Research Article

Guofeng Che and Tsung-fang Wu*

Multiple positive solutions for a class of Kirchhoff type equations with indefinite nonlinearities

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Abstract: We study the following Kirchhoff type equation:

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + u = k(x)|u|^{p-2}u + m(x)|u|^{q-2}u \text{ in } \mathbb{R}^N,$$

where $N \geq 3$, $a, b > 0$, $1 < q < 2 < p < \min\{4, 2^*\}$, $2^* = 2N/(N-2)$, $k \in C(\mathbb{R}^N)$ is bounded and $m \in L^{p/(p-q)}(\mathbb{R}^N)$. By imposing some suitable conditions on functions $k(x)$ and $m(x)$, we firstly introduce some novel techniques to recover the compactness of the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ ($2 \leq r < 2^*$); then the Ekeland variational principle and an innovative constraint method of the Nehari manifold are adopted to get three positive solutions for the above problem.

Keywords: Multiple positive solutions; Kirchhoff type equation; Indefinite nonlinearities**MSC:** 35J20; 35B09

1 Introduction

We investigate the existence of multiple positive solutions to the Kirchhoff type equation with indefinite nonlinearities:

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + u = k(x)|u|^{p-2}u + m(x)|u|^{q-2}u \text{ in } \mathbb{R}^N, \quad (P)$$

where $N \geq 3$, $a, b > 0$, $1 < q < 2 < p < \min\{4, 2^*\}$, $2^* = 2N/(N-2)$, and functions $k(x)$ and $m(x)$ satisfy the following conditions.

(H_1) $k \in C(\mathbb{R}^N)$ is a bounded function in $\mathbb{R}^N$;

(H_2) k is sign-changing in $\mathbb{R}^N$ and $\Omega_1 = \{x \in \mathbb{R}^N : k(x) > 0\}$ is a bounded domain;

(H_3) $m \in L^{q^*}(\mathbb{R}^N)$ and $m^+ = \max\{m(x), 0\} \not\equiv 0$, where $q^* = \frac{p}{p-q}$.

Guofeng Che, School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou 510006, Guangdong, P. R. China, E-mail: cheguofeng222@163.com

***Corresponding Author: Tsung-fang Wu**, Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan, E-mail: tfwu@nuk.edu.tw

Problem (P) is a variant type of the Kirchhoff problem as follows:

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

which is related to the stationary analogue of the equation:

$$\begin{cases} u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where Ω is a bounded domain in $\mathbb{R}^N$. Such problems are nonlocal owing to the appearance of $\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u$, which makes Eq.(1.1) and Eq.(1.2) are no longer pointwise identities. In [19], Kirchhoff firstly introduced Eq.(1.2) as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The changes in length of the string produced by transverse vibrations are considered by the Kirchhoff's model. Moreover, such nonlocal problem are also appeared in other fields, for instance, biological systems. For more details about the backgrounds of problems (1.1) and (1.2), one can be referred to [2, 4, 12, 20].

In [22], Lions firstly proposed an abstract framework of Eq.(1.1), from then on, lots of researchers began to study Eq.(1.1) in general dimension, see [11, 16, 25, 26, 28, 39] and the references therein. More precisely, Zhang and Perera [39] obtained the existence of a positive solution, a negative solution and a sign-changing solutions for Eq.(1.1) with $N \geq 1$ by using invariant sets of descent flow. Pei and Ma [26] got three positive solutions for Eq.(1.1) with $N = 1, 2, 3$ via the minimax method and the Morse theory. By using the theory developed in [27], Ricceri [28] obtained three positive solutions for Eq.(1.1) with $N \geq 4$.

Recently, many papers [6, 8, 15, 29–32, 38] study the Kirchhoff equation on the whole space:

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(x)u = f(x, u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), & \text{in } \mathbb{R}^N. \end{cases} \quad (1.3)$$

When the potential function $V(x)$ satisfies the following assumptions:

(V_1) $V \in C(\mathbb{R}^N)$ with $V(x) \geq 0$ in $\mathbb{R}^N$ and there exists $c_0 > 0$ such that the set $\{V < c_0\} := \{x \in \mathbb{R}^N \mid V(x) < c_0\}$ has finite positive Lebesgue measure, where $|\cdot|$ is the Lebesgue measure;

(V_2) $\Omega = \text{int}\{x \in \mathbb{R}^N \mid V(x) = 0\}$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

Sun and Wu [29] obtained the existence, nonexistence and concentration of nontrivial solutions for Eq.(1.3) with $N = 3$ and V being replaced by λV , $\lambda > 0$. Later, when $N \geq 4$, V is replaced by λV , $\lambda > 0$, and f satisfies the superlinear condition, Sun *et al.* [30] proved that Eq.(1.3) possessed two positive solutions. When $N = 3$, V satisfies $(V_1) - (V_2)$ and f satisfies the classical Ambrosetti–Rabinowitz type condition, Xie and Ma [38] proved that Eq.(1.3) has at least a positive solution. Moreover, the concentration behavior is also studied by the authors. In [32], Sun and Zhang obtained the uniqueness of the positive ground state solution for Eq.(1.3) with $N = 3$, $f(x, u) = d|u|^{p-1}u$, $3 < p < 5$, and $V(x) \equiv c$, where c and d are positive constants. Besides, by using the uniqueness result and the concentration–compactness lemma [23], the authors also got the existence and concentration theorems for the following Kirchhoff problem:

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = K(x)|u|^{p-1}u, \quad \text{in } \mathbb{R}^3. \quad (1.4)$$

In [15], by using the Schwartz symmetric arrangement, Guo proved that Eq.(1.3) possesses a positive ground state solution when V is continuous and f does not satisfies the classical Ambrosetti–Rabinowitz type condition.

For the elliptic problems with concave–convex nonlinearities, there are also several results. For example, one can be referred to [1, 24, 35–37] and the references therein. Indeed, in [1], Ambrosetti *et al.* firstly introduced the elliptic problem involving the concave–convex nonlinearities:

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \quad (1.5)$$

where $1 < q < 2 < p \leq 2^*$ ($2^* = \infty$ if $N = 1, 2$; $2^* = 2N/(N-2)$ if $N \geq 3$), $\lambda > 0$ and $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is bounded. With the aid of variational methods, the authors established the existence, nonexistence and multiplicity of solutions for Eq.(1.5). Later, Wu [37] obtained the existence of multiple positive solutions for the following problem:

$$\begin{cases} -\Delta u + u = f(x)u^{q-1} + g(x)u^{p-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (1.6)$$

where $1 < q < 2 < p \leq 2^*$ ($2^* = \infty$ if $N = 1, 2$; $2^* = 2N/(N-2)$ if $N \geq 3$), f is sign-changing and g is positive in $\mathbb{R}^N$.

To the best of our knowledge, for the Kirchhoff problem with concave–convex terms, there are few results [5, 8–10, 21]. In [9], Chen *et al.* considered a class of Kirchhoff problem on the bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 1$):

$$\begin{cases} -\left(a \int_{\Omega} |\nabla u|^2 dx + b\right) \Delta u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.7)$$

where $1 < q < 2 < p \leq 2^*$ ($2^* = \infty$ if $N = 1, 2$; $2^* = 2N/(N-2)$ if $N \geq 3$). By giving different scopes on a and λ , the authors proved that Eq.(1.7) possesses multiple positive solutions. By using the Nehari manifold technique, Liao *et al.* [21] obtained two nonnegative solutions for Eq.(1.7). When $f(x) = g(x) \equiv 1$ in Eq.(1.7), by constraining the energy functional of the problem on a subset of the nodal Nehari set, Chen and Ou [10] proved that there exists a constant $\lambda^* > 0$ such that for any $\lambda < \lambda^*$, Eq.(1.7) has a nodal solution u_λ with positive energy. In [8], we obtained three positive solutions for the Kirchhoff type problem with steep potential well.

Motivated by the above facts, more precisely by [9], it is quite natural for us to ask that whether Eq.(P) can have multiple positive solutions when $1 < q < 2 < p \leq \min\{4, 2^*\}$ and the domain is unbounded? As far as we know, such problem has never been discussed in the available literature. In our paper, we will give a definite answer to the above question. Under some suitable conditions on functions k and m , we will obtain three positive solutions for Eq.(P). It is worthy pointing out, in [8], we considered Eq.(P) with steep potential well and positive nonlinearities, which we overcome the compactness of the Sobolev embedding by giving some inequalities. While, in the present paper, since the potential function is a constant and the nonlinearities are sign-changing, the standard method of recovering the compactness does not work, motivated by [17], a novel method will be used to get the Palais–Smale (PS for short) condition.

Before introducing our main conclusions, we first recall a well known result (c.f.[34]). Suppose that w_{Ω_1} is the positive ground state solution to the nonlinear elliptic equation:

$$\begin{cases} -\Delta u + u = k(x)|u|^{p-2}u & \text{in } \Omega_1, \\ u \in H_0^1(\Omega_1), \end{cases} \quad (E_{\Omega_1})$$

where Ω_1 is defined by the condition (H_2) .

