Research Article

Gui-Dong Li, Yong-Yong Li, and Chun-Lei Tang*

Infinitely many radial and non-radial sign-changing solutions for Schrödinger equations

<https://doi.org/10.1515/anona-2021-0221>

Received September 11, 2021; accepted December 18, 2021.

Abstract: In the present paper, a class of Schrödinger equations is investigated, which can be stated as

$$-\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N.$$

If the external potential V is radial and coercive, then we give the local Ambrosetti-Rabinowitz super-linear condition on the nonlinearity term $f \in C(\mathbb{R}, \mathbb{R})$ which assures the problem has not only infinitely many radial sign-changing solutions, but also infinitely many non-radial sign-changing solutions.

Keywords: Schrödinger equation, Radial sign-changing solutions, Non-radial sign-changing solutions, Variational methods, Principle of symmetric criticality, Invariant sets of descending flow

MSC: 35A15, 35D30, 35J10, 35J20

1 Introduction

In this paper, a class of Schrödinger equations is studied. It can be stated as

$$-\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N, \quad (1.1)$$

where V is the external potential and the nonlinearity term $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. For the appliance to the nonlinear optics, the nonlinear Schrödinger equation is receiving increasingly amount of attention from the field of mathematics. Indeed, some nonlinear Schrödinger equations arose from some simplified models. Bergé [10], C. Sulem and P.-L. Sulem [33] studied them from the modelization aspects. On the other hand, quantum field theory, especially the Hartree-Fock theory, is also origin fields of Schrödinger equations. Many researchers, Avron, Herbst and Simon [1], Bialynicki-Birula and Mycielski [11], Gogny and Lions [16], Kato [18], and C. Sulem and P.-L. Sulem [33] has been focusing on it since decades ago. What's more, the nonlinear Schrödinger equation is also a good model dispersion equation, because it is usually technically simpler than other dispersion equations (such as the wave or KdV). In the mind of mathematicians, this kind of problem has been concerned for a long time. Of course, there are rich results about it (e.g. [8, 9, 15, 19, 21]). What we care about is whether there are multiple solutions to this problem. Let's review some of the work in this area.

For the case of autonomy, that is, $V > 0$ is a constant, to handle the existence of infinitely many solutions to problem (1.1), researchers in [14, 15, 26, 30]) considered it with some special nonlinearities

Gui-Dong Li, School of Mathematics and Statistics, Guizhou University, Guiyang, Guizhou 550025, People's Republic of China

Yong-Yong Li, College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu 730070, People's Republic of China

***Corresponding Author: Chun-Lei Tang**, School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China, E-mail: tangcl@swu.edu.cn

$u^p, p \in (1, 2^* - 1)$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ or $2^* = +\infty$ if $N = 1, 2$. We wish to point out an interesting result by Berestycki and Lions [8], in which they showed that under the so-called "Berestycki-Lions condition" on f , there exist infinitely many radial solutions and the action of these solutions is arbitrarily large. Here the Berestycki-Lions condition is not only necessary but also sufficient for guaranteeing the existence of solution. Under the same hypotheses on f , Hirata, Ikoma and Tanaka gave a different method to show that problem (1.1) have infinitely many radial solutions. Mederski [25] further explained that problem (1.1), under "Berestycki-Lions condition", has infinitely many non-radial solutions. There is another interesting result introduced by J. Liu, X. Liu and Wang [21] where a general critical point theory was developed to tackle the existence and locations of multiple critical points created by minimax methods related to the multiple invariant sets of the associated gradient flow, and then it is proved that problem (1.1) possesses a family of infinitely many radial sign-changing solutions.

Notice that the authors in [21] made the standard Ambrosetti-Rabinowitz super-linear condition (there is $\mu > 2$ satisfying $sf(s) \geq \mu F(s) > 0$ for all $s \in \mathbb{R} \setminus \{0\}$, where $F(s) = \int_0^s f(t)dt$) on the nonlinearity f , and we say (AR) condition for short. We will prove that this condition on f can be further weakened in this paper. There are many interesting results in the case of autonomy, but due to the limited space, we will not introduce them one by one. Let us now turn our attention to the case of non-autonomy, namely, V is a non-constant potential.

For the case where V is continuous and 1-periodic in $x_1, \dots, x_N$, it has been considered by Szulkin and Weth [35], and Tang et al. [37]. In [12, 31] the authors showed that problem (1.1) have infinitely many solutions where V is an asymptotically autonomous potential, namely, $\lim_{|x| \rightarrow \infty} V(x) = V_\infty > 0$. For the case where V is radial potential, i.e., $V(x) = V(|x|)$, provided that $r^2 V(r)$ has a local maximum point, or local minimum point $r_0 > 0$ with $V(r_0) > 0$, Chen, Wei and Yan [13] then proved that problem (1.1) with critical growth has infinitely many non-radial solutions. If $V(y) = (y', y'')$, $(y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$ has a stable critical point, then problem (1.1) with critical growth has infinitely many solutions by using the standard reduction method to locate the concentrating points of the solutions (see [28]).

Hong [17] required that $V(x) = \lambda a(x) + 1$ and f satisfies the standard Ambrosetti-Rabinowitz super-linear condition, thanks for the principle of symmetric criticality and the fountain theorem, the authors proved that the existence of a sequence of solutions to equation (1.1) and proved the existence of an unbounded sequence of solutions, which is non-radial and sign-changing. As we know, there also exists many similar researches concerning the existence of the nontrivial solutions to problem (1.1). We refer the interested reader to [15, 23, 29, 34, 36, 38] and references therein.

However, it can be seen from the above literature that few articles start from two aspects, that is, to prove the existence of two families of solutions. There is a natural guess: it exists infinitely many radial and non-radial sign-changing solutions for problem (1.1). Fortunately, we will endorse this guess in this work. We introduce our work space given by

$$E = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\},$$

endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx.$$

This is a Hilbert space, and its inner product is denoted by $\langle \cdot, \cdot \rangle$. Here V satisfies

- (V₁) $V \in C(\mathbb{R}^N)$, $\inf_{x \in \mathbb{R}^N} V(x) > 0$ and $V(x) = V(|x|)$,
 (V₂) there exists a constant $r > 0$ such that, for any $M > 0$, $\text{meas} \{x \in \mathbb{R}^N : |x - y| \leq r, V(x) \leq M\} \rightarrow 0$ as $|y| \rightarrow +\infty$.

