Research Article

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Continuous flows driving branching processes and their nonlinear evolution equations

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Abstract: We consider on $M(\mathbb{R}^d)$ (the set of all finite measures on $\mathbb{R}^d$) the evolution equation associated with the nonlinear operator $F \mapsto \Delta F' + \sum_{k \geq 1} b_k F^k$, where F' is the variational derivative of F and we show that it has a solution represented by means of the distribution of the d -dimensional Brownian motion and the non-local branching process on the finite configurations of $M(\mathbb{R}^d)$, induced by the sequence $(b_k)_{k \geq 1}$ of positive numbers such that $\sum_{k \geq 1} b_k \leq 1$. It turns out that the representation also holds with the same branching process for the solution to the equation obtained replacing the Laplace operator by the generator of a Markov process on $\mathbb{R}^d$ instead of the d -dimensional Brownian motion; more general, we can take the generator of a right Markov process on a Lusin topological space. We first investigate continuous flows driving branching processes. We show that if the branching mechanism of a superprocess is independent of the spatial variable, then the superprocess is obtained by introducing the branching in the time evolution of the right continuous flow on measures, canonically induced by a right continuous flow as spatial motion. A corresponding result holds for non-local branching processes on the set of all finite configurations of the state space of the spatial motion.

Keywords: Nonlinear evolution equation, superprocess, non-local branching process, right continuous flow, weak generator, log-potential

MSC: 35J60, 60J35, 60J80, 60J68, 60J45, 47D07

1 Introduction

Let $(b_k)_{k \geq 1}$ be a sequence of positive numbers such that $\sum_{k \geq 1} b_k \leq 1$ and consider the following parabolic equation on measures

$$\frac{d}{dt} F_t = \Delta F'_t - c F_t + c \sum_{k \geq 1} b_k F_t^k \text{ on } M(\mathbb{R}^d), t \geq 0, \quad (1.1)$$

with the initial condition $F_0 = F$, where $c > 0$ and for a function $F : M(\mathbb{R}^d) \rightarrow [0, 1]$ we denoted by F' its variational derivative.

The first goal of this paper is to show that equation (1.1) has a mild solution having a probabilistic representation by means of the distribution of the d -dimensional Brownian motion and the non-local pure

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branching process on the finite configurations of $M(\mathbb{R}^d)$, induced by the sequence $(b_k)_{k \geq 1}$. Recall that the measure-valued branching processes intend to be stochastic models for the population dynamics. The time evolution of a system of particles is characterised by a *branching mechanism* and a *spatial motion* which describes the movement of the particles between the branching moments. The spatial motion is a right Markov space with state space a Lusin topological space E , while the state space of the branching process is a set of measures on E , namely, either $M(E)$ ($:=$ the set of all finite measures on E) for the superprocesses (see [1, 2, 15, 16, 18, 19, 22, 29, 30]), or $\widehat{E}$ ($:=$ the set of all finite configurations of E) for the non-local branching processes; see e.g. [8, 10, 11, 26, 31].

If in (1.1) we replace the Laplace operator with a second order integro-differential operator generating a Markov process with state space $\mathbb{R}^d$, then the obtained equation has a solution with a similar probabilistic representation, and notice that it is based on the same pure branching process. In this way we complete the answer given in [8] to the challenging suggestion of T.E. Harris from [23], page 50, to investigate branching processes for which "each object is a very complicated entity; e.g., an object may itself be a population". In fact, in Section 5 we consider equation (1.1) on $M(E)$, with the generator of a right Markov process on the Lusin topological space E instead of the Laplacian on $\mathbb{R}^d$.

Our second goal is to emphasize a class of branching processes which are driven by a right continuous flow Φ on E , in the sense that such a measure-valued process $\widehat{X}$ admits a representation by means of a second branching process $\widehat{X}^0$ and of the flow on measures induced by Φ ,

$$\widehat{X}_t = \Phi_t(\widehat{X}_t^0) \text{ for all } t \geq 0. \quad (1.2)$$

It turns out that the branching process $\widehat{X}^0$ has the same branching mechanism as $\widehat{X}$, however $\widehat{X}^0$ has no spatial motion and therefore we call it *pure branching process*.

The representation (1.2) holds in the case of a superprocess for which the branching mechanism Ψ is independent of the spatial variable, $\Psi : [0, \infty) \rightarrow \mathbb{R}$ is of the form

$$\Psi(\lambda) = -b\lambda - c\lambda^2 + \int_0^\infty (1 - e^{-\lambda s} - \lambda s)N(ds),$$

where $b, c \in \mathbb{R}$, $c \geq 0$, and N is a measure on $(0, \infty)$ such that $N(u \wedge u^2) < \infty$, and the spatial motion is a right continuous flow on E , regarded as a deterministic Markov process. Recall that a superprocess $\widehat{X}$ provides a stochastic solution for the nonlinear evolution equation written formally as

$$\frac{d}{dt}v_t = Dv_t + \Psi(v_t), \quad t \geq 0, \quad (1.3)$$

with the initial condition $v_0 = f$, where D is the generator of the spatial motion and Ψ is the branching mechanism. The nonlinear evolution equation associated with the pure branching process $\widehat{X}^0$ is the particular case of (1.3) with no spatial motion term on the right hand side, that is, with $D \equiv 0$. Actually, we consider the integral version of equation (1.3), or equivalently, we work with the *mild solutions* of it.

A basic step of our approach is to solve equation (1.3) with D the generator of a right continuous flow Φ on E , to find a solution v_t , $t \geq 0$, having the following two probabilistic representations:

$$v_t(x) = -\ln \mathbb{E}_0^{\delta_x} e_{f \circ \Phi_t}(\widehat{X}_t^0) = -\ln \mathbb{E}_0^{\delta_{\Phi_t(x)}} e_f(\widehat{X}_t^0), \quad x \in E, \quad t \geq 0, \quad (1.4)$$

where e_f is the *exponential mapping* on $M(E)$, $e_f(\mu) := e^{-\int f d\mu}$ for all $\mu \in M(E)$; with the terminology of [18], page 133, v_t is written as the "log-potential of $\widehat{X}_t^0$ and Φ_t ". We prove analogous results for non-local branching processes, provided that the flow and the distributions of the branching are compatible.

If $\mathcal{L}$ and $\mathcal{L}^0$ are the extended weak generators of the superprocesses $\widehat{X}$ and respectively $\widehat{X}^0$ on $M(E)$, then we have

$$\mathcal{L} = \widehat{D} + \mathcal{L}^0,$$

where $\widehat{D}$ is the generator of the continuous flow on $M(E)$ induced by Φ . In this way, the representation (1.2) may be interpreted as a consequence of regarding $\mathcal{L}$ as a modification of $\mathcal{L}^0$ with the first order operator $\widehat{D}$,

which is a substitute for a "drift type" operator acting in the considered infinite dimensional frame. This is an exemplification of a general strategy developed in [3]; for more details see assertion (iv) of Remark 3.5 below.

A key point in obtaining the claimed probabilistic solution to (1.1) is the representation (1.2) applied on $M(\mathbb{R}^d)$ as state space for the spatial motion.

We present now the structure of the paper.

The first main result (Theorem 3.2) is stated in Section 3. It is preceded by the description of the weak generator of a superprocess (Proposition 3.1), which completes results from [2, 19, 22, 30]. Results related to equation (1.3) and (1.4) are gathered in assertion (iii) of Theorem 3.2, Corollary 3.4, and Remark 3.5.

The non-local branching processes on configuration spaces are investigated in Section 4. First, we highlight relevant absorbing sets of finite configurations with respect to a pure branching process on $\hat{E}$ (in Proposition 4.1). Theorem 4.2 is the second main statement of the work. Results similar to (1.4) and Corollary 3.4, but valid on $\hat{E}$, are collected in Remark 4.3, which also indicates absorbing sets for the pure branching process, when the branching mechanism is the classical one.

The proofs of the two theorems use a unifying argument (Lemma 3.3), which is applicable on both spaces of measures $(M(E) \text{ and } \hat{E})$ and we put its proof in the Appendix. Recall that in [9] it was developed a method of proving the path regularity for both superprocesses and non-local branching processes. The Appendix also presents briefly complements on the right continuous flows and the extended weak generator of a Markov process.

The application in solving equation (1.1) is given in Section 5, Corollary 5.2.

In Section 2 we present basic notations and preliminary facts on the right Markov processes, branching processes, and potential theoretical tools we shall use further: transition semigroups and resolvents of kernels, excessive functions, the fine topology, absorbing sets, and right continuous flows on measures.

2 Preliminaries

For a σ -algebra $\mathcal{G}$ we denote by $[\mathcal{G}]$ (resp. $p\mathcal{G}$) the vector space of all real-valued (resp. the set of all positive, numerical) $\mathcal{G}$ -measurable functions on E . Also, for a set of real-valued functions $\mathcal{C}$ we denote by $\sigma(\mathcal{C})$ the σ -algebra generated by $\mathcal{C}$, by $[\mathcal{C}]$ the vector space spanned by $\mathcal{C}$, and by $p\mathcal{C}$ (resp. $b\mathcal{C}$) the set of all positive (resp. bounded) functions from $\mathcal{C}$.

Excessive functions. Let E be a Lusin topological space (i.e., E is homeomorphic to a Borel subset of a compact metric space) with Borel σ -algebra $\mathcal{B}(E)$. Let further $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B}(E))$. We denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{B}(E)$ -measurable $\mathcal{U}$ -excessive functions, i.e., $u \in \mathcal{E}(\mathcal{U})$ if and only if u is a nonnegative, numerical, $\mathcal{B}(E)$ -measurable function, $\alpha U_\alpha u \leq u$ for all $\alpha > 0$, and $\lim_{\alpha \rightarrow \infty} \alpha U_\alpha u(x) = u(x)$, $x \in E$.

