T]} \leq R/3$. For the second term, we have

$$\begin{aligned} t^{\frac{1}{\mu}} \left\| \int_{T^*}^t S(t-s)f(v(s))ds \right\|_{L^r(\mathbb{R}^N)} &\leq Ct^{\frac{1}{\mu}} \int_{T^*}^t (t-s)^{-\frac{1}{\mu}} \|f(v(s))\|_{L^p(\mathbb{R}^N)} ds \\ &\leq CKt^{1-\theta} \int_{T^*/t}^1 (1-s)^{-\frac{1}{\mu}} s^{-\theta} ds \|v\|_{\widetilde{Y_{pr}}[T]}^{\sigma+1} \rightarrow 0, \end{aligned}$$

as $t \rightarrow T_*$ by the property of incomplete Beta function for $\vartheta = (\sigma + 1)/\mu$. Similarly,

$$\begin{aligned} & \left\| t^{\frac{1-\alpha}{2}} \int_{T_*}^t (-\Delta)^{\frac{1}{2}} S(t-s) f(v(s)) ds \right\|_{L^p(\mathbb{R}^N)} \\ & \leq C t^{\frac{1-\alpha}{2}} \int_{T_*}^t (t-s)^{-\frac{1-\alpha}{2}} \|f(v(s))\|_{L^p(\mathbb{R}^N)} ds \\ & \leq CK t^{1-\vartheta} \int_{T_*/t}^1 (1-s)^{-\frac{1-\alpha}{2}} s^{-\vartheta} ds \|v\|_{\widetilde{Y}_{pr}[T]}^{\sigma+1} \rightarrow 0, \end{aligned}$$

as $t \rightarrow T_*$. Therefore, as for the second term, we know that it belongs $\widetilde{Y}_{pr}[T]$ and we can choose T_b so close to T_* such that its norm is less than $R/3$. As similar arguments, the last term of the right hand side of (3.5) belongs $\widetilde{Y}_{pr}[T]$, moreover from Lemma 2.5 and Lebesgue's dominated convergence theorem can be applied to prove that its norm is less than $R/3$ when we choose T_c so close to T_* . Now, let $T = \min\{T_a, T_b, T_c\}$, it follows that

$$\sup_{t \in [T_*, T]} t^{\frac{1}{\mu}} \|\Phi(v)(t) - u(T)\|_{L^r(\mathbb{R}^N)} + \sup_{t \in [T_*, T]} t^{\frac{1-\alpha}{2}} \|(-\Delta)^{\frac{1}{2}} (\Phi(v)(t) - u(T))\|_{L^p(\mathbb{R}^N)} \leq R.$$

In the same way, we can prove that Φ is a contraction on $\widetilde{Y}_{pr}[T]$. Thus, Φ has a unique fixed point by Banach fixed point theorem, which is a mild solution that extends to $[0, T]$.

Next, set

$$T_{max} := \sup \{T \in (0, +\infty) : \exists \text{ unique local solution } u \text{ to problem (1.1) on } [0, T]\}.$$

Suppose by contradiction that $T_{max} < \infty$ and there exists a constant $\widetilde{C} > 0$ such that $t^{\frac{1}{\mu}} \|u(t)\|_{L^r(\mathbb{R}^N)} \leq \widetilde{C}$ and $t^{\frac{1-\alpha}{2}} \|(-\Delta)^{\frac{1}{2}} u(t)\|_{L^p(\mathbb{R}^N)} \leq \widetilde{C}$ for all $t \in (0, T_{max})$. Next, consider a sequence of positive real number $\{t_n\}_{n=1}^\infty$ satisfying $t_n \rightarrow T_{max}$ as $n \rightarrow \infty$, we will verify that the sequence $\{u(t_n)\}_{n=1}^\infty$ belongs to $L^r(\mathbb{R}^N)$. Let us show that the sequence $\{u(t_n)\}_{n=1}^\infty$ is a Cauchy sequence in $L^r(\mathbb{R}^N)$. Indeed, for $0 < t_i < t_j < T_{max}$, we get

$$\begin{aligned} \|u(t_j) - u(t_i)\|_{L^r(\mathbb{R}^N)} & \leq \|S(t_j)\varphi - S(t_i)\varphi\|_{L^r(\mathbb{R}^N)} + \int_{t_i}^{t_j} \|S(t_j-s)f(v(s))\|_{L^r(\mathbb{R}^N)} ds \\ & \quad + \int_0^{t_i} \|S(t_j-s) - S(t_i-s)\|_{L^r(\mathbb{R}^N)} \|f(u(s))\|_{L^r(\mathbb{R}^N)} ds. \end{aligned}$$

