Research Article

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Optimal decay rate for higher-order derivatives of solution to the 3D compressible quantum magnetohydrodynamic model

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Abstract: We investigate optimal decay rates for higher-order spatial derivatives of strong solutions to the 3D Cauchy problem of the compressible viscous quantum magnetohydrodynamic model in the $H^5 \times H^4 \times H^4$ framework, and the main novelty of this work is three-fold: First, we show that fourth order spatial derivative of the solution converges to zero at the L^2 -rate $(1+t)^{-\frac{11}{4}}$, which is same as that of the heat equation, and particularly faster than the L^2 -rate $(1+t)^{-\frac{5}{4}}$ in Pu–Xu [Z. Angew. Math. Phys., 68:1, 2017] and the L^2 -rate $(1+t)^{-\frac{9}{4}}$ in Xi–Pu–Guo [Z. Angew. Math. Phys., 70:1, 2019]. Second, we prove that fifth-order spatial derivative of density ρ converges to zero at the L^2 -rate $(1+t)^{-\frac{13}{4}}$, which is same as one of the heat equation, and particularly faster than ones of Pu–Xu [Z. Angew. Math. Phys., 68:1, 2017] and Xi–Pu–Guo [Z. Angew. Math. Phys., 70:1, 2019]. Third, we show that the high-frequency part of the fourth order spatial derivatives of the velocity u and magnetic B converge to zero at the L^2 -rate $(1+t)^{-\frac{13}{4}}$, which are faster than ones of themselves, and totally new as compared to Pu–Xu [Z. Angew. Math. Phys., 68:1, 2017] and Xi–Pu–Guo [Z. Angew. Math. Phys., 70:1, 2019].

Keywords: Compressible quantum magnetohydrodynamic model; Optimal decay rate

MSC: 35B40; 35Q35; 76W05

1 Introduction

In this paper, we consider optimal decay rates for higher-order spatial derivatives of strong solutions to the following 3D compressible viscous quantum magnetohydrodynamic (vQMHD) model:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u + \nabla P(\rho) - \frac{\vartheta^2}{2} \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = (\nabla \times B) \times B, \\ B_t - \nabla \times (u \times B) = -\nabla \times (v \nabla \times B), \quad \operatorname{div} B = 0, \end{cases} \quad (1.1)$$

where $t \geq 0$ is time and $x \in \mathbb{R}^3$ is the spatial coordinate, and the symbol $\otimes$ is the Kronecker tensor product. The unknown functions $\rho = \rho(x, t)$ is the density, $u = (u_1, u_2, u_3)(x, t)$ denotes the velocity, and $B = (B_1, B_2, B_3)(x, t)$ represents the magnetic field. $P = P(\rho) = a\rho^\gamma$ ($a > 0, \gamma \geq 1$) stands for the pressure. The constant viscosity coefficients μ and λ satisfy the physical conditions: $\mu > 0$, $\frac{2}{3}\mu + \lambda \geq 0$, and $v > 0$ denotes the magnetic diffusion coefficient. The constant $\vartheta > 0$ represents the Planck constant. The expression $\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}$ is

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the so-called Bohm quantum potential satisfying:

$$2\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = \operatorname{div}(\rho \nabla^2 \rho) = \nabla \Delta \rho + \frac{|\nabla \rho|^2 \nabla \rho}{\rho^2} - \frac{\nabla \rho \Delta \rho}{\rho} - \frac{\nabla \rho \cdot \nabla^2 \rho}{\rho} = \nabla \Delta \rho - 4 \operatorname{div}(\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}).$$

The system (1.1) is supplemented with the following initial data

$$(\rho, u, B)|_{t=0} = (\rho_0(x), u_0(x), B_0(x)). \quad (1.2)$$

Moreover, we assume that when the space variable goes to infinity, the initial perturbation satisfies

$$\lim_{|x| \rightarrow \infty} (\rho_0 - 1, u_0, B_0)(x) = 0. \quad (1.3)$$

1.1 History of the problem

The quantum fluid model can provide many pieces of information for the particles in the semiconductor simulation, and it could be used to describe quantum semiconductors [6], weakly interacting Bose gases [8], and quantum trajectories of Bohmian mechanics [34]. The quantum magnetohydrodynamic (QMHD) model plays an important role in modeling and simulating electron transport, which was extended by Hass [10] later from a Wigner–Maxwell system. Moreover, this model could be used to describe global properties of quantum plasmas. It is worth mentioning that system (1.1) will reduce to the compressible MHD equations without quantum effects. There is a vast literature addressing the decay and other asymptotic behaviors of compressible fluid equations, we refer to previous studies [1–5, 11, 13–22, 24, 25, 29–32] and references therein.

In what follows, we only review some results closely related for simplicity. The study for decay rates of solutions to the QMHD model has attracted much attention of mathematicians. Pu and Guo [25] established the optimal decay rates of classical solutions near constant states via the spectral method in $\mathbb{R}^3$. In [27], Pu and Xu showed that the classical solution of the vQMHD model has the following decay rate:

$$\left\| \nabla^k (\rho - 1)(t) \right\|_{H^{5-k}} + \left\| \nabla^k u(t) \right\|_{H^{4-k}} + \left\| \nabla^k B(t) \right\|_{H^{4-k}} \leq C(1+t)^{-\frac{3+2k}{4}}, \quad k = 0, 1.$$

Pu–Xu [28] employed the pure energy method developed by Guo–Wang [9] to obtain the optimal decay rates of higher-order spatial derivatives of solutions to the full hydrodynamic equations with quantum effects under the condition that the initial perturbation belongs to $(H^{N+2} \cap \dot{H}^{-s}) \times (H^{N+1} \cap \dot{H}^{-s}) \times (H^N \cap \dot{H}^{-s})$ for $N \geq 3$ and $s \in [0, \frac{3}{2})$. Recently, by making full use of Fourier splitting method, Xi–Pu–Guo [35] improved the work of Pu–Xu [27], and particularly they got

$$\left\| \nabla^k (\rho - 1)(t) \right\|_{H^{5-k}} + \left\| \nabla^k u(t) \right\|_{H^{4-k}} + \left\| \nabla^k B(t) \right\|_{H^{4-k}} \leq C(1+t)^{-\frac{3+2k}{4}}, \quad k = 0, 1, 2, 3. \quad (1.4)$$

For more results about the large time behavior of solutions to the quantum fluid model, readers can refer to [26, 36] and references therein.

It is clear that decay rate of k ($k = 0, 1, 2, 3$) order spatial derivative of solution in (1.4) is optimal in the sense that it coincides with decay rate of solution to the heat equation. However, decay rate in (1.4) implies that fourth order spatial derivative of solution converges to zero at L^2 -rate $(1+t)^{-\frac{9}{4}}$, which is slower than the L^2 -rate $(1+t)^{-\frac{11}{4}}$ of solution to the heat equation. Moreover, fifth order spatial derivative of density ρ converges to zero at L^2 -rate $(1+t)^{-\frac{9}{4}}$, which is slower than the L^2 -rate $(1+t)^{-\frac{13}{4}}$ of solution to the heat equation. Therefore, decay rates of fourth order spatial derivative of solution and fifth order spatial derivative of density ρ are not optimal in this sense. The main purpose of this paper is to give a clear answer to this issue. More precisely, our main results can be outlined as follows. Firstly, we show that fourth order spatial derivative of the solution converges to zero at the L^2 -rate $(1+t)^{-\frac{11}{4}}$. Secondly, we prove that fifth-order spatial derivative of density ρ converges to zero at the L^2 -rate $(1+t)^{-\frac{13}{4}}$. Thirdly, we show that the high-frequency part of the fourth order spatial derivatives of velocity u and magnetic B converge to zero at the L^2 -rate $(1+t)^{-\frac{13}{4}}$, which are faster than ones of themselves, and totally new as compared to Pu–Xu [27] and Xi–Pu–Guo [35].

1.2 Notation

In this paper, we use $H^k(\mathbb{R}^3)$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^k}$. Generally, we use L^p , $1 \leq p \leq \infty$ to denote the usual $L^p(\mathbb{R}^3)$ spaces with norm $\|\cdot\|_{L^p}$. The notation $a \lesssim b$ means that $a \leq Cb$, where the universal constant $C > 0$ may be from line to line but independent of time t . Similarly, the notation $a \gtrsim b$ means that $a \geq Cb$ for a universal positive constant which is independent of time t . We define

$$\nabla^k u = \partial_x^\alpha u_i \quad \alpha = k, \quad i = 1, 2, 3, \quad u = (u_1, u_2, u_3).$$

For a radial function $\phi \in C_0^\infty(\mathbb{R}_\xi^3)$ such that $\phi(\xi) = 1$ when $|\xi| \leq 1$ and $\phi(\xi) = 0$ when $|\xi| \geq 2$, we define the low-frequency part and the high-frequency part of f as follows

$$f^l = \mathfrak{F}^{-1}[\phi(\xi)\widehat{f}], \quad \text{and} \quad f^h = \mathfrak{F}^{-1}[(1 - \phi(\xi))\widehat{f}].$$

1.3 Main results

Before stating our main results, let us recall the following results obtained in previous studies [28, 35], which will be used in this paper frequently.