If $\beta > 0$, we denote by $\mathcal{U}_\beta$ the sub-Markovian resolvent of kernels $\mathcal{U}_\beta = (U_{\beta+\alpha})_{\alpha>0}$. A $\mathcal{U}_\beta$ -excessive function is also called β -excessive.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x, \zeta)$ be the right Markov process on E having $\mathbb{T} = (T_t)_{t \geq 0}$ as transition function, $T_t f(x) = \mathbb{E}^x(f(X_t), t < \zeta)$, $t \geq 0$, $f \in p\mathcal{B}(E)$ ($:=$ the set of all positive $\mathcal{B}(E)$ -measurable functions on E). The resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}(E))$ associated with $(T_t)_{t \geq 0}$ is defined as $U_\alpha f := \int_0^\infty e^{-\alpha t} T_t f dt$, $f \in p\mathcal{B}(E)$.

Fine topology. The *fine topology* is the coarsest topology on E making continuous all β -excessive functions for some (and equivalently for all) $\beta > 0$. Recall that a function f from $p\mathcal{B}(E)$ is finely continuous if and only if $t \mapsto f(X_t)$ is a.s. right continuous on $[0, \zeta)$. Consequently, due to the right continuity of the paths of the process X , each continuous function on E is finely continuous; see [4, 32] for details on the potential theory for Markov processes with a general state space.

Absorbing set. A set $A \in \mathcal{B}(E)$ is called *absorbing* (with respect to the Markov process X) if $R_\beta^{E \setminus A} 1 = 0$ on A , where $R_\beta^A 1 := \inf\{u \in \mathcal{E}(\mathcal{U}_\beta) : u \geq 1_A\}$ for some $\beta > 0$. If A is absorbing then it is finely open, $U_\beta 1_{E \setminus A} = 0$ on A , and the restriction of X to A is again a right process; see e.g. [32] or [13]. Basic examples of absorbing sets

are $[v = 0]$ and $[v < \infty]$ for some $v \in \mathcal{E}(\mathcal{U}_\beta)$. Examples of absorbing sets for branching processes will be given in Proposition 4.1 and Remark 4.3 (iii) below.

Trivial Markov process. Let $X^0 = (X_t^0)_{t \geq 0}$ be the Markov process on E for which each point is a trap, i.e., $\mathbb{P}^x(X_t^0 = x) = 1$ for all $t \geq 0$ and $x \in E$, or equivalently, each kernel from its transition function $(T_t^0)_{t \geq 0}$ is the identity operator, $T_t^0 f = f$ for every $t \geq 0$ and $f \in \mathcal{P}\mathcal{B}(E)$. Clearly, any Borel subset of E is absorbing with respect to the trivial Markov process on E .

Branching processes. We endow $M(E)$ with the weak topology and let $\mathcal{M}(E)$ denote the Borel σ -algebra on $M(E)$.

A second measure state space we will consider is the set $\widehat{E} \subseteq M(E)$ of all *finite configurations* of E , $\widehat{E} := \left\{ \sum_{i \leq i_0} \delta_{x_i} : i_0 \in \mathbb{N}, i_0 \geq 1, x_i \in E \text{ for all } 1 \leq i \leq i_0 \right\} \cup \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero measure. The set $\widehat{E}$ is endowed with the weak topology on the finite measures on E and the corresponding Borel σ -algebra $\mathcal{B}(\widehat{E})$; see e.g. [14].

Further $(M, \mathcal{M})$ will denote either $(M(E), \mathcal{M}(E))$ or $(\widehat{E}, \mathcal{B}(\widehat{E}))$. Recall that a right Markov process with state space M is called *branching process* provided that for any two independent copies X and X' of the given process on M , starting respectively from two measures μ and μ' , $X + X'$ and the process starting from $\mu + \mu'$ are equal in distribution. Recall also that if p_1, p_2 are two finite measures on M , then their convolution $p_1 * p_2$ is the finite measure on M defined for every $h \in \mathcal{P}\mathcal{M}$ by

$$\int_M p_1 * p_2(d\nu) h(\nu) := \int_M p_1(d\nu_1) \int_M p_2(d\nu_2) h(\nu_1 + \nu_2).$$

A bounded kernel Q on $(M, \mathcal{M})$ is called *branching kernel* provided that

$$Q_{\mu+\nu} = Q_\mu * Q_\nu \text{ for all } \mu, \nu \in M,$$

where Q_μ denotes the measure on M such that $\int_M h dQ_\mu = Qh(\mu)$ for all $h \in \mathcal{P}\mathcal{M}$. It turns out that: *a right Markov process with state space M is a branching process if and only if its transition function is formed by branching kernels*. For a function $f \in \mathcal{B}\mathcal{P}\mathcal{B}(E)$ we shall consider the mappings $l_f : M \rightarrow \mathbb{R}_+$ and $e_f : M \rightarrow [0, 1]$ defined as

$$l_f(\mu) := \langle \mu, f \rangle := \int_E f d\mu, \mu \in M, \quad e_f := e^{-l_f}.$$

Notice that $\mathcal{M}$ is generated by $\{l_f : f \in \mathcal{B}\mathcal{P}\mathcal{B}(E)\}$ and also by $\{e_f : f \in \mathcal{B}\mathcal{P}\mathcal{B}(E)\}$.

Right continuous flows on measures. We present in Appendix (A3) the basic facts on the right continuous flows.

If $t \geq 0$ define the kernel Q_t^0 on $M(E)$ as

$$Q_t^0 F(\mu) := F(\mu \circ T_t), \quad F \in \mathcal{P}\mathcal{M}(E), \quad \mu \in M(E).$$

We may consider the right continuous flow on $M(E)$ induced by the right Markov process X . More precisely, let $\widehat{\Phi} = (\widehat{\Phi}_t)_{t \geq 0}$ be the family of mappings on $M(E)$ defined as $\widehat{\Phi}_t(\mu) := X_t(\mathbb{P}^\mu)$, $\mu \in M(E)$; recall the notation $X_t(\mathbb{P}^\mu)(A) := \int_E \mathbb{P}^x(X_t^{-1}(A)) \mu(dx)$ for all $A \in \mathcal{B}(E)$. Then $\widehat{\Phi}$ is a right continuous flow on $M(E)$ and its transition function is $\mathbb{Q}^0 = (Q_t^0)_{t \geq 0}$. It is easy to check that Q_t^0 , $t \geq 0$, is a branching kernel on $M(E)$ and consequently, the (deterministic) Markov process induced by $\widehat{\Phi}$ is actually a branching process on $M(E)$, it is the $(X, 0)$ -superprocess.

Let $\Phi = (\Phi_t)_{t \geq 0}$ be a right continuous flow on E . Recall that Φ may be regarded as a deterministic right Markov process on E with infinite lifetime, $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$: $\Omega = E$, $\mathcal{F} = \mathcal{F}_t = \mathcal{B}(E)$, $X_t(x) := \Phi_t(x) =: \theta_t(x)$ for all $x \in \Omega$, and $\mathbb{P}^x = \delta_x$. In this particular case, the right continuous flow on $M(E)$ induced by X , as described before, is also denoted by $\Phi = (\Phi_t)_{t \geq 0}$: if $\mu \in M(E)$ then $\Phi_t(\mu)$ is the transport of μ by means of Φ_t , i.e.,

$$\Phi_t(\mu)(A) := \mu(\Phi_t^{-1}(A)) \text{ for all } A \in \mathcal{B}(E) \text{ and } t \geq 0.$$

If $\mu \in \widehat{E}$, $\mu = \sum_{i=1}^k \delta_{x_i}$, then $\Phi_t(\mu)$ also belongs to $\widehat{E}$, $\Phi_t(\sum_{i=1}^k \delta_{x_i}) = \sum_{i=1}^k \delta_{\Phi_t(x_i)}$, $t \geq 0$. Consequently, the right continuous flow Φ on E also induces a right continuous flow on $\widehat{E}$, we shall denote it by Φ too; here we put $\Phi_t(\mathbf{0}) = \mathbf{0}$.

Let $A \in \mathcal{B}(E)$ be an absorbing set with respect to the right continuous flow Φ on E , i.e., $\Phi_t(x) \in A$ for all $x \in A$. Then $\widehat{A}$ is an absorbing set with respect to the right continuous flow on $\widehat{E}$ induced by Φ .

3 Right continuous flows driving superprocesses

In the sequel the fixed right Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$ with state space E and infinite lifetime is called *spatial motion*, and let $\mathcal{U}$ be its resolvent. We also fix a *branching mechanism*, that is, a function $\Psi : E \times [0, \infty) \rightarrow \mathbb{R}$ of the form

$$\Psi(x, \lambda) = -b(x)\lambda - c(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda s} - \lambda s)N(x, ds),$$

where $c \geq 0$ and b are bounded $\mathcal{B}(E)$ -measurable functions, and $N : \mathcal{p}\mathcal{B}((0, \infty)) \rightarrow \mathcal{p}\mathcal{B}(E)$ is a kernel such that $N(u \wedge u^2) \in \text{bp}\mathcal{B}(E)$. Examples of branching mechanisms are $\Psi(\lambda) = -\lambda^\alpha$ for $1 < \alpha \leq 2$ and $\Psi(\lambda) = \lambda^\alpha$ for $0 < \alpha < 1$.

We present now briefly the construction of the measure-valued branching Markov process associated with X and Ψ , the (X, Ψ) -superprocess; cf. [2, 22, 30].