Obviously,

$$\|w_{\Omega_1}\|_{H^1}^2 := \int_{\Omega_1} (|\nabla w_{\Omega_1}|^2 + w_{\Omega_1}^2) dx = \int_{\Omega_1} k(x)w_{\Omega_1}^p dx, \quad (1.8)$$

and

$$\inf_{u \in \mathcal{M}_{\Omega_1}} J_{\Omega_1}(u) = J_{\Omega_1}(w_{\Omega_1}) = \frac{p-2}{2p} \int_{\Omega_1} k(x)w_{\Omega_1}^p dx,$$

where J_{Ω_1} is the energy functional of problem (E_{Ω_1}) , defined by

$$J_{\Omega_1}(u) = \frac{1}{2} \int_{\Omega_1} (|\nabla u|^2 + u^2) dx - \frac{1}{p} \int_{\Omega_1} k(x)|u|^p dx, \quad (1.9)$$

and the corresponding Nehari manifold is given by:

$$\mathcal{M}_{\Omega_1} = \{u \in H_0^1(\Omega_1) \setminus \{0\} \mid \langle J'_{\Omega_1}(u), u \rangle = 0\}.$$

In order to simplify the calculation, in the present paper, we hypothesize that $a = 1$ in problem (P) .

Denote $k_{\max} := \sup_{x \in \Omega_1} k(x)$ and $A(p) = \left(\frac{2}{4-p}\right)^{1/(p-2)}$. Let S_r , S_{r,Ω_1} and S be the best Sobolev constants for the embedding of $H^1(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$, $H_0^1(\Omega_1)$ in $L^r(\Omega_1)$ and $D^{1,2}(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$ with $2 \leq r \leq 2^*$, respectively. For any $2 \leq r \leq +\infty$, we shall also denote by $|\cdot|_r$ the L^r -norm. If we take a subsequence of a sequence $\{u_n\}$, we may denote it again by $\{u_n\}$,

We now summarize our main results below.

Theorem 1.1. Assume that functions k and m satisfy hypotheses (H_1) – (H_3) . Then there exists a constant $\Pi_1 > 0$ such that for each $0 < b + |m|_q < \Pi_1$, Eq.(P) admits two positive solutions $u_b^{(1)}, u_b^{(*)}$, satisfying

$$\|u_b^{(1)}\|_{H^1} < \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}}\right)^{1/(p-2)} < \|u_b^{(*)}\|_{H^1} < A(p)\|w_{\Omega_1}\|_{H^1},$$

and

$$I_b(u_b^{(1)}) < 0 < I_b(u_b^{(*)}),$$

where I_b is the corresponding energy functional of Eq.(P) defined in Section 2.

Theorem 1.2. Assume that functions k and m satisfy hypotheses (H_1) – (H_3) . In addition, we assume that

(H_4) The weight function m changes sign in $\mathbb{R}^N$ and $\Omega_2 = \{x \in \mathbb{R}^N : m(x) > 0\}$ is a bounded domain.

Then there exists a constant $\Pi_2 > 0$ such that for each $0 < b + |m|_q < \Pi_2$, Eq.(P) admits three positive solutions $u_b^{(1)}, u_b^{(2)}$ and $u_b^{(*)}$ satisfying

$$\|u_b^{(1)}\|_{H^1} < \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}}\right)^{1/(p-2)} < \|u_b^{(*)}\|_{H^1} < \|u_b^{(2)}\|_{H^1},$$

and

$$\max\{I_b(u_b^{(1)}), I_b(u_b^{(2)})\} < 0 < I_b(u_b^{(*)}).$$

Remark 1.1. Theorem 1.2 seems to be the first result about the Kirchhoff type equation with constant potential and indefinite nonlinearities, which has three positive solutions in the whole space $\mathbb{R}^N$, $N \geq 3$. We also remark that, in the previous papers, to obtain three positive solutions for the Kirchhoff problem, the domain is usually required to be bounded. For the unbounded domain $\mathbb{R}^N$, $N \geq 3$, as far as we are concerned, there are few results except [8], which we obtained three positive solutions for the Kirchhoff type equation with steep potential well.

In order to obtain our main results, we will make use of variational methods. Since we consider Eq.(P) in the whole space $\mathbb{R}^N$, the embedding from $H^1(\mathbb{R}^N)$ into $L^s(\mathbb{R}^N)$ ($2 < s < 2^*$) is not compact. Moreover, the appearance of the nonlocal term $\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u$ makes it hard to prove that the (PS) condition holds.

Precisely, for any (PS) sequence $\{u_n\}$, if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, we do not know whether there holds

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} \nabla u_n \nabla v dx \rightarrow \int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v dx, \quad \text{for all } v \in H^1(\mathbb{R}^N).$$

Besides, the weight function k is sign-changing, which also makes it much more complicated to recover the compactness. Note that in order to overcome the lack of compactness of the Sobolev embedding, there are some existed strategies, such as constructing a convergent Cerami sequence [18]. While in this paper, inspired by [17, 20], we will make use of a novel method to verify that the (PS) condition holds. It is worthy pointing out, due to $2 < p < \min\{4, 2^*\}$, the energy functional of Eq.(P) is usually not coercive and bounded below, while, in this paper, under the hypotheses (H_1) – (H_4) , we can prove that I_b is coercive and bounded below in $\mathbb{R}^N$, thus, two positive solutions are obtained. To get the third positive solution for Eq.(P), motivated by [31], the filtration of the Nehari manifold will be utilized:

$$\mathcal{M}_b(c) = \{u \in \mathcal{M}_b : I_b(u) < c\} \text{ for some } c > 0 \text{ depending on } p, q, k, m,$$

where $\mathcal{M}_b$ is the corresponding Nehari manifold that can be divided into $\mathcal{M}_b^{(1)}(c)$ and $\mathcal{M}_b^{(2)}(c)$, where

$$\mathcal{M}_b^{(1)}(c) = \{u \in \mathcal{M}_b(c) : \|u\|_{H^1} < c_1\} \text{ and } \mathcal{M}_b^{(2)}(c) = \{u \in \mathcal{M}_b(c) : \|u\|_{H^1} > c_2\},$$

for some $c_i > 0$, $i = 1, 2$. By a simple computation, we know that I_b is bounded from below on $\mathcal{M}_b^{(1)}(c)$ under the assumptions (H_1) – (H_3) , and then the third positive solution for Eq.(P) is obtained.

The remainder of this paper is organized as follows. In Section 2, we recall some preliminaries. In Section 3, we first propose some novel techniques to recover the compactness of the Sobolev embedding, then obtain two positive solutions with negative energy for Eq.(P). After introducing the filtration of Nehari manifold in Section 4, we obtain the third positive solution with positive energy for Eq.(P) and prove our results in Section 5.

2 Preliminaries

For any $u \in H^1(\mathbb{R}^N)$, let

$$I_b(u) = \frac{1}{2}\|u\|_{H^1}^2 + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 - \frac{1}{p} \int_{\mathbb{R}^N} k(x)|u|^p dx - \frac{1}{q} \int_{\mathbb{R}^N} m(x)|u|^q dx. \quad (2.1)$$

Then I_b is a well defined C^1 functional with the following derivative:

$$\begin{aligned} \langle I'_b(u), v \rangle &= \langle u, v \rangle_{H^1} + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v dx - \int_{\mathbb{R}^N} k(x)|u|^{p-2} u v dx \\ &\quad - \int_{\mathbb{R}^N} m(x)|u|^{q-2} u v dx. \end{aligned} \quad (2.2)$$

Define the Nehari manifold

$$\mathcal{M}_b := \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle I'_b(u), u \rangle = 0 \right\}.$$

Thus $u \in \mathcal{M}_b$ if and only if

$$\|u\|_{H^1}^2 + b \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 = \int_{\mathbb{R}^N} k(x)|u|^p dx + \int_{\mathbb{R}^N} m(x)|u|^q dx.$$

Note that $\mathcal{M}_b$ is closely related to the behavior of the fibering map [3, 13]: $h_{b,u} : t \rightarrow I_b(tu)$, $t > 0$. From (2.1), it is easy to know that

$$h_{b,u}(t) = I_b(tu) = \frac{t^2}{2}\|u\|_{H^1}^2 + \frac{bt^4}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 - \frac{t^p}{p} \int_{\mathbb{R}^N} k(x)|u|^p dx - \frac{t^q}{q} \int_{\mathbb{R}^N} m(x)|u|^q dx.$$

Thus

$$h'_{b,u}(t) = t\|u\|_{H^1}^2 + bt^3 \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 - t^{p-1} \int_{\mathbb{R}^N} k(x)|u|^p dx - t^{q-1} \int_{\mathbb{R}^N} m(x)|u|^q dx,$$

and

$$\begin{aligned} h''_{b,u}(t) &= \|u\|_{H^1}^2 + 3bt^2 \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 - (p-1)t^{p-2} \int_{\mathbb{R}^N} k(x)|u|^p dx \\ &\quad - (q-1)t^{q-2} \int_{\mathbb{R}^N} m(x)|u|^q dx. \end{aligned}$$

A straightforward calculation shows that

$$th'_{b,u}(t) = \|tu\|_{H^1}^2 + b \left(\int_{\mathbb{R}^N} |\nabla(tu)|^2 dx \right)^2 - \int_{\mathbb{R}^N} k(x)|tu|^p dx - \int_{\mathbb{R}^N} m(x)|tu|^q dx.$$

Obviously, $tu \in \mathcal{M}_b$ holds if and only if $h'_{b,u}(t) = 0$. Hence, points in $\mathcal{M}_b$ correspond to the stationary points of the maps $h_{b,u}$, then it is natural to divide $\mathcal{M}_b$ into three parts corresponding to the local minima, local maxima and points of inflection. Following [33], we define

$$\mathcal{M}_b^+ := \left\{ u \in \mathcal{M}_b : h''_{b,u}(1) > 0 \right\},$$

$$\mathcal{M}_b^0 := \left\{ u \in \mathcal{M}_b : h''_{b,u}(1) = 0 \right\},$$

and

$$\mathcal{M}_b^- := \left\{ u \in \mathcal{M}_b : h''_{b,u}(1) < 0 \right\}.$$

Similar to the arguments of Brown and Zhang [3, Theorem 2.3], we may get the following conclusion.