We investigate Schrödinger equations as follows:

$$\begin{cases} -\Delta u + V(x)u = f(u), & x \in \mathbb{R}^N, \\ u \in E, \end{cases} \quad (1.2)$$

where $N \geq 1$ and f satisfies

- (f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd and $\lim_{s \rightarrow 0} \frac{f(s)}{s} = 0$,
 (f_2) $\lim_{s \rightarrow +\infty} \frac{sf(s)}{s^p} = 0$, $p \in (2, 2^*)$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ or $2^* = +\infty$ if $N = 1, 2$,
 (f_3) $f(s) \geq 0$ for any $s \geq 0$ and $\lim_{s \rightarrow +\infty} \frac{F(s)}{s^2} = +\infty$, where $F(s) = \int_0^s f(t)dt$,
 (f_4) there exist $L > 0, \mu > 2$ such that

$$sf(s) - \mu F(s) \geq 0, \quad s \in [L, +\infty).$$

Our result reads as

Theorem 1.1. *Let $(V_1), (V_2)$ and (f_1) – (f_4) hold. If $N = 2k, k \in \mathbb{Z}^+$ then problem (1.2) has not only infinitely many radial sign-changing solutions, but also infinitely many non-radial sign-changing solutions.*

Remark 1.2. To our knowledge, our result seems to be the first attempt to illustrate that problem (1.2) has two families of infinitely many sign-changing solutions. One can easily see that the local (AR) condition (f_4) is weaker than the (AR) condition. For example,

$$f(u) = \begin{cases} |u|^{q-2}u, & |u| \leq 1, q \in (2, p), \\ u, & 1 < |u| \leq 2, \\ \frac{2}{2^{q-1}}|u|^{q-2}u, & 2 < |u|. \end{cases} \quad (1.3)$$

Clearly f does not satisfy the (AR) condition, but satisfies (f_4). Therefore, to a certain extent, this work can also be regarded as an extension and supplement to previous research results.

The current work consists of four parts. Let's end the first section here. The second part gives some preliminary results. In Section 3, we introduce a constraint problem. The proof of Theorem 1.1 will be given in Section 4.

2 Preliminaries

Here we give notations which will be used in the rest of this paper.

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the usual norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx.$$

- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm

$$|u|_p^p = \int_{\mathbb{R}^N} |u|^p dx \text{ and } |u|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^N} |u(x)| \text{ for all } p \in [1, +\infty).$$

- $\operatorname{meas} \Omega$ denotes the Lebesgue measure of the set Ω .
- $u^\pm := \max\{\pm u, 0\}$.
- $C, C_i (i = 1, 2, \dots)$ denote positive constants and are possibly different from line to line.

The embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ ($2 \leq s < 2^*$) is compact (see [6, 7]). Now, we have to emphasize that the existence of solutions for problem (1.2) can be proved by using variational methods. As for problem (1.2), the energy functional $\mathcal{J} : E \rightarrow \mathbb{R}$ can be constructed as

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \int_{\mathbb{R}^N} F(u) dx.$$

Thus, based on the conditions of (f_1) – (f_3) , by a standard argument we can affirm that $\mathcal{J}$ is well defined and of class C^1 , and its derivative in E is

$$\langle \mathcal{J}'(u), v \rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv dx - \int_{\mathbb{R}^N} f(u)v dx \quad \text{for all } v \in E.$$

Formally, all critical points of $\mathcal{J}$ are solutions of problem (1.2).

Lemma 2.1. Assume that (V_1) , (V_2) and (f_1) – (f_4) hold, then $\mathcal{J}$ satisfies the Palais-Smale condition.

Proof. Suppose that $\{v_n\}$ is a Palais-Smale (Shortly: (PS)) sequence of $\mathcal{J}$ for some $c \in \mathbb{R}$, that is,

$$\mathcal{J}(v_n) \rightarrow c, \quad \mathcal{J}'(v_n) \rightarrow 0.$$

We claim that the sequence $\{v_n\}$ is bounded in E , otherwise there should be a subsequence, still denoted v_n , such that $\|v_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Let $u_n = \frac{v_n}{\|v_n\|}$, so that there exists $u \in E$ satisfying $u_n \rightarrow u$ in $L^q(\mathbb{R}^N)$ ($2 \leq q < 2^*$), $u_n(x) \rightarrow u(x)$ a.e. $x \in \mathbb{R}^N$. Let

$$\Omega = \{x \in \mathbb{R}^N : u(x) \neq 0\}.$$

If $\text{meas}(\Omega)$ is not equal to 0, then $\lim_{n \rightarrow \infty} |v_n(x)| = +\infty$ almost everywhere in Ω . With applying (f_2) and Fatou's lemma, we calculate directly

$$\liminf_{n \rightarrow \infty} \frac{\int_{\mathbb{R}^N} F(v_n) dx}{\|v_n\|^2} \geq \liminf_{n \rightarrow \infty} \int_{\Omega} \frac{F(v_n) u_n^2}{|v_n|^2} dx \geq \int_{\Omega} \liminf_{n \rightarrow \infty} \frac{F(v_n) u_n^2(x)}{|v_n(x)|^2} dx = +\infty.$$

Then we obtain that

$$0 = \limsup_{n \rightarrow \infty} \frac{c}{\|v_n\|^2} = \limsup_{n \rightarrow \infty} \frac{\mathcal{J}(v_n)}{\|v_n\|^2} = \frac{1}{2} - \liminf_{n \rightarrow \infty} \frac{\int_{\mathbb{R}^N} F(v_n) dx}{\|v_n\|^2} = -\infty.$$

This contradiction indicates that $\text{meas}(\Omega) = 0$, so that $u(x) = 0$ a.e. $x \in \mathbb{R}^N$ and $u_n \rightarrow 0$ in $L^s(\mathbb{R}^N)$ ($2 \leq s < 2^*$). By (f_1) , we have

$$|sf(s) - \mu F(s)| \leq C_1 |s|^2 \quad \text{for all } |s| \in [0, L].$$

Combining with (f_3) , we deduce

$$sf(s) - \mu F(s) \geq -C_2 |s|^2, \quad \text{for all } s \in \mathbb{R}.$$

Hence,

$$\begin{aligned} o(1) &= \frac{1}{\|v_n\|^2} (\mu \mathcal{J}(v_n) - \langle \mathcal{J}'(v_n), v_n \rangle) \\ &= \left(\frac{\mu}{2} - 1 \right) + \frac{\int_{\mathbb{R}^N} f(v_n) v_n - \mu F(v_n) dx}{\|v_n\|^2} \\ &\geq \left(\frac{\mu}{2} - 1 \right) - C_2 |u_n|_2^2 \\ &\rightarrow \frac{\mu}{2} - 1. \end{aligned}$$

This is a contradiction with $\mu > 2$. So $\{v_n\}$ is bounded in E and then there exists $v \in E$ satisfying $v_n \rightarrow v$ in $L^q(\mathbb{R}^N)$ ($2 \leq q < 2^*$), $v_n(x) \rightarrow v(x)$ a.e. $x \in \mathbb{R}^N$. Generalized control convergence theorem gives us that (f_1) and (f_2) imply that

$$\int_{\mathbb{R}^N} f(v_n) v_n dx \rightarrow \int_{\mathbb{R}^N} f(v) v dx.$$

Therefore $v_n \rightarrow v$ in E , and hence $\mathcal{J}$ satisfies the Palais-Smale condition. $\square$

Lemma 2.2. Assume that (V_1) , (V_2) and (f_1) – (f_4) hold. Then

- (i) $\inf_{\|u\|=\rho} \mathcal{J}(u) > 0$ holds for some positive constant ρ ,
(ii) Let E_m be a subspace of E with $\dim E_m < +\infty$, then there is $R_m = R(E_m) > 0$ satisfying

$$\sup_{u \in B_{R_m}^c \cap E_m} \mathcal{J}(u) < 0,$$

where $B_R := \{u \in E : \|u\| \leq R\}$ and $B_R^c := E \setminus B_R$.