Equation (1.3) has a unique mild solution, more precisely, for each $f \in \text{bp}\mathcal{B}(E)$ the equation

$$v_t(x) = T_t f(x) + \int_0^t T_s(x, \Psi(\cdot, v_{t-s})) ds, \quad t \geq 0, \quad x \in E, \quad (3.1)$$

has a unique solution $(t, x) \mapsto V_t f(x)$ jointly measurable in (t, x) such that $\sup_{0 \leq s \leq t} \|V_s f\|_\infty < \infty$ for all $t > 0$ and $[0, \infty) \ni t \mapsto V_t f(x)$ is right continuous for all $x \in E$, provided that $t \mapsto T_t f(x)$ has this property.

The mappings $f \mapsto V_t f$ form a nonlinear semigroup of operators on $\text{bp}\mathcal{B}(E)$.

For each $t \geq 0$ there exists a unique Markovian kernel $\widehat{T}_t$ on $(M(E), \mathcal{M}(E))$ such that

$$\widehat{T}_t(e_f) = e_{V_t f}, \quad f \in \text{bp}\mathcal{B}(E). \quad (3.2)$$

Since the family $(V_t)_{t \geq 0}$ is a (nonlinear) semigroup on $\text{bp}\mathcal{B}(E)$, it follows that $\widehat{\mathbb{T}} = (\widehat{T}_t)_{t \geq 0}$ is a transition function $(M(E), \mathcal{M}(E))$. By Theorem 4.9 from [2], under a Feller-type regularity condition, there exists a Borel right process $\widehat{X}$, with state space $M(E)$, having the transition function $\widehat{\mathbb{T}}$, called (X, Ψ) -superprocess.

Let $\widehat{\mathcal{U}} = (\widehat{U}_\alpha)_{\alpha > 0}$ be the Markovian resolvent of kernels on $(M(E), \mathcal{M}(E))$ induced by $\widehat{\mathbb{T}}$.

Let

$$\beta := \|b^-\|_\infty \text{ and } b' := b + \beta' \text{ with } \beta' \geq \beta.$$

Then $b' \geq 0$ and consider the transition function $\mathbb{T}_{b'} = (T_t^{b'})_{t \geq 0}$ of the process obtained by killing X with the multiplicative functional induced by b' ,

$$T_t^{b'} f(x) = \mathbb{E}^x(e^{-\int_0^t b'(X_s) ds} f(X_t)), \quad f \in \mathcal{p}\mathcal{B}(E), \quad x \in E.$$

If $u \in \text{bp}\mathcal{B}(E)$ then by Corollary 4.3 in [2] we have

$$u \in \mathcal{E}(\mathbb{T}_{b'}) \iff l_u \in \mathcal{E}(\widehat{\mathbb{T}}_{b'}) \iff 1 - e_{au} \in \mathcal{E}(\widehat{\mathbb{T}}_{b'}) \text{ for all } \alpha > 0. \quad (3.3)$$

We also have

$$\widehat{T}_t(l_u) = l_{T_t^{b'} u}. \quad (3.4)$$

The extended weak generator of a superprocess. Let $(L, \mathcal{D}(L))$ be the extended weak generator of the spatial motion X on E and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ the extended weak generator of the (X, Ψ) -superprocess on $M(E)$; see Appendix (A4) for the basic notions about the extended weak generator of a Markov process.

Let $(\widehat{D}, \mathcal{D}(\widehat{D}))$ be the extended weak generator of the right continuous flow $\widehat{\Phi}$ on $M(E)$ induced by the right Markov process X , $\widehat{\Phi}_t(\mu) := X_t(\mathbb{P}^\mu)$, $\mu \in M(E)$, that is, the extended weak generator of the transition function $\mathbb{Q}^0 = (Q_t^0)_{t \geq 0}$, $Q_t^0 F(\mu) := F(\mu \circ T_t)$ for all $F \in \text{bp}\mathcal{M}(E)$, $\mu \in M(E)$, and $t \geq 0$.

Proposition 3.1. (i) If $u \in \text{b}\mathcal{D}_o^c(L)$ is such that Lu is a bounded function then l_u belongs to the domain $\mathcal{D}(\mathcal{L})$ of the extended weak generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ of $\widehat{X}$ and

$$\mathcal{L}(l_u) = l_{Lu-bu}.$$

(ii) Let $n \geq 1$, $u_1, \dots, u_n \in \text{b}\mathcal{D}(L)$ such that all Lu_i are bounded functions, $\psi \in C_b^1(\mathbb{R}^n)$, and consider the function F on $M(E)$, defined as $F(\mu) := \psi(\langle \mu, u_1 \rangle, \dots, \langle \mu, u_n \rangle)$, $\mu \in M(E)$. Then F belongs to $\mathcal{D}(\widehat{D})$, $F'(\mu, \cdot) \in \mathcal{D}(L)$ and we have

$$\widehat{D}F(\mu) := \int_E LF'(\mu, x)\mu(dx) \text{ for all } \mu \in M(E), \quad (3.5)$$

where recall that the variational derivative of a function $F : M(E) \rightarrow \mathbb{R}$ is $F'(\mu, x) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (F(\mu + \varepsilon \delta_x) - F(\mu))$, $\mu \in M(E)$, $x \in E$.

We present the proof of Proposition 3.1 in Appendix (A1). Notice that assertion (ii) is a particular case of a version of a result from [22]; see also Remark 3.5 (i) below.

Further, we consider branching processes with no spatial motion.

Pure branching superprocesses. The (X^0, Ψ) -superprocess is named *pure branching*. It is a right Markov process $\widehat{X}^0$ with state space $M(E)$ and its transition function $(T_t^0)_{t \geq 0}$ is obtained as follows. The particular case of the equation (3.1),

$$v_t(x) = f(x) + \int_0^t \Psi(x, v_s(x))ds, \quad t \geq 0, x \in E, \quad (3.6)$$

has a unique solution $(t, x) \mapsto V_t^0 f(x)$ jointly measurable in (t, x) . For each $t \geq 0$ there exists a unique Markovian kernel $\widehat{T}_t^0$ on $(M(E), \mathcal{M}(E))$ such that $\widehat{T}_t^0(e_f) = e_{V_t^0 f}$, $f \in \text{bp}\mathcal{B}(E)$. So, the time evolution of the process $\widehat{X}^0$ is governed only by the branching procedure.

For a nonlinear evolution equation which is equivalent to (3.6) see Remark 3.5 below.

We can state now the first main result of this paper, involving the pure branching process and the right continuous flow driving it. In particular, this is an illustration of Section 4 from [3].

Theorem 3.2. Consider a branching mechanism Ψ which is independent of the spatial variable, that is, with b and c constant functions and $N(x, ds) = N(ds)$ for all $x \in E$. Assume that the spatial motion $X = (X_t)_{t \geq 0}$ is the deterministic process associated with a right continuous flow $\Phi = (\Phi_t)_{t \geq 0}$ on E and suppose that the mapping $[0, \infty) \times E \ni (t, x) \mapsto \Phi_t(x)$ is continuous.

Let $\widehat{X} = (\widehat{X}_t)_{t \geq 0}$ be the (X, Ψ) -superprocess induced by the spatial motion X and the branching mechanism Ψ and let $\widehat{X}^0 = (\widehat{X}_t^0)_{t \geq 0}$ be the pure branching (X^0, Ψ) -superprocess. Then the following assertions hold.

(i) The (X, Ψ) -superprocess $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}^\mu)$ has the representation (1.2) as the pure branching process $\widehat{X}^0 = (\widehat{X}_t^0, \widehat{\mathbb{P}}^{\mathbb{P}^\mu})$ driven by the continuous flow Φ on $M(E)$, where the equality is in the distribution sense and we also have

$$\widehat{\mathbb{P}}^\mu(\widehat{X}_t \in \Gamma) = \widehat{\mathbb{P}}^{\mathbb{P}^\mu}(\widehat{X}_t^0 \in \Gamma) \text{ for all } \mu \in M(E) \text{ and } \Gamma \in \mathcal{M}(E). \quad (3.7)$$

(ii) Let $(D, \mathcal{D}(D))$ (resp. $(\widehat{D}, \mathcal{D}(\widehat{D}))$, $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$, and $(\mathcal{L}^0, \mathcal{D}(\mathcal{L}^0))$) be the extended weak generator of the spatial motion X (resp. of the flow Φ on $M(E)$, $\widehat{X}$, and $\widehat{X}^0$). Let further $\mathcal{D}_o := U_\alpha V_\beta(\text{b}C(M(E)))$, $\alpha, \beta > 0$, where $(U_\alpha)_{\alpha > 0}$ (resp. $(V_\alpha)_{\alpha > 0}$) is the resolvent of $\widehat{X}^0$ (resp. the resolvent of the flow Φ on $M(E)$). Then $\mathcal{D}_o \subset \mathcal{D}_o^c(\mathcal{L}) \cap \mathcal{D}_o(\widehat{D}) \cap \mathcal{D}(\mathcal{L}^0)$ and

$$\mathcal{L} = \widehat{D} + \mathcal{L}^0 \text{ on } \mathcal{D}_o.$$

(iii) Equation (1.3) has a mild solution v_t , $t \geq 0$, which is given by the log-potential type formula (1.4). More precisely, v_t is the solution of the nonlinear integral evolution equation (3.1).

Proof. We use the construction of $V_t^0 f$ from [2], the proof of Proposition 3.2. Observe that, using Proposition 3.1 from [2], equation (3.6) is equivalent to

$$v_t(x) = e^{-at}f(x) + \int_0^t e^{-a(t-s)}\Psi^0(v_s(x))ds, \quad t \geq 0, x \in E \quad (3.8)$$

where $\Psi^0(\lambda) := \Psi(\lambda) + a\lambda$, and let $V_t^0 f$, $t \geq 0$, be its unique solution for $f \in \text{bp}\mathcal{B}(E)$.

