Research Article

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Application of Capacities to Space-Time Fractional Dissipative Equations II: Carleson Measure Characterization for $L^q(\mathbb{R}_+^{n+1}, \mu)$ –Extension

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Abstract: This paper provides the Carleson characterization of the extension of fractional Sobolev spaces and Lebesgue spaces to $L^q(\mathbb{R}_+^{n+1}, \mu)$ via space-time fractional equations. For the extension of fractional Sobolev spaces, preliminary results including estimates, involving the fractional capacity, measures, the non-tangential maximal function, and an estimate of the Riesz integral of the space-time fractional heat kernel, are provided. For the extension of Lebesgue spaces, a new L^p –capacity associated to the spatial-time fractional equations is introduced. Then, some basic properties of the L^p –capacity, including its dual form, the L^p –capacity of fractional parabolic balls, strong and weak type inequalities, are established.

Keywords: Space-time fractional equations, capacities, extension, fractional Laplacian, Lebesgue spaces, Sobolev spaces

MSC: Primary: 31C15; Secondary: 26A33; 46E30

1 Introduction

In this paper, we study the Carleson characterization of $L^q(\mathbb{R}_+^{n+1}, \mu)$ –extensions of Sobolev spaces and Lebesgue spaces via the following space-time fractional equation:

$$\begin{cases} \partial_t^\beta u(x, t) = -\nu(-\Delta)^{\alpha/2} u(x, t), & (x, t) \in \mathbb{R}_+^{n+1}; \\ u(x, 0) = \varphi(x), & x \in \mathbb{R}^n. \end{cases} \quad (1.1)$$

with $\beta \in (0, 1)$ and $\alpha > 0$. Here the symbol ∂_t^β denotes the Caputo fractional derivative defined as

$$\partial_t^\beta u(x, t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \partial_r u_r(x) \frac{dr}{(t-r)^\beta}.$$

Carleson measures were introduced by L. Carleson [5] to characterize the interpolating sequences in the algebra H^∞ of bounded analytic functions in the open unit disc and to give a solution to the corona problem. In the viewpoint of geometry, a Carleson measure on a domain Ω can be seen as a measure that does not vanish at the boundary $\partial\Omega$ when compared to the surface measure on the boundary of Ω . In the field of

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harmonic analysis and the potential theory, Carleson measures are widely applied to the extension and the trace problem of function spaces. Let D denote the unit disc in the complex plane $\mathbb{C}$ and μ be a Borel measure on D . For $1 \leq p < \infty$, let $H^p(\partial D)$ denote the Hardy space on the boundary of D and let $L^p(D, \mu)$ denote the Lebesgue space on D with respect to the measure μ . It is well known that the Poisson operator is bounded from $H^p(\partial D)$ to $L^p(D, \mu)$ if and only if μ is a Carleson measure, i.e., there exists a constant $C > 0$ such that, for every point $x \in \partial D$ and every radius $r > 0$,

$$\mu(\Omega \cap B(x, r))/|\partial\Omega \cap B(x, r)| \leq C, \quad (1.2)$$

where $B(x, r)$ denotes the ball centered at x with radius r . In the setting of the upper half space $\mathbb{R}_+^{n+1}$, an analogue conclusion holds, see for instance [12, Theorem 7.37].

In 1960s, the capacities related with function spaces were introduced to characterize the embedding or trace inequalities. Let $C_0^\infty(\mathbb{R}^n)$ stand for all infinitely smooth functions with compact support in $\mathbb{R}^n$. For a nonnegative Borel measure μ on $\mathbb{R}^n$, to investigate the spectral problems for Schrödinger operators, Maz'ya first proved (cf. [14, 15]) that if $1 < p \leq q$ and $pl < n$, then there exists a constant C such that

$$\|u\|_{L^q(\mathbb{R}^n, \mu)} \leq C \|u\|_{h_p^l(\mathbb{R}^n)}, \quad \forall u \in h_p^l(\mathbb{R}^n) \quad (1.3)$$

holds if and only if

$$\sup \left\{ \frac{(\mu(E))^{p/q}}{\text{cap}(E, h_p^l)} : E \subset \mathbb{R}^n, \text{cap}(E, h_p^l) > 0 \right\} < \infty. \quad (1.4)$$