Theorem 1.1. (see [28, 35]) Suppose that the initial data $(\rho_0 - 1, u_0, B_0) \in H^5 \times H^4 \times H^4$. There exists a small constant $\delta > 0$ such that if

$$\|\rho_0 - 1\|_{H^5} + \|u_0\|_{H^4} + \|B_0\|_{H^4} \leq \delta, \quad (1.5)$$

then the solution (ρ, u, B) of (1.1)–(1.3) satisfy for all $T \geq 0$

$$\begin{aligned} & \|(\rho - 1, u, B)(t)\|_{H^4}^2 + \|\partial \nabla \rho(t)\|_{H^4}^2 + \int_0^t \|\nabla(u, B, \partial \rho)(s)\|_{H^4}^2 ds \\ & \leq C(\|\rho_0 - 1\|_{H^5}^2 + \|u_0\|_{H^4}^2 + \|B_0\|_{H^4}^2). \end{aligned} \quad (1.6)$$

Moreover, provided that $\|(\rho_0 - 1, u_0, B_0)\|_{L^1}$ is finite additionally, then the solution (ρ, u, B) satisfy

$$\|\nabla^k(\rho - 1)(t)\|_{H^{5-k}} + \|\nabla^k u(t)\|_{H^{4-k}} + \|\nabla^k B(t)\|_{H^{4-k}} \leq C(1+t)^{-\frac{3+2k}{4}}, \quad (1.7)$$

with $k = 0, 1, 2, 3$. Here, the positive constant C is independent of time.

Our main purpose in this paper is to establish the upper optimal decay rates for fourth order spatial derivative of the solution and fifth order spatial derivatives of the density ρ which are the same as those of the heat equation. Our main results are stated in the following theorems:

Theorem 1.2. Under all the assumptions in Theorem 1.1, the global solution (ρ, u, B) has the following time decay rate for all $t \geq T$

$$\|\nabla^4(\rho - 1)(t)\|_{H^1} + \|\nabla^4 u(t)\|_{L^2} + \|\nabla^4 B(t)\|_{L^2} \lesssim (1+t)^{-\frac{11}{4}}. \quad (1.8)$$

Here T is a positive large time.

Theorem 1.3. Under all the assumptions in Theorem 1.1, the global solution (ρ, u, B) has the following time decay rate for all $t \geq T$

$$\|\nabla^5(\rho - 1)(t)\|_{L^2} + \|\nabla^4 u^h(t)\|_{L^2} + \|\nabla^4 B^h(t)\|_{L^2} \lesssim (1+t)^{-\frac{13}{4}}. \quad (1.9)$$

Here T is a positive large time.

Remark 1.4. It is interesting to make a comparison between Theorem 1.2 and Theorem 1.1, where the authors derived decay rate of fourth order spatial derivative of solution is the same as one of third order spatial derivative. We show that fourth order spatial derivative of the solution converges to zero at the L^2 -rate $(1+t)^{-\frac{11}{4}}$, which is same as that of the heat equation, and particularly faster than the L^2 -rate $(1+t)^{-\frac{5}{4}}$ in Pu–Xu [27] and the L^2 -rate $(1+t)^{-\frac{9}{4}}$ in Xi–Pu–Guo [35]. Therefore, our decay rate is optimal, and improve the results of [28, 35].

Remark 1.5. Compared to the main results (1.7) in [28, 35], decay rate in (1.9) is totally new and gives optimal decay rate of fifth order spatial derivative of density ϱ , which is same as that of the heat equation. In addition, it also gives optimal decay rates of the high-frequency part of fourth order spatial derivatives of the velocity u and magnetic B , which are faster than ones of themselves.

Now, let's sketch the strategy of proving Theorem 1.2–Theorem 1.3 and explain some main difficulties and techniques involved in the process. Roughly speaking, our methods mainly involve low-frequency and high-frequency decomposition technique and delicate energy estimates. Our strategy can be outlined as follows.

For the proof of Theorem 1.2, we hope to establish the optimal decay rate for fourth order spatial derivatives of solution for the compressible viscous quantum magnetohydrodynamic equations (1.1). For the convenience of calculation, we linearize the system (1.1) around the equilibrium state, and then rewrite the linearized system in terms of the variables ϱ , u , B , which is stated in (2.1). We only need to make full use of the benefit of low-frequency and high-frequency decomposition to deal with high-frequency part of $\nabla^4(\varrho, u, B)$. The proof mainly involves the following three steps. First, we derive high-frequency L^2 energy estimate of fourth order spatial derivative of the solution to the compressible vQMHD model:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^4 \varrho^h|^2 + |\nabla^4 u^h|^2 + |\nabla^4 B^h|^2 + \frac{g^2}{4} |\nabla^5 \varrho^h|^2 dx + (2\mu + \lambda) \int_{\mathbb{R}^3} |\nabla^5 u^h|^2 dx + \nu \int_{\mathbb{R}^3} |\nabla^5 B^h|^2 dx \\ & \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} (\|\nabla^4 \varrho^h\|_{H^1}^2 + \|\nabla^5 B^h\|_{L^2}^2) + (1+t)^{-\frac{3}{2}} (\|\nabla^5 u^h\|_{L^2}^2 + \|\nabla^6 \varrho^h\|_{L^2}^2). \end{aligned}$$

Second, by noticing that the above energy inequality only involves dissipative estimates of u^h and B^h . In order to explore the dissipative estimate of ϱ^h , the main idea here is to employ the new interactive energy functional to get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^4 u^h \nabla^5 \varrho^h dx + \frac{1}{2} \|\nabla^5 \varrho^h\|_{L^2}^2 + \frac{g^2}{4} \|\nabla^6 \varrho^h\|_{L^2}^2 \\ & \lesssim (1+t)^{-\frac{11}{2}} + \|\nabla^5 u^h\|_{L^2}^2 + (1+t)^{-\frac{3}{2}} (\|\nabla^6 \varrho^h\|_{L^2}^2 + \|\nabla^5 u^h\|_{L^2}^2). \end{aligned}$$

Last, we choose two sufficiently large positive constants D_0 and T_1 , and define the temporal energy functional

$$\mathcal{E}(t) = D_0 (\|\nabla^4(\varrho, u, B)^h\|_{L^2}^2 + \frac{g^2}{2} \|\nabla^5 \varrho^h\|_{L^2}^2) + \int_{\mathbb{R}^3} \nabla^4 u^h \nabla^5 \varrho^h dx,$$

for $t \geq 0$, where it is mentioned that $\mathcal{E}(t)$ is equivalent to $\|\nabla^4 \varrho^h\|_{H^1}^2 + \|\nabla^4(u, B)^h\|_{L^2}^2$. Combining the above estimates, we can obtain

$$\frac{d}{dt} \mathcal{E}(t) + \|\nabla^4 \varrho^h(t)\|_{H^1}^2 + \|\nabla^4 u^h(t)\|_{L^2}^2 + \|\nabla^4 B^h(t)\|_{L^2}^2 \lesssim (1+t)^{-\frac{11}{2}},$$

which together with Gronwall's argument and low-frequency decay rate gives rise to (1.8), and thus completes the proof of Theorem 1.2.

For the proof of Theorem 1.3, we hope to establish the fifth order spatial derivatives of the density ρ . Our main idea is to find the optimal decay rate at low-frequency and high-frequency of the fifth order spatial derivative of density, and then use the properties of high-frequency and low-frequency decomposition to obtain the optimal decay rates of the fifth order spatial derivative of density. For the low-frequency part

$\|\nabla^5 \varrho^l\|_{L^2}$, by using the equation (2.1)₁, Theorem 1.2, Duhamel's principle, Plancherel theorem, Hölder's inequality and Hausdorff-Young's inequality, we have

$$\|\nabla^5 \varrho^l(t)\|_{L^2} \lesssim (1+t)^{-\frac{13}{4}} \|(\varrho, u, B)(0)\|_{L^1} + \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{13}{4}} \|G(\tau)\|_{L^1} d\tau + \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{5}{4}} \| |\xi|^4 \widehat{G}^l(\tau) \|_{L^\infty} d\tau.$$

After some detailed calculations, we can get

$$\|\nabla^5 \varrho^l\|_{L^2} \lesssim (1+t)^{-\frac{13}{4}}. \quad (1.10)$$

For the high-frequency part $\|\nabla^5 \varrho^h\|_{L^2}$, it is easy to find that the calculation in this part is similar to that of section 2, we only need to replace the decay rate of fourth order space derivative from $(1+t)^{-\frac{9}{4}}$ to $(1+t)^{-\frac{11}{4}}$. We finally arrive at

$$\|\nabla^4 \varrho^h(t)\|_{H^1} + \|\nabla^4 u^h(t)\|_{L^2} + \|\nabla^4 B^h(t)\|_{L^2} \lesssim (1+t)^{-\frac{13}{4}}. \quad (1.11)$$

Combining the estimates (1.10) and (1.11), we conclude the proof of Theorem 1.3.