We need the following lemma, we present its proof in Appendix (A2).

Lemma 3.3. *Let $J : E \rightarrow E$ be a $\mathcal{B}(E)/\mathcal{B}(E)$ -measurable map and denote by $\mathcal{K}(J)$ the set of all operators $(W, \mathcal{D}(W))$ on $\text{bp}\mathcal{B}(E)$, $W : \mathcal{D}(W) \rightarrow \text{bp}\mathcal{B}(E)$, such that for all $f \in \text{bp}\mathcal{D}(W)$ we have $f \circ J \in \mathcal{D}(W)$ and $W(f \circ J) = Wf \circ J$. Convention: If the domain $\mathcal{D}(W)$ of W is not indicated, then this means that $\mathcal{D}(W) = \text{bp}\mathcal{B}(E)$.*

Then the following assertions hold.

(i) *The set $\mathcal{K}(J)$ has the following properties.*

(p1) *If $(W, \mathcal{D}(W))$ is an operator on $\text{bp}\mathcal{B}(E)$ such that there exists a sequence $(W_n)_n$ in $\mathcal{K}(J)$ with $\mathcal{D}(W_n) = \mathcal{D}(W)$ for all n , which is converging pointwise to W (i.e. $\lim_n W_n f(x) = Wf(x)$ for all $f \in \mathcal{D}(W)$ and $x \in E$), then W belongs to $\mathcal{K}(J)$;*

(p2) *If $(W, \mathcal{D}(W))$ and $(V, \mathcal{D}(V))$ are two operators on $\text{bp}\mathcal{B}(E)$ such that $W(\mathcal{D}(W)) \subset \mathcal{D}(V)$ and $V, W \in \mathcal{K}(J)$ then $V \circ W \in \mathcal{K}(J)$;*

(p3) *Let $(\mathbf{K}, \mathcal{D}(\mathbf{K}))$ be an operator on $\text{bp}\mathcal{B}(E)$ such that $\mathbf{K} \in \mathcal{K}(J)$. If $Wf = \int_0^t \mathbf{K}(W_s f) \nu(ds)$, where ν is a finite measure on $[0, t]$ and $W_s \in \mathcal{K}(J)$, $\mathcal{D}(W_s) = \mathcal{D}(\mathbf{K})$, and $W_s(\mathcal{D}(W_s)) \subset \mathcal{D}(\mathbf{K})$ for all s , then W also belongs to $\mathcal{K}(J)$.*

(ii) *Assume that $(\mathbf{K}, \mathcal{D}(\mathbf{K}))$ is Lipschitz with respect to the supremum norm and $\mathbf{K}(0) = 0$. Let $a > 0$, $t_0 > 0$ and consider the integral equation*

$$w_t = e^{-at}f + \int_0^t e^{-a(t-s)}\mathbf{K}(w_s)ds, \quad 0 \leq t \leq t_0, \quad (3.9)$$

where $f \in \mathcal{D}(\mathbf{K})$.

(ii.1) *The equation (3.9) has a unique solution $[0, t_0] \ni t \mapsto W_t f \in \mathcal{D}(\mathbf{K})$ such that the function $(t, x) \mapsto W_t f(x)$ is jointly measurable, provided that one of the following two conditions is verified:*

(I) $\mathcal{D}(\mathbf{K}) = \text{bp}\mathcal{B}(E)$;

(II) $\mathcal{D}(\mathbf{K}) = \{f \in \text{bp}\mathcal{B}(E) : f \leq 1\}$ and $\mathbf{K}f \leq a$ for all $f \in \mathcal{D}(\mathbf{K})$.

(ii.2) *Assume that $\mathbf{K}$ preserves the pointwise convergence, that is, if $(f_n)_n \subset \mathcal{D}(\mathbf{K})$ is pointwise converging to $f \in \mathcal{D}(\mathbf{K})$, then $\mathbf{K}(f_n) \rightarrow \mathbf{K}(f)$ pointwise on E . Then the function $[0, t_0] \ni t \mapsto W_t f(x)$ is differentiable for each $x \in E$ and $u_t := W_t f$ is the unique solution of the nonlinear evolution equation*

$$\frac{du_t}{dt} = -au_t + \mathbf{K}(u_t), \quad 0 \leq t < t_0, \quad (3.10)$$

with the initial condition $u_0 = f$.

(iii) *Suppose in addition that $\mathbf{K} \in \mathcal{K}(J)$, then $W_t \in \mathcal{K}(J)$ for all $t \in [0, t_0]$.*

Let J be a map as in Lemma 3.3. We claim that

$$V_t^0 \in \mathcal{K}(J) \text{ for all } t \geq 0. \quad (3.11)$$

Assume first that $c = 0$ and $\int_0^1 N(ds) < \infty$. Let $\mathbf{K}f(x) := \Psi^0(f(x))$ for all $f \in \text{bp}\mathcal{B}(E)$ and $x \in E$. Then $\mathbf{K}$ is Lipschitz and one can see that $\mathbf{K} \in \mathcal{K}(J)$ and for $t \in [0, t_0]$ equation (3.8) is precisely (3.9). We can apply Lemma 3.3 (iii) to conclude that $V_t^0 \in \mathcal{K}(J)$.

In the general case for c and N , V_t^0 is obtained after an additional pointwise approximation and using again property (p1) we conclude that (3.11) holds.

Applying (3.11) for $J = \Phi_{t'}$, $t' \geq 0$, it follows that

$$V_t^0(f \circ \Phi_{t'}) = V_t^0 f \circ \Phi_{t'} \text{ for all } f \in \text{bp}\mathcal{B}(E) \text{ and } t, t' \geq 0. \quad (3.12)$$

We can prove now assertion (i). Equality (1.2) is equivalent to

$$\widehat{T}_t F(\mu) = \widehat{\mathbb{E}}^0{}^\mu F(\Phi_t(\widehat{X}_t^0)) \text{ for all } F \in \text{bp}\mathcal{M}(E), t \geq 0, \text{ and } \mu \in M(E), \quad (3.13)$$

where recall that $(\widehat{T}_t)_{t \geq 0}$ is the transition function of $\widehat{X}$. By a monotone class argument we may suppose that $F = e_f$ with $f \in \text{bp}\mathcal{B}(E)$ and in this case the right hand side of the equality (3.13) equals $\widehat{T}_t(e_{f \circ \Phi_t})(\mu)$, where $(\widehat{T}_t^0)_{t \geq 0}$ is the transition function of $\widehat{X}^0$. By (3.2) applied to $\widehat{T}_t$ and $\widehat{T}_t^0$ it follows now that (3.13) is equivalent to

$$V_t f = V_t^0(f \circ \Phi_t) \text{ for all } f \in \text{bp}\mathcal{B}(E) \text{ and } t \geq 0.$$

Let $(S_t)_{t \geq 0}$ be the transition function of the right continuous flow Φ , $S_t f := f \circ \Phi_t$ for all $f \in \text{bp}\mathcal{B}(E)$ and $t \geq 0$. So, by the uniqueness of the solution to equation (3.1), we have to check that $(t, x) \mapsto V_t^0(f \circ \Phi_t)(x)$ verifies (3.1), written with S_t instead of T_t . Since $(t, x) \mapsto V_t^0 f(x)$ is the solution of (3.8) we get

$$V_t^0 f(\Phi_t(x)) = f(\Phi_t(x)) + \int_0^t \Psi(V_s^0 f(\Phi_t(x))) \, ds. \quad (3.14)$$

But by (3.12) we have $V_t^0(f \circ \Phi_{t'}) (x) = V_t^0 f(\Phi_{t'}(x))$ and therefore $V_s^0 f(\Phi_t(x)) = V_s^0(f \circ \Phi_s)(\Phi_{t-s}(x))$, hence $\Psi(V_s^0 f(\Phi_t(x))) = S_{t-s}(\Psi(V_s^0(f \circ \Phi_s)))(x)$. We conclude that equality (3.14) becomes

$$V_t^0(f \circ \Phi_t)(x) = S_t f(x) + \int_0^t S_{t-s}(\Psi(V_s^0(f \circ \Phi_s)))(x) \, ds,$$

hence (3.1) is verified as claimed.

To prove (3.7) observe that it is equivalent to

$$\widehat{T}_t F(\mu) = \widehat{T}_t^0 F(\Phi_t(\mu)) \text{ for all } F \in \text{bp}\mathcal{M}(E), t \geq 0, \text{ and } \mu \in M(E).$$

Arguing as before, the equality is equivalent to

$$V_t f = V_t^0 f \circ \Phi_t \text{ for all } f \in \text{bp}\mathcal{B}(E) \text{ and } t \geq 0.$$

But taking into account (3.12) this last equality holds.

Assertion (ii) is a consequence of Section 4 from [3], applied to Φ and $\widehat{X}^0$ on $M(E)$. Notice that as a consequence of (3.12) the kernels $\widehat{T}_t^0$ and $Q_{t'}^0$ commute for all $t, t' \geq 0$.

(iii) The fact that v_t given by (1.4) is a solution to equation (3.1) and the second equality of (1.4) follow from (3.2) and (3.7). $\square$

Corollary 3.4. *Let Ψ be a branching mechanism. Then the following assertions hold.*

(i) *Consider the pure branching (X^0, Ψ) -superprocess, the right Markov process $\widehat{X}^0 = (\widehat{X}_t^0, \mathbb{P}^{0\mu})$ with state space $M(E)$, and let $u \in \text{bp}\mathcal{B}(E)$. Let*

$$V_t^0 u(x) := -\ln \widehat{\mathbb{E}}^{0\delta_x} e_u(\widehat{X}_t^0), \quad t \geq 0, x \in E,$$

then $w_t := V_t^0 u$, $t \geq 0$, solves the following nonlinear evolution equation:

$$\frac{d}{dt} w_t = \Psi(\cdot, w_t), \quad t \geq 0, \quad (3.15)$$

with the initial condition $w_0 = u$.