Here $h_p^l(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to $\|f\|_{h_p^l(\mathbb{R}^n)} := \|(-\Delta)^{1/2} f\|_{L^p(\mathbb{R}^n)}$, and $\text{cap}(E, h_p^l)$ is the capacity of E associated with $h_p^l(\mathbb{R}^n)$. Such embeddings like (1.3) are referred to as trace inequalities (cf. [3]). Meanwhile, (1.4) is called the isocapacitary inequality, see Maz'ya [16]. Since the pioneer work of Maz'ya in [14], other equivalent conditions of trace inequalities were established in [1, 3, 6, 15, 17, 20]. Especially, when $1 < p < q < \infty$, E in (1.3) can be replaced by balls, see [3, Theorem 7.2.2]. When $0 < q < p$ and $1 < p < n/l$, (1.3) holds if and only if the Wolff potential $W_\alpha^\mu(\cdot) \in L^{q(p-1)/(p-q)}(\mu)$, see Cascante-Ortega-Verbitsky [6]. There exist other conditions involving no capacity, which are equivalent to (1.3), see, e.g., [17, 20]. These equivalent conditions were widely applied to harmonic analysis, the operator theory, function spaces, linear and nonlinear partial differential equations.

When extending a function φ from $\mathbb{R}^n$ to $\mathbb{R}_+^{n+1}$ through PDEs, Xiao in [21] studied the Carleson measure characterization of embeddings of Sobolev spaces:

$$\|u(x, t^2)\|_{L^q(\mathbb{R}_+^{n+1}, \mu)} \lesssim \|\nabla \varphi\|_{L^p(\mathbb{R}^n)}, \quad (1.5)$$

where $u(x, t)$ is the solution to the classical heat equation with the initial data $\varphi(x)$. In [22], Xiao established the Carleson characterization of embeddings similar to (1.5) for the fractional Besov space $\dot{\Lambda}_\alpha^{1,1}(\mathbb{R}^n)$. On the other hand, Xiao [22] also proved that $\dot{\Lambda}_\alpha^{1,1}(\mathbb{R}^n)$ embeds the Choquet space $L^1(\tilde{H}_{n-\alpha})$. In [23], Xiao proved that some isocapacitary inequalities are equivalent to strong versions of classical Sobolev inequalities. In [24], Xiao established the equivalence of the fractional version of (1.3) to several geometric inequalities in terms of fractional capacities. Motivated by Xiao [21, 22], for the solution of the spatial fractional heat equations, Zhai in [25] provided the Carleson characterization of embeddings similar to (1.5) for fractional Sobolev spaces by fractional capacities. The Carleson characterization of embeddings similar to (1.5) for Lebesgue spaces was established by Chang-Xiao in [8] and Shi-Xiao in [18] using L^p -capacities. For the fractional Poisson extension, Li et al. [13] established the Carleson characterization for the embeddings of fractional Sobolev spaces and Lebesgue spaces via fractional capacities and L^p -capacities.

Motivated by the above mentioned works, in this paper, we provide the Carleson characterization of the $L^q(\mathbb{R}_+^{n+1}, \mu)$ -extension of Sobolev spaces and Lebesgue spaces via the solution to (1.1). For equation (1.1), the solution can be written as

$$u(x, t) = R_{\alpha, \beta}(\varphi)(x, t) := \int_{\mathbb{R}^n} G_t(x - y) \varphi(y) dy,$$

where $G_t(\cdot)$ is the space-time fractional heat kernel. Recently, there has been an increasing interest in the fractional calculus. This is because time fractional operators are proving to be very useful for modeling purposes. For example, while the classical heat equation $\partial_t u(x, t) = \Delta u(x, t)$ is used for modeling heat diffusion in the homogeneous media, the fractional heat equations

$$\partial_t^\beta u(x, t) = \Delta u(x, t) \quad (1.6)$$

are used to describe heat propagation in the inhomogeneous media. It is known that as opposed to the classical heat equation, the equation (1.6) is known to exhibit the sub diffusive behaviour and is related with anomalous diffusion or diffusion in the non-homogeneous media, with random fractal structures.