Lemma 2.1. Suppose that u_0 is a local minimizer for I_b on $\mathcal{M}_b$ and $u_0 \notin \mathcal{M}_b^0$. Then $I'_b(u_0) = 0$.

For each $u \in \mathcal{M}_b$, there holds

$$h''_{b,u}(1) = -2\|u\|_{H^1}^2 + (4-p) \int_{\mathbb{R}^N} k(x)|u|^p dx + (4-q) \int_{\mathbb{R}^N} m(x)|u|^q dx \quad (2.3)$$

$$= (2-q)\|u\|_{H^1}^2 + (4-q)b \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 - (p-q) \int_{\mathbb{R}^N} k(x)|u|^p dx \quad (2.4)$$

$$= -(p-2)\|u\|_{H^1}^2 + (4-p)b \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 + (p-q) \int_{\mathbb{R}^N} m(x)|u|^q dx.$$

For each $u \in \mathcal{M}_b^-$, in view of (2.4), there holds

$$\begin{aligned} (2-q)\|u\|_{H^1}^2 &< (2-q)\|u\|_{H^1}^2 + (4-q)b \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 \\ &< (p-q)k_{\max}S_p^{-p}\|u\|_{H^1}^p, \end{aligned} \quad (2.5)$$

since $1 < q < 2 < p < \min\{4, 2^*\}$. From (2.5), for every $u \in \mathcal{M}_b^-$, we derive

$$\|u\|_{H^1} > \rho_0 := \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}} \right)^{1/(p-2)}. \quad (2.6)$$

From (2.3) and (2.6), for any $u \in \mathcal{M}_b^-$, there holds

$$\begin{aligned} I_b(u) &> \frac{p-2}{4p} \|u\|_{H^1}^2 - \frac{(4-q)(p-q)}{4pq} \int_{\mathbb{R}^N} m(x)|u|^q dx \\ &\geq \left[\frac{p-2}{4p} \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}} \right)^{(2-q)/(p-2)} - \frac{(4-q)(p-q)|m|_{q^*}}{4pqS_p^q} \right] \|u\|_{H^1}^q \\ &> 0, \end{aligned}$$

provided that

$$|m|_{q^*} < \Gamma_* := \frac{q(p-2)}{4-q} \left(\frac{S_p^2}{p-q} \right)^{(p-q)/(p-2)} \left(\frac{2-q}{k_{\max}} \right)^{(2-q)/(p-2)}. \quad (2.7)$$

Therefore, we may have the following result.

Lemma 2.2. Assume that hypotheses (H_1) – (H_3) hold. Then for every $0 < |m|_{q^*} < \Gamma_*$, I_b is coercive and bounded from below on $\mathcal{M}_b^-$, where Γ_* is defined by (2.7). Moreover, for every $u \in \mathcal{M}_b^-$, there holds $I_b(u) > C_0$ for some $C_0 := C_0(p, q, k, m) > 0$.

3 Existence of two negative–energy positive solutions

Lemma 3.1. Under the conditions (H_1) – (H_3) , there exists a constant $\Gamma_0 > 0$ such that for every $0 < |m|_{q^*} \leq \Gamma_0$, we obtain

$$\inf \{ I_b(u) \mid u \in H^1(\mathbb{R}^N) \text{ with } \|u\|_{H^1} = \rho_0 \} \geq 0.$$

Proof. For any $u \in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1} = \rho_0$, in view of (2.1) and the Hölder inequality, there holds

$$\begin{aligned} I_b(u) &\geq \frac{1}{2} \|u\|_{H^1}^2 - \frac{k_{\max}}{pS_p^p} \|u\|_{H^1}^p - \frac{|m|_{q^*}}{qS_p^q} \|u\|_{H^1}^q \\ &= \rho_0^q \left(\frac{1}{2} \rho_0^{2-q} - \frac{k_{\max}}{pS_p^p} \rho_0^{p-q} - \frac{|m|_{q^*}}{qS_p^q} \right). \end{aligned} \quad (3.1)$$

Define a function $l : [0, +\infty) \rightarrow \mathbb{R}$ as follows:

$$l(t) = \frac{1}{2} t^{2-q} - \frac{k_{\max}}{pS_p^p} t^{p-q}.$$

A straightforward calculations implies that

$$\max_{t \geq 0} l(t) = l(t_0) = \frac{1}{2} \left(\frac{p-2}{p-q} \right) \left(\frac{p(2-q)S_p^p}{2(p-q)k_{\max}} \right)^{(2-q)/(p-2)},$$

where

$$t_0 = \left[\frac{p(2-q)S_p^p}{2(p-q)k_{\max}} \right]^{1/(p-2)} > 0.$$

Note that

$$\rho_0 = \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}} \right)^{1/(p-2)} < t_0,$$

and

$$l(\rho_0) = \frac{(p-2)(p+2-q)}{2p(p-q)} \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}} \right)^{(2-q)/(p-2)} > 0.$$

Thus, for every $u \in H^1(\mathbb{R}^N)$ satisfying $\|u\|_{H^1} = \rho_0$, there hold

$$I_b(u) \geq \rho_0^q \left(l(\rho_0) - \frac{|m|_{q^*}}{qS_p^q} \right) \geq 0,$$

and

$$|m|_{q^*} \leq \Gamma_0 := \frac{q(p-2)(p+2-q)S_p^q}{2p(p-q)} \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}} \right)^{(2-q)/(p-2)}. \quad (3.2)$$

Consequently, the proof is complete. $\square$

Lemma 3.2. Assume that hypotheses $(H_1) - (H_4)$ hold. Then I_b is bounded from below and coercive on $H^1(\mathbb{R}^N)$. Furthermore, for every $b > 0$, there exists $R_b > \rho_0$ such that $\inf\{I_b(u) \mid u \in H^1(\mathbb{R}^N) \text{ with } \|u\|_{H^1} \geq R_b\} \geq 0$.

Proof. Denote $|u|_{r, \Omega_i}^r = \int_{\Omega_i} |u|^r dx$ for $i = 1, 2$ and $1 \leq r \leq +\infty$, where Ω_1 and Ω_2 are given by conditions (H_2) and (H_4) . From the Sobolev and Hölder inequalities, for any $2 \leq r \leq 2^*$ and $i = 1, 2$, we obtain

$$\int_{\Omega_i} |u|^r dx \leq |u|_{2^*, \Omega_i}^r |\Omega_i|^{(2^*-r)/2^*} \leq S^{-r} |\Omega_i|^{(2^*-r)/2^*} \|u\|_{D^{1,2}}^r. \quad (3.3)$$

In view of (3.3) and the Hölder inequality again, we derive

$$\begin{aligned} I_b(u) &= \frac{b}{4} \|u\|_{D^{1,2}}^4 + \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{p} \int_{\mathbb{R}^N} k(x) |u|^p dx - \frac{1}{q} \int_{\mathbb{R}^N} m(x) |u|^q dx \\ &\geq \frac{b}{4} \|u\|_{D^{1,2}}^4 + \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{p} \int_{\Omega_1} k(x) |u|^p dx - \frac{1}{q} \int_{\Omega_2} m(x) |u|^q dx \\ &\geq \frac{b}{4} \|u\|_{D^{1,2}}^4 + \frac{1}{2} \|u\|_{H^1}^2 - \frac{k_{\max}}{p} \int_{\Omega_1} |u|^p dx - \frac{|m|_{q^*}}{q} \left(\int_{\Omega_2} |u|^p dx \right)^{q/p} \\ &\geq \frac{b}{4} \|u\|_{D^{1,2}}^4 + \frac{1}{2} \|u\|_{H^1}^2 - \frac{k_{\max} |\Omega_1|^{(2^*-p)/2^*}}{pS^p} \|u\|_{D^{1,2}}^p - \frac{|m|_{q^*} |\Omega_2|^{q(2^*-p)/p2^*}}{qS^q} \|u\|_{D^{1,2}}^q. \end{aligned} \quad (3.4)$$

Thus, for every $b > 0$, there exists $\widehat{R}_b > 0$ such that

$$I_b(u) \geq \frac{1}{2} \|u\|_{H^1}^2 \quad \text{for every } \|u\|_{D^{1,2}} \geq \widehat{R}_b.$$