(ii) *Let $(\mathcal{L}^0, \mathcal{D}(\mathcal{L}^0))$ be the extended weak generator of the pure branching (X^0, Ψ) -superprocess on $M(E)$. Let $u \in \text{bp}\mathcal{B}^0$, then $F := e_u$ belongs to $\mathcal{D}(\mathcal{L}^0)$, $\mathcal{L}^0 F = -F \cdot l_{\Psi(\cdot, u)}$, and for $\mu \in M(E)$ we have*

$$\begin{aligned} \mathcal{L}^0 F(\mu) = & \int_E c(x) F''(\mu, x) \mu(dx) - \int_E b(x) F'(\mu, x) \mu(dx) + \\ & \int_E \int_0^\infty [F(\mu + s\delta_x) - F(\mu) - sF'(\mu, x)] N(x, ds) \mu(dx). \end{aligned} \quad (3.16)$$

Proof. (i) By (3.2) applied to the transition function of $\widehat{X}^0$, we know that $V_t^0 u$, $t \geq 0$, is the solution to the integral equation (3.6). On the other hand, we already observed in the proof of Theorem 3.2 that equation (3.6) is equivalent to (3.8). The assertion follows now by Lemma 3.3 (ii.2); notice that we have $\mathbf{K}u(x) = \Psi^0(x, u(x))$, $x \in E$, and therefore $\mathbf{K}$ preserves the pointwise convergence, the property required by the hypothesis of the lemma.

(ii) Let $u \in \mathcal{B}^0$, then $\lim_{t \searrow 0} V_t^0 u(x) = u(x)$ for all $x \in E$. From dominated convergence and (3.2) we get $\lim_{t \searrow 0} \widehat{T}_t^0 F(\mu) = \lim_{t \searrow 0} e^{-\langle \mu, V_t^0 u \rangle} = F(\mu)$ for each $\mu \in M(E)$. By Lemma 3.1 (iii) from [3] we conclude that $e_u \in \mathcal{B}_o(\widehat{\mathbb{T}}^0)$.

Let $\alpha(t) := \langle \mu, w_t \rangle$, $t \geq 0$. Arguing as in the proof of Proposition 3.2 from [2], if we fix $t_o > 0$ and put $\lambda_1 := e^{\beta t_o} \|u\|_\infty$ we get that $\|w_t\|_\infty \leq \lambda_1$ for all $t \leq t_o$ and that Ψ is bounded on $E \times [0, \lambda_1]$. Therefore the map $t \mapsto \langle \mu, \Psi(\cdot, w_t) \rangle$ is bounded and continuous on $[0, t_o]$. We have $w_t = f + \int_0^t \Psi(\cdot, w_s) ds$ and thus $\alpha(t) = \langle \mu, f \rangle + \int_0^t \langle \mu, \Psi(\cdot, w_s) \rangle ds$. We deduce that the function α is differentiable on $[0, t_o]$ and $\alpha'(0) = \langle \mu, \Psi(\cdot, u) \rangle$. Using again (3.2) we obtain $\mathcal{L}^0 F(\mu) := \lim_{t \searrow 0} \frac{e_{V_t^0 u}(\mu) - e_u(\mu)}{t} = \frac{d}{dt} e^{-\alpha(t)}|_{t=0} = -e^{-\langle \mu, u \rangle} \langle \mu, \Psi(\cdot, u) \rangle$, hence $\mathcal{L}^0 F(\mu) = -F(\mu) \cdot l_{\Psi(\cdot, u)}(\mu)$.

We can show now that (3.16) holds. Indeed, the last equality we proved means that $\mathcal{L}^0 F(\mu) = -F(\mu) \int_E \mu(dx) [-b(x)u(x) - c(x)u(x)^2 + \int_0^\infty (1 - e^{-u(x)s}) -u(x)s N(x, ds)]$ and since $F'(\cdot, x) = -u(x)F$ and $F''(\cdot, x) = u^2(x)F$, a straightforward computation leads to (3.16). As in the proof of Proposition 3.1 (ii) we also have $\widehat{T}_t^0 \mathcal{L}^0 F = e_{w_t} l_{\Psi(\cdot, w_t)}$ and letting $t \searrow 0$, by dominated convergence we get that $\mathcal{L}^0 F \in \mathcal{B}_o(\widehat{\mathbb{T}}^0)$. In addition $\frac{|\widehat{T}_t^0 F(\mu) - F(\mu)|}{t} \leq \sup_{(x, \lambda) \in E \times [0, \lambda_1]} |\Psi(x, \lambda)| \cdot l_1(\mu)$. Since by (3.3) we have $l_1 \in \mathcal{E}(\widehat{\mathbb{T}}^0_{\beta'})$, we conclude that $F \in \mathcal{D}(\mathcal{L}^0)$. $\square$

Remark 3.5. (i) According to [22], Theorem (4.1) (a) and its proof, if $n \geq 1$, $f_1, \dots, f_n \in \mathcal{D}(D_w)$, and $\psi \in C_0^\infty(\mathbb{R}^n)$, then the function F on $M(E)$, defined as $F(\mu) := \psi(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle)$, $\mu \in M(E)$, belongs to $\mathcal{D}(\widehat{\mathcal{L}})$ and the following formula holds

$$\begin{aligned} \widehat{\mathcal{L}}F(\mu) = & \int_E c(x)F''(\mu, x)\mu(dx) + \int_E [LF'(\mu, \cdot)(x) - b(x)F'(\mu, x)]\mu(dx) + \\ & \int_E \int_0^\infty [F(\mu + s\delta_x) - F(\mu) - sF'(\mu, x)]N(x, ds)\mu(dx). \end{aligned} \quad (3.17)$$

In the case when the spatial motion is given by a right continuous flow, taking into account (3.5) and (3.16), the equality $\mathcal{L} = \widehat{D} + \mathcal{L}^0$ from assertion (ii) of Theorem 3.2 becomes a version of the above representation (3.17) of $\mathcal{L}$.

(ii) If E has d points then $M(E)$ is identified with $\mathbb{R}_+^d$ and in this case the pure branching (X^0, Ψ) -superprocess from Theorem 3.2, with Ψ which is independent of the spatial variable, is precisely a continuous-state branching process (CB-process), as considered and studied in [21, 30].

(iii) Comparing equations (1.3) and (3.15), it follows that a pure branching superprocess has no spatial motion term in the associated evolution equation. Corollary 3.4 (ii) and (3.17) underline the nonlinear PDE counterpart of the pure branching superprocesses.

(iv) The operator $\widehat{D}$ occurring in assertion (ii) of Theorem 3.2 is the extended weak generator of a continuous flow on $M(E)$, therefore one can prove that $\widehat{D}(F^2) = 2F \cdot \widehat{D}F$, provided that $F \in \mathcal{B}\mathcal{D}_o^c(\widehat{D})$ and $\widehat{D}F$ is a bounded function. Hence $\widehat{D}$ acts as a derivation, it is a first order operator in an infinite dimensional frame. Consequently, $\mathcal{L}$ can be seen as the modification of $\mathcal{L}^0$ with a "drift type" operator and the representation (1.2) may be obtained as a consequence; see [3] for details.

4 Right continuous flows driving non-local branching processes on finite configurations

Let $(b_k)_{k \geq 1}$ be a sequence of functions from $\text{bp}\mathcal{B}(E)$ such that $\sum_{k \geq 1} b_k \leq 1$, let $m_1 := \|\sum_{k \geq 1} k b_k\|_\infty$ and assume that $1 < m_1 < \infty$. We fix also a constant c such that $0 < c \leq \frac{m_1}{m_1 - 1}$.

Recall that the set $\widehat{E}$ of all finite configurations of E is identified with the union of all symmetric k -th powers $E^{(k)}$ of E , i.e., if $k \geq 1$ then $E^{(k)}$ is the factorisation of the Cartesian product E^k by the equivalence relation induced by the permutation group σ^k . Hence $\widehat{E} = \bigcup_{k \geq 0} E^{(k)}$, where $E^{(0)} := \{\mathbf{0}\}$; see e.g. [11, 26]. For each $k \geq 1$, let B_k be a Markovian kernel from $E^{(k)}$ to E .

If $\varphi \in \text{p}\mathcal{B}(E)$, $0 \leq \varphi \leq 1$, then by Proposition 4.1 from [8] (see also Theorem 3.1 in [10]), the integral evolution equation

$$h_t = e^{-ct} T_t \varphi + c \int_0^t e^{-c(t-s)} T_{t-s} \sum_{k \geq 1} b_k B_k(h_s^{(k)}) ds, \quad t \geq 0, \quad (4.1)$$

has a unique solution $t \mapsto H_t \varphi$, jointly measurable in $(t, x) \in \mathbb{R}_+ \times E$, such that $0 \leq H_t \varphi \leq 1$. Here, for a function $h \in \text{bp}\mathcal{B}(E)$ we have denoted by $h^{(k)}$, $k \geq 1$, the function on $E^{(k)}$ defined as $h^{(k)}(\mathbf{x}) := h(x_1) \cdots h(x_k)$ for all $\mathbf{x} = (x_1, \dots, x_k) \in E^{(k)}$. We also denote by $\widehat{h} : \widehat{E} \rightarrow \mathbb{R}_+$ the function defined as $\widehat{h}|_{E^{(k)}} = h^{(k)}$ if $k \geq 1$ and $\widehat{h}(\mathbf{0}) = 1$.