2 Proof of Theorem 1.2

In this section, we will devote ourselves to proving Theorem 1.2. We suppose that all the conditions of Theorem 1.1 are in force in this and next section. Without loss of generality, we assume that $P'(1) = 1$. Let us setting $\varrho = \rho - 1$, we can rewrite the system (1.1) in the perturbation form as

$$\begin{cases} \varrho_t + \operatorname{div} u = G_1, \\ u_t - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u + \nabla \varrho - \frac{g^2}{4} \nabla \Delta \varrho = G_2, \\ B_t - \nu \Delta B = G_3. \end{cases} \quad (2.1)$$

Here the nonlinear source terms G_1 , G_2 and G_3 are given by

$$G_1 = -\varrho \operatorname{div} u - u \cdot \nabla \varrho,$$

$$\begin{aligned} G_2 = & -u \cdot \nabla u + \frac{\varrho}{(\varrho + 1)} (\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u) - \left(\frac{P'(\varrho + 1)}{\varrho + 1} - 1 \right) \nabla \varrho - \frac{g^2}{4} \frac{\varrho}{\varrho + 1} \nabla \Delta \varrho \\ & + \frac{g^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1 + \varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1 + \varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1 + \varrho)^2} \right) + \frac{1}{1 + \varrho} ((\nabla \times B) \times B), \end{aligned}$$

$$G_3 = \nabla \times (u \times B).$$

The initial data are given as

$$(\varrho, u, B)(x, t)|_{t=0} = (\varrho_0, u_0, B_0)(x) \rightarrow (0, 0, 0) \text{ as } |x| \rightarrow \infty. \quad (2.2)$$

First, we recall the L^2 time decay rates on the linearized system for the first two equations (2.1).

Lemma 2.1. (see [27]) *Let $s \geq 0$ be an integer. Assume that (ϱ, u) is the solution of the linearized system for the first two equations in (2.1) with initial data $\varrho \in H^{s+1} \cap L^1$, $u_0 \in H^s \cap L^1$. Then, for any $t \geq 0$, it holds that*

$$\|\varrho(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}} (\|(\varrho, u_0)\|_{L^1} + \|(\varrho_0, u_0)\|_{L^2}),$$

$$\left\| \nabla^{k+1} \varrho(t) \right\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k+1}{2}} (\|(\varrho_0, u_0)\|_{L^1} + \|(\nabla^{k+1} \varrho, \nabla^k u_0)\|_{L^2}), \quad (2.3)$$

$$\left\| \nabla^k u(t) \right\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} (\|(\varrho_0, u_0)\|_{L^1} + \|(\nabla^{k+1} \varrho, \nabla^k u_0)\|_{L^2}), \quad (2.4)$$

for $0 \leq k \leq s$.

Next, we recall the L^2 time decay rate on the linearized system for the third equation in (2.1).

Lemma 2.2. (see [33]) *Let $s \geq 0$ be an integer. Assume that B is the solution of the linearized system for the third equation in (2.1) with initial data $B_0 \in H^s \cap L^1$. Then, for any $t \geq 0$, it holds that*

$$\left\| \nabla^k B \right\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \|B_0\|_{L^1}, \quad (2.5)$$

for $0 \leq k \leq s$.

Finally, we deduce the L^2 time decay rate on the nonlinear system (2.1).

Lemma 2.3. *Assume that the assumptions of Theorem 1.1 are in force, the solution (ϱ, u, B) of the nonlinear system (2.1) satisfies the following decay estimate:*

$$\left\| \nabla^4 (\varrho^l, u^l, B^l)(t) \right\|_{L^2} \lesssim (1+t)^{-\frac{11}{4}}. \quad (2.6)$$

Proof. By virtue of the equation (2.1), Lemma 2.1, Lemma 2.2, Duhamel's principle, Plancherel theorem, Hölder's inequality and Hausdorff–Young's inequality, we have

$$\left\| \nabla^4 (\varrho^l, u^l, B^l)(t) \right\|_{L^2} \lesssim (1+t)^{-\frac{11}{4}} \|(\varrho, u, B)(0)\|_{L^1} + \int_0^t (1+t-\tau)^{-\frac{11}{4}} \|G(\tau)\|_{L^1} d\tau + \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{5}{4}} \|\xi|^3 \widehat{G^l}(\tau)\|_{L^\infty} d\tau. \quad (2.7)$$

Next, we shall estimate the second term and the third term on the right-hand side of (2.7). To begin with, by virtue of Lemma 4.2, we can bound the term by

$$\begin{aligned} \|G(\tau)\|_{L^1} &\lesssim \|\operatorname{div}(\varrho u)(\tau)\|_{L^1} + \|u \cdot \nabla u(\tau)\|_{L^1} + \left\| \frac{\varrho}{\varrho+1} [\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u](\tau) \right\|_{L^1} \\ &\quad + \left\| \left[\frac{P'(\varrho+1)}{\varrho+1} - P'(1) \right] \nabla \varrho(\tau) \right\|_{L^1} + \left\| \frac{g^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho(\tau) \right\|_{L^1} \\ &\quad + \left\| \frac{g^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1+\varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1+\varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1+\varrho)^2} \right)(\tau) \right\|_{L^1} + \left\| \frac{1}{1+\varrho} ((\nabla \times B) \times B)(\tau) \right\|_{L^1} + \|\nabla \times (u \times B)(\tau)\|_{L^1} \\ &\lesssim \|(\varrho, u)(\tau)\|_{L^2} \|\nabla(\varrho, u)(\tau)\|_{L^2} + \|\varrho(\tau)\|_{L^2} \|\nabla^2 u(\tau)\|_{L^2} + \|\varrho(\tau)\|_{L^2} \|\nabla^3 \varrho(\tau)\|_{L^2} \\ &\quad + \|\nabla \varrho(\tau)\|_{L^2} \|\nabla^2 \varrho(\tau)\|_{L^2} + \|B(\tau)\|_{L^2} \|\nabla^2 B(\tau)\|_{L^2} + \|(u, B)(\tau)\|_{L^2} \|\nabla(u, B)(\tau)\|_{L^2} \\ &\lesssim (1+t)^{-\frac{3}{2}}, \end{aligned} \quad (2.8)$$

and

$$\begin{aligned}
 \left\| |\xi|^3 \widehat{G}^l(\tau) \right\|_{L^\infty} &\lesssim \left\| \nabla^3 G^l(\tau) \right\|_{L^1} \\
 &\lesssim \left\| \nabla^3 (\operatorname{div}(\varrho u))(\tau) \right\|_{L^1} + \left\| \nabla^3 (u \cdot \nabla u)(\tau) \right\|_{L^1} \\
 &\quad + \left\| \nabla^2 \left(\frac{\varrho}{\varrho+1} [\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u] \right)(\tau) \right\|_{L^1} + \left\| \nabla^3 \left(\left[\frac{P'(\varrho+1)}{\varrho+1} - P'(1) \right] \nabla \varrho \right)(\tau) \right\|_{L^1} \\
 &\quad + \left\| \nabla^2 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)(\tau) \right\|_{L^1} + \left\| \nabla^3 \left(\frac{\varrho^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1+\varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1+\varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1+\varrho)^2} \right) \right)(\tau) \right\|_{L^1} \\
 &\quad + \left\| \nabla^3 \left(\frac{1}{1+\varrho} ((\nabla \times B) \times B) \right)(\tau) \right\|_{L^1} + \left\| \nabla^3 (\nabla \times (u \times B)(\tau)) \right\|_{L^1} \\
 &\lesssim \left\| (\varrho, u)(\tau) \right\|_{L^2} \left\| \nabla^4 (\varrho, u)(\tau) \right\|_{L^2} + \left\| \nabla^2 u(\tau) \right\|_{L^2}^2 + \left\| \varrho(\tau) \right\|_{L^2} \left\| \nabla^5 \varrho(\tau) \right\|_{L^2} \\
 &\quad + \left\| \nabla \varrho(\tau) \right\|_{L^2} \left\| \nabla^5 \varrho(\tau) \right\|_{L^2} + \left\| \nabla \varrho(\tau) \right\|_{L^2} \left\| \nabla^4 \varrho(\tau) \right\|_{L^2} + \left\| \nabla^2 \varrho(\tau) \right\|_{L^2} \left\| \nabla^3 \varrho(\tau) \right\|_{L^2} \\
 &\quad + \left\| \nabla B(\tau) \right\|_{L^2} \left\| \nabla^3 B(\tau) \right\|_{L^2} + \left\| (u, B)(\tau) \right\|_{L^2} \left\| \nabla^4 (u, B)(\tau) \right\|_{L^2} \\
 &\lesssim (1+t)^{-3}.
 \end{aligned} \tag{2.9}$$