Let

$$R_b = \widehat{R}_b^2 + 4 \left(\frac{k_{\max} |\Omega_1|^{(2^*-p)/2^*}}{pS^p} \widehat{R}_b^p + \frac{|m|_{q^*} |\Omega_2|^{q(2^*-p)/p2^*}}{qS^q} \widehat{R}_b^q \right). \quad (3.5)$$

Next, we prove that

$$I_b(u) \geq \frac{1}{4} \|u\|_{H^1}^2 \quad \text{for every } \|u\|_{H^1} \geq R_b. \quad (3.6)$$

For any $u \in H^1(\mathbb{R}^N)$ satisfying $\|u\|_{H^1} \geq R_b$. If $\|u\|_{D^{1,2}} \geq \widehat{R}_b$, (3.6) holds. If $\|u\|_{D^{1,2}} < \widehat{R}_b$, then it suffices to prove that $I_b(u) \geq \frac{1}{4} \|u\|_{H^1}^2$ when

$$\int_{\mathbb{R}^N} u^2 dx \geq 4 \left(\frac{k_{\max} |\Omega_1|^{(2^*-p)/2^*}}{pS^p} \widehat{R}_b^p + \frac{|m|_{q^*} |\Omega_2|^{q(2^*-p)/p2^*}}{qS^q} \widehat{R}_b^q \right). \quad (3.7)$$

Note that, by (3.3), for any $r \in [2, 2^*]$ and $i = 1, 2$, there holds

$$\int_{\Omega_i} |u|^r dx \leq S^{-r} |\Omega_i|^{(2^*-r)/2^*} \|u\|_{D^{1,2}}^r \leq S^{-r} |\Omega_i|^{(2^*-r)/2^*} \widehat{R}_b^r. \quad (3.8)$$

Then from (3.7) and (3.8), we obtain

$$\begin{aligned}
 I_b(u) &\geq \frac{b}{4}\|u\|_{D^{1,2}}^4 + \frac{1}{2}\|u\|_{D^{1,2}}^2 + \frac{1}{2}\int_{\mathbb{R}^N} u^2 dx - \frac{1}{p}\int_{\Omega_1} k(x)|u|^p dx - \frac{1}{q}\int_{\Omega_2} m(x)|u|^q dx \\
 &\geq \frac{1}{4}\|u\|_{H^1}^2 + \frac{1}{4}\int_{\mathbb{R}^N} u^2 dx - \frac{k_{\max}}{p}\int_{\Omega_1} |u|^p dx - \frac{|m|_{q^*}}{q}\left(\int_{\Omega_2} |u|^p dx\right)^{q/p} \\
 &\geq \frac{1}{4}\|u\|_{H^1}^2 + \frac{1}{4}\int_{\mathbb{R}^N} u^2 dx - \frac{k_{\max}|\Omega_1|^{(2^*-p)/2^*}}{pS^p}\widehat{R}_b^p - \frac{|m|_{q^*}|\Omega_2|^{q(2^*-p)/p2^*}}{qS^q}\widehat{R}_b^q \\
 &\geq \frac{1}{4}\|u\|_{H^1}^2.
 \end{aligned} \tag{3.9}$$

From (3.9), it is obvious that I_b is bounded from below and coercive on $H^1(\mathbb{R}^N)$ and $\inf\{I_b(u) \mid u \in H^1(\mathbb{R}^N) \text{ with } \|u\|_{H^1} \geq R_b\} \geq 0$. The proof is complete. $\square$

Lemma 3.3. *Under the conditions $(H_1) - (H_3)$, if $\{u_n\}$ is a bounded (PS) sequence, then $\{u_n\}$ has a convergent sequence.*

Proof. Let $\{u_n\}$ be a (PS) sequence satisfying $\|u_n\|_{H^1} \leq M$, where M is a positive constant. Then there exists $u \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^N), \tag{3.10}$$

$$u_n \rightarrow u \text{ in } L_{loc}^r(\mathbb{R}^N) \text{ for } 2 \leq r < 2^*, \tag{3.11}$$

$$u_n \rightarrow u \text{ a.e. in } \mathbb{R}^N. \tag{3.12}$$

Denote $\omega_n = k(x)|u_n|^{p-2}u_n$, and $\omega = k(x)|u|^{p-2}u$. Then $\omega_n \rightarrow \omega$ a.e. in $\mathbb{R}^N$. Moreover, since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^N)$ for $2 < p < \min\{4, 2^*\}$ and $k(x)$ is bounded in $\mathbb{R}^N$, thus $\{\omega_n\}_{n \in \mathbb{N}}$ is bounded in $L^{p/(p-1)}(\mathbb{R}^N)$ and $\omega_n \rightharpoonup \omega$ in $L^{p/(p-1)}(\mathbb{R}^N)$ for $2 < p < \min\{4, 2^*\}$. Then for any $v \in H^1(\mathbb{R}^N)$, it follows from (H_3) and (3.10) that

$$\int_{\mathbb{R}^N} k(x)|u_n|^{p-2}u_n v dx \rightarrow \int_{\mathbb{R}^N} k(x)|u|^{p-2}u v dx, \tag{3.13}$$

and

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla v + u_n v) dx \rightarrow \int_{\mathbb{R}^N} (\nabla u \nabla v + u v) dx, \tag{3.14}$$

as $n \rightarrow \infty$. Similarly, we obtain

$$\int_{\mathbb{R}^N} m(x)|u_n|^{q-2}u_n v dx \rightarrow \int_{\mathbb{R}^N} m(x)|u|^{q-2}u v dx \text{ as } n \rightarrow \infty. \tag{3.15}$$

Note that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, thus we obtain

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \rightarrow A \text{ as } n \rightarrow \infty,$$

for some $A > 0$. Define

$$J_b(u) = \frac{1}{2}\|u\|_{H^1}^2 + \frac{bA}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} k(x)|u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} m(x)|u|^q dx. \tag{3.16}$$

Then it follows from (3.15)–(3.16) that

$$o(1) = \langle J'_b(u_n), v \rangle$$

$$\begin{aligned}
& \rightarrow (1 + bA) \int_{\mathbb{R}^N} \nabla u \nabla v \, dx + \int_{\mathbb{R}^N} u v \, dx - \int_{\mathbb{R}^N} k(x) |u|^{p-2} u v \, dx \\
& \quad - \int_{\mathbb{R}^N} m(x) |u|^{q-2} u v \, dx \\
& = \langle J'_b(u_n), v \rangle = 0 \text{ for any } v \in H^1(\mathbb{R}^N).
\end{aligned} \tag{3.17}$$

Note that from (2.1) and (3.16), we can also easily obtain

$$\langle J'_b(u_n), u_n \rangle + o(1) = \langle J'_b(u_n), u_n \rangle = o(1). \tag{3.18}$$

Denote $v_n = u_n - u$, then by (3.10) $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, it follows from Brézis–Lieb Lemma that

$$\|u_n\|_{H^1}^2 = \|u\|_{H^1}^2 + \|v_n\|_{H^1}^2 + o(1), \tag{3.19}$$

$$\int_{\mathbb{R}^N} k(x) |u_n|^p \, dx = \int_{\mathbb{R}^N} k(x) |u|^p \, dx + \int_{\mathbb{R}^N} k(x) |v_n|^p \, dx + o(1), \tag{3.20}$$

and

$$\int_{\mathbb{R}^N} m(x) |u_n|^q \, dx = \int_{\mathbb{R}^N} m(x) |u|^q \, dx + \int_{\mathbb{R}^N} m(x) |v_n|^q \, dx + o(1). \tag{3.21}$$

In view of (3.19)–(3.21), there holds

$$\begin{aligned}
\langle J'_b(u_n), u_n \rangle &= \langle J'_b(u), u \rangle + \|v_n\|_{H^1}^2 + bA \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx - \int_{\mathbb{R}^N} k(x) |v_n|^p \, dx \\
&\quad - \int_{\mathbb{R}^N} m(x) |v_n|^q \, dx + o(1),
\end{aligned}$$

which, together with (3.17)–(3.18), implies that

$$\begin{aligned}
& \|u_n - u\|_{H^1}^2 + bA \int_{\mathbb{R}^N} |\nabla (u_n - u)|^2 \, dx - \int_{\mathbb{R}^N} k(x) |u_n - u|^p \, dx \\
& - \int_{\mathbb{R}^N} m(x) |u_n - u|^q \, dx \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned} \tag{3.22}$$

From $2 < p < \min\{4, 2^*\}$, (3.11) and (H_2) , we have

$$k(x) \leq 0 \text{ for any } x \in \mathbb{R}^N \setminus \Omega_1 \text{ and } \int_{\Omega_1} k(x) |u_n - u|^p \, dx \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.23}$$

On the other hand, since $m \in L^{q^*}(\mathbb{R}^N)$, thus for every $\varepsilon > 0$, there exists $R_* > 0$ such that

$$\int_{|x| > R_*} |m|^{q^*} \, dx < \varepsilon^{q^*}. \tag{3.24}$$

Then from (3.24), we obtain

$$\int_{|x| > R_*} m(x) |u_n - u|^q \, dx \leq \left(\int_{|x| > R_*} |m|^{q^*} \, dx \right)^{\frac{1}{q^*}} \left(\int_{|x| > R_*} |u_n - u|^p \, dx \right)^{\frac{q}{p}} \leq C\varepsilon, \tag{3.25}$$

since u_n and u are bounded in $L^p(\mathbb{R}^N)$ for $2 \leq p < \min\{4, 2^*\}$.