Proof. (i). From (f_1) – (f_3) , for any fixed $\delta > 0$ we can find $C_\delta > 0$ such that

$$|F(s)| \leq \delta |s|^2 + C_\delta |s|^p \quad \text{for any } s \in \mathbb{R},$$

where p is given in (f_2) . Then we have

$$\mathcal{J}(u) \geq \frac{\|u\|^2}{2} - \delta \int_{\mathbb{R}^N} |u|^2 dx - C_\delta \int_{\mathbb{R}^N} |u|^p dx \geq \frac{\|u\|^2}{2} - C_1 \delta \|u\|^2 - C_2 \|u\|^p. \quad (2.1)$$

Noting that $p > 2$, taking δ small enough we conclude that there is $0 < \rho < 1$ satisfying

$$\inf_{u \in \partial B_\rho} \mathcal{J}(u) \geq \frac{\rho^2}{8} > 0.$$

(ii). It follows from (f_1) – (f_3) that there exists $C_1, C_2 > 0$ such that

$$F(s) \geq C_1 |s|^\mu - C_2 |s|^2 \quad \text{for all } s \in \mathbb{R}.$$

So we deduce

$$\mathcal{J}(u) \leq \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(u) dx \leq \frac{1}{2} \|u\|^2 - C_1 \int_{\mathbb{R}^N} |u|^\mu dx + C_2 \int_{\mathbb{R}^N} |u|^2 dx.$$

For $\mu > 2$ and because of the equivalence of norms in finite dimensional space, we see that $\mathcal{J}(u) = -\infty$ as $\|u\| \rightarrow \infty$, $u \in E_m$. Therefore the result follows. $\square$

3 Radial sign-changing solutions

Set

$$E_r := \{u \in E : u \text{ is a radial function}\}.$$

Clearly, E_r is a nonempty complete subspace of E . Now, let problem (1.2) constrain on E_r , namely,

$$\begin{cases} -\Delta u + V(x)u = f(u), & x \in \mathbb{R}^N, \\ u \in E_r. \end{cases} \quad (3.1)$$

Notice that $\mathcal{J}|_{E_r}$ is the corresponding energy function of problem (3.1). For simplicity, we denote $\mathcal{J}|_{E_r}$ by $\mathcal{J}_r$. Next we introduce an auxiliary operator B , which plays an important role in constructing the descending flow for $\mathcal{J}_r$. Let the operator B be defined as follows: for a fixed $u \in E_r$, $v = B(u) \in E_r$ is the unique solution of

$$-\Delta v + V(x)v = f(u). \quad (3.2)$$

Clearly, the following three expressions are equivalent:

- u is a solution of problem (3.1),
- u is a critical point of $\mathcal{J}_r$,

- u is a fixed point of B .

In the following lemmas, we introduce some properties of B , which can be similarly proved in [21, 24, 34]. For better viewing, we provide the proof here.

Lemma 3.1. Assume that (V_1) and (f_1) – (f_4) hold. Then

- (i) the operator B is well defined as well as continuous,
- (ii) $\langle \mathcal{J}'_r(u), u - B(u) \rangle = \|u - B(u)\|^2$ for all $u \in E_r$,
- (iii) $\|\mathcal{J}'_r(u)\| \leq C\|u - B(u)\|$ for some $C > 0$ and any $u \in E_r$,
- (iv) for $c < d$ and $a > 0$, there exists $b > 0$ such that $\|u - B(u)\| \geq a$ if $u \in E_r$, $\mathcal{J}_r(u) \in [c, d]$ and $\|\mathcal{J}'_r(u)\| \geq b$.

Proof. (i) Let $u \in E_r$, and define for any $v \in E_r$,

$$\mathcal{E}(v) = \frac{1}{2}\|v\|^2 - \int_{\mathbb{R}^N} f(u)v dx,$$

so that $\mathcal{E} \in C^1(E_r, \mathbb{R})$ and $\mathcal{E}$ is weakly lower semicontinuous. By (f_1) – (f_2) and the Sobolev embedding we then obtain

$$\int_{\mathbb{R}^N} f(u)v dx \leq C\|v\|,$$

which implies

$$\mathcal{E}(v) \geq \frac{1}{2}\|v\|^2 - C\|v\|,$$

and this means that $\mathcal{E}(v) \rightarrow +\infty$ as $\|v\| \rightarrow \infty$, where $C = C(u)$ is a constant depending on u . This shows that $\mathcal{E}$ is coercive. Moreover, we know that $\mathcal{E}$ is bounded below and it keeps the boundness of sets as a mapping. We now estimate

$$\langle \mathcal{E}'(v) - \mathcal{E}'(w), v - w \rangle \geq \|v - w\|^2,$$

so that $\mathcal{E}$ is also strictly convex. Hence, from [2, Theorems 1.5.6 and 1.5.8], $\mathcal{E}$ admits a unique minimizer $v = B(u) \in E_r$, which is the unique solution to problem (3.2). Clearly, B also maps bounded sets into bounded sets by (f_1) – (f_3) .

Next, we give the proof of the continuity of B . Assume that $\{u_n\} \subset E_r$ with $u_n \rightarrow u$ in E_r . Denote $v = B(u)$ and $v_n = B(u_n)$. Since $\langle \mathcal{E}'(v_n), v_n \rangle = 0$ we have

$$\|v_n\|^2 = \int_{\mathbb{R}^N} f(u_n)v_n dx.$$

Combining with (f_1) – (f_2) , we can obtain the boundness of $\{v_n\}$ in E_r . Assume that $v_n \rightharpoonup v^*$ in E_r , after extracting a subsequence, then for any $w \in E_r$

$$\int_{\mathbb{R}^N} \nabla v^* \cdot \nabla w + V(x)v^* w dx = \int_{\mathbb{R}^N} f(u)w dx,$$

which implies that $v^* = B(u)$. So $v = v^*$ by the uniqueness.

We show that $\|v_n - v\| \rightarrow 0$ as $n \rightarrow \infty$. From $\langle \mathcal{E}'(v) - \mathcal{E}'(v_n), v - v_n \rangle = 0$, we can write

$$\|v_n - v\|^2 = \int_{\mathbb{R}^N} (f(u_n) - f(u)) (v_n - v) dx.$$

So, by the general Lebesgue control convergence theorem, $v_n \rightarrow v$ in E_r . Hence, B is continuous.

(ii) By using the fact that $B(u)$ is a solution of problem (3.2), one obtains

$$\langle \mathcal{J}'_r(u), u - B(u) \rangle = \|u - B(u)\|^2,$$

for all $u \in E_r$.