Substituting (2.8)–(2.9) into (2.7) gives (2.6). $\square$

Proof of Theorem 1.2 The first step is to establish the energy estimate for the fourth order spatial derivative of solution. Taking

$$\langle \mathcal{F}^{-1}(1 - \phi(\xi)) \nabla^4 (2.1)_1, \nabla^4 \varrho^h \rangle + \langle \mathcal{F}^{-1}(1 - \phi(\xi)) \nabla^4 (2.1)_2, \nabla^4 u^h \rangle + \langle \mathcal{F}^{-1}(1 - \phi(\xi)) \nabla^4 (2.1)_3, \nabla^4 B^h \rangle,$$

and using integration by parts, we can obtain

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^4 \varrho^h|^2 + |\nabla^4 u^h|^2 + |\nabla^4 B^h|^2 + \frac{\varrho^2}{4} |\nabla^5 \varrho^h|^2 dx \\
 &\quad + (2\mu + \lambda) \int_{\mathbb{R}^3} |\nabla^5 u^h|^2 dx + \nu \int_{\mathbb{R}^3} |\nabla^5 B^h|^2 dx = \langle \nabla^4 G_1^h, \nabla^4 \varrho^h \rangle \\
 &\quad + \langle \nabla^4 G_2^h, \nabla^4 u^h \rangle + \langle \nabla^4 G_3^h, \nabla^4 B^h \rangle + \frac{h^2}{4} \langle \nabla^5 G_1^h, \nabla^5 \varrho^h \rangle.
 \end{aligned} \tag{2.10}$$

The right-hand side of the above equation can be estimated as follows. Firstly, we have

$$\begin{aligned}
 \langle \nabla^4 G_1^h, \nabla^4 \varrho^h \rangle &= -\langle \nabla^4 (\varrho \operatorname{div} u + u \cdot \nabla \varrho)^h, \nabla^4 \varrho^h \rangle \\
 &= -\langle \nabla^4 (\varrho \operatorname{div} u)^h, \nabla^4 \varrho^h \rangle - \langle \nabla^4 (u \cdot \nabla \varrho)^h, \nabla^4 \varrho^h \rangle \\
 &:= I_1 + I_2.
 \end{aligned} \tag{2.11}$$

For term I_1 , by using Hölder's inequality, Lemma 4.2, Lemma 4.3, Lemma 4.1, Young's inequality and Sobolev interpolation theorem, we arrive at

$$\begin{aligned}
 |I_1| &\lesssim \left\| \nabla^3 (\varrho \operatorname{div} u)^h \right\|_{L^2} \left\| \nabla^5 \varrho^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^3 (\varrho \operatorname{div} u) \right\|_{L^2} \left\| \nabla^5 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^4 u \right\|_{L^2} + \|\nabla u\|_{L^3} \left\| \nabla^3 \varrho \right\|_{L^6}) \left\| \nabla^5 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 u \right\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}) \left\| \nabla^5 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^5 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^5 \varrho^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.12}$$

For term I_2 , we can rewrite it as follows

$$\begin{aligned}
 I_2 &= \langle -\nabla^4(u \cdot \nabla \varrho)^h, \nabla^4 \varrho^h \rangle \\
 &= \langle -\nabla^4(u \cdot \nabla \varrho) + \nabla^4(u \cdot \nabla \varrho)^l, \nabla^4 \varrho^h \rangle \\
 &= \langle -\nabla^4(u \cdot \nabla \varrho^h) - \nabla^4(u \cdot \nabla \varrho^l) + \nabla^4(u \cdot \nabla \varrho)^l, \nabla^4 \varrho^h \rangle \\
 &= \langle -\nabla^4(u \cdot \nabla \varrho^h), \nabla^4 \varrho^h \rangle + \langle -\nabla^4(u \cdot \nabla \varrho^l), \nabla^4 \varrho^h \rangle + \langle \nabla^4(u \cdot \nabla \varrho)^l, \nabla^4 \varrho^h \rangle \\
 &:= I_{2,1} + I_{2,2} + I_{2,3}.
 \end{aligned} \tag{2.13}$$

By routine checking, one may show that

$$\nabla^4(u \cdot \nabla \varrho^h) = u \cdot \nabla^5 \varrho^h + 4 \nabla u \cdot \nabla^4 \varrho^h + 6 \nabla^2 u \cdot \nabla^3 \varrho^h + 4 \nabla^3 u \cdot \nabla^2 \varrho^h + 4 \nabla^4 u \cdot \nabla \varrho^h.$$

The integration by part yields directly

$$\int_{\mathbb{R}^3} u \cdot \nabla(\nabla^4 \varrho^h) \cdot \nabla^4 \varrho^h dx = -\frac{1}{2} \int_{\mathbb{R}^3} \operatorname{div} u |\nabla^4 \varrho^h|^2 dx,$$

and hence, we show that

$$\begin{aligned}
 |\langle \operatorname{div} u \nabla^4 \varrho^h, \nabla^4 \varrho^h \rangle| &\lesssim \|\operatorname{div} u\|_{L^\infty} \|\nabla^4 \varrho^h\|_{L^2}^2 \\
 &\lesssim \|\operatorname{div} u\|_{L^\infty} \|\nabla^4 \varrho^h\|_{L^2}^2 \\
 &\lesssim \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\nabla^3 u\|_{L^2}^{\frac{1}{2}} \|\nabla^4 \varrho^h\|_{L^2}^2 \\
 &\lesssim (1+t)^{-\frac{17}{4}} \|\nabla^4 \varrho^h\|_{L^2}^2 \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-3} \|\nabla^4 \varrho^h\|_{L^2}^2.
 \end{aligned} \tag{2.14}$$

Similar to the estimate (2.14), we have

$$|\langle \nabla u \cdot \nabla^4 \varrho^h, \nabla^4 \varrho^h \rangle| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-3} \|\nabla^4 \varrho^h\|_{L^2}^2, \tag{2.15}$$

and

$$\begin{aligned}
 |\langle -6 \nabla^2 u \cdot \nabla^3 \varrho^h, \nabla^4 \varrho^h \rangle| &\lesssim \|\nabla^2 u\|_{L^3} \|\nabla^3 \varrho^h\|_{L^6} \|\nabla^4 \varrho^h\|_{L^2} \\
 &\lesssim \|\nabla^2 u\|_{L^3} \|\nabla^3 \varrho^h\|_{L^6} \|\nabla^4 \varrho^h\|_{L^2} \\
 &\lesssim \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\nabla^3 u\|_{L^2}^{\frac{1}{2}} \|\nabla^4 \varrho^h\|_{L^2}^2 \\
 &\lesssim (1+t)^{-\frac{17}{4}} \|\nabla^4 \varrho^h\|_{L^2}^2 \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-3} \|\nabla^4 \varrho^h\|_{L^2}^2,
 \end{aligned} \tag{2.16}$$

$$\begin{aligned}
 |\langle -4 \nabla^3 u \cdot \nabla^2 \varrho^h, \nabla^4 \varrho^h \rangle| &\lesssim \|\nabla^2 \varrho^h\|_{L^3} \|\nabla^3 u\|_{L^6} \|\nabla^4 \varrho^h\|_{L^2} \\
 &\lesssim \|\nabla^2 \varrho^h\|_{L^3} \|\nabla^3 u\|_{L^6} \|\nabla^4 \varrho^h\|_{L^2} \\
 &\lesssim \|\nabla^2 \varrho^h\|_{L^2}^{\frac{1}{2}} \|\nabla^3 \varrho^h\|_{L^2}^{\frac{1}{2}} \|\nabla^4 u\|_{L^2} \|\nabla^4 \varrho^h\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{17}{4}} \|\nabla^4 \varrho^h\|_{L^2}^2 \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-3} \|\nabla^4 \varrho^h\|_{L^2}^2,
 \end{aligned} \tag{2.17}$$

$$\begin{aligned}
|(-\nabla^4 u \cdot \nabla \varrho^h, \nabla^4 \varrho^h)| &\lesssim \|\nabla \varrho^h\|_{L^\infty} \|\nabla^4 u\|_{L^2} \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim \|\nabla^2 \varrho\|_{L^2}^{\frac{1}{2}} \|\nabla^3 \varrho\|_{L^2}^{\frac{1}{2}} \|\nabla^4 u\|_{L^2} \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (1+t)^{-\frac{17}{4}} \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-3} \|\nabla^4 \varrho^h\|_{L^2}^2,
\end{aligned} \tag{2.18}$$

and hence, it follows that

$$|I_{2,1}| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-3} \|\nabla^4 \varrho^h\|_{L^2}^2. \tag{2.19}$$

For the term $I_{2,2}$, we have

$$\begin{aligned}
|I_{2,2}| &\lesssim \|\nabla^4(u \cdot \nabla \varrho^l)\|_{L^2} \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (\|u\|_{L^\infty} \|\nabla^5 \varrho^l\|_{L^2} + \|\nabla \varrho^l\|_{L^\infty} \|\nabla^4 u\|_{L^2}) \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\nabla^4 \varrho^l\|_{L^2} + \|\nabla^2 \varrho^l\|_{L^2}^{\frac{1}{2}} \|\nabla^3 \varrho^l\|_{L^2}^{\frac{1}{2}} \|\nabla^4 u\|_{L^2}) \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\nabla^4 \varrho^l\|_{L^2} + \|\nabla \varrho^l\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \varrho^l\|_{L^2}^{\frac{1}{2}} \|\nabla^4 u\|_{L^2}) \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (1+t)^{-\frac{15}{4}} \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \|\nabla^4 \varrho^h\|_{L^2}^2.
\end{aligned} \tag{2.20}$$