Note that from (3.11), we may also have

$$\int_{|x| \leq R^*} m(x)|u_n - u|^q dx \leq \left(\int_{|x| \leq R^*} |m|^{q^*} dx \right)^{\frac{1}{q^*}} \left(\int_{|x| \leq R^*} |u_n - u|^p dx \right)^{\frac{q}{p}} \rightarrow 0 \quad (3.26)$$

as $n \rightarrow \infty$. In view of (3.25) and (3.26), there holds

$$\begin{aligned} \int_{\mathbb{R}^N} m(x)|u_n - u|^q dx &= \int_{|x| \leq R^*} m(x)|u_n - u|^q dx + \int_{|x| > R^*} m(x)|u_n - u|^q dx \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.27)$$

It is now deduced from (3.21)–(3.23) and (3.27) that

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} \|u_n - u\|_{H^1}^2 \leq \limsup_{n \rightarrow \infty} \left(\|u_n - u\|_{H^1}^2 + bA \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx \right) \\ &\leq \liminf_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} k(x)|u_n - u|^p dx + \int_{\mathbb{R}^N} m(x)|u_n - u|^q dx \right) \\ &\leq \lim_{n \rightarrow \infty} \left(\int_{\Omega_1} k(x)|u_n - u|^p dx + \int_{\mathbb{R}^N} m(x)|u_n - u|^q dx \right) \\ &= 0. \end{aligned} \quad (3.28)$$

Thus, $u_n \rightarrow u$ in $H^1(\mathbb{R}^N)$ by (3.28). The proof is complete. $\square$

Theorem 3.4. Assume that hypotheses $(H_1) - (H_3)$ hold. Then the functional I_b has a local minimizer $u_b^{(1)} \in H^1(\mathbb{R}^N)$ for each $0 < |m|_{q^*} \leq \Gamma_0$, where Γ_0 is given by (3.2). Moreover,

(i) $u_b^{(1)}$ is a positive solution of Eq.(P);

(ii) $I_b(u_b^{(1)}) < 0$ and $\|u_b^{(1)}\|_{H^1} < \rho_0$.

Proof. In view of the hypothesis (H_3) , there exists $\omega \in H^1(\mathbb{R}^N) \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^N} m(x)|\omega|^q dx > 0.$$

When $t > 0$ is sufficiently small, there holds

$$\begin{aligned} I_b(t\omega) &= \frac{t^2}{2} \|\omega\|_{H^1}^2 + \frac{bt^4}{4} \left(\int_{\mathbb{R}^N} |\nabla \omega|^2 dx \right)^2 - \frac{t^p}{p} \int_{\mathbb{R}^N} k(x)|\omega|^p dx \\ &\quad - \frac{t^q}{q} \int_{\mathbb{R}^N} m(x)|\omega|^q dx \\ &< 0. \end{aligned} \quad (3.29)$$

Thus, from (3.29) and Lemma 3.1, we obtain

$$\theta_m := \inf\{I_b(u) \mid u \in \overline{B}_{\rho_0}(0)\} < 0. \quad (3.30)$$

From the Ekeland variational principle [14], there exists a sequence $\{u_n\} \subset \overline{B}_{\rho_0}(0)$ satisfying

$$I_b(u_n) = \theta_m + o(1) \text{ and } I'_b(u_n) = o(1).$$

It is easy to know that I_b satisfies the $(PS)_{\theta_m}$ -condition in $\bar{B}_{\rho_0}(0)$ by Lemma 3.3. As a result, there exists $u_b^{(1)} \in \bar{B}_{\rho_0}(0)$ satisfying $u_n \rightarrow u_b^{(1)}$ as $n \rightarrow +\infty$. This implies that $u_b^{(1)}$ is a local minimizer on $B_{\rho_0}(0)$ that satisfies $I_b(u_b^{(1)}) = \theta_m < 0$ and $\|u_b^{(1)}\|_{H^1} < \rho_0$ by Lemma 3.1. Since $I_b(u_b^{(1)}) = I_b(|u_b^{(1)}|) = \theta_m$, we may hypothesize that $u_b^{(1)}$ is a positive solution of Eq.(P). The proof is complete. $\square$

Theorem 3.5. Assume that hypotheses $(H_1) - (H_4)$ hold. Then there exists a constant $b_0 > 0$ such that for any $0 < b < b_0$, Eq.(P) admits a positive solution $u_b^{(2)}$ satisfying $I_b(u_b^{(2)}) < 0$ and $\rho_0 < \|u_b^{(2)}\|_{H^1} < R_b$.

Proof. Since $k \in C(\mathbb{R}^N)$ and $k_{\max} := \sup_{x \in \bar{\Omega}_1} k(x)$, thus there exists a domain $\Omega_0 \subset \Omega_1$ such that for every $x \in \Omega_0$, there holds $k(x) \geq \frac{k_{\max}}{2}$. Let $I_0(u) := I_b(u)$ with $b = 0$. Then for every $\xi \in H_0^1(\Omega_1)$, we know that $\lim_{t \rightarrow +\infty} \frac{I_0(t\xi)}{t^p} \leq -\frac{k_{\max}}{2p} \int_{\Omega_0} |\xi|^p dx < 0$, which shows that $I_0(t\xi) \rightarrow -\infty$ as $t \rightarrow +\infty$. Hence, there exists $\zeta \in H^1(\mathbb{R}^N)$ with $\|\zeta\|_{H^1} > \rho_0$ such that $I_0(\zeta) < 0$. Since $I_b(\zeta) \rightarrow I_0(\zeta)$ as $b \rightarrow 0^+$, then there exists $b_0 > 0$ such that for every $0 < b < b_0$, there holds $I_b(\zeta) < 0$. It is easy to know that $\|\zeta\|_{H^1} < R_b$ by Lemma 3.2. The rest proof is analogous to that of Theorem 3.4, we omit it here. $\square$

4 The filtration of Nehari manifold

Set

$$B(p, q, k, m) = \frac{p(4-q)2^{q/(p-2)}|m|_{q^*}}{(p-2)qS_{p,\Omega_1}^q \left(\int_{\Omega_1} k(x)w_{\Omega_1}^p dx \right)^{(2-q)/2}}.$$

For any $u \in \mathcal{M}_b$ with

$$\|u\|_{H^1} > \rho_0 = \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}} \right)^{1/(p-2)}, \quad (4.1)$$

and

$$I_b(u) < \frac{p-2}{2p} A^2(p)(1+B(p, q, k, m)) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx,$$

the Sobolev and Hölder inequalities imply that

$$\begin{aligned} & \frac{p-2}{2p} A^2(p)(1+B(p, q, k, m)) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx \\ & > I_b(u) = \frac{1}{2} \|u\|_{H^1}^2 + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 - \frac{1}{p} \int_{\mathbb{R}^N} k(x)|u|^p dx \\ & \quad - \frac{1}{q} \int_{\mathbb{R}^N} m(x)|u|^q dx \\ & \geq \frac{p-2}{2p} \|u\|_{H^1}^2 - \frac{(4-p)b}{4p} \|u\|_{H^1}^4 - \frac{p-q}{pqS_p^q} |m|_{q^*} \|u\|_{H^1}^q \\ & > \frac{p-2}{2p} \|u\|_{H^1}^2 - \left[\frac{(4-p)b}{4p} + \frac{p-q}{pqS_p^q} \rho_0^{q-4} |m|_{q^*} \right] \|u\|_{H^1}^4, \end{aligned}$$

since $\|u\|_{H^1}^q < \rho_0^{q-4} \|u\|_{H^1}^4$. This implies that for

$$0 < \frac{(4-p)b}{4p} + \frac{p-q}{pqS_p^q} \rho_0^{q-4} |m|_{q^*} < \Gamma_1,$$

there are two constants C_1, C_2 satisfying $0 < C_1 < A(p)\|w_{\Omega_1}\|_{H^1} < C_2$ such that

$$\|u\|_{H^1} < C_1 \text{ or } \|u\|_{H^1} > C_2,$$

where

$$\Gamma_1 := \min \left\{ \frac{(p-2)}{8pA^2(p)(1+B(p,q,k,m)) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx}, \frac{p-2}{4pA^2(p) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx} \right\}. \quad (4.2)$$

Thus, it holds

$$\begin{aligned} & \mathcal{M}_b \left[\frac{p-2}{2p} A^2(p)(1+B(p,q,k,m)) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx \right] \\ &= \left\{ u \in \mathcal{M}_b \mid \|u\|_{H^1} > \rho_0 \text{ and } I_b(u) < \frac{p-2}{2p} A^2(p)(1+B(p,q,k,m)) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx \right\} \\ &= \mathcal{M}_b^{(1)} \cup \mathcal{M}_b^{(2)}, \end{aligned} \quad (4.3)$$

where

$$\mathcal{M}_b^{(1)} := \left\{ u \in \mathcal{M}_b \left[\frac{p-2}{2p} A^2(p)(1+B(p,q,k,m)) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx \right] \mid \|u\|_{H^1} < C_1 \right\},$$

and

$$\mathcal{M}_b^{(2)} := \left\{ u \in \mathcal{M}_b \left[\frac{p-2}{2p} A^2(p)(1+B(p,q,k,m)) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx \right] \mid \|u\|_{H^1} > C_2 \right\}.$$