Definition 1.1. Let $\gamma \in (0, n)$ and $p \in [1, n/\gamma]$. The homogeneous Sobolev space $\dot{W}^{\gamma,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} := \begin{cases} \|(-\Delta)^{\gamma/2} \varphi\|_{L^p(\mathbb{R}^n)}, & p \in (1, n/\gamma); \\ \left(\int_{\mathbb{R}^n} \frac{\|\Delta_h^k \varphi\|_{L^p(\mathbb{R}^n)}^p}{|h|^{n+p\gamma}} dh \right)^{1/p}, & p = 1 \text{ or } p = n/\gamma, \gamma \in (0, n), \end{cases}$$

where $k = 1 + [\gamma]$, $\gamma = [\gamma] + \{\gamma\}$ with $[\gamma] \in \mathbb{Z}_+$, $\{\gamma\} \in (0, 1)$ and

$$\Delta_h^k \varphi(x) = \begin{cases} \Delta_h^1 \Delta_h^{k-1} \varphi(x), & k > 1; \\ \varphi(x+h) - \varphi(x), & k = 1. \end{cases}$$

Denote by $T(O)$ the tent based on an open subset O of $\mathbb{R}^n$, i.e.,

$$T(O) = \{(x, r) \in \mathbb{R}_+^{n+1} : B(x, r) \subseteq O\}$$

with $B(x, r)$ being the open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. The fractional Sobolev capacities are defined as follows.

Definition 1.2. Let $\gamma \in (0, n)$ and $p \in [1, n/\gamma]$.

(i) The fractional capacity of an arbitrary set $S \subset \mathbb{R}^n$, denoted by $\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(S)$, is defined as

$$\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(S) := \inf \left\{ \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}^p : \varphi \in C_0^\infty(\mathbb{R}^n) \text{ \& } \varphi \geq 1_S \right\},$$

where 1_S denotes the characteristic function of S .

(ii) For $t \in (0, \infty)$, the (p, γ) -fractional capacity minimizing function associated with both $\dot{W}^{\gamma,p}(\mathbb{R}^n)$ and a nonnegative measure μ on $\mathbb{R}_+^{n+1}$, denoted by $c_p^\gamma(\mu, t)$, is defined as

$$c_p^\gamma(\mu, t) := \inf \left\{ \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(O) : \text{bounded open } O \subseteq \mathbb{R}^n, \mu(T(O)) > t \right\}.$$

In Lemma 2.5, we provide four standard estimates involving the capacity $\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(\cdot)$, measures and non-tangential maximal functions. Based on Lemma 2.2, we obtain an elementary Riesz integral upper estimate of the space-time fractional heat kernel $G_t(\cdot)$, see Lemma 2.6. In Section 3, given $\alpha > n$, $\beta \in (0, 1)$ and a nonnegative Radon measure μ on $\mathbb{R}_+^{n+1}$, we establish several characterizations of the extension:

$$\|R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})\|_{L^q(\mathbb{R}_+^{n+1}, \mu)} \lesssim \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} \quad (1.7)$$

for $0 < \gamma < n$, $1 \leq p < n/\gamma$ and $1 < q < \infty$. Our results exhibit that (1.7) holds if and only if

$$\begin{cases} \sup_{t>0} \frac{t^{p/q}}{c_p^\gamma(\mu; t)} < \infty, & p \leq q; \\ \int_0^\infty \left(\frac{t^{p/q}}{c_p^\gamma(\mu; t)} \right)^{q/(p-q)} \frac{dt}{t} < \infty, & p > q, \end{cases} \quad (1.8)$$

see Theorems 3.1, 3.2, 3.5 and 3.7, respectively. Specially, for $\gamma = 1$, our results obtained in Section 3 characterize the $L^q(\mathbb{R}_+^{n+1}, \mu)$ -extension via the p -variational capacity and generalize [21, Theorem 1.1 & 1.2], see Theorems 3.4 & 3.6.

However, it can be seen from Lemma 2.6 that the above characterizations (1.8) are invalid for the case $\dot{W}^{0,p}(\mathbb{R}^n)$, i.e., Lebesgue spaces $L^p(\mathbb{R}^n)$. To investigate the extension:

$$R_{\alpha,\beta} : L^p(\mathbb{R}^n) \mapsto L^q(\mathbb{R}_+^{n+1}, \mu)$$

with μ being a nonnegative Radon measure on $\mathbb{R}_+^{n+1}$. We introduce the L^p -capacity related to $R_{\alpha,\beta}$ which is defined as follows.