For the term $I_{2,3}$, we get

$$\begin{aligned}
|I_{2,3}| &\lesssim \|\nabla^4(u \cdot \nabla \varrho)\|_{L^2} \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim \|\nabla^3(u \cdot \nabla \varrho)\|_{L^2} \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (\|u\|_{L^\infty} \|\nabla^4 \varrho\|_{L^2} + \|\nabla \varrho\|_{L^3} \|\nabla^3 u\|_{L^6}) \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\nabla^4 \varrho\|_{L^2} + \|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \varrho\|_{L^2}^{\frac{1}{2}} \|\nabla^4 u\|_{L^2}) \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (1+t)^{-\frac{15}{4}} \|\nabla^4 \varrho^h\|_{L^2} \\
&\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \|\nabla^4 \varrho^h\|_{L^2}^2.
\end{aligned} \tag{2.21}$$

Summing up estimates (2.19)–(2.21), we arrive at

$$|I_2| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \|\nabla^4 \varrho^h\|_{H^1}^2. \tag{2.22}$$

Applying equation (2.1), it holds that

$$\begin{aligned}
\langle \nabla^4 G_2^h, \nabla^4 u^h \rangle &= \langle -\nabla^4(u \cdot \nabla u)^h, \nabla^4 u^h \rangle + \left\langle \nabla^4 \left\{ \frac{\varrho}{(\varrho+1)} [\mu \Delta v + (\mu + \lambda) \nabla \operatorname{div} u] \right\}^h, \nabla^4 u^h \right\rangle \\
&+ \left\langle -\nabla^4 \left\{ \left[\frac{P'(\varrho+1)}{\varrho+1} - P'(1) \right] \nabla \varrho \right\}^h, \nabla^4 u^h \right\rangle + \left\langle -\nabla^4 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^h, \nabla^4 u^h \right\rangle \\
&+ \left\langle \nabla^4 \left[\frac{\varrho^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1+\varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1+\varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1+\varrho)^2} \right) \right]^h, \nabla^4 u^h \right\rangle \\
&+ \left\langle \nabla^4 \left[\frac{1}{1+\varrho} ((\nabla \times B) \times B) \right]^h, \nabla^4 u^h \right\rangle := \sum_{j=3}^8 I_j.
\end{aligned} \tag{2.23}$$

The first term on the right-hand side of the above equation can be estimated as follows

$$\begin{aligned}
 |I_3| &\lesssim \left\| \nabla^3 (u \cdot \nabla u)^h \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^3 (u \cdot \nabla u) \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|u\|_{L^\infty} \left\| \nabla^4 u \right\|_{L^2} + \left\| \nabla^3 u \right\|_{L^6} \|\nabla u\|_{L^3}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim \|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 u \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^5 u^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.24}$$

Similarly, using the integration by parts, the terms $I_4 \sim I_8$ can be estimated as follows

$$\begin{aligned}
 |I_4| &\lesssim \left\| \nabla^3 \left\{ \frac{\varrho}{(\varrho+1)} [\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u] \right\}^h \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^3 \left\{ \frac{\varrho}{(\varrho+1)} [\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u] \right\} \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^5 u \right\|_{L^2} + \left\| \nabla^2 u \right\|_{L^3} \left\| \nabla^3 \varrho \right\|_{L^6}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 u \right\|_{L^2} + \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^3 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^5 u^h \right\|_{L^2} + (1+t)^{-\frac{17}{4}} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^5 u^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.25}$$

$$\begin{aligned}
 |I_5| &\lesssim \left\| \nabla^3 \left\{ \left[\frac{P'(\varrho+1)}{\varrho+1} - P'(1) \right] \nabla \varrho \right\}^h \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^3 \left\{ \left[\frac{P'(\varrho+1)}{\varrho+1} - P'(1) \right] \nabla \varrho \right\} \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^4 \varrho \right\|_{L^2} + \|\nabla \varrho\|_{L^3} \left\| \nabla^3 \varrho \right\|_{L^6}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim \|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^5 u^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.26}$$

For term I_6 , we can rewrite it as follows

$$\begin{aligned}
 I_6 &= \left\langle -\nabla^4 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^h, \nabla^4 u^h \right\rangle \\
 &= \left\langle -\nabla^4 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right) + \nabla^4 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^l, \nabla^4 u^h \right\rangle \\
 &= \left\langle -\nabla^4 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho^h \right) - \nabla^4 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho^l \right) + \nabla^4 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^l, \nabla^4 u^h \right\rangle \\
 &:= I_{6,1} + I_{6,2} + I_{6,3}.
 \end{aligned} \tag{2.27}$$

Similar to the proof of (2.19)–(2.21), for $I_{6,1}$, $I_{6,2}$ and $I_{6,3}$, it holds that

$$\begin{aligned}
 |I_{6,1}| &\lesssim \left\| \nabla^3 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho^h \right) \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^6 \varrho^h \right\|_{L^2} + \left\| \nabla^3 \varrho^h \right\|_{L^6} \left\| \nabla^3 \varrho \right\|_{L^3}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^6 \varrho^h \right\|_{L^2} + \left\| \nabla^3 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{3}{2}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} + (1+t)^{-\frac{9}{2}} \left\| \nabla^5 u^h \right\|_{L^2}^2 \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} (\left\| \nabla^6 \varrho^h \right\|_{L^2}^2 + \left\| \nabla^5 u^h \right\|_{L^2}^2). \\
 |I_{6,2}| &\lesssim \left\| \nabla^3 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho^l \right) \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^6 \varrho^l \right\|_{L^2} + \left\| \nabla^3 \varrho^l \right\|_{L^6} \left\| \nabla^3 \varrho \right\|_{L^3}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^5 \varrho \right\|_{L^2} + \left\| \nabla^3 \varrho^l \right\|_{L^6} \left\| \nabla^3 \varrho \right\|_{L^3}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 \varrho \right\|_{L^2} + \left\| \nabla^3 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^5 u^h \right\|_{L^2} + (1+t)^{-\frac{9}{2}} \left\| \nabla^5 u^h \right\|_{L^2}^2 \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^5 u^h \right\|_{L^2}^2.
 \end{aligned}$$

Using the properties of low-frequency decomposition, it holds that

$$\begin{aligned}
 |I_{6,3}| &\lesssim \left\| \nabla^3 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^l \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^2 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right) \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^5 \varrho \right\|_{L^2} + \left\| \nabla^3 \varrho \right\|_{L^6} \left\| \nabla^2 \varrho \right\|_{L^3}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 \varrho \right\|_{L^2} + \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^3 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^5 u^h \right\|_{L^2}^2.
 \end{aligned}$$

Putting the above estimates into (2.27) yields directly

$$|I_6| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} (\left\| \nabla^6 \varrho^h \right\|_{L^2}^2 + \left\| \nabla^5 u^h \right\|_{L^2}^2). \quad (2.28)$$

For the terms I_7 , I_8 , we have

$$\begin{aligned}
 |I_7| &\lesssim \left\| \nabla^3 \left[\frac{\varrho^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1+\varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1+\varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1+\varrho)^2} \right) \right]^h \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^3 \left[\frac{\varrho^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1+\varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1+\varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1+\varrho)^2} \right) \right] \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^\infty} \left\| \nabla^5 \varrho \right\|_{L^2} + \left\| \nabla^4 \varrho \right\|_{L^6} \left\| \nabla^2 \varrho \right\|_{L^3}) \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^3 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 \varrho \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{17}{4}} \left\| \nabla^5 u^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-3} \left\| \nabla^5 u^h \right\|_{L^2}^2,
 \end{aligned} \quad (2.29)$$

$$\begin{aligned}
|I_8| &\lesssim \left\| \nabla^3 \left[\frac{1}{1+\varrho} ((\nabla \times B) \times B) \right]^h \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
&\lesssim \left\| \nabla^3 \left[\frac{1}{1+\varrho} ((\nabla \times B) \times B) \right] \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
&\lesssim (\|B\|_{L^\infty} \left\| \nabla^4 B \right\|_{L^2} + \|\nabla B\|_{L^3} \left\| \nabla^3 B \right\|_{L^6}) \left\| \nabla^5 u^h \right\|_{L^2} \\
&\lesssim \|\nabla B\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 B \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 B \right\|_{L^2} \left\| \nabla^5 u^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^5 u^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^5 u^h \right\|_{L^2}^2.
\end{aligned} \tag{2.30}$$