Moreover, we get

$$\|u\|_{H^1} < C_1 < A(p)\|w_{\Omega_1}\|_{H^1} \text{ for all } u \in \mathcal{M}_b^{(1)}, \quad (4.4)$$

and

$$\|u\|_{H^1} > C_2 > A(p)\|w_{\Omega_1}\|_{H^1} \text{ for all } u \in \mathcal{M}_b^{(2)}.$$

It follows from (2.3), (4.1) and (4.4) that

$$\begin{aligned} h_{b,u}''(1) &\leq -(p-2)\|u\|_{H^1}^2 + (4-p)b\|u\|_{H^1}^4 + \frac{p-q}{S_p^q} |m|_{q^*} \|u\|_{H^1}^q \\ &< -(p-2)\|u\|_{H^1}^2 + \left[(4-p)b + \frac{p-q}{S_p^q} \rho_0^{q-4} |m|_{q^*} \right] \|u\|_{H^1}^4 \\ &< 0, \end{aligned} \quad (4.5)$$

provided that

$$\frac{(4-p)b}{4} + \frac{p-q}{qS_p^q} \rho_0^{q-4} |m|_{q^*} < \frac{p-2}{4A^2(p)\|w_{\Omega_1}\|_{H^1}^2}.$$

Hence, we get the following result.

Lemma 4.1. *Under the conditions $(H_1) - (H_3)$, there is a constant $\Pi_0 > 0$ such that for every $0 < b + |m|_{q^*} < \Pi_0$, $\mathcal{M}_b^{(1)} \subset \mathcal{M}_b$ is a C^1 sub-manifold. Furthermore, the local minimizer of the functional I_b in $\mathcal{M}_b^{(1)}$ is a critical point of I_b in $H^1(\mathbb{R}^N)$.*

Lemma 4.2. *Under the conditions $(H_1) - (H_3)$, there exists a constant $\Pi_1 \in (0, \Pi_0)$ such that for any $0 < b + |m|_{q^*} < \Pi_1$, there is a constant $\bar{t}_b$ satisfying $t_b^- < \bar{t}_b$ such that $t_b^- w_{\Omega_1}(x) \in \mathcal{M}_b^{(1)}$, where*

$$\bar{t}_b = \begin{cases} \left(\frac{2}{4-p} \right)^{1/(p-2)} & \text{if } \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \geq 0, \\ \left[\frac{2}{4-p} \left(1 + \frac{\left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right|}{\int_{\Omega_1} k(x)w_{\Omega_1}^p dx} \right) \right]^{1/(p-2)} & \text{if } \int_{\Omega_1} m(x)w_{\Omega_1}^q dx < 0. \end{cases}$$

Proof. Let

$$f(t) = t^{-2} \|w_{\Omega_1}\|_{H^1}^2 - t^{p-4} \int_{\Omega_1} k(x) w_{\Omega_1}^p dx - t^{q-4} \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \text{ for } t > 0.$$

Then

$$\begin{aligned} h'_{b,w_{\Omega_1}}(t) &= t \|w_{\Omega_1}\|_{H^1}^2 + b t^3 \left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2 - t^{p-1} \int_{\Omega_1} k(x) w_{\Omega_1}^p dx - t^{q-1} \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \\ &= t^3 \left[f(t) + b \left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2 \right]. \end{aligned}$$

It is obvious that $tw_{\Omega_1} \in \mathcal{M}_b$ if and only if $h'_{b,w_{\Omega_1}}(t) = 0$, i.e., $f(t) + b \left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2 = 0$. In view of (1.8),

we obtain

$$\begin{aligned} f(t) &= t^{-2} \int_{\Omega_1} (|\nabla w_{\Omega_1}|^2 + w_{\Omega_1}^2) dx - t^{p-4} \int_{\Omega_1} k(x) w_{\Omega_1}^p dx - t^{q-4} \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \\ &= t^{q-4} (t^{2-q} - t^{p-q} - C_{k,m}) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx \text{ for } t > 0, \end{aligned}$$

where

$$C_{k,m} := \frac{\int_{\Omega_1} m(x) w_{\Omega_1}^q dx}{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx} \leq \frac{|m|_{q_*}}{S_{p,\Omega_1}^q \left(\int_{\Omega_1} k(x) w_{\Omega_1}^p dx \right)^{(2-q)/2}}.$$

Define $f_1(t) = t^{2-q} - t^{p-q} - C_{k,m}$, then $f(t) > 0$ if and only if $f_1(t) > 0$, and $f'_1(t) = 0$ exactly at

$$t_1 = \left(\frac{2-q}{p-q} \right)^{1/(p-2)},$$

and

$$\begin{aligned} f(t_1) &= t_1^{q-4} (t_1^{2-q} - t_1^{p-q} - C_{k,m}) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx \\ &= t_1^{q-4} \left[\frac{p-2}{p-q} \left(\frac{2-q}{p-q} \right)^{(2-q)/(p-2)} - C_{k,m} \right] \int_{\Omega_1} k(x) w_{\Omega_1}^p dx \\ &> 0, \end{aligned}$$

provided that

$$|m|_{q_*} < \Gamma_2 := \frac{(p-2)S_{p,\Omega_1}^q}{p-q} \left(\frac{2-q}{p-q} \right)^{(2-q)/(p-2)} \left(\int_{\Omega_1} k(x) w_{\Omega_1}^p dx \right)^{(2-q)/2}.$$

Next, we consider the problem in the following two cases.

Case (I) : $\int_{\Omega_1} m(x) w_{\Omega_1}^q dx \geq 0$. Let

$$\tilde{f}(t) = t^{-2} \|w_{\Omega_1}\|_{H^1}^2 - t^{p-4} \int_{\Omega_1} k(x) w_{\Omega_1}^p dx, \quad t > 0.$$

Obviously, $f(t) \leq \tilde{f}(t)$ for $t > 0$. In view of (1.8), there holds

$$\begin{aligned}\tilde{f}(t) &= t^{-2} \int_{\Omega_1} (|\nabla w_{\Omega_1}|^2 + w_{\Omega_1}^2) dx - t^{p-4} \int_{\Omega_1} k(x) w_{\Omega_1}^p dx \\ &= (t^{-2} - t^{p-4}) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx.\end{aligned}$$

Evidently,

$$\tilde{f}(1) = 0, \quad \lim_{t \rightarrow 0^+} \tilde{f}(t) = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \tilde{f}(t) = 0.$$

Moreover

$$\inf_{t>0} \tilde{f}(t) = \tilde{f}(\tilde{t}_0) = -\frac{p-2}{4-p} \left(\frac{4-p}{2} \right)^{2/(p-2)} \int_{\Omega_1} k(x) w_{\Omega_1}^p dx < 0,$$

where

$$\tilde{t}_0 := \left(\frac{2}{4-p} \right)^{1/(p-2)} > 1.$$

Observe that

$$\begin{aligned}& \frac{p-2}{4-p} \left(\frac{4-p}{2} \right)^{2/(p-2)} \frac{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx}{\left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2} \\ & \geq \frac{p-2}{4-p} \left(\frac{4-p}{2} \right)^{2/(p-2)} \frac{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx}{\|w_{\Omega_1}\|_{H^1}^4} \\ & = \frac{p-2}{4-p} \left(\frac{4-p}{2} \right)^{2/(p-2)} \frac{1}{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx}.\end{aligned}$$

Thus, for each

$$0 < b < b_*^{(1)} := \frac{p-2}{4-p} \left(\frac{4-p}{2} \right)^{2/(p-2)} \frac{1}{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx},$$

we obtain

$$\inf_{t>0} \tilde{f}(t) = -\frac{p-2}{4-p} \left(\frac{4-p}{2} \right)^{2/(p-2)} \int_{\Omega_1} k(x) w_{\Omega_1}^p dx < -b \left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2.$$

This shows that there exist two constants t_b^+ and t_b^- that satisfy

$$t_b^- < \left(\frac{2}{4-p} \right)^{1/(p-2)} < t_b^+$$

such that

$$f(t_b^\pm) + b \left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2 = 0, \quad f'(t_b^-) < 0 \quad \text{and} \quad f'(t_b^+) > 0.$$

That is, $t_b^\pm w_{\Omega_1} \in \mathcal{M}_b$. Besides, it is evident that

$$h''_{b, t_b^- w_{\Omega_1}}(1) = (t_b^-)^5 f'(t_b^-) < 0,$$

and

$$h''_{b, t_b^+ w_{\Omega_1}}(1) = (t_b^+)^5 f'(t_b^+) > 0.$$

These imply that $t_b^\pm w_{\Omega_1} \in \mathcal{M}_b^\pm$.