(iii) Notice that for all $\phi \in C_0^\infty(\mathbb{R}^N) \cap E_r$,

$$\langle \mathcal{J}'_r(u), \phi \rangle = \int_{\mathbb{R}^N} [\nabla(u - B(u)) \cdot \nabla \phi + V(x)(u - B(u)) \phi] dx.$$

Then we have

$$\|\mathcal{J}'_r(u)\| \leq C\|u - B(u)\| \quad \text{for all } u \in E_r.$$

(iv) Contrarily, suppose that it exists $\{u_n\} \subset E_r$ with

$$\mathcal{J}_r(u_n) \in [c, d] \quad \text{and} \quad \|\mathcal{J}'_r(u_n)\| \geq \alpha,$$

such that

$$\|u_n - B(u_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and hence from (iii) we deduce that $\|\mathcal{J}'_r(u_n)\| \rightarrow 0$. But it is impossible, so the proof is complete. $\square$

Since the Hilbert space $L^2(\mathbb{R}^N)$ is a separable and E_r can be embedded continuously in it, we can conclude that E_r possesses a countable orthogonal basis $\{l_i\}$. For any $m \in \mathbb{N}$, we denote

$$H_m := \text{span}\{l_1, l_2, \dots, l_m\}.$$

In order to prove our results, as in many references such as [21, 23, 34], we introduce the positive and negative cones, which is used to determine the position of the solution. Exactly, set

$$Y^+ := \{u \in E_r : u(x) \geq 0, x \in \mathbb{R}^N\} \quad \text{and} \quad Y^- := \{u \in E_r : u(x) \leq 0, x \in \mathbb{R}^N\}.$$

For $\varepsilon > 0$, let

$$Y_\varepsilon^+ := \left\{ u \in E_r : \text{dist}(u, Y^+) = \inf_{v \in Y^+} \|v - u\| < \varepsilon \right\}.$$

Similarly, set $Y_\varepsilon^- := \{u \in E_r : \text{dist}(u, Y^-) < \varepsilon\}$. It is easy to see that $W := Y_\varepsilon^+ \cup Y_\varepsilon^-$ is an open and symmetric subset of E_r and $E_r^W := E_r \setminus W$ only contains sign-changing functions. Define $\mathcal{B}_a(G) := \{u \in E_r : \text{dist}(u, G) < a\}$ for $G \subset E_r$ and $a > 0$.

Lemma 3.2. Assume that (V_1) and (f_1) – (f_4) hold. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,

- $B(\partial Y_\varepsilon^-) \subset Y_\varepsilon^-$ and any solution $v \in Y_\varepsilon^- \setminus \{0\}$ is negative,
- $B(\partial Y_\varepsilon^+) \subset Y_\varepsilon^+$ and any solution $v \in Y_\varepsilon^+ \setminus \{0\}$ is positive.

Proof. Let $u \in E_r$ and $v = B(u)$. Conditions (f_1) and (f_2) imply that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ depending on ε to satisfy

$$\int_{\mathbb{R}^N} f(u)v^+ dx \leq \varepsilon \int_{\mathbb{R}^N} u^+ v^+ dx + C_\varepsilon \int_{\mathbb{R}^N} |u^+|^{p-1} v^+ dx. \quad (3.3)$$

Based on $\text{dist}(v, Y^-) \leq \|v^+\|$, combined with (3.3) and Sobolev's inequality, it is true that

$$\begin{aligned} \text{dist}(v, Y^-) \|v^+\| &\leq \langle v, v^+ \rangle = \int_{\mathbb{R}^N} f(u)v^+ dx \\ &\leq \varepsilon \int_{\mathbb{R}^N} u^+ v^+ dx + C_\varepsilon \int_{\mathbb{R}^N} |u^+|^{p-1} v^+ dx \end{aligned}$$

$$\leq C \left[\epsilon \operatorname{dist}(u, Y^-) + |\operatorname{dist}(u, Y^-)|^{p-1} \right] \|v^+\|. \quad (3.4)$$

Let's make ϵ and ε small enough, then

$$\operatorname{dist}(v, Y^-) \leq \frac{1}{2} \operatorname{dist}(u, Y^-),$$

and thus one can find $\varepsilon_0 > 0$ such that

$$\operatorname{dist}(B(u), Y^-) = \operatorname{dist}(v, Y^-) < \varepsilon, \quad \varepsilon \in (0, \varepsilon_0).$$

As a result, it gives us that $B(u) \in Y_\varepsilon^-$ for any $u \in Y_\varepsilon^-$. Suppose that there exists $u \in Y_\varepsilon^-$ with $B(u) = u$, hence we can say that $u \in Y^-$. What's more, if u is nontrivial, it follows from the maximum principle that $u < 0$ in $\mathbb{R}^N$. Similarly, the second conclusion is also true. $\square$

Lemma 3.2 illustrates that every sign-changing solutions of problem (3.1) should be contained in E_r^w for ε small. At the same time, we note that B itself is not suitable for constructing the descending flow for $\mathcal{J}_r$, because B is merely continuous. But fortunately, the local Lipschitz continuous operator can overcome it. By the same way as [5, Lemma 2.1] (also see [3, 21]), a locally Lipschitz continuous operator, without loss of the main properties of B , can be constructed on $E_r^k := E_r \setminus K$, denote it as A . Here K is the set of fixed points of B , which is also the set of critical points of $\mathcal{J}_r$.

Lemma 3.3. Assume that (V_1) and (f_1) – (f_4) hold. Then there exists a operator $A : E_r^k \rightarrow E_r$ which is locally Lipschitz continuous and satisfies

- (i) $A(\partial Y_\varepsilon^+) \subset Y_\varepsilon^+$ and $A(\partial Y_\varepsilon^-) \subset Y_\varepsilon^-$ for every $\varepsilon \in (0, \varepsilon_0)$,
- (ii) $\frac{1}{2}\|u - A(u)\| \leq \|u - B(u)\| \leq 2\|u - A(u)\|$ for any $u \in E_r$,
- (iii) $\langle \mathcal{J}_r'(u), u - A(u) \rangle \geq \frac{1}{2}\|u - B(u)\|^2$ for every $u \in E_r$,
- (iv) if f is an odd function, then A is odd.

For a fixed $u \in E_r$, we introduce the following initial value problem

$$\begin{cases} \frac{d}{dt} \tau(t, u) = -A(\tau(t, u)), \\ \tau(0, u) = u, \end{cases} \quad (3.5)$$

where $A(u) = u - B(u)$. Because A is a locally Lipschitz continuous operator, problem (3.5) has a unique solution by the existence and uniqueness theory of ODE. Denote by $\tau(t, u)$ with maximal interval of existence $[0, T(u)]$. It follows from Lemma 3.3 that $\mathcal{J}_r(\tau(t, u))$ is strictly decreasing in $[0, T(u)]$. According to [5, Lemma 3.2], through Lemma 3.3 we have the fact that for any $\varepsilon \in (0, \varepsilon_0]$ and $u \in \overline{Y_\varepsilon^+} \setminus K$, $\tau(t, u) \in Y_\varepsilon^+$ for any $t \in (0, T(u))$, where ε_0 is given in Lemma 3.2.

For a fixed $c \in \mathbb{R}$, set $K_c = \{u \in E_r : \mathcal{J}_r(u) = c \text{ and } \mathcal{J}_r'(u) = 0\}$ and $\mathcal{J}_r^c = \{u \in E_r : \mathcal{J}_r(u) \leq c\}$. Inspired by [21, 23, 24, 34], it is a deformation lemma about invariant sets that is needed, which plays a critical role in the process of obtaining the existence and location of multiple solutions.

Lemma 3.4. Assume that (V_1) , (V_2) and (f_1) – (f_4) hold, and $\mathcal{N}$ is an open symmetric neighborhood of $K_c \setminus W$. Then one can find $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon' < \varepsilon_0$, there is $\sigma \in C([0, 1] \times E_r, E_r)$ satisfying:

- (a) $\sigma(t, u) = u$ if $t = 0$ or $u \notin \mathcal{J}_r^{-1}([c - \varepsilon_0, c + \varepsilon_0])$,
- (b) $\sigma(1, \mathcal{J}_r^{c+\varepsilon} \setminus (W \cup \mathcal{N})) \subset \mathcal{J}_r^{c-\varepsilon}$,
- (c) if $K_c \setminus W = \emptyset$, then $\sigma(1, \mathcal{J}_r^{c+\varepsilon} \setminus W) \subset \mathcal{J}_r^{c-\varepsilon}$,
- (d) $\sigma(t, -u) = -\sigma(t, u)$ for all $(t, u) \in [0, 1] \times E_r$,
- (e) $\sigma(t, \overline{Y_\varepsilon^+}) \subset \overline{Y_\varepsilon^+}$ and $\sigma(t, \overline{Y_\varepsilon^-}) \subset \overline{Y_\varepsilon^-}$ for any $t \in [0, 1]$.