Summing up (2.23)–(2.26), (2.28)–(2.30), we obtain

$$|\langle \nabla^4 G_2^h, \nabla^4 u^h \rangle| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} (\left\| \nabla^5 u^h \right\|_{L^2}^2 + \left\| \nabla^6 \varrho^h \right\|_{L^2}^2), \tag{2.31}$$

and we have

$$\begin{aligned}
|\langle \nabla^4 G_3^h, \nabla^4 B^h \rangle| &\lesssim \left\| \nabla^3 (\nabla \times (u \times B))^h \right\|_{L^2} \left\| \nabla^5 B^h \right\|_{L^2} \\
&\lesssim \left\| \nabla^3 (\nabla \times (u \times B)) \right\|_{L^2} \left\| \nabla^5 B^h \right\|_{L^2} \\
&\lesssim (\|u\|_{L^\infty} \left\| \nabla^4 B \right\|_{L^2} + \|B\|_{L^\infty} \left\| \nabla^4 u \right\|_{L^2}) \left\| \nabla^5 B^h \right\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 B \right\|_{L^2} + \|\nabla B\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 B \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 u \right\|_{L^2}) \left\| \nabla^5 B^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^5 B^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^5 B^h \right\|_{L^2}^2.
\end{aligned} \tag{2.32}$$

For the term $\langle \nabla^5 G_1^h, \nabla^5 \varrho^h \rangle$, we have

$$\begin{aligned}
\frac{\vartheta^2}{4} \langle \nabla^5 G_1^h, \nabla^5 \varrho^h \rangle &= -\frac{\vartheta^2}{4} \langle \nabla^5 (\varrho \operatorname{div} u + u \cdot \nabla \varrho)^h, \nabla^5 \varrho^h \rangle \\
&= -\frac{\vartheta^2}{4} \langle \nabla^5 (\varrho \operatorname{div} u)^h, \nabla^5 \varrho^h \rangle + \frac{\vartheta^2}{4} \langle -\nabla^5 (u \cdot \nabla \varrho)^h, \nabla^5 \varrho^h \rangle \\
&:= I_9 + I_{10}.
\end{aligned} \tag{2.33}$$

For the term I_9 , by using the properties of low-frequency and high-frequency decomposition, we get

$$\begin{aligned}
I_9 &= \frac{\vartheta^2}{4} \langle -\nabla^5 (\varrho \operatorname{div} u)^h, \nabla^5 \varrho^h \rangle \\
&= \frac{\vartheta^2}{4} \langle -\nabla^5 (\varrho \operatorname{div} u) + \nabla^5 (\varrho \operatorname{div} u)^l, \nabla^5 \varrho^h \rangle \\
&= \frac{\vartheta^2}{4} \langle -\nabla^5 (\varrho \operatorname{div} u)^h - \nabla^5 (\varrho \operatorname{div} u)^l + \nabla^5 (\varrho \operatorname{div} u)^l, \nabla^5 \varrho^h \rangle \\
&= I_{9,1} + I_{9,2} + I_{9,3}.
\end{aligned} \tag{2.34}$$

For the term $I_{9,1}$, by using integration by parts, we obtain

$$\begin{aligned}
|I_{9,1}| &\lesssim \left\| \nabla^4 (\varrho \operatorname{div} u)^h \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^5 u^h \right\|_{L^2} + \left\| \nabla u^h \right\|_{L^3} \left\| \nabla^4 \varrho \right\|_{L^6}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 u^h \right\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 \varrho \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{3}{2}} \left\| \nabla^5 u^h \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} + (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} (\left\| \nabla^6 \varrho^h \right\|_{L^2}^2 + \left\| \nabla^5 u^h \right\|_{L^2}^2).
\end{aligned} \tag{2.35}$$

For the term $I_{9,2}$, we have

$$\begin{aligned}
 |I_{9,2}| &\lesssim \left\| \nabla^4(\varrho \operatorname{div} u^l) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^5 u^l \right\|_{L^2} + \left\| \nabla u^l \right\|_{L^3} \left\| \nabla^4 \varrho \right\|_{L^6}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 u \right\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 \varrho \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.36}$$

For the term $I_{9,3}$, we get

$$\begin{aligned}
 |I_{9,3}| &\lesssim \left\| \nabla^4(\varrho \operatorname{div} u) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^3(\varrho \operatorname{div} u) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^4 u \right\|_{L^2} + \|\nabla u\|_{L^3} \left\| \nabla^3 \varrho \right\|_{L^6}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 u \right\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.37}$$

Putting estimates (2.35)–(2.37) into (2.34), it arrives at

$$|I_9| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} (\left\| \nabla^6 \varrho^h \right\|_{L^2}^2 + \left\| \nabla^5 u^h \right\|_{L^2}^2). \tag{2.38}$$

Similarly, for the term I_{10} , we can rewrite it as follows

$$\begin{aligned}
 I_{10} &= \frac{\vartheta^2}{4} \langle -\nabla^5(u \cdot \nabla \varrho)^h, \nabla^5 \varrho^h \rangle \\
 &= \frac{\vartheta^2}{4} \langle -\nabla^5(u \cdot \nabla \varrho) + \nabla^5(u \cdot \nabla \varrho)^l, \nabla^5 \varrho^h \rangle \\
 &= \frac{\vartheta^2}{4} \langle -\nabla^5(u \cdot \nabla \varrho^h) - \nabla^5(u \cdot \nabla \varrho^l) + \nabla^5(u \cdot \nabla \varrho)^l, \nabla^5 \varrho^h \rangle \\
 &= I_{10,1} + I_{10,2} + I_{10,3}.
 \end{aligned} \tag{2.39}$$

Similar to the proof of (2.35), we have

$$\begin{aligned}
 |I_{10,1}| &\lesssim \left\| \nabla^4(u \cdot \nabla \varrho^h) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|u\|_{L^\infty} \left\| \nabla^5 \varrho^h \right\|_{L^2} + \left\| \nabla \varrho^h \right\|_{L^\infty} \left\| \nabla^4 u \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 \varrho^h \right\|_{L^2} + \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^3 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 u \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{17}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} + (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2,
 \end{aligned} \tag{2.40}$$

$$\begin{aligned}
|I_{10,2}| &\lesssim \left\| \nabla^4(u \cdot \nabla \varrho^l) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (\|u\|_{L^\infty} \left\| \nabla^5 \varrho^l \right\|_{L^2} + \left\| \nabla \varrho^l \right\|_{L^\infty} \left\| \nabla^4 u \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2} + \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^3 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 u \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2,
\end{aligned} \tag{2.41}$$

and

$$\begin{aligned}
|I_{10,3}| &\lesssim \left\| \nabla^4(u \cdot \nabla \varrho) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim \left\| \nabla^3(u \cdot \nabla \varrho) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (\|u\|_{L^\infty} \left\| \nabla^4 \varrho \right\|_{L^2} + \|\nabla \varrho\|_{L^3} \left\| \nabla^3 u \right\|_{L^6}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2} + \|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 u \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2.
\end{aligned} \tag{2.42}$$

It follows from the estimates (2.40)–(2.42) and (2.39) that

$$|I_{10}| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2. \tag{2.43}$$

Combining with (2.12), (2.22), (2.31)–(2.32), (2.38) and (2.43), it arrives at

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^4 \varrho^h|^2 + |\nabla^4 u^h|^2 + |\nabla^4 B^h|^2 + \frac{g^2}{4} |\nabla^5 \varrho^h|^2 dx \\
&\quad + (2\mu + \lambda) \int_{\mathbb{R}^3} |\nabla^5 u^h|^2 dx + \nu \int_{\mathbb{R}^3} |\nabla^5 B^h|^2 dx \lesssim (1+t)^{-\frac{11}{2}} \\
&\quad + (1+t)^{-2} (\left\| \nabla^4 \varrho^h \right\|_{H^1}^2 + \left\| \nabla^5 B^h \right\|_{L^2}^2) + (1+t)^{-\frac{3}{2}} (\left\| \nabla^5 u^h \right\|_{L^2}^2 + \left\| \nabla^6 \varrho^h \right\|_{L^2}^2).
\end{aligned} \tag{2.44}$$

In order to close the estimate, we will establish the dissipation estimate for $\nabla^4 \varrho^h$. Applying the operator $\mathfrak{F}^{-1}(1 - \phi(\xi))$ to $\nabla^4(2.1)_2$, multiplying the resulting equality by $\nabla^5 \varrho^h$, integrating over $\mathbb{R}^3$, we can deduce that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^3} \nabla^4 u^h \nabla^5 \varrho^h dx + \left\| \nabla^5 \varrho^h \right\|_{L^2}^2 + \frac{g^2}{4} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2 = \left\| \nabla^5 u^h \right\|_{L^2}^2 - \mu \langle \nabla^5 u^h, \nabla^6 \varrho^h \rangle \\
&\quad - (\mu + \lambda) \langle \nabla^4 \operatorname{div} u^h, \nabla^6 \varrho^h \rangle - \langle \nabla^4 \operatorname{div}(\varrho u)^h, \nabla^6 u^h \rangle + \langle \nabla^4 G_2^h, \nabla^5 \varrho^h \rangle.
\end{aligned} \tag{2.45}$$