Definition 1.3. Let $1 \leq p < \infty$. For any subset $K \subseteq \mathbb{R}_+^{n+1}$, one defines

$$C_p^{(\alpha,\beta)}(E) := \inf \left\{ \|f\|_{L^p(\mathbb{R}^n)}^p : f \geq 0 \quad \& \quad R_{\alpha,\beta} f \geq 1_E \right\}.$$

In Section 4, we study some fundamental properties of the $L^p(\mathbb{R}_+^{n+1})$ -capacity $C_p^{(\alpha,\beta)}(\cdot)$, and further, estimate the capacities of fractional parabolic balls $B_{r_0}^{(\alpha,\beta)}(x_0, t_0)$, see Proposition 4.3. The strong and weak type inequalities corresponding to $C_p^{(\alpha,\beta)}(\cdot)$ are also derived in Lemma 4.8. Finally, in Section 5, for $\lambda > 0$, define

$$\kappa(\mu; \lambda) := \inf \left\{ C_p^{(\alpha,\beta)}(K) : \text{compact } K \subset \mathbb{R}_+^{n+1}, \quad \mu(K) \geq \lambda \right\}.$$

We prove that the extension

$$R_{\alpha,\beta} : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}_+^{n+1}, \mu)$$

is bounded if and only if

$$\begin{cases} \sup_{\lambda \in (0, \infty)} \frac{\lambda^{p/q}}{\kappa(\mu; \lambda)} < \infty, & 1 < p \leq q < \infty; \\ \int_0^\infty \left(\frac{\lambda^{p/q}}{\kappa(\mu; \lambda)} \right)^{q/(p-q)} \frac{d\lambda}{\lambda} < \infty, & 1 < q < p < \infty, \end{cases} \quad (1.9)$$

see Theorems 5.1 and 5.2, respectively.

We point out that the characterizations obtained in Sections 3 & 5 exhibit the relation between the differential property of functions and the order of capacities. Let $1 < p < q < \infty$. For the case $\dot{W}^{\gamma,p}(\mathbb{R}^n)$, we prove in Theorem 3.1 that one of equivalent conditions of (1.7) is

$$\sup_{\text{open } O \subset \mathbb{R}^n} \frac{(\mu(T(O)))^{p/q}}{\text{Cap}_{\mathbb{R}^n}^\gamma(O)} < \infty.$$

Specially, letting $O = B(x_0, r_0^\beta)$ the ball centered at x_0 with radius r_0^β , then

$$\text{Cap}_{\mathbb{R}^n}^\gamma(B(x_0, r_0^\beta)) \simeq r_0^{\beta(n-\gamma)}.$$

On the other hand, for the case $L^p(\mathbb{R}^n)$, Theorem 5.1 shows that, under the assumption that $t_0 \lesssim r^\alpha$, the extension

$$\|R_{\alpha,\beta}(\varphi)\|_{L^q(\mathbb{R}_+^{n+1}, \mu)} \lesssim \|\varphi\|_{L^p(\mathbb{R}^n)}$$

holds if and only if

$$(\mu(B_r^{(\alpha,\beta)}(x_0, t_0)))^{p/q} \lesssim r^{\beta n}, \quad \forall (r, t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n.$$

The above analysis indicates that, in the $L^q(\mathbb{R}_+^{n+1}, \mu)$ -extensions of Sobolev / Lebesgue spaces via (1.1), the loss of derivatives of functions need to be compensated via enhancing the order of capacities.

Some notations:

- Let $\Omega \subseteq \mathbb{R}^n$. Throughout this article, we use $C(\Omega)$ to denote the spaces of all continuous functions on Ω . Let $k \in \mathbb{N}_+ \cup \{\infty\}$. The symbol $C^k(\Omega)$ denotes the class of all functions $f : \Omega \rightarrow \mathbb{R}$ with k continuous partial derivatives. Let $C_0^\infty(\Omega)$ stand for all infinitely smooth functions with compact support in Ω .

- For $1 \leq p \leq \infty$, denote by p' the conjugate number of p , i.e., $1/p + 1/p' = 1$. $U \simeq V$ represents that there is a constant $c > 0$ such that $c^{-1}V \leq U \leq cV$ whose right inequality is also written as $U \lesssim V$. Similarly, one writes $V \gtrsim U$ for $V \geq cU$.
- For convenience, the positive constant C may change from one line to another and usually depends on the dimension n , α , β and other fixed parameters. For $f \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{f}$ means the Fourier transform of f .