Proof. Set $K_c^1 := K_c \cap W$ and $K_c^2 := K_c \setminus W$. Lemma 2.1 tells us that K_c is compact and then $d := \text{dist}(K_c^1, E_r^W) > 0$, where $E_r^W = E_r \setminus W$. Furthermore, choose $\delta \in \left(0, \frac{d}{8}\right)$ so small that $\mathcal{B}_{3\delta}(K_c^2) \subset \mathcal{N}$ and $\mathcal{B}_{3\delta}(K_c^1) \subset W$. In fact, there are $a, \varepsilon_0 > 0$ satisfying

$$\|\mathcal{J}'_r(u)\| \geq a \quad \text{for any } u \in \mathcal{J}_r^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \setminus \mathcal{B}_{2\delta}(K_c).$$

Thanks to Lemma 3.1, one can always find $b > 0$ satisfying

$$\|u - A(u)\| \geq b \quad \text{for any } u \in \mathcal{J}_r^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \setminus \mathcal{B}_{2\delta}(K_c).$$

Taking generality into account, we can still assume $\varepsilon_0 \leq \frac{b\delta}{32}$. Take a locally Lipschitz continuous cut-off function $\chi : E_r \rightarrow [0, 1]$ such that

$$\chi(u) = \begin{cases} 1, & u \notin \mathcal{B}_{2\delta}(K_c) \text{ and } u \in \mathcal{J}_r^{-1}([c - \varepsilon, c + \varepsilon]), \\ 0, & u \in \mathcal{B}_\delta(K_c) \text{ or } u \notin \mathcal{J}_r^{-1}([c - \varepsilon', c + \varepsilon']). \end{cases}$$

Decreasing ε_0 if necessary, by Lemma 2.1 there is $\delta' > 0$ such that

$$\mathcal{J}_r^{-1}[c - \varepsilon_0, c + \varepsilon_0] \cap \mathcal{B}_{\delta'}(K) \subset \mathcal{B}_\delta(K_c).$$

Thus, $\chi(u) = 0$ for any $u \in \mathcal{B}_{\delta'}(K)$. From Lemma 3.3, we know that $\chi(\cdot)A(\cdot)$ is locally Lipschitz continuous on E_r , where $A(u) = u - A(u)$. We study the following initial value problem

$$\begin{cases} \frac{d}{dt}\eta(t, u) = -\frac{\chi(\eta(t, u))A(\eta(t, u))}{\|A(\eta(t, u))\|}, \\ \eta(0, u) = u, \end{cases} \quad (3.6)$$

which deduces that $\eta(t, u)$ is well-defined and continuous on $\mathbb{R}^+ \times E_r$.

Let $\sigma(t, u) = \eta\left(\frac{16\varepsilon}{b}t, u\right)$. It is sufficient to easily show (a) and (d). For (b), conversely, we suppose that there exists $u \in \mathcal{J}_r^{c+\varepsilon} \setminus (\mathcal{N} \cup W)$ such that $\sigma(1, u) \notin \mathcal{J}_r^{c-\varepsilon}$. It is a fact that $u \notin \mathcal{B}_{3\delta}(K_c)$, since $\mathcal{B}_{3\delta}(K_c) \subset \mathcal{B}_{3\delta}(K_c^1) \cup \mathcal{B}_{3\delta}(K_c^2) \subset W \cup \mathcal{N}$. Observe that for any $t \in [0, 1]$

$$\|\sigma(t, u) - u\| = \left\| \eta\left(\frac{16\varepsilon}{b}t, u\right) - u \right\| = \left\| \int_0^{\frac{16\varepsilon}{b}t} \eta'(s, u) ds \right\| \leq \frac{16\varepsilon}{b} \leq \frac{\delta}{2}.$$

This says that for any $t \in [0, 1]$, $\sigma(t, u) \notin \mathcal{B}_{2\delta}(K_c)$. Lemma 3.3 tells us that

$$\frac{\mathcal{J}_r(\sigma(t, u))}{\partial t} = \left\langle \mathcal{J}'_r(\sigma(t, u)), \frac{\partial \sigma(t, u)}{\partial t} \right\rangle \leq 0,$$

so that $\mathcal{J}_r(\sigma(t, u))$ is not increased for $t \geq 0$, and thus $\sigma(t, u) \notin \mathcal{J}_r^{c-\varepsilon}$ for any $t \in [0, 1]$. Therefore,

$$\sigma(t, u) \in \mathcal{J}_r^{-1}([c - \varepsilon, c + \varepsilon]) \setminus \mathcal{B}_{2\delta}(K_c) \quad \text{for all } t \in [0, 1].$$

This shows that for any $t \in [0, 1]$ we have

$$\chi(\sigma(t, u)) = 1 \quad \text{and} \quad \|\sigma(t, u) - A(\sigma(t, u))\| \geq b.$$

Therefore, from Lemma 3.3, we have

$$\begin{aligned} \mathcal{J}_r\left(\eta\left(\frac{16\varepsilon}{b}, u\right)\right) &= \mathcal{J}_r(u) - \int_0^{\frac{16\varepsilon}{b}} \langle \mathcal{J}'_r(\eta(s, u)), \eta'(s, u) \rangle ds \\ &\leq \mathcal{J}_r(u) - \frac{1}{8} \int_0^{\frac{16\varepsilon}{b}} \|\eta(s, u) - A(\eta(s, u))\| ds \end{aligned}$$

$$\leq c + \varepsilon - \frac{16\varepsilon}{b} \frac{b}{8} = c - \varepsilon,$$

which is a contradiction suggesting that (b) holds.

If $K_c \setminus W = \emptyset$, we can see that $\mathcal{B}_{3\delta}(K_c) \subset W$. According to the proof of (b), we can derive (c). Finally, (e) is the result of Lemma 3.3 (see [5, 21, 22]). This proves the lemma. $\square$

Proposition 3.5. *Assume that (V_1) , (V_2) and (f_1) – (f_4) hold. Then problem (3.1) possesses infinitely many sign-changing solutions.*

Proof. Define

$$\Psi_k := \{ \psi (B_{R_m} \cap H_m \setminus P) : \psi \in G_m, m \geq k, P = -P \text{ is an open set and } \gamma(P) \leq m - k \},$$

where $k \geq 2$, $\gamma(P)$ denotes the Krasnoselskir's genus of P (see [32]) and

$$G_m := \{ \psi \in C(H_m \cap B_{R_m}, E_r) : \psi \text{ is an odd function } \psi(u) = u \text{ if } u \in \partial B_{R_m} \cap H_m \}.$$

Define

$$c_k := \inf_{\Psi \in \Psi_k} \sup_{u \in \Psi \cap E_r^w} \mathcal{J}_r(u).$$

We claim that $0 < a \leq c_k < +\infty$. In fact, $G_m \neq \emptyset$ since $\text{id} \in G_m$, $m \in \mathbb{N}$. Thus, $c_k < +\infty$. Now, we seek to prove $0 < a \leq c_k$. From [29, Proposition 9.18], Ψ_k satisfies:

- (1) $\Psi_k \neq \emptyset$ and $\Psi_{k+1} \subset \Psi_k$ for all $k \geq 2$,
- (2) Let $\phi \in C(E_r, E_r)$ be an odd function and $\phi(u) = u$, $u \in \partial B_{R_m} \cap H_m$, then $\phi(\Psi) \in \Psi_k$ if $\Psi \in \Psi_k$ for every $k \geq 2$,
- (3) if $\Psi \in \Psi_k$, $X = -X$ is an open set and $\gamma(X) \leq s < k$ with $k - s \geq 2$, then $\Psi \setminus X \in \Psi_{k-s}$.