Using the Young inequality, we can obtain

$$|\mu \langle \nabla^5 u^h, \nabla^6 \varrho^h \rangle| \leq \mu^2 \left\| \nabla^5 u^h \right\|_{L^2}^2 + \frac{1}{4} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2. \tag{2.46}$$

Similarly, we have

$$|(\mu + \lambda) \langle \nabla^4 \operatorname{div} u^h, \nabla^6 \varrho^h \rangle| \leq (\mu + \lambda)^2 \left\| \nabla^4 \operatorname{div} u^h \right\|_{L^2}^2 + \frac{1}{4} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2. \tag{2.47}$$

For $\langle \nabla^4 \operatorname{div}(\varrho u)^h, \nabla^5 u^h \rangle$, we obtain

$$|\langle \nabla^4 \operatorname{div}(\varrho u)^h, \nabla^5 u^h \rangle| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} \left\| \nabla^5 u^h \right\|_{L^2}^2. \tag{2.48}$$

We notice that

$$\begin{aligned}
 \langle \nabla^4 G_2^h, \nabla^5 \varrho^h \rangle &= \langle -\nabla^4 (u \cdot \nabla u)^h, \nabla^5 \varrho^h \rangle + \left\langle \nabla^4 \left\{ \frac{\varrho}{(\varrho+1)} [\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u] \right\}^h, \nabla^5 \varrho^h \right\rangle \\
 &+ \left\langle -\nabla^4 \left\{ \left[\frac{P'(\varrho+1)}{\varrho+1} - P'(1) \right] \nabla \varrho \right\}^h, \nabla^5 \varrho^h \right\rangle + \left\langle -\nabla^4 \left(\frac{\vartheta^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^h, \nabla^5 \varrho^h \right\rangle \\
 &+ \left\langle \nabla^4 \left[\frac{\vartheta^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1+\varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1+\varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1+\varrho)^2} \right) \right]^h, \nabla^5 \varrho^h \right\rangle \\
 &+ \left\langle \nabla^4 \left[\frac{1}{1+\varrho} ((\nabla \times B) \times B) \right]^h, \nabla^5 \varrho^h \right\rangle := \sum_{j=1}^6 J_j.
 \end{aligned} \tag{2.49}$$

The right-hand side of the above equation can be estimated as follows:

$$\begin{aligned}
 |J_1| &\lesssim \left\| \nabla^3 (\nabla u \cdot u)^h \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^3 (\nabla u \cdot u) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|u\|_{L^\infty} \left\| \nabla^4 u \right\|_{L^2} + \left\| \nabla^3 u \right\|_{L^6} \|\nabla u\|_{L^3}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim \|\nabla u\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 u \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.50}$$

Similar to the proof of (2.38), we have

$$|J_2| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} (\left\| \nabla^6 \varrho^h \right\|_{L^2}^2 + \left\| \nabla^5 u^h \right\|_{L^2}^2). \tag{2.51}$$

For J_3 , we get

$$\begin{aligned}
 |J_3| &\lesssim \left\| \nabla^3 \left\{ \left[\frac{P'(\varrho+1)}{\varrho+1} - P'(1) \right] \nabla \varrho \right\}^h \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^3 \left\{ \left[\frac{P'(\varrho+1)}{\varrho+1} - P'(1) \right] \nabla \varrho \right\} \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^4 \varrho \right\|_{L^2} + \|\nabla \varrho\|_{L^3} \left\| \nabla^3 \varrho \right\|_{L^6}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim \|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.52}$$

For J_4 , we can rewrite it as follows

$$\begin{aligned}
 J_4 &= \frac{\vartheta^2}{4} \langle \nabla^3 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^h, \nabla^6 \varrho^h \rangle \\
 &= \frac{\vartheta^2}{4} \langle \nabla^3 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right) - \nabla^3 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^l, \nabla^6 \varrho^h \rangle \\
 &= \frac{\vartheta^2}{4} \langle \nabla^3 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho^h \right) + \nabla^3 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho^l \right) - \nabla^3 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^l, \nabla^6 \varrho^h \rangle \\
 &= J_{4,1} + J_{4,2} + J_{4,3}.
 \end{aligned} \tag{2.53}$$

For $J_{4,1}$, we obtain

$$\begin{aligned}
 |J_{4,1}| &\lesssim \left\| \nabla^3 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho^h \right) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^6 \varrho^h \right\|_{L^2} + \left\| \nabla^3 \varrho^h \right\|_{L^6} \left\| \nabla^3 \varrho \right\|_{L^3}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^6 \varrho^h \right\|_{L^2} + \left\| \nabla^3 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho^h \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{3}{2}} \left\| \nabla^6 \varrho^h \right\|_{L^2} + (1+t)^{-\frac{9}{2}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.54}$$

For $J_{4,2}$, we get

$$\begin{aligned}
 |J_{4,2}| &\lesssim \left\| \nabla^3 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho^l \right) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^6 \varrho^l \right\|_{L^2} + \left\| \nabla^3 \varrho \right\|_{L^6} \left\| \nabla^3 \varrho^l \right\|_{L^3}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 \varrho \right\|_{L^2} + \left\| \nabla^3 \varrho^l \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho^l \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.55}$$

For $J_{4,3}$, we have

$$\begin{aligned}
 |J_{4,3}| &\lesssim \left\| \nabla^3 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)^l \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^2 \left(\frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right) \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\varrho\|_{L^\infty} \left\| \nabla^5 \varrho \right\|_{L^2} + \left\| \nabla^3 \varrho \right\|_{L^6} \left\| \nabla^2 \varrho \right\|_{L^3}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^2}^{\frac{1}{2}} \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 \varrho \right\|_{L^2} + \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^3 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^4 \varrho \right\|_{L^2}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} + (1+t)^{-\frac{17}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2.
 \end{aligned} \tag{2.56}$$

The combination of (2.53)–(2.56) give rises to

$$|J_4| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2, \tag{2.57}$$

$$\begin{aligned}
 |J_5| &\lesssim \left\| \nabla^3 \left[\frac{\varrho^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1+\varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1+\varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1+\varrho)^2} \right) \right]^h \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^3 \left[\frac{\varrho^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1+\varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1+\varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1+\varrho)^2} \right) \right] \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (\|\nabla \varrho\|_{L^\infty} \left\| \nabla^5 \varrho \right\|_{L^2} + \left\| \nabla^4 \varrho \right\|_{L^6} \left\| \nabla^2 \varrho \right\|_{L^3}) \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim \left\| \nabla^2 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^3 \varrho \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla^5 \varrho \right\|_{L^2} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{17}{4}} \left\| \nabla^6 \varrho^h \right\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-3} \left\| \nabla^6 \varrho^h \right\|_{L^2}^2,
 \end{aligned} \tag{2.58}$$

and

$$\begin{aligned}
 |J_6| &\lesssim \left\| \nabla^3 \left[\frac{1}{1+\varrho} ((\nabla \times B) \times B) \right]^h \right\|_{L^2} \|\nabla^6 \varrho^h\|_{L^2} \\
 &\lesssim \left\| \nabla^3 \left[\frac{1}{1+\varrho} ((\nabla \times B) \times B) \right] \right\|_{L^2} \|\nabla^6 \varrho^h\|_{L^2} \\
 &\lesssim (\|B\|_{L^\infty} \|\nabla^4 B\|_{L^2} + \|\nabla B\|_{L^3} \|\nabla^3 B\|_{L^6}) \|\nabla^6 \varrho^h\|_{L^2} \\
 &\lesssim \|\nabla B\|_{L^2}^{\frac{1}{2}} \|\nabla^2 B\|_{L^2}^{\frac{1}{2}} \|\nabla^4 B\|_{L^2} \|\nabla^6 \varrho^h\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{15}{4}} \|\nabla^6 \varrho^h\|_{L^2} \\
 &\lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-2} \|\nabla^6 \varrho^h\|_{L^2}^2.
 \end{aligned} \tag{2.59}$$

It follows from the estimates (2.50)–(2.52) and (2.57)–(2.59) that

$$|\langle \nabla^3 G_2^h, \nabla^4 \varrho^h \rangle| \lesssim (1+t)^{-\frac{11}{2}} + (1+t)^{-\frac{3}{2}} (\|\nabla^6 \varrho^h\|_{L^2}^2 + \|\nabla^5 u^h\|_{L^2}^2), \tag{2.60}$$

which together with (2.46)–(2.48) implies immediately

$$\begin{aligned}
 &\frac{d}{dt} \int_{\mathbb{R}^3} \nabla^4 u^h \nabla^5 \varrho^h dx + \frac{1}{2} \|\nabla^5 \varrho^h\|_{L^2}^2 + \frac{g^2}{4} \|\nabla^6 \varrho^h\|_{L^2}^2 \\
 &\lesssim (1+t)^{-\frac{11}{2}} + \|\nabla^5 u^h\|_{L^2}^2 + (1+t)^{-\frac{3}{2}} (\|\nabla^6 \varrho^h\|_{L^2}^2 + \|\nabla^5 u^h\|_{L^2}^2).
 \end{aligned} \tag{2.61}$$