The integral evolution equation (4.1) associated with a non-local branching process on $\widehat{E}$ is formally equivalent to the equation

$$\frac{d}{dt} h_t = (L - c) h_t + c \sum_{k \geq 1} b_k B_k(h_t^{(k)}), \quad t \geq 0, \quad (4.2)$$

with the initial condition $h_0 = \varphi$, where L is the generator of the spatial motion; see e.g. Remark 4.2 (ii) from [8].

It turns out that the nonlinear semigroup $(H_t)_{t \geq 0}$ induces a branching semigroup of kernels $(\widehat{\mathbf{H}}_t)_{t \geq 0}$ on $\widehat{E}$ such that $\widehat{\mathbf{H}}_t \widehat{\varphi} = \widehat{H}_t \widehat{\varphi}$ for all $\varphi \in \text{bp}\mathcal{B}(E)$, $0 \leq \varphi \leq 1$. According to Theorem 4.10 from [8], under some additional assumptions, there exists a branching right Markov process $\widehat{X}$ with state space $\widehat{E}$ (depending on the spatial motion X , $(b_k)_{k \geq 1}$, the sequence of kernels $(B_k)_{k \geq 1}$, and c) having the transition function $(\widehat{\mathbf{H}}_t)_{t \geq 0}$.

Non-local pure branching processes on $\widehat{E}$. We consider now the non-local pure branching process $\widehat{X}^0$ on $\widehat{E}$ for which the spatial motion is the trivial Markov process X^0 . In this case equation (4.1) becomes

$$h_t = e^{-ct} \varphi + c \int_0^t e^{-c(t-s)} \sum_{k \geq 1} b_k B_k(h_s^{(k)}) ds, \quad t \geq 0. \quad (4.3)$$

Let $(H_t^0 \varphi)_{t \geq 0}$ be the solution to equation (4.3), then the transition function of $\widehat{X}^0$ is $(\widehat{\mathbf{H}}_t^0)_{t \geq 0}$.

The next proposition emphasises absorbing sets with respect to a pure branching processes on $\widehat{E}$, it is a version of a result from [6].

Proposition 4.1. *Let $M \in \mathcal{B}(E)$ and suppose that*

$$b_k(x) B_{k,x}(\widehat{M} \cap E^{(k)}) = 0 \text{ for every } k \geq 1 \text{ and } x \in E \setminus M, \quad (4.4)$$

that is, the sub-probability measure on $E^{(k)}$ induced by the sub-Markovian kernel $b_k B_k$ at x is carried by $E^{(k)} \setminus \widehat{M}$. Then $\widehat{E} \setminus \widehat{M}$ is a finely closed absorbing subset of $\widehat{E}$ with respect to the pure branching process $\widehat{X}^0$. The restriction of the pure branching process $\widehat{X}^0$ to $\widehat{M}$ is still a pure branching process, it is induced by the trivial process on M and the restrictions of B_k , $k \geq 1$, to M .

Proof. The assertion follows arguing as in the proof of assertion (iii) of Theorem 3.2 from [6], applied for $A := E \setminus M$ and observing that A is an absorbing set with respect to the trivial process X^0 on E . $\square$

For each $k \geq 1$ let $(\mathbf{B}_k, \mathcal{D}(\mathbf{B}_k))$ be the operator on $\text{bp}\mathcal{B}(E)$ defined as $\mathbf{B}_k\varphi := B_k(\varphi^{(k)})$, with $\mathcal{D}(\mathbf{B}_k) := \{\varphi \in \text{p}\mathcal{B}(E) : \varphi \leq 1\}$.

Further, as in [7, 25], for the non-local pure branching process $\widehat{X}^0 = (\widehat{X}_t^0)_{t \geq 0}$ we shall use the notation $\widehat{X}_t^0 = (x_t^1, \dots, x_t^{N(t)}) \in E^{(N(t))}$, where $N(t)$ is the number of the particles at time t .

We assume further that each $b_k, k \geq 1$, is a constant function on E , so, $(b_k)_{k \geq 1}$ is a sequence of positive numbers.

Theorem 4.2. Let $\widehat{X} = (\widehat{X}_t)_{t \geq 0}$ be the non-local branching process on $\widehat{E}$, depending on a spatial motion given by a right continuous flow $\Phi = (\Phi_t)_{t \geq 0}$ on E , $(b_k)_{k \geq 1}$, the sequence of kernels $(B_k)_{k \geq 1}$, and c . Assume that the mapping $[0, \infty) \times E \ni (t, x) \mapsto \Phi_t(x)$ is continuous and suppose that

$$\mathbf{B}_k \in \mathcal{K}(\Phi_t) \text{ for all } t \geq 0 \text{ and } k \geq 1. \quad (4.5)$$

Then the following assertions hold.

(i) The branching process $\widehat{X} = (\widehat{X}_t, \mathbb{P}^\mu)$ has the representation (1.2) on $\widehat{E}$ as the pure branching process $\widehat{X}^0 = (\widehat{X}_t^0, \mathbb{P}^{0^\mu})$ driven by the continuous flow Φ and (3.7) holds for all $\mu \in \widehat{E}$ and $\Gamma \in \mathcal{B}(\widehat{E})$.

(ii) Equation (4.2), with L as the generator of the right continuous flow Φ , has a mild solution, namely, the unique solution $h_t, t \geq 0$, to equation (4.1) given by

$$h_t(x) = \widehat{\mathbb{E}}^{0^\mu} \prod_{k=1}^{N(t)} \varphi(\Phi_t(x_t^k)) = \widehat{\mathbb{E}}^{0^\mu} \prod_{k=1}^{N(t)} \varphi(x_t^k), \quad x \in E. \quad (4.6)$$

Proof. By (A3.3) from [8], the operator $\mathbf{B} := c \sum_{k \geq 1} b_k \mathbf{B}_k$ is Lipschitz; notice that (4.3) is precisely of the form (3.9) from Lemma 3.3 with $a := c$ and condition (II) holds. From (4.5) it follows that we also have $\mathbf{B} \in \mathcal{K}(\Phi_t)$ for all $t \geq 0$ and therefore we can apply Lemma 3.3 (iii) to conclude that

$$H_t^0 \in \mathcal{K}(\Phi_{t'}) \text{ for all } t' \geq 0. \quad (4.7)$$

(i) Arguing as in the proof of Theorem 3.2, to prove (1.2) on $\widehat{E}$ we have to show that $\widehat{\mathbf{H}}_t \widehat{\varphi}(\mu) = \widehat{\mathbb{E}}^{0^\mu} \widehat{\varphi}(\Phi_t(\widehat{X}_t^0))$ for all $\varphi \in \text{p}\mathcal{B}(E)$, $\varphi \leq 1$, and $\mu \in \widehat{E}$. Since $\widehat{\mathbf{H}}_t$ and $\widehat{\mathbf{H}}_t^0$ are branching kernels on $\widehat{E}$ and $\widehat{\varphi} \circ \Phi_t = \widehat{\varphi} \circ \Phi_t$, it is enough to show that

$$H_t \varphi = H_t^0(\varphi \circ \Phi_t) \text{ for all } \varphi \in \text{p}\mathcal{B}(E), \varphi \leq 1, \text{ and } t \geq 0. \quad (4.8)$$

By the uniqueness of solutions to equation (4.1), it is sufficient to prove that $t \mapsto H_t^0(\varphi \circ \Phi_t)$ also satisfies (4.1) with S_t instead of T_t . Because $(H_t^0 \varphi)_{t \geq 0}$ is the solution to equation (4.3) it follows that

$$H_t^0 \varphi \circ \Phi_t = e^{-ct} \varphi \circ \Phi_t + \int_0^t e^{-c(t-s)} \mathbf{B}(H_s^0 \varphi) \circ \Phi_t \, ds \text{ for all } t \geq 0.$$

Since H_t^0 and $\mathbf{B}$ belong to $\mathcal{K}(\Phi_{t'})$, $t' \geq 0$, and using property (p2) from Lemma 3.3, the last equality becomes

$$H_t^0(\varphi \circ \Phi_t) = e^{-ct} S_t \varphi + \int_0^t e^{-c(t-s)} S_{t-s} \mathbf{B}(H_s^0(\varphi \circ \Phi_s)) \, ds \text{ for all } t \geq 0,$$

which means that indeed $(H_t^0(\varphi \circ \Phi_t))_{t \geq 0}$ is a solution to (4.1) and therefore (4.8) holds and the proof of assertion (i) is complete.

(ii) The first equality of (4.6) follows from (1.2) since $(\widehat{\mathbf{H}}_t)_{t \geq 0}$ is the transition function of $\widehat{X}$, while the second one is a consequence of (4.8) and (4.7). $\square$

Remark 4.3. (i) **Example of kernels satisfying condition (4.5).** For each $k \geq 1$ define the kernel B_k as $B_k F(x) := F(x, \dots, x)$ for all $x \in E$ and $F \in \text{bp}\mathcal{B}(E^{(k)})$. Then $\mathbf{B}_k \varphi = \varphi^k$ for all $\varphi \in \text{p}\mathcal{B}(E)$, $\varphi \leq 1$, and clearly (4.5) is verified. Notice that equation (4.1) becomes

$$h_t = e^{-ct} T_t \varphi + c \int_0^t e^{-c(t-s)} T_{t-s} \left(\sum_{k \geq 1} b_k h_s^k \right) \, ds, \quad t \geq 0.$$

(ii) Recall the classical probabilistic meaning of the branching procedure: if $k \geq 1$ and $x \in E$, then $b_k(x)$ represents the probability that a particle destroyed at the point $x \in E$ has precisely k descendants, and $B_{k,x}$ is the distribution of the k descendants. As in [6], taking into account this probabilistic description of a branching process, one can explain the additional condition from Proposition 4.1, imposed to the kernels B_k : (4.4) means that when a particle is destroyed at a point of $E \setminus M$ then not all the direct descendants are distributed inside M .