Define the attracting domain of 0 in E_r as

$$\mathcal{O} := \{u \in E_r : \tau(t, u) \rightarrow 0 \text{ as } t \rightarrow \infty\},$$

where τ is defined in (3.5). Since 0 is a local minimum of $\mathcal{J}_r$, combining with the continuous dependence of ODE on initial data, we have seen that $\mathcal{O}$ is open. In fact, $Y_\varepsilon^+ \cap Y_\varepsilon^- \subset \mathcal{O}$ and $\partial \mathcal{O}$ is an invariant set. As in [4, Lemma 3.4], one obtains

$$\mathcal{J}_r(u) > 0, \quad u \in \overline{Y_\varepsilon^+} \cap \overline{Y_\varepsilon^-} \setminus \{0\}.$$

We choose $\Psi \in \Psi_k$ such that

$$\Psi = \psi (B_{R_m} \cap H_m \setminus P) \quad \text{with } \gamma(P) \leq m - k.$$

Define

$$\mathcal{D} := \{u \in B_{R_m} \cap H_m : \Psi(u) \in \mathcal{O}\}.$$

So that $\mathcal{D}$ is a bounded open symmetric set with $0 \in \mathcal{D}$ and $\overline{\mathcal{D}} \subset B_{R_m} \cap H_m$. The Borsuk-Ulam theorem gives us that $\gamma(\partial \mathcal{D}) = m$. In addition, $\psi(\partial \mathcal{D}) \subset \partial \mathcal{D}$ from the continuity of ψ . Consequently,

$$\psi(\partial \mathcal{D} \setminus P) \subset \Psi \cap \partial \mathcal{O}.$$

According to [32, Proposition 5.4], one can see that

$$\gamma(\Psi \cap \partial \mathcal{O}) \geq \gamma(\psi(\partial \mathcal{D} \setminus P)) \geq \gamma(\partial \mathcal{D} \setminus P) \geq \gamma(\partial \mathcal{D}) - \gamma(P) \geq k.$$

Since $Y_\varepsilon^+ \cap Y_\varepsilon^- \cap \partial \mathcal{O} = \emptyset$, we obtain $\gamma(W \cap \partial \mathcal{O}) \leq 1$, and then one can derive

$$\gamma(\Psi \cap E_r^w \cap \partial \mathcal{O}) \geq \gamma(\Psi \cap \partial \mathcal{O}) - \gamma(W \cap \partial \mathcal{O}) \geq k - 1 \geq 1,$$

so that

$$\Psi \cap E_r^w \cap \partial \mathcal{O} \neq \emptyset. \quad (3.7)$$

As a result, $\Psi \cap E_r^W \neq \emptyset$ for every $\Psi \in \Psi_k$ with $k \geq 2$. We can choose ρ given in Lemma 2.2 so small that $\partial B_\rho \subset \mathcal{O}$. Therefore, from Lemma 2.2 and the property of τ , we have

$$\sup_{u \in \Psi \cap E_r^W} \mathcal{J}_r(u) \geq \inf_{u \in \partial \mathcal{O}} \mathcal{J}_r(u) \geq \inf_{u \in \partial B_\rho} \mathcal{J}_r(u) \geq a > 0.$$

Thus, $c_k \geq a > 0$.

Next, we will show that $K_{c_k} \cap E_r^W \neq \emptyset$. By the contradiction, we may assume that $K_{c_k} \cap E_r^W = \emptyset$. From the definition of c_k , we can find $\Psi \in \Psi_k$ satisfying

$$\sup_{u \in \Psi \cap E_r^W} \mathcal{J}_r(u) \leq c_k + \varepsilon.$$

Then $\Psi \cap E_r^W \subset \mathcal{J}_r^{c_k+\varepsilon}$. It follows from Lemma 3.4 that there is a map $\sigma \in C([0, 1] \times E_r, E_r)$ and $\varepsilon > 0$ such that $\sigma(1, \cdot)$ is an odd function, $\sigma(1, u) = u$, $u \in \mathcal{J}_r^{c_k-2\varepsilon}$ and

$$\sigma(1, \mathcal{J}_r^{c_k+\varepsilon} \setminus W) \subset \mathcal{J}_r^{c_k-\varepsilon}. \quad (3.8)$$

Combining with the definition of W , it follows that $\sigma(1, W) \setminus W = \emptyset$ from (e) of Lemma 3.4. Then, by (3.8), one has see that

$$\begin{aligned} \sigma(1, \Psi) \cap E_r^W &\subset [\sigma(1, \Psi \setminus W) \cup \sigma(1, W)] \setminus W \subset \sigma(1, \Psi \setminus W) \setminus W \\ &\subset \sigma(1, \mathcal{J}_r^{c_k+\varepsilon}) \setminus \sigma(1, W) \subset \sigma(1, \mathcal{J}_r^{c_k+\varepsilon} \setminus W) \subset \mathcal{J}_r^{c_k-\varepsilon}, \end{aligned}$$

which is contrary to the definition of c_k , since $\sigma(1, \Psi) \in \Psi_k$. So $K_{c_k} \cap E_r^W \neq \emptyset$.

We also note that $c_{k+1} \geq c_k$ for any $k \geq 2$ because of the property (1) of Ψ_k . Finally, we're arguing to prove that $\mathcal{J}_r$ has infinitely many sign-changing critical points by proving that $c_k \rightarrow \infty$ as $k \rightarrow \infty$. By using reduction to absurdity, we assume that $c_k \rightarrow c < \infty$ as $k \rightarrow \infty$. As a result of Lemma 2.2, one can see that K_c is compact and nonempty. In addition, we claim that

$$K_c^W := K_c \setminus W \neq \emptyset.$$

Let $\{u^k\}$ be a sequence of sign-changing solutions of problem (3.1) such that $\mathcal{J}_r(u^k) \rightarrow c$ as $k \rightarrow \infty$. Since Lemma 2.2, up to a subsequence, there exists $u \in E_r^W$ such that $u^k \rightarrow u$ as $k \rightarrow \infty$. In fact, $\langle \mathcal{J}_r'(u^k), (u^k)^\pm \rangle = 0$ implies that

$$\|(u^k)^\pm\|^2 \leq \int_{\mathbb{R}^N} f(u^k)(u^k)^\pm dx \leq \frac{1}{2} \int_{\mathbb{R}^N} |(u^k)^\pm|^2 dx + C \int_{\mathbb{R}^N} |(u^k)^\pm|^p dx,$$

which yields that $\|(u^k)^\pm\| \geq \delta_0 > 0$, where δ_0 is a constant independent of k . Consequently, u is still a sign-changing solution and then $K_c^W \neq \emptyset$.