Case (II) : $\int_{\Omega_1} m(x)w_{\Omega_1}^q dx < 0$. Note that

$$\begin{aligned} f(t) &= t^{-2} \int_{\Omega_1} (|\nabla w_{\Omega_1}|^2 + w_{\Omega_1}^2) dx - t^{p-4} \int_{\Omega_1} k(x)w_{\Omega_1}^p dx + t^{q-4} \left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right| \\ &= (t^{-2} - t^{p-4}) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx + t^{q-4} \left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right|. \end{aligned}$$

Define

$$\widehat{f}(t) = t^{-2} \left(\int_{\Omega_1} k(x)w_{\Omega_1}^p dx + \left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right| \right) - t^{p-4} \int_{\Omega_1} k(x)w_{\Omega_1}^p dx.$$

It is obvious that

$$f(t) < \widehat{f}(t) \text{ for } t > 1, \quad (4.6)$$

and $\widehat{f}(1) = f(1) = \left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right| > 0$ and $\lim_{t \rightarrow \infty} \widehat{f}(t) = 0$. Furthermore, we obtain

$$\inf_{t>0} \widehat{f}(t) = \widehat{f}(\widehat{t}_0) = -\frac{p-2}{(4-p)\widehat{t}_0^2} \left(\int_{\Omega_1} k(x)w_{\Omega_1}^p dx + \left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right| \right) < 0, \quad (4.7)$$

where

$$\widehat{t}_0 := \left[\frac{2}{4-p} \left(1 + \frac{\left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right|}{\int_{\Omega_1} k(x)w_{\Omega_1}^p dx} \right) \right]^{1/(p-2)} > 1. \quad (4.8)$$

Observe that

$$\begin{aligned} & \frac{p-2}{(4-p)\widehat{t}_0^2} \left(\int_{\Omega_1} k(x)w_{\Omega_1}^p dx + \left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right| \right) \frac{1}{\left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2} \\ & \geq \frac{p-2}{(4-p)\widehat{t}_0^2} \left(\int_{\Omega_1} k(x)w_{\Omega_1}^p dx + \left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right| \right) \frac{1}{\|w_{\Omega_1}\|_{H^1}^4} \\ & = \frac{p-2}{(4-p)\widehat{t}_0^2} \left(\int_{\Omega_1} k(x)w_{\Omega_1}^p dx + \left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right| \right) \frac{1}{\left(\int_{\Omega_1} k(x)w_{\Omega_1}^p dx \right)^2} \\ & = \frac{p-2}{4-p} \left(\frac{4-p}{2} \right)^{2/(p-2)} \left(1 + \frac{\left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right|}{\int_{\Omega_1} k(x)w_{\Omega_1}^p dx} \right)^{(p-4)/(p-2)} \frac{1}{\int_{\Omega_1} k(x)w_{\Omega_1}^p dx} \\ & := b_*^{(2)}. \end{aligned}$$

It is obvious that $b_*^{(2)} < b_*^{(1)}$ since $2 < p < \min\{4, 2^*\}$. Thus, for each $0 < b < b_*^{(2)}$, it follows from (4.7) that

$$\widehat{f}(\widehat{t}_0) = -\frac{p-2}{(4-p)\widehat{t}_0^2} \left(\int_{\Omega_1} k(x)w_{\Omega_1}^p dx + \left| \int_{\Omega_1} m(x)w_{\Omega_1}^q dx \right| \right) < -b \left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2. \quad (4.9)$$

From (4.6), (4.8) and (4.9), we derive

$$f(\widehat{t}_0) < \widehat{f}(\widehat{t}_0) < -b \left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2. \quad (4.10)$$

In view of (4.6) and (4.10), there are two constants t_b^+ and t_b^- that satisfy

$$1 < t_b^- < \widehat{t}_0 < t_b^+ \quad (4.11)$$

such that

$$f(t_b^\pm) + b \left(\int_{\Omega_1} |\nabla w_{\Omega_1}|^2 dx \right)^2 = 0, \quad f'(t_b^-) < 0 \quad \text{and} \quad f(t_b^+) > 0.$$

That is, $t_b^\pm w_{\Omega_1} \in \mathcal{M}_b$. A straightforward calculation shows that

$$h''_{b, t_b^- w_{\Omega_1}}(1) = (t_b^-)^5 f'(t_b^-) < 0,$$

and

$$h''_{b, t_b^+ w_{\Omega_1}}(1) = (t_b^+)^5 f'(t_b^+) > 0.$$

These imply that $t_b^\pm w_{\Omega_1} \in \mathcal{M}_b^\pm$.

Next we prove that $t_b^- w_{\Omega_1} \in \mathcal{M}_b^{(1)}$.

If $\int_{\Omega_1} m(x) w_{\Omega_1}^q dx < 0$, it follows from (4.11) that

$$\begin{aligned} & I_b(t_b^- w_{\Omega_1}) \\ &= \frac{1}{4} (t_b^-)^2 \left[1 - \frac{4-p}{p} (t_b^-)^{p-2} \right] \int_{\Omega_1} k(x) w_{\Omega_1}^p dx + \frac{4-q}{4q} (t_b^-)^q \left| \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \right| \\ &\leq \frac{1}{4} (t_b^-)^2 \left[1 - \frac{4-p}{p} (t_b^-)^{p-2} \right] \int_{\Omega_1} k(x) w_{\Omega_1}^p dx \\ &\quad + \frac{4-q}{4q} \left[\frac{2}{4-p} \left(1 + \frac{\left| \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \right|}{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx} \right) \right]^{q/(p-2)} \left| \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \right| \\ &:= I_1 + I_2, \end{aligned} \quad (4.12)$$

where

$$I_1 = \frac{1}{4} (t_b^-)^2 \left[1 - \frac{4-p}{p} (t_b^-)^{p-2} \right] \int_{\Omega_1} k(x) w_{\Omega_1}^p dx,$$

and

$$I_2 = \frac{4-q}{4q} \left[\frac{2}{4-p} \left(1 + \frac{\left| \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \right|}{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx} \right) \right]^{q/(p-2)} \left| \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \right|.$$

Define a function by

$$g(t) = \frac{t^2}{4} \left(1 - \frac{4-p}{p} t^{p-2} \right), \quad \text{for } 2 < p < \min\{4, 2^*\},$$

where

$$1 < t < \widehat{t}_0 := \left[\frac{2}{4-p} \left(1 + \frac{\left| \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \right|}{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx} \right) \right]^{1/(p-2)}.$$

By a direct calculation, we obtain

$$\max_{0 < t \leq \hat{t}_0} g(t) = g(t_*) = \frac{p-2}{4p} \left(\frac{2}{4-p} \right)^{2/(p-2)},$$

where

$$t_* = \left(\frac{2}{4-p} \right)^{1/(p-2)} \in (1, \hat{t}_0). \quad (4.13)$$

Hence,

$$\max_{0 < t \leq \hat{t}_0} g(t) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx \leq \frac{(p-2)A^2(p)}{2p} \|w_{\Omega_1}\|_{H^1}^2,$$

that is,

$$I_1 \leq \frac{(p-2)A^2(p)}{2p} \|w_{\Omega_1}\|_{H^1}^2. \quad (4.14)$$

On the other hand, by $0 < b + |m|_{q^*} < \Pi_1$ and (4.11), it holds

$$\begin{aligned} I_2 &= \frac{4-q}{4q} \left[\frac{2}{4-p} \left(1 + \frac{\left| \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \right|}{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx} \right) \right]^{q/(p-2)} \left| \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \right| \\ &\leq \frac{4-q}{4q} \left[\frac{2}{4-p} \left(1 + \frac{|m|_{q^*} |w_{\Omega_1}|_p^q}{\int_{\Omega_1} k(x) w_{\Omega_1}^p dx} \right) \right]^{q/(p-2)} |m|_{q^*} |w_{\Omega_1}|_p^q \\ &\leq \begin{cases} \frac{4-q}{2q} \left(\frac{2}{4-p} \right)^{q/(p-2)} |m|_{q^*} S_{p, \Omega_1}^{-q} \left(\int_{\Omega_1} k(x) w_{\Omega_1}^p dx \right)^{q/2} & \text{if } q \leq p-2 \\ \frac{4-q}{4q} \left(\frac{4}{4-p} \right)^{q/(p-2)} |m|_{q^*} S_{p, \Omega_1}^{-q} \left(\int_{\Omega_1} k(x) w_{\Omega_1}^p dx \right)^{q/2} & \text{if } q > p-2 \end{cases} \\ &\leq \frac{4-q}{2q} \left(\frac{4}{4-p} \right)^{q/(p-2)} |m|_{q^*} S_{p, \Omega_1}^{-q} \left(\int_{\Omega_1} k(x) w_{\Omega_1}^p dx \right)^{q/2} \\ &\leq \frac{p-2}{2p} A^2(p) B(p, q, k, m) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx. \end{aligned} \quad (4.15)$$

It follows from (4.12), (4.14) and (4.15) that $t_b^- w_{\Omega_1} \in \mathcal{M}_b^{(1)}$.