(iii) Consider the classical kernels B_k , $k \geq 1$, from assertion (i) and observe that condition (4.4) is satisfied in this case for every set $M \in \mathcal{B}(E)$. Consequently, applying Proposition 4.1 we get that: the set $\widehat{E} \setminus \widehat{M}$ is absorbing with respect to any pure branching process $\widehat{X}^0$ on $\widehat{E}$, for all $M \in \mathcal{B}(E)$. In particular, we highlighted absorbing sets for the induced non-local pure branching process from [7], occurring in the study of the nonlinear evolution equation considered there.

(iv) We can state for non-local branching processes on $\widehat{E}$ results similar to (1.4) and Corollary 3.4. Let $\widehat{X}^0 = (\widehat{X}_t^0, \widehat{\mathbb{P}}^{\mu_0})$ be the non-local pure branching process on $\widehat{E}$, for which the spatial motion is the trivial Markov process X^0 , and let $\varphi \in \mathcal{P}(E)$, $\varphi \leq 1$. If we put

$$h_t(x) := \widehat{\mathbb{E}}^{\delta_x} \widehat{\varphi}(\widehat{X}_t^0), \quad x \in E, t \geq 0,$$

then h_t , $t \geq 0$, solves the nonlinear evolution equation

$$\frac{d}{dt} h_t = -c h_t + c \sum_{k \geq 1} b_k B_k(h_t^{(k)}), \quad t \geq 0, \quad (4.9)$$

with the initial condition $h_0 = \varphi$; observe that equation (4.9) is precisely (4.2) with $L = 0$. Indeed, h_t , $t \geq 0$, is the solution to the integral equation (4.3). The assertion follows by Lemma 3.3 (ii.2), applied for $\mathbf{K}\varphi = c \sum_{k \geq 1} b_k B_k(\varphi^{(k)})$ and observing that $\mathbf{K}$ preserves the pointwise convergence.

(v) In [28] it is studied an equation analogue to (4.9) (see also [27]) on the locally finite configurations of a locally compact metrizable space, however, with no associated branching Markov process.

5 Nonlinear evolution equations on measures

Let $(b_k)_{k \geq 1}$ be a sequence of positive numbers such that $\sum_{k \geq 1} b_k \leq 1$ and $1 < \sum_{k \geq 1} k b_k < \infty$ as in the previous section. Let $\widehat{Y}^0 = (\widehat{Y}_t^0)_{t \geq 0}$ be the non-local pure branching process from Remark 4.3 (i) with the state space $\widehat{M}(E)$ instead of $\widehat{E}$, so, it depends on the sequence $(b_k)_{k \geq 1}$, the constant c , and its spatial motion is the trivial Markov process Y^0 on $M(E)$. The corresponding equation (4.1) becomes

$$h_t = e^{-ct} f + c \int_0^t e^{-c(t-s)} \sum_{k \geq 1} b_k h_s^k ds \quad \text{on } M(E), \quad t \geq 0.$$

Let $(L, \mathcal{D}(L))$ be the extended weak generator of a spatial motion X on E and let $\mathbb{T} = (T_t)_{t \geq 0}$ be its transition function.

We consider on $M(E)$ the following nonlinear evolution equation

$$\frac{d}{dt} F_t = L F_t - c F_t + c \sum_{k \geq 1} b_k F_t^k \quad \text{on } M(E), \quad t \geq 0, \quad (5.1)$$

with the initial condition $F_0 = F$, where for a function $F : M(E) \rightarrow [0, 1]$ we denoted by F' its variational derivative and $L F'(\mu) := \int_E L F'(\mu, x) \mu(dx)$.

Let $\widehat{\Phi} = (\widehat{\Phi}_t)_{t \geq 0}$ be the right continuous flow on $M(E)$ induced by the right Markov process X , $\widehat{\Phi}_t(\mu) := \mu \circ T_t = X_t(\mathbb{P}^\mu)$, $\mu \in M(E)$. Recall that by Proposition 3.1 (ii) the extended weak generator of $\widehat{\Phi}$ coincides with $F \mapsto L F'$ on a large class of functions F on $M(E)$.

Further, we need to consider the following condition:

$$\text{the mapping } [0, \infty) \times M(E) \ni (t, \mu) \mapsto \mu \circ T_t \in M(E) \text{ is continuous} \quad (5.2)$$

Let $\mathcal{G} \subset \text{bC}(E)$ be a family of functions which is *convergence determining*, i.e., for every sequence $(\mu_n)_n$ of probabilities on E we have that $(\mu_n)_n$ is weakly convergent to μ if and only if $\mu_n(g) \rightarrow \mu(g)$ for all $g \in \mathcal{G}$; see [20], page 112.

Remark 5.1. The following Feller type property of the transition function $\mathbb{T}$ implies that condition (5.2) holds: There exists a convergence determining family $\mathcal{G}$ such that $T_t(\mathcal{G}) \subset \text{bC}(E)$ for every $t > 0$ and $\lim_{t \searrow 0} \|T_t g - g\|_\infty = 0$ for all $g \in \mathcal{G}$. In particular, if E is a locally compact space with countable base and $\mathbb{T} = (T_t)_{t \geq 0}$ is a Feller semigroup on E , that is, T_t maps $C_0(E)$ into itself for every $t > 0$, then (5.2) is satisfied.

The next corollary follows from Theorem 4.2 (ii) and the above considerations.

Corollary 5.2. Equation (1.1) has a mild solution F_t , $t \geq 0$, with $F_0 = F \in \text{pM}(\mathbb{R}^d)$, $F \leq 1$, uniquely determined by the pure branching process $\widehat{Y}^0$ on $\widehat{M}(\mathbb{R}^d)$, the d -dimensional Brownian motion $(B_t)_{t \geq 0}$, and the constant c ,

$$F_t(\mu) = \widehat{\mathbb{E}}^0 \delta_\mu \prod_{k=1}^{N(t)} F(B_t(\mathbb{P}^{y_t^k})) = \widehat{\mathbb{E}}^0 \delta_{B_t(\mathbb{P}^\mu)} \prod_{k=1}^{N(t)} F(y_t^k), \quad \mu \in M(\mathbb{R}^d), \quad (5.3)$$

where $\widehat{Y}_t^0 = (y_t^1, \dots, y_t^{N(t)})$ and $N(t)$ is the number of the particles in $M(\mathbb{R}^d)$ at time t .

More generally, if we assume that condition (5.2) holds then equation (5.1) has a mild solution uniquely determined by the pure branching process $\widehat{Y}^0$ on $\widehat{M}(E)$, the spatial motion X , and the constant c , namely, for every $F \in \text{pM}(E)$, $F \leq 1$, the nonlinear evolution equation on measures

$$F_t = e^{-ct} F \circ \widehat{\Phi}_t + c \int_0^t e^{-c(t-s)} \sum_{k \geq 1} b_k F_s^k ds \quad \text{on } M(E), \quad t \geq 0,$$

has a unique solution $t \mapsto F_t$, jointly measurable in $(t, \mu) \in \mathbb{R}_+ \times M(E)$, such that $0 \leq F_t \leq 1$ and (5.3) holds with X_t instead of B_t and $M(E)$ instead of $M(\mathbb{R}^d)$.

Remark. In [12], in solving a nonlinear Schrödinger equation of a non-local branching process, the solution is obtained from an L^p -weak solution for a stationary linear equation similar to (5.1), associated to a branching semigroup on $\widehat{E}$ and using a Feynman-Kac formula stated in [5].

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Appendix

(A1) Proof of Proposition 3.1. (i) Let $u \in \text{bD}_0^c(L)$ be such that Lu is a bounded function and $t \mapsto LT_t u(x)$ is continuous. Replacing $\mathbb{T}_b$ by $\mathbb{T}_{b+\alpha}$, we may assume that $u = U_b f$ with f bounded and $t \mapsto T_t^b f(x)$ continuous, where U_b is the potential kernel of $\mathbb{T}_b$, $U_b := \int_0^\infty T_t^b dt$. Then we have $\frac{T_t^b u - u}{t} = \frac{-1}{t} \int_0^t T_s^b f ds$ and $|\frac{T_t^b u - u}{t}| \leq \|f\|_\infty$. By dominated convergence we obtain $\lim_{t \searrow 0} \int_E \frac{T_t^b u - u}{t} d\mu = -\mu(f)$ for every $\mu \in M(E)$. From (3.4) we have $|\frac{\widehat{T}_t(l_u) - l_u}{t}| = |\frac{l_{T_t^b u - u}}{t}| \leq \|f\|_\infty l_1$. On the other hand from (3.3) we get $\widehat{U}_\beta l_1 < \infty$ for some $\beta > 0$. We deduce that $l_u \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}(l_u) = l_{Lu - bu}$.