Assume that $\gamma(K_c^W) = n$. One can always find an open neighborhood $\mathcal{N}$ in E_r satisfying $K_c^W \in \mathcal{N}$ and $\gamma(\overline{\mathcal{N}}) = n$ on account of $0 \notin K_c^W$, the compactness of K_c^W and the continuous property of the genus. Lemma 3.4 then says that there exists $\varepsilon > 0$ and a map $\sigma \in C([0, 1] \times E_r, E_r)$ such that $\sigma(1, \cdot)$ is an odd function, $\sigma(1, u) = u$, $u \in \mathcal{J}_r^{c-2\varepsilon}$ and

$$\sigma(1, \mathcal{J}_r^{c+\varepsilon} \setminus (\mathcal{N} \cup W)) \subset \mathcal{J}_r^{c-\varepsilon}.$$

Because $c_k \rightarrow c$ as $k \rightarrow \infty$, one may choose k large enough to get

$$c + \frac{1}{4}\varepsilon > c_k \geq c - \frac{1}{2}\varepsilon. \quad (3.9)$$

In fact, $c_{k+n} \geq c_k$ for any $k \geq 2$. The definition of c^{k+n} brings us that there exists $\Psi \in \Psi_{k+n}$, i.e.,

$$\Psi = \psi(B_{R_m} \cap H_m \setminus P),$$

where $\psi \in G_m$, $m \geq k + n$, $\gamma(\bar{P}) \leq m - (k + n)$, so that

$$\mathcal{J}_r(u) \leq c_{k+n} + \frac{1}{4}\epsilon < c + \epsilon \quad \text{for any } u \in \Psi \cap E_r^w.$$

Therefore, $\Psi \cap E_r^w \subset \mathcal{J}_r^{c+\epsilon}$. Set $P_1 = P \cup \psi^{-1}(\mathcal{N})$. Undoubtedly, P_1 is symmetric and open, and then

$$\gamma(P_1) \leq \gamma(P) + \gamma(\psi^{-1}(\mathcal{N})) \leq m - (k + n) + n = m - k.$$

Thus

$$\tilde{\Psi} := \sigma(1, \psi(B_{R_m} \cap H_m \setminus P_1)) \in \Psi_k.$$

Notice that

$$\begin{aligned} \tilde{\Psi} \cap E_r^w &\subset \sigma(1, \psi(B_{R_m} \cap H_m \setminus P_1)) \setminus W \subset \sigma(1, \psi(B_{R_m} \cap H_m \setminus P) \setminus \mathcal{N}) \setminus W \\ &\subset [\sigma(1, \Psi \setminus \mathcal{N})] \setminus W \subset \sigma(1, \mathcal{J}_r^{c+\epsilon} \setminus (\mathcal{N} \cup W)) \subset \mathcal{J}_r^{c-\epsilon}, \end{aligned}$$

where the fact $\sigma(1, W) \setminus W = \emptyset$ also plays a role. Hence, $c_k \leq \sup_{u \in \tilde{\Psi} \cap E_r^w} \mathcal{J}_r(u) \leq c - \epsilon$. This contradicts (3.9) and proves to be complete. $\square$

4 Proof of Theorem 1.1

Set

$$E_o := \{u \in E : u \text{ is an odd function}\}.$$

Clearly, E_o is a nonempty complete subspace of E . Also because $E_o \hookrightarrow L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ is separable, E_o admits a countable orthogonal basis $\{e_i^o\}$. For any $m \in \mathbb{N}$, set

$$E_m^o := \text{span}\{e_1^o, e_2^o, \dots, e_m^o\}.$$

Similarly, let problem (1.2) constrain on E_o , namely,

$$\begin{cases} -\Delta u + V(x)u = f(u), & x \in \mathbb{R}^N, \\ u \in E_o. \end{cases} \quad (4.1)$$

Proposition 4.1. Assume that (V_1) , (V_2) and (f_1) – (f_4) hold. Then problem (4.1) has infinitely many solutions in E_o .

Proof. For $k \geq 2$, set

$$\Gamma_k^o := \{g(B_{R_m} \cap E_m^o \setminus P) : g \in G_m^o, m \geq k, P = -P \text{ is an open set and } \gamma(P) \leq m - k\},$$

and

$$G_m^o := \{g \in C(B_{R_m} \cap E_m^o, E_o) : g \text{ is an odd function and } g(u) = u, u \in \partial B_{R_m} \cap E_m^o\}.$$

Define

$$c_k^o := \inf_{G \in \Gamma_k^o} \sup_{u \in G} \mathcal{J}_o(u),$$

where $\mathcal{J}_o$ stands for $\mathcal{J}|_{E_o}$. Certainly, $G_m^o \neq \emptyset$, based on the fact that $\text{id} \in G_m^o$ for all $m \in \mathbb{N}$. Thus, $c_k^o < +\infty$. Notice that $0 < \alpha \leq c_k^o < +\infty$ from Lemma 3.1. Now, we seek to prove that $\mathcal{J}_o$ has one nontrivial critical point u_k and $\mathcal{J}_o(u_k) = c_k^o$. From the definition of c_k^o , one can find a sequence $\{u_{n,k}\} \subset E_o$ such that

$$\mathcal{J}_o(u_{n,k}) \rightarrow c_k^o \quad \text{and} \quad \mathcal{J}_o'(u_{n,k}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Lemma 2.1 gives us that $\mathcal{J}_o$ satisfies the (PS) condition. Therefore, $u_{n,k} \rightarrow u_k$ in E_o , u_k is one nontrivial critical point of $\mathcal{J}_o$ and $\mathcal{J}_o(u_k) = c_k^o$.

As in [29, Proposition 9.33], one can prove that $c_k^o \rightarrow \infty$ as $k \rightarrow \infty$. This brings us that $\mathcal{J}_o(u)$ admits infinitely many critical points and hence completes the proof. $\square$

Proof of Theorem 1.1. Recall $Z_2 = \{-1, 1\}$. Define the action of a topological group Z_2 on a normed space E as

$$-u(-x) = -1 \circ u(x), \quad u = 1 \circ u \quad \text{for any } u \in E.$$

Hence, the action is a continuous map such that, for any $g, h \in Z_2, u \in E, u \mapsto hu$ is linear and $(hg)u = h(gu)$. Thus, $\|hu\| = \|u\|$ for any $h \in Z_2$, which implies that the action is isometric. The space of invariant points is defined by

$$\text{Fix}(Z_2) := \{u \in E : hu = u \text{ for all } h \in Z_2\} = \{u \in E : u(x) = -u(-x)\}.$$

If $N = 2k, k \in \mathbb{Z}^+$, then $\mathcal{J}(-1 \circ u) = \mathcal{J}(u)$, so that $\mathcal{J}$ is invariant. In fact, $\text{Fix}(Z_2) = E_o$, thus, by principle of symmetric criticality [27, 38], we conclude that if u is a critical point of $\mathcal{J}$ restricted to E_o then u is also a critical point of $\mathcal{J}$. Due to the properties of E_o one can see that u is non-radial sign-changing for any $u \in E_o \setminus \{0\}$. So problem (1.2) admits infinitely many non-radial sign-changing solutions from proposition 4.1.