2 Preliminaries on fractional heat kernels and Sobolev capacities

We first state some preliminaries on the space-time fractional heat kernel which will be used in the sequel. Let X_t denote a symmetric α stable process with density function denote by $K_{\alpha,t}(\cdot)$. This is characterized through the Fourier transform which is given by

$$\widehat{K_{\alpha,t}}(\xi) = e^{-v|\xi|^\alpha}.$$

Let $D = \{D_r, r \geq 0\}$ denote a β -stable subordinator and E_t be its first passage time. It is known that the density of the time changed X_{E_t} is given by the $G_t(x)$. By conditioning, we have

$$G_t(x) = \int_0^\infty K_{\alpha,t}(s, x) f_{E_t}(s) ds, \quad (2.1)$$

where

$$f_{E_t}(x) = t\beta^{-1}x^{-1-1/\beta}g_\beta(tx^{-1/\beta}),$$

where $g_\beta(\cdot)$ is the density function of D_1 , and is infinitely differentiable on the entire real line, with $g_\beta(u) = 0$ for $u \leq 0$. Moreover,

$$g_\beta(u) \sim K(\beta/u)^{(1-\beta/2)/(1-\beta)} \exp\{-|1-\beta|(u/\beta)^{\beta/(\beta-1)}\} \text{ as } u \rightarrow 0+,$$

and

$$g_\beta(u) \sim \frac{\beta}{\Gamma(1-\beta)} u^{-\beta-1} \text{ as } u \rightarrow \infty.$$

While the above expressions will be very important, we will also need the Fourier transform of $G_t(x)$:

$$\widehat{G_t}(\xi) = E_\beta(-v|\xi|^\beta),$$

where

$$E_\beta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\beta k)}$$

and

$$\frac{1}{1+\Gamma(1-\beta)} \leq E_\beta(-x) \leq \frac{1}{1+\Gamma(1+\beta)^{-1}x} \text{ for } x > 0.$$

Even though, we will be mainly using the representation given by (2.1), we also have another explicit description of the heat kernel. Using the convention $\sim$ to denote the Laplace transform, we get

$$\widetilde{\widehat{G_t}}(x) = \frac{\lambda^{\beta-1}}{\lambda^\beta + v|\xi|^\alpha}.$$

Inverting the Laplace transform yields

$$\widehat{G_t}(\xi) = E_\beta(-v|\xi|^\beta t^\beta),$$

where

$$E_\beta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\beta k)},$$

is the Mittag-Leffler function. In order to invert the Fourier transform, we will make use of the integral

$$\int_0^\infty \cos(ks) E_{\beta, \alpha}(-as^\mu) ds = \frac{\pi}{k} H_{3,3}^{2,1} \left[\frac{k^\mu}{a} \middle| \begin{matrix} (1,1), (\alpha, \beta), (1, \mu/2) \\ (1, \mu), (1,1), (1, \mu/2) \end{matrix} \right],$$

where $\Re(\alpha) > 0$, $\beta > 0$, $k > 0$, $a > 0$, $H_{p,q}^{m,n}$ is the H-function given in [10, Definition 1.9.1, page 55] and the formula

$$\frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\xi x} f(\xi) d\xi = \frac{1}{\pi} \int_0^\infty f(\xi) \cos(\xi x) d\xi.$$

This gives the function as

$$G_t(x) = \frac{1}{|x|} H_{3,3}^{2,1} \left[\frac{|x|^\alpha}{vt^\beta} \middle| \begin{matrix} (1,1), (1,\beta), (1,\alpha/2) \\ (1,\alpha), (1,1), (1,\alpha/2) \end{matrix} \right].$$

Remark 2.1. Specially, taking $\alpha = 2$, we can get, by the reduction formula for the H-function,

$$G_t(x) = \begin{cases} \frac{1}{|x|} H_{1,1}^{1,0} \left[\frac{|x|^2}{vt^\beta} \middle| \begin{matrix} (1,\beta) \\ (1,2) \end{matrix} \right], & \beta \in (0, 1); \\ \frac{1}{(4v\pi t)^{1/2}} \exp\left(-\frac{|x|^2}{4vt}\right), & \beta = 1. \end{cases}$$

In [11], M. Foondun and E. Nane obtained the following estimate for $G_t(\cdot)$.