Therefore, the same reasoning used to estimate (3.5) gives that

$$\|t_j^{\frac{1}{\mu}} u(t_j) - t_i^{\frac{1}{\mu}} u(t_i)\|_{L^r(\mathbb{R}^N)} \rightarrow 0, \quad \text{as } t_i \rightarrow t_j.$$

Hence, the limit $\lim_{n \rightarrow \infty} u(t_n) := u(T_{max})$ exists in $L^r(\mathbb{R}^N)$. Similarly, we can check the limit $\lim_{n \rightarrow \infty} (-\Delta)^{\frac{1}{2}} u(t_n)$ in $L^p(\mathbb{R}^N)$. Therefore, $u(T_{max})$ and $(-\Delta)^{\frac{1}{2}} u(T_{max})$ exist in $\widetilde{Y}_{pr}[T]$. As the before results in this theorem, the mild solution of problem (1.1) contradicts the maximality of T_{max} . Thus, we may define the maximal mild solution u of problem (1.1) on the interval $[0, T_{max})$.

If we consider $u(t_0)$ as the initial value for some $0 < t_0 < T_{max}$, so $\|u(t_0)\|_{L^r(\mathbb{R}^N)} < +\infty$ for $p = r = 2$, by above arguments we can prolong this solution u at least on $[t_0, t_1]$, it follows from (3.4) and the fixed point argument, we have for some $C > 0$,

$$2C\|u(t_0)\|_{L^r(\mathbb{R}^N)} + 2CKM_\sigma(t_1 - t_0)^{1-\vartheta} R^{\sigma+1} \leq R,$$

for some $t_1 < T_{max}$. Observe that if $0 \leq t_0 < T_{max}$ and $2C\|u(t_0)\|_{L^r(\mathbb{R}^N)} < R$, then

$$(T_{max} - t_0)^{1-\vartheta} > \frac{R - 2C\|u(t_0)\|_{L^r(\mathbb{R}^N)}}{2CKM_\sigma R^{\sigma+1}}.$$

In fact, otherwise for some $R > 2C\|u(t_0)\|_{L^r(\mathbb{R}^N)}$ and all $t \in (t_0, T_{max})$ we would have

$$(t - t_0)^{1-\theta} \leq \frac{R - 2C\|u(t_0)\|_{L^r(\mathbb{R}^N)}}{2CKM_\sigma R^{\sigma+1}}.$$

which implies $2C\|u(t)\|_{L^r(\mathbb{R}^N)} < R$ for all $t \in (t_0, T_{max})$ by the previous arguments. However, this is impossible since $\|u(t)\|_{L^r(\mathbb{R}^N)} \rightarrow +\infty$ as $t \rightarrow T_{max}$. Hence, choosing for example, $R = 4C\|u(t_0)\|_{L^r(\mathbb{R}^N)}$, we see that for $0 < t_0 < T_{max}$

$$(T_{max} - t_0)^{1-\theta} \|u(t_0)\|_{L^r(\mathbb{R}^N)}^\sigma > \widehat{C},$$

where $\widehat{C} > 0$ is some new fixed constant. Therefore, for the arbitrariness of t_0 we obtain the desired blow-up rate estimate. $\square$

3.1 Integrability of Solution

In this section, we will present the integrability of the global mild solution for current problem. For this purpose, we first discuss the properties of the solution operator in $L^r(0, \infty; L^q(\mathbb{R}^N))$.

Lemma 3.1. *Let $\alpha \in (0, 1)$ and (p, q, r) be an admissible triplet such that $1 < p < +\infty$. Assume that*

i) $q \in (p, +\infty)$ if $1 \leq N \leq \frac{2}{1-\alpha}$ and

ii) $q \in \left(p, \frac{(1-\alpha)Np}{(1-\alpha)N-2}\right)$ if $N > \frac{2}{1-\alpha}$.