Let $\mathcal{H}$ be a subgroup of $O(N)$ ([20, 38]) and the action of E on E is defined by

$$hu(x) := u(h^{-1}x), \quad h \in \mathcal{H}.$$

Notice that the subspace of invariant functions is

$$E_r := \{u \in E : hu = u \text{ for all } h \in \mathcal{H}\}.$$

We also note that $\mathcal{J}$ is invariant since (V_1) and (f_1) . Then, also by principle of symmetric criticality, we conclude that if u is a critical point of $\mathcal{J}$ restricted to E_r then u is also a critical point of $\mathcal{J}$. So problem (1.2) has infinitely many radial sign-changing solutions from proposition 3.5. This completes the proof of Theorem 1.1. $\square$

Acknowledgments: The author states no conflict of interest, and would like to express their heartfelt thanks to the anonymous referees for his/her valuable suggestions and comments.

This work was supported by the special (special post) scientific research fund of natural science of Guizhou University (No. (2021)43) and National Natural Science Foundation of China (No.11971393).

References

- [1] J. E. Avron, I. W. Herbst and B. Simon, Separation of center of mass in homogeneous magnetic fields, *Ann. Physics*, **114** (1978), 431–451.
- [2] M. Badiale and E. Serra, *Semilinear elliptic equations for beginners*, Universitext, Springer, London, 2011, Existence results via the variational approach.
- [3] T. Bartsch and Z. Liu, On a superlinear elliptic p -Laplacian equation, *J. Differential Equations*, **198** (2004), 149–175.
- [4] T. Bartsch, Z. Liu and T. Weth, Sign changing solutions of superlinear Schrödinger equations, *Comm. Partial Differential Equations*, **29** (2004), 25–42.
- [5] T. Bartsch, Z. Liu and T. Weth, Nodal solutions of a p -Laplacian equation, *Proc. London Math. Soc. (3)*, **91** (2005), 129–152.
- [6] T. Bartsch, A. Pankov and Z.-Q. Wang, Nonlinear Schrödinger equations with steep potential well, *Commun. Contemp. Math.*, **3** (2001), 549–569.
- [7] T. Bartsch and Z.-Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, *Comm. Partial Differential Equations*, **20** (1995), 1725–1741.
- [8] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.*, **82** (1983), 313–345.
- [9] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. II. Existence of infinitely many solutions, *Arch. Rational Mech. Anal.*, **82** (1983), 347–375.
- [10] L. Bergé, Wave collapse in physics: principles and applications to light and plasma waves, *Phys. Rep.*, **303** (1998), 259–370.
- [11] I. Białynicki-Birula and J. Mycielski, Wave equations with logarithmic nonlinearities, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, **23** (1975), 461–466.
- [12] G. Cerami, G. Devillanova and S. Solimini, Infinitely many bound states for some nonlinear scalar field equations, *Calc. Var. Partial Differential Equations*, **23** (2005), 139–168.

- [13] W. Chen, J. Wei and S. Yan, Infinitely many solutions for the Schrödinger equations in $\mathbb{R}^N$ with critical growth, *J. Differential Equations*, **252** (2012), 2425–2447.
- [14] C. V. Coffman, Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, *Arch. Rational Mech. Anal.*, **46** (1972), 81–95.
- [15] V. Coti Zelati and P. H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$, *Comm. Pure Appl. Math.*, **45** (1992), 1217–1269.
- [16] D. Gogny and P.-L. Lions, Hartree-Fock theory in nuclear physics, *RAIRO Modél. Math. Anal. Numér.*, **20** (1986), 571–637.
- [17] M. Hong, Infinitely many sign-changing solutions for a Schrödinger equation in $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **354** (2009), 459–468.
- [18] T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag New York, Inc., New York, 1966.
- [19] G.-D. Li, Y.-Y. Li and C.-L. Tang, Existence and asymptotic behavior of ground state solutions for Schrödinger equations with Hardy potential and Berestycki-Lions type conditions, *J. Differential Equations*, **275** (2021), 77–115.
- [20] P.-L. Lions, Symétrie et compacité dans les espaces de Sobolev, *J. Funct. Anal.*, **49** (1982), 315–334.
- [21] J. Liu, X. Liu and Z.-Q. Wang, Multiple mixed states of nodal solutions for nonlinear Schrödinger systems, *Calc. Var. Partial Differential Equations*, **52** (2015), 565–586.
- [22] Z. Liu and J. Sun, Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations, *J. Differential Equations*, **172** (2001), 257–299.
- [23] Z. Liu, Z.-Q. Wang and J. Zhang, Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system, *Ann. Mat. Pura Appl. (4)*, **195** (2016), 775–794.
- [24] Z. Liu, Z. Ouyang and J. Zhang, Existence and multiplicity of sign-changing standing waves for a gauged nonlinear Schrödinger equation in $\mathbb{R}^2$, *Nonlinearity*, **32** (2019), 3082–3111.
- [25] J. Mederski, Nonradial solutions of nonlinear scalar field equations, *Nonlinearity*, **33** (2020), 6349–6380.
- [26] Z. Nehari, On a nonlinear differential equation arising in nuclear physics, *Proc. Roy. Irish Acad. Sect. A*, **62** (1963), 117–135 (1963).
- [27] R. S. Palais, The principle of symmetric criticality, *Comm. Math. Phys.*, **69** (1979), 19–30.
- [28] S. Peng, C. Wang and S. Yan, Construction of solutions via local Pohozaev identities, *J. Funct. Anal.*, **274** (2018), 2606–2633.
- [29] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, vol. 65 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.
- [30] G. H. Ryder, Boundary value problems for a class of nonlinear differential equations, *Pacific J. Math.*, **22** (1967), 477–503.
- [31] Y. Sato and M. Shibata, Infinitely many solutions for a nonlinear Schrödinger equation with general nonlinearity, *Calc. Var. Partial Differential Equations*, **57** (2018), Paper No. 137, 31.
- [32] M. Struwe, *Variational methods*, vol. 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 3rd edition, Springer-Verlag, Berlin, 2000, Applications to nonlinear partial differential equations and Hamiltonian systems.
- [33] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation*, vol. 139 of Applied Mathematical Sciences, Springer-Verlag, New York, 1999, Self-focusing and wave collapse.
- [34] J. Sun, L. Li, M. Cencelj and B. Gabrovšek, Infinitely many sign-changing solutions for Kirchhoff type problems in $\mathbb{R}^3$, *Nonlinear Anal.*, **186** (2019), 33–54.
- [35] A. Szulkin and T. Weth, Ground state solutions for some indefinite variational problems, *J. Funct. Anal.*, **257** (2009), 3802–3822.
- [36] X. Tang, Infinitely many solutions for semilinear Schrödinger equations with sign-changing potential and nonlinearity, *J. Math. Anal. Appl.*, **401** (2013), 407–415.
- [37] X. Tang, S. Chen, X. Lin and J. Yu, Ground state solutions of Nehari-Pankov type for Schrödinger equations with local super-quadratic conditions, *J. Differential Equations*, **268** (2020), 4663–4690.
- [38] M. Willem, *Minimax theorems*, vol. 24 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston, Inc., Boston, MA, 1996.