(ii) Observe first that if $u \in \text{bD}(L)$ is such that Lu is bounded and we fix $\alpha > 0$ then $u = U_\alpha g$ for some function $g \in \text{bB}^0(\mathbb{T})$. If $\mu \in M(E)$ then by dominated convergence $\lim_{t \searrow 0} \langle \mu, \frac{T_t u - u}{t} \rangle = l_{Lu}(\mu)$. Let $\mathbf{u} := (u_1, \dots, u_n)$, $T_t \mathbf{u} := (T_t u_1, \dots, T_t u_n)$, $L \mathbf{u} := (Lu_1, \dots, Lu_n)$, and $l_{\mathbf{u}} := (l_{u_1}, \dots, l_{u_n})$. Hence $\lim_{t \searrow 0} Q_t^0 F(\mu) = \lim_{t \searrow 0} \psi(l_{T_t \mathbf{u}}(\mu)) = \psi(l_{\mathbf{u}}(\mu)) = F(\mu)$, $F \in \text{bB}^0(\mathbb{Q}^0) = \text{bB}_o(\mathbb{Q}^0)$. We have also $\lim_{t \searrow 0} \frac{Q_t^0 F(\mu) - F(\mu)}{t} = \nabla \psi(l_{\mathbf{u}}(\mu)) \cdot l_{L \mathbf{u}}(\mu) = \langle \mu, LF'(\mu, \cdot) \rangle =: G(\mu)$. Then $Q_t^0 G(\mu) = \nabla \psi(l_{T_t \mathbf{u}}(\mu)) \cdot l_{LT_t \mathbf{u}}(\mu)$

and therefore $G \in \mathcal{B}_o(\mathbb{Q}^0)$. It remains to show that $F \in \mathcal{D}(\widehat{D})$. Let $t_o > 0$ such that for all $t < t_o$ we have $\|\frac{T_t u - u}{t}\|_\infty \leq C_1$ for some constant $C_1 > 0$. Since ψ is Lipschitz we get $|\frac{Q_t^0 F(\mu) - F(\mu)}{t}| \leq C_2 \frac{|l_{T_t u}(\mu) - l_u(\mu)|}{t} \leq C_2 \|\frac{T_t u - u}{t}\|_\infty \mu(1) \leq C_2 C_1 l_1(\mu)$ for all $t < t_o$ and $\mu \in M(E)$, with $C_2 > 0$. By (3.3), arguing as in last part of the proof of assertion (i) we conclude that $F \in \mathcal{D}(\widehat{D})$ and $\widehat{D}F(\mu) = \langle \mu, LF'(\mu, \cdot) \rangle$. $\square$

(A2) Proof of Lemma 3.3. The proof of assertion (i) is straightforward and we omit it.

(ii.1) We argue as in the proof of Proposition 3.2 from [2] and show the existence of a solution to equation (3.9) by Picard approximation. Let $w_t^0 = 0$ and for $k \geq 0$ and $x \in E$ define inductively

$$w_t^{k+1}(x) = e^{-at} f(x) + \int_0^t e^{-a(t-s)} \mathbf{K}(w_s^k)(x) ds, \quad 0 \leq t \leq t_o.$$

Observe first that if condition (II) holds and we take $f \in \text{bp}\mathcal{B}(E)$, $f \leq 1$, then we have $w_t^k \leq 1$ for all $k \geq 0$. Indeed, we reason by induction. If we suppose that $w_s^k \leq 1$ then, since $\mathbf{K}f \leq a$, we get $w_t^{k+1} \leq e^{-at} + a \int_0^t e^{-as} ds = 1$. So, in this case w_s^k belongs to $\mathcal{D}(\mathbf{K})$ for all s and k .

One can see that $0 \leq w_t^k(x) \leq e^{Ct_o} \|f\|_\infty$ for all $x \in E$ and $k \geq 0$, where C is the Lipschitz constant of $\mathbf{K}$. We have also $\|w_t^{k+1} - w_t^k\|_\infty \leq \frac{(Ct)^k}{k!} \|f\|_\infty$ and therefore the limit $\lim_k w_t^k(x) =: W_t f(x)$ exists uniformly in $(t, x) \in [0, t_o] \times E$ and in particular, $t \mapsto W_t f$ is a solution to (3.9).

The uniqueness of the solution to (3.9) follows by Gronwall's lemma, using the Lipschitz property of $\mathbf{K}$.

(ii.2) First, using the hypothesis that $\mathbf{K}$ preserves the pointwise convergence, one can prove inductively that for every $k \geq 0$ and $x \in E$ the function $[0, t_o] \ni t \mapsto w_t^k(x)$ is continuous. Since $u_t = \lim_k w_t^k$ uniformly in t , it follows that $t \mapsto u_t(x)$ is also continuous on $[0, t_o]$. Consequently, (3.10) is now a direct consequence of (3.9).

(iii) By the properties (p3) and (p2) of $\mathcal{K}(J)$ and since we assumed that $\mathbf{K} \in \mathcal{K}(J)$, one can see inductively that $w_t^k \in \mathcal{K}(J)$ for all k and t . We have $W_t f(x) = \lim_k w_t^k(x)$ for all $x \in E$ and therefore, by property (p1) of $\mathcal{K}(J)$ we deduce that $W_t \in \mathcal{K}(J)$ for all t . $\square$

(A3) Right continuous flow. Assume that E is a Lusin topological space and $\mathcal{B}(E)$ is the Borel σ -algebra. A family $\Phi = (\Phi_t)_{t \geq 0}$ of mappings on E is called *right continuous flow* (cf. [32], page 41) provided that:

- (1) $\Phi_{t+s}(x) = \Phi_t(\Phi_s(x))$ for all $s, t > 0$ and $x \in E$;
- (2) $\Phi_0(x) = x$ for all $x \in E$;
- (3) For each $t > 0$ the function $E \ni x \mapsto \Phi_t(x)$ is $\mathcal{B}(E)/\mathcal{B}(E)$ -measurable;
- (4) For each $x \in E$ the function $t \mapsto \Phi_t(x)$ is right continuous on $[0, \infty)$.

(A4) The extended weak generator. We recall several facts about the weak generator of a right Markov process X with state space E , transition function $\mathbb{T} = (T_t)_{t \geq 0}$, and $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ as associated resolvent of kernels; for details see [3], where the classical weak generator acting on bounded functions, considered by E.B. Dynkin in [17], was extended to unbounded real-valued functions.

Define $\mathcal{B}^0 = \mathcal{B}^0(\mathbb{T}) := \{f \in [\mathcal{B}(E)] : T_t(|f|) < \infty \text{ for all } t > 0 \text{ and } f = \lim_{s \searrow 0} T_s f \text{ pointwise on } E\}$. Clearly, we have $[\mathcal{E}(\mathcal{U}_\alpha)] \subset \mathcal{B}^0$ for every $\alpha \geq 0$ and every bounded finely continuous function belongs to $\mathcal{B}^0$, in particular, $b\mathcal{C}(E) \subset \mathcal{B}^0$.

Define also $\mathcal{B}_o = \mathcal{B}_o(\mathbb{T}) := \{f \in \mathcal{B}^0 : \text{for each } \alpha > 0 \text{ there exist } t_o > 0 \text{ and } h_\alpha \in p\mathcal{B}(E) \text{ such that } \sup_{0 < s < t_o} T_s |f| \leq h_\alpha \text{ with } U_\alpha h_\alpha < \infty\}$.

Let $\mathcal{D}(L)$ be the set of all $u \in \mathcal{B}_o$ such that for each $\alpha > 0$ there exist $t_o > 0$ and $h_\alpha \in p\mathcal{B}(E)$ with $\sup_{0 < t < t_o} |\frac{T_t u - u}{t}| \leq h_\alpha$, $U_\alpha h_\alpha < \infty$ on E , and there exists pointwise $\lim_{t \searrow 0} \frac{T_t u - u}{t} \in \mathcal{B}_o$. Clearly, $\mathcal{B}_o$ and $\mathcal{D}(L)$ are vector spaces and define the linear operator $L : \mathcal{D}(L) \rightarrow \mathcal{B}_o$ as $Lu(x) := \lim_{t \searrow 0} \frac{T_t u(x) - u(x)}{t}$, $f \in \mathcal{D}(L)$, $x \in E$.

The operator $(L, \mathcal{D}(L))$ is called the *extended weak generator* of $\mathbb{T} = (T_t)_{t \geq 0}$ and one can prove that if $\alpha > 0$ then $\mathcal{D}(L) = U_\alpha(\mathcal{B}_o)$.

In [24] it was considered an *extended generator* $(\bar{L}, \mathcal{D}(\bar{L}))$ of $\mathbb{T} = (T_t)_{t \geq 0}$, taking into account unbounded real-valued functions, as follows: let $u, g \in \mathcal{B}^0$, then u belongs to the domain $\mathcal{D}(\bar{L})$ of $\bar{L}$ and $g = \bar{L}u$ provided

that for all $t > 0$ and $x \in E$ we have $\int_0^t T_s(|g|)(x) ds < \infty$ and $T_t u(x) = u(x) + \int_0^t T_s g(x) ds$. According to [24],

Proposition 4.1 (see also [22], page 354, the proof of Theorem (4.1)), we have the following equivalent definition for the extended generator: if $u, g \in \mathcal{B}^0$ then $u \in \mathcal{D}(\bar{L})$ and $\bar{L}u = g$ if and only if for all $x \in E$ we have $\int_0^t T_s(|g|)(x) ds < \infty$ for all $t > 0$ and $\left(u(X_t) - u(X_0) - \int_0^t g(X_s) ds\right)_{t \geq 0}$ is a $(\mathcal{F}_t)$ -martingale under $\mathbb{P}^x$.

Let $\mathcal{B}_{oo} := \{f \in \mathcal{B}_o : \text{for each } t > 0 \text{ there exist } t_o > 0 \text{ and } h_t \in p\mathcal{B} \text{ such that } \sup_{0 < s < t_o} T_s |f| \leq h_t \text{ with } T_t h_t < \infty\}$ and define $\mathcal{D}_o(L) := \{u \in \mathcal{D}(L) : Lu \in \mathcal{B}_{oo}\}$ and $\mathcal{D}_o^c(L) := \{u \in \mathcal{D}_o(L) : [0, \infty) \ni t \mapsto LT_t u(x) \text{ is continuous for each } x \in E\}$.

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