It holds that $S(t)v \in L^r(0, \infty; L^q(\mathbb{R}^N))$, for any $v \in L^p(\mathbb{R}^N)$. Moreover, there exists $M_\alpha = M(\alpha, q, N) > 0$ such that

$$\int_0^\infty \|S(t)v\|_{L^q(\mathbb{R}^N)}^r dt \leq M_\alpha \|v\|_{L^p(\mathbb{R}^N)}^r, \quad v \in L^p(\mathbb{R}^N).$$

Proof. Fix q, ϱ with $p < q < \infty$ and $1 \leq \varrho < p$. Consider the operator $\mathcal{T}$ defined in $L^\varrho(\mathbb{R}^N)$ given by

$$\mathcal{T}(v)(t) := \|S(t)v\|_{L^q(\mathbb{R}^N)}, \quad v \in L^\varrho(\mathbb{R}^N), \quad t > 0.$$

Observe that for each $v \in L^\varrho(\mathbb{R}^N)$, Lemma 2.5 (ii) guarantees the inclusion

$$\{t > 0 : |\mathcal{T}(v)(t)| > \lambda\} \subset \{t > 0 : Ct^{-\frac{N(1-\alpha)}{2}\left(\frac{1}{\varrho}-\frac{1}{q}\right)} \|v\|_{L^\varrho(\mathbb{R}^N)} > \lambda\}, \quad \text{for } \lambda > 0,$$

which ensures the inequality

$$\begin{aligned} \mu(\{t > 0 : |\mathcal{T}(v)(t)| > \lambda\}) &\leq \mu(\{t > 0 : Ct^{-\frac{N(1-\alpha)}{2}\left(\frac{1}{\varrho}-\frac{1}{q}\right)} \|v\|_{L^\varrho(\mathbb{R}^N)} > \lambda\}) \\ &= \mu\left(\left\{t > 0 : t < \left(\frac{C\|v\|_{L^\varrho(\mathbb{R}^N)}}{\lambda}\right)^{\frac{2\varrho q}{N(1-\alpha)(q-\varrho)}}\right\}\right) \\ &\leq \left(\frac{C\|v\|_{L^\varrho(\mathbb{R}^N)}}{\lambda}\right)^{\frac{2\varrho q}{N(1-\alpha)(q-\varrho)}} \end{aligned}$$

Therefore, operator $\mathcal{T}$ is of the weak-type $\left(\varrho, \frac{2\varrho q}{N(1-\alpha)(q-\varrho)}\right)$. For considering the operator $\mathcal{T}$ defined in $L^q(\mathbb{R}^N)$. Thus, we also have that

$$\{t > 0 : |\mathcal{T}(v)(t)| > \lambda\} \subset \{t > 0 : C\|v\|_{L^q(\mathbb{R}^N)} > \lambda\}, \quad \lambda > 0,$$

which guarantees $\mathcal{T}$ is of the weak-type (q, ∞) .

In the sequel, we divide the proof into two steps.

Step 1: Since $N \leq \frac{2}{1-\alpha}$ we see that $\varrho \leq \frac{2\varrho q}{(1-\alpha)N(q-\varrho)}$ with $\varrho < q$, Marcinkiewicz interpolation theorem implies that $\mathcal{T}$ is of type (a_θ, b_θ) where

$$\frac{1}{a_\theta} = \frac{1-\theta}{\varrho} + \frac{\theta}{q}, \quad \text{and} \quad \frac{1}{b_\theta} = \frac{(1-\theta)(1-\alpha)N(q-\varrho)}{2\varrho q},$$

for any $\theta \in (0, 1)$. Therefore, there exists $M_\theta > 0$ such that for any $v \in L^{a_\theta}(\mathbb{R}^N)$

$$\|S(t)v\|_{L^{b_\theta}(0, \infty; L^q(\mathbb{R}^N))} \leq M_\theta \|v\|_{L^{a_\theta}(\mathbb{R}^N)}.$$

Particularly, taking $\theta = \frac{q(p-\varrho)}{p(q-\varrho)}$ we obtain $a_\theta = p$ and $b_\theta = r$, this means that $S(t)v \in L^r(0, \infty; L^q(\mathbb{R}^N))$ for any $v \in L^p(\mathbb{R}^N)$.

Step 2: Assume that $N > \frac{2}{1-\alpha}$. Fix q such that $p < q < \frac{(1-\alpha)Np}{(1-\alpha)N-2}$ and choose ϱ satisfying $\frac{((1-\alpha)N-2)q}{(1-\alpha)N} < \varrho < p$. It follows that the operator $\mathcal{T}$ is the weak-type $(\varrho, \frac{2\varrho q}{(1-\alpha)N(q-\varrho)})$ as well as the weak-type (q, ∞) . The proof is similar to that of Step 1 for $\varrho < q$ and $\varrho \leq \frac{2\varrho q}{(1-\alpha)N(q-\varrho)}$, so we omit it. $\square$

The next is the last main theorem of this section, concerning integrability for the solutions of problem (1.1).