Last, we will close the estimate to prove the decay rate (1.8). Choosing sufficient large time T_1 and positive constant D_0 , we define the temporary energy functional

$$\mathcal{E}(t) = D_0 (\|\nabla^4(\varrho, u, B)^h\|_{L^2}^2 + \frac{g^2}{2} \|\nabla^5 \varrho^h\|_{L^2}^2) + \int_{\mathbb{R}^3} \nabla^4 u^h \nabla^5 \varrho^h dx, \tag{2.62}$$

for $t \geq 0$, where it is clear that $\mathcal{E}(t)$ is equivalent to $\|\nabla^4 \varrho^h\|_{H^1}^2 + \|\nabla^4(u, B)^h\|_{L^2}^2$ since D_0 is large enough. Combining the estimates (2.44) with (2.61) and using Lemma 2.3, for $t \geq T_1$, one may deduce that

$$\frac{d}{dt} \mathcal{E}(t) + \|\nabla^4 \varrho^h(t)\|_{H^1}^2 + \|\nabla^4 u^h(t)\|_{L^2}^2 + \|\nabla^4 B^h(t)\|_{L^2}^2 \lesssim (1+t)^{-\frac{11}{2}}, \tag{2.63}$$

where we have used the fact that T_1 is large enough. On the other hand, it is easy to see that

$$\|\nabla^4 \varrho^h(t)\|_{H^1}^2 + \|\nabla^4 u^h(t)\|_{L^2}^2 + \|\nabla^4 B^h(t)\|_{L^2}^2 \geq C \mathcal{E}(t). \tag{2.64}$$

Therefore, (1.8) follows from (2.63), (2.64), Lemma 2.3 and Gronwall's argument immediately.

3 Proof of Theorem 1.3

This section is concerned with the optimal convergence rate of the solution, which is stated on Theorem 1.3. First, we will derive optimal convergence rate on $\|\nabla^5 \varrho^l\|_{L^2}$.

Step 1. Low-frequency L^2 energy estimate. By virtue of the equation (2.1)₁, Theorem 1.2, Duhamel's principle, Plancherel theorem, Hölder's inequality and Hausdorff–Young's inequality, we have

$$\|\nabla^5 \varrho^l(t)\|_{L^2} \lesssim (1+t)^{-\frac{13}{4}} \|(\varrho, u, B)(0)\|_{L^1} + \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{13}{4}} \|G(\tau)\|_{L^1} d\tau + \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{5}{4}} \|\xi^4 \widehat{G}^l(\tau)\|_{L^\infty} d\tau. \tag{3.1}$$

For $\|G(\tau)\|_{L^1}$, we can see the proof of (2.8). On the other hand, we can use (1.8) to bound the third term on the right-hand side of (3.1) as follows

$$\begin{aligned}
\| |\xi|^4 \widehat{G}^l(\tau) \|_{L^\infty} &\lesssim \| \nabla^4 G^l(\tau) \|_{L^1} \\
&\lesssim \| \nabla^3 (\operatorname{div}(\varrho u))(\tau) \|_{L^1} + \| \nabla^3 (u \cdot \nabla u)(\tau) \|_{L^1} \\
&\quad + \| \nabla^2 \left(\frac{\varrho}{\varrho+1} [\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u] \right)(\tau) \|_{L^1} + \| \nabla^3 \left(\left[\frac{P'(\varrho+1)}{\varrho+1} - P'(1) \right] \nabla \varrho \right)(\tau) \|_{L^1} \\
&\quad + \| \nabla^2 \left(\frac{\varrho^2}{4} \frac{\varrho}{\varrho+1} \nabla \Delta \varrho \right)(\tau) \|_{L^1} + \| \nabla^3 \left(\frac{\varrho^2}{4} \left(\frac{|\nabla \varrho|^2 \nabla \varrho}{(1+\varrho)^3} - \frac{\nabla \varrho \Delta \varrho}{(1+\varrho)^2} - \frac{\nabla \varrho \cdot \nabla^2 \varrho}{(1+\varrho)^2} \right) \right)(\tau) \|_{L^1} \\
&\quad + \| \nabla^3 \left(\frac{1}{1+\varrho} ((\nabla \times B) \times B) \right)(\tau) \|_{L^1} + \| \nabla^3 (\nabla \times (u \times B)(\tau)) \|_{L^1} \\
&\lesssim \| (\varrho, u)(\tau) \|_{L^2} \| \nabla^4 (\varrho, u)(\tau) \|_{L^2} + \| \nabla^2 u(\tau) \|_{L^2}^2 + \| \varrho(\tau) \|_{L^2} \| \nabla^5 \varrho(\tau) \|_{L^2} \\
&\quad + \| \nabla \varrho(\tau) \|_{L^2} \| \nabla^5 \varrho(\tau) \|_{L^2} + \| \nabla \varrho(\tau) \|_{L^2} \| \nabla^4 \varrho(\tau) \|_{L^2} + \| \nabla^2 \varrho(\tau) \|_{L^2} \| \nabla^3 \varrho(\tau) \|_{L^2} \\
&\quad + \| \nabla B(\tau) \|_{L^2} \| \nabla^3 B(\tau) \|_{L^2} + \| (u, B)(\tau) \|_{L^2} \| \nabla^4 (u, B)(\tau) \|_{L^2} \\
&\lesssim (1+t)^{-\frac{7}{2}}.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), it arrives at

$$\| \nabla^5 \varrho^l \|_{L^2} \lesssim (1+t)^{-\frac{13}{4}}. \tag{3.3}$$

Step 2. High-frequency L^2 energy estimate. In this part, we will make full use of (1.8) to achieve this goal. It is not difficult to find that the following calculations are consistent with the second section except replacing the L^2 decay rate $(1+t)^{-\frac{9}{4}}$ by $(1+t)^{-\frac{11}{4}}$ for the fourth order spatial derivative of solution. Similar to the proof of (2.48), we have

$$\frac{d}{dt} \mathcal{E}(t) + \| \nabla^4 \varrho^h(t) \|_{H^1}^2 + \| \nabla^4 u^h(t) \|_{L^2}^2 + \| \nabla^4 B^h(t) \|_{L^2}^2 \lesssim (1+t)^{-\frac{13}{2}}. \tag{3.4}$$

Finally, we conclude that

$$\| \nabla^4 \varrho^h(t) \|_{H^1} + \| \nabla^4 u^h(t) \|_{L^2} + \| \nabla^4 B^h(t) \|_{L^2} \lesssim (1+t)^{-\frac{13}{4}}, \tag{3.5}$$

which together with (3.3) implies (1.9) directly.

4 Analytic tools

Now, we state some auxiliary lemmas, which will be frequently used in this paper.

Lemma 4.1. (Gagliardo-Nirenberg's inequality) *Given $2 \leq p \leq +\infty$ and $0 \leq k, m \leq \ell$, then for any $f \in H^\ell(\mathbb{R}^3)$, we have*

$$\| \nabla^k f \|_{L^p} \lesssim \| \nabla^m f \|_{L^2}^\alpha \| \nabla^\ell f \|_{L^2}^{1-\alpha},$$

where $\alpha \in [0, 1]$ satisfies

$$\frac{k}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{2} \right) \alpha + \left(\frac{\ell}{3} - \frac{1}{2} \right) (1 - \alpha).$$

Proof. This is the special case of [23]. □

Lemma 4.2. *Let $k \geq 1$ be an integer, then it holds that*

$$\| \nabla^k (fg) \|_{L^p} \lesssim \| f \|_{L^{p_1}} \| \nabla^k g \|_{L^{p_2}} + \| g \|_{L^{p_3}} \| \nabla^k f \|_{L^{p_4}},$$

where $p, p_2, p_3 \in (1, +\infty)$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Proof. For $p = p_2 = p_3 = 2$, it can be proved by using Lemma 4.1. For the general case, one may refer to [12] $\square$

Lemma 4.3. *If $f \in L^r(\mathbb{R}^3)$ for any $2 \leq r \leq \infty$, then we have*

$$\|f^l\|_{L^r} + \|f^h\|_{L^r} \lesssim \|f\|_{L^r}.$$

Proof. For $2 \leq r \leq \infty$, by Young inequality's for convolutions, for the low frequency, it holds

$$\|f^l\|_{L^r} \lesssim \|\mathfrak{F}^{-1}\phi\|_{L^1} \|f\|_{L^r} \lesssim \|f\|_{L^r},$$

and hence

$$\|f^h\|_{L^r} \lesssim \|f\|_{L^r} + \|f^l\|_{L^r} \lesssim \|f\|_{L^r}.$$

$\square$

Lemma 4.4. *Let $r_1, r_2 > 0$, then it holds that*

$$\int_0^t (1+t-s)^{-r_1} (1+s)^{-r_2} ds \leq C(1+t)^{-\min\{r_1, r_2, r_1+r_2-1-\epsilon\}},$$

for any arbitrarily small $\epsilon > 0$.

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