Moreover, if $\int_{\Omega_1} m(x) w_{\Omega_1}^q dx \geq 0$, then by (H_2) , we have

$$\begin{aligned} &I_b(t_b^- w_{\Omega_1}) \\ &= \frac{1}{4} (t_b^-)^2 \|w_{\Omega_1}\|_{H^1}^2 - \frac{4-p}{4p} (t_b^-)^p \int_{\Omega_1} k(x) w_{\Omega_1}^p dx - \frac{4-q}{4q} (t_b^-)^q \int_{\Omega_1} m(x) w_{\Omega_1}^q dx \\ &\leq \frac{1}{4} (t_b^-)^2 \|w_{\Omega_1}\|_{H^1}^2 - \frac{4-p}{4p} (t_b^-)^p \int_{\Omega_1} k(x) w_{\Omega_1}^p dx \\ &= \frac{1}{4} (t_b^-)^2 \left[1 - \frac{4-p}{p} (t_b^-)^{p-2} \right] \int_{\Omega_1} k(x) w_{\Omega_1}^p dx. \end{aligned}$$

Let

$$\eta(t) = \frac{t^2}{4} \left(1 - \frac{4-p}{p} t^{p-2} \right), \quad 2 < p < \min\{4, 2^*\},$$

where

$$1 < t \leq \tilde{t}_0 := \left(\frac{2}{4-p} \right)^{1/(p-2)}.$$

A simple calculation shows that

$$\max_{0 < t \leq \tilde{t}_0} \eta(t) = \eta(\tilde{t}_0) = \frac{p-2}{4p} \left(\frac{2}{4-p} \right)^{2/(p-2)}.$$

Then,

$$\max_{0 < t \leq \tilde{t}_0} \eta(t) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx < \frac{p-2}{2p} A^2(p) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx,$$

that is,

$$I_b(t_b^- w_{\Omega_1}) < \frac{p-2}{2p} A^2(p) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx \leq \frac{p-2}{2p} A^2(p) (1 + B(p, q, k, m)) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx.$$

The proof is complete. $\square$

5 Proofs of main results

Lemma 5.1. *Under the conditions $(H_1) - (H_3)$, there exist a constant $\sigma > 0$ and a differentiable function $t^* : B(0, \sigma) \subset H^1(\mathbb{R}^N) \rightarrow \mathbb{R}^+$ such that for any $u \in \mathcal{M}_b^{(1)}$, there holds*

$$t^*(0) = 1 \text{ and } t^*(v)(u - v) \in \mathcal{M}_b^{(1)}$$

for all $v \in B(0, \sigma)$, and

$$\langle (t^*)'(0), \varphi \rangle = \frac{\psi(u, \varphi)}{\Gamma(u)},$$

where

$$\begin{aligned} \psi(u, \varphi) &= 2 \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + u \varphi) dx + 4b \int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla \varphi dx \\ &\quad - p \int_{\mathbb{R}^N} k(x) |u|^{p-2} u \varphi dx - q \int_{\mathbb{R}^N} m(x) |u|^{q-2} u \varphi dx, \end{aligned} \quad (5.1)$$

and

$$\Gamma(u) = \|u\|_{H^1}^2 - (p-1) \int_{\mathbb{R}^N} k(x) |u|^p dx - (q-1) \int_{\mathbb{R}^N} m(x) |u|^q dx \quad (5.2)$$

for all $\varphi \in H^1(\mathbb{R}^N)$.

Proof. The proof of Lemma 5.1 is analogous to [7, Lemma 3.1] and [24, Lemma 3.1], we omit it here. $\square$

Define

$$\alpha_b^- = \inf_{u \in \mathcal{M}_b^{(1)}} I_b(u).$$

In view of Lemma 2.2 and $\mathcal{M}_b^{(1)} \subset \mathcal{M}_b^-$, we derive

$$0 < C_0 \leq \alpha_b^- < \frac{p-2}{2p} A^2(p) (1 + B(p, q, k, m)) \int_{\Omega_1} k(x) w_{\Omega_1}^p dx. \quad (5.3)$$

Proposition 5.2. *Assume that hypotheses $(H_1) - (H_3)$ hold. Then there exists a sequence $\{u_n\} \subset \mathcal{M}_b^{(1)}$ such that*

$$I_b(u_n) = \alpha_b^- + o(1) \text{ and } I'_b(u_n) = o(1). \quad (5.4)$$

Proof. The proof of Proposition 5.2 is similar to [7, Proposition 3.2] and [24, Proposition 2], we omit it here. $\square$

By similar arguments to those of Lemma 3.3, we may get the following conclusion.

Proposition 5.3. Assume that hypotheses $(H_1) - (H_3)$ hold. Then for any $0 < d < \frac{p-2}{2p}A^2(p)(1 + B(p, q, k, m)) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx$, I_b satisfies the $(PS)_d$ -condition in $\mathcal{M}_b^{(1)}$.

Theorem 5.4. Assume that hypotheses $(H_1) - (H_3)$ hold. Then for any $0 < b + |m|_{q^*} < \Pi_1$, where $\Pi_1 > 0$ is given by Lemma 4.2, Eq.(P) admits a positive solution $u_b^{(*)} \in \mathcal{M}_b^{(1)}$ satisfying $I_b(u_b^{(*)}) > 0$ and

$$\left(\frac{(2-q)S_p^p}{(p-q)k_{\max}} \right)^{1/(p-2)} < \|u_b^{(*)}\|_{H^1} < A(p) \|w_{\Omega_1}\|_{H^1}.$$

Proof. By Proposition 5.2, there is a sequence $\{u_n\} \subset \mathcal{M}_b^{(1)}$ satisfying

$$I_b(u_n) = \alpha_b^- + o(1) \text{ and } I'_b(u_n) = o(1).$$

Clearly, $0 < \alpha_b^- < \frac{p-2}{2p}A^2(p)(1+B(p, q, k, m)) \int_{\Omega_1} k(x)w_{\Omega_1}^p dx$, and for small b , I_b satisfies the $(PS)_{\alpha_b^-}$ -condition in $\mathcal{M}_b^{(1)}$ by Proposition 5.3. Thus there exists $u_b^{(*)} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $u_n \rightarrow u_b^{(*)}$ in $H^1(\mathbb{R}^N)$. Observe that $\{u_n\} \subset \mathcal{M}_b^{(1)}$ and

$$\|u_n\|_{H^1} < C_1 < A(p) \|w_{\Omega_1}\|_{H^1} \text{ for all } n = 1, 2, \dots,$$

then from Fatou's lemma, we have

$$\|u_b^{(*)}\|_{H^1} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{H^1} \leq C_1 < A(p) \|w_{\Omega_1}\|_{H^1}. \quad (5.5)$$

(4.5) and (5.5) imply that

$$h''_{b, u_b^{(*)}}(1) < 0,$$

provided that

$$\frac{4-p}{4}b + \frac{p-q}{qS_p^q \rho_0^{4-q}} |m|_{q^*} < \Gamma_1,$$

where Γ_1 is defined by (4.2). Thus, $u_b^{(*)} \in \mathcal{M}_b^-$. On the other hand, observe that

$$\alpha_b^- = I_b(u_b^{(*)}) \leq I_b(t_b^- w_{\Omega_1}) \leq \frac{p-2}{2p}A^2(p) (1 + B(p, q, k, m)) \|w_{\Omega_1}\|_{H^1}^2,$$

then $u_b^{(*)} \in \mathcal{M}_b^{(1)}$. Furthermore, $I_b(u_b^{(*)}) = I_b(|u_b^{(*)}|) = \alpha_b^-$. Thus, from Lemma 2.1, we may assume that $u_b^{(*)}$ is a positive solution of Eq.(P). The proof is complete. $\square$

Proof of Theorem 1.1: From Theorems 3.4 and 5.4, there exists a constant $\Pi_1 > 0$ such that for any $0 < b + |m|_{q^*} < \Pi_1$, Eq.(P) admits two positive solutions $u_b^{(1)}$ and $u_b^{(*)}$ that satisfy

$$\|u_b^{(1)}\|_{H^1} < \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}} \right)^{1/(p-2)} < \|u_b^{(*)}\|_{H^1} < A(p) \|w_{\Omega_1}\|_{H^1},$$

and

$$I_b(u_b^{(1)}) < 0 < I_b(u_b^{(*)}).$$

The proof is complete. $\square$

Proof of Theorem 1.2: From Theorems 3.4, 3.5 and 5.4, there exists a constant $\Pi_2 = \min\{b_0, \Pi_1\} > 0$ such that for any $0 < b + |m|_{q^*} < \Pi_2$, Eq.(P) admits three positive solutions $u_b^{(1)}$, $u_b^{(2)}$ and $u_b^{(*)}$ that satisfy

$$\|u_b^{(1)}\|_{H^1} < \rho_0 := \left(\frac{(2-q)S_p^p}{(p-q)k_{\max}} \right)^{1/(p-2)} < \min \left\{ \|u_b^{(*)}\|_{H^1}, \|u_b^{(2)}\|_{H^1} \right\},$$

and

$$\max \{I_b(u_b^{(1)}), I_b(u_b^{(2)})\} < 0 < I_b(u_b^{(*)}).$$

Then by Lemma 2.2 and (4.3), $u_b^{(*)} \in \mathcal{M}_b^{(2)}$. Hence,

$$\|u_b^{(1)}\|_{H^1} < \rho_0 < \|u_b^{(*)}\|_{H^1} < \|u_b^{(2)}\|_{H^1}.$$

The proof is complete. $\square$

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