Theorem 3.4. For $\alpha \in (0, 1)$ and $\varphi \in L^p(\mathbb{R}^N)$. Let (p, r, μ) be the admissible triplet given in X_{pr} and let $\frac{1}{\mu} = \frac{1}{\sigma} \left(1 - \frac{N(1-\alpha)}{2} \left(1 - \frac{2}{r}\right)\right)$. If (Hf) holds and there exists $\lambda > 0$ such that $\|\varphi\|_{L^p(\mathbb{R}^N)} \leq \lambda$, then we get the following conclusions

i) for $1 \leq N \leq \frac{2}{1-\alpha}$, problem (1.1) has a unique global mild solution which belongs to $L^\mu(0, \infty; L^r(\mathbb{R}^N))$, for $1 < p < r < +\infty$;

ii) for $\frac{2}{1-\alpha} < N \leq \frac{2}{(1-\alpha)(1-2/r)}$, problem (1.1) has a unique global mild solution which belongs to $L^\mu(0, \infty; L^r(\mathbb{R}^N))$, for $1 < p < r < \frac{(1-\alpha)Np}{(1-\alpha)N-2}$.

Proof. We first prove i). Define a Banach space by $Z_{pr} := X_{pr} \cap L^\mu(0, \infty; L^r(\mathbb{R}^N))$ where X_{pr} is the Banach space considered in Theorem 3.1. We consider the norm of space Z_{pr} given by

$$\|v\|_{Z_{pr}} := \|v\|_{X_{pr}} + \|v\|_{L^\mu(0, \infty; L^r(\mathbb{R}^N))}.$$

By Theorem 3.1 and Lemma 3.1, there exists $M = M(N, p, r, \alpha) > 0$ such that

$$\|S(t)\varphi\|_{Z_{pr}} \leq M \|\varphi\|_{L^p(\mathbb{R}^N)}.$$

It remains to prove the another term of (3.2) in Z_{pr} . It follows that

$$\begin{aligned} \left\| \int_0^t S(t-s)f(u(s))ds \right\|_{L^r(\mathbb{R}^N)} &\leq C \int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r}-\frac{1}{r}\right)} \|f(u(s))\|_{L^{r'}(\mathbb{R}^N)} ds \\ &\leq CK \int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r}-\frac{1}{r}\right)} \|u(s)\|_{L^{r'}(\mathbb{R}^N)}^{\sigma+1} ds \\ &\leq CK \|u\|_{X_{pr}}^\sigma \int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r}-\frac{1}{r}\right)} s^{-\frac{\sigma}{\mu}} \|u(s)\|_{L^{r'}(\mathbb{R}^N)} ds, \end{aligned}$$

The assumption $\frac{1}{\mu} = \frac{1}{\sigma} \left(1 - \frac{N(1-\alpha)}{2} \left(1 - \frac{2}{r}\right)\right)$ and the doubly weighted Hardy-Littlewood-Sobolev inequality imply that

$$\left\| \int_0^t (t-s)^{-\frac{N(1-\alpha)}{2} \left(\frac{1}{r}-\frac{1}{r}\right)} s^{-\frac{\sigma}{\mu}} \|u(s)\|_{L^{r'}(\mathbb{R}^N)} ds \right\|_{L^\mu(0, \infty)} \leq L_\alpha \|u\|_{L^\mu(0, \infty; L^{r'}(\mathbb{R}^N))},$$

for a constant $L_\alpha = L(\alpha, p, r) > 0$. Hence, there exists constant $C = C(N, p, \alpha, r) > 0$ such that

$$\left\| \int_0^t S(t-s)f(u(s))ds \right\|_{L^\mu(0,\infty;L^r(\mathbb{R}^N))} \leq C \|u\|_{X_{pr}}^\sigma \|u\|_{L^\mu(0,\infty;L^r(\mathbb{R}^N))} \leq C \|u\|_{Z_{pr}}^{\sigma+1}.$$

Together above arguments, we conclude that there is a unique global solution which belongs to $L^\mu(0, \infty; L^r(\mathbb{R}^N))$. The proof of ii) is analogous, so we omit it. The proof is completed. $\square$

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