Proposition 2.2. ([11, Lemma 2.1])

(i) There exists a positive constant C_1 such that for all $x \in \mathbb{R}^n$,

$$G_t(x) \geq C_1 \min \left\{ t^{-\beta n/\alpha}, \frac{t^\beta}{|x|^{n+\alpha}} \right\}.$$

(ii) If we further suppose that $\alpha > n$, then there exists a positive constant C_2 such that for all $x \in \mathbb{R}^n$,

$$G_t(x) \leq C_2 \min \left\{ t^{-\beta n/\alpha}, \frac{t^\beta}{|x|^{n+\alpha}} \right\}.$$

Below we always assume that $\alpha > n$. It can be deduced from (i) of Lemma 2.2 that

$$G_t(x) \sim \frac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}}. \quad (2.2)$$

Also, a direct computation, together with change of variables, gives

$$\int_{\mathbb{R}^n} G_t(x) dx \sim \int_{\mathbb{R}^n} \frac{t^\beta}{(|x| + t^{\beta/\alpha})^{n+\alpha}} dx \lesssim 1. \quad (2.3)$$

The following estimate is an immediate corollary of Lemma 2.2.

Lemma 2.3. Let $1 \leq r \leq p \leq \infty$ and $\varphi \in L^r(\mathbb{R}^n)$. For $\alpha > n$ & $t > 0$, we have

$$\|R_{\alpha, \beta}(\varphi)\|_{L^p(\mathbb{R}^n)} \lesssim t^{-n\beta(1/r-1/p)/\alpha} \|\varphi\|_{L^r(\mathbb{R}^n)}.$$

Proof. Let q obey $1/r + 1/q = 1/p + 1$. By Young's inequality,

$$\|R_{\alpha, \beta}(\varphi)\|_{L^p(\mathbb{R}^n)} = \|G_t * \varphi\|_{L^p(\mathbb{R}^n)} \lesssim \|\varphi\|_{L^r(\mathbb{R}^n)} \|G_t(\cdot)\|_{L^q(\mathbb{R}^n)}.$$

Applying Lemma 2.2 and a direct computation, we get

$$\|G_t(\cdot)\|_{L^q(\mathbb{R}^n)} \sim \left(\int_{\mathbb{R}^n} \frac{t^{\beta q}}{(|x| + t^{\beta/\alpha})^{q(n+\alpha)}} dx \right)^{1/q} \lesssim t^{\beta n(1/q-1)/\alpha},$$

which implies that

$$\|R_{\alpha,\beta}(\varphi)\|_{L^p(\mathbb{R}^n)} \lesssim t^{\beta n(1/q-1)/\alpha} \|\varphi\|_{L^r(\mathbb{R}^n)} = t^{-\beta n(1/r-1/p)/\alpha} \|\varphi\|_{L^r(\mathbb{R}^n)}.$$

□

Below we state some basic properties of the capacities $\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(\cdot)$ and refer the reader to [3, Section 2] for the details.

Proposition 2.4. *The following properties are valid.*

- (i) $\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(\emptyset) = 0$.
- (ii) If $K_1 \subseteq K_2 \subset \mathbb{R}^n$, then $\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(K_1) \leq \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(K_2)$.
- (iii) For any sequence $\{K_j\}_{j=1}^\infty$ of subsets of $\mathbb{R}_+^{n+1}$,

$$\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}\left(\bigcup_{j=1}^\infty K_j\right) \leq \sum_{j=1}^\infty \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(K_j).$$

- (iv) For any decreasing sequence $\{K_j\}_{j=1}^\infty$ in $\mathbb{R}^n$ with $K = \bigcap_j K_j$ and any increasing sequence $\{E_j\}_{j=1}^\infty$ in $\mathbb{R}^n$ with $E = \bigcup_j E_j$, one has

$$\begin{cases} \lim_{j \rightarrow \infty} \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(K_j) = \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(K), \\ \lim_{j \rightarrow \infty} \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(E_j) = \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(E). \end{cases}$$

Let $\mathcal{M}_+(\mathbb{R}_+^{n+1})$ represent the class of all nonnegative Radon measures on $\mathbb{R}_+^{n+1}$.

Lemma 2.5. *Let $\alpha > n$, $\beta \in (0, 1)$ and $\gamma \in (0, n)$. Given $\varphi \in \dot{W}^{\gamma,p}(\mathbb{R}^n)$, $p \geq 1$, $s > 0$, and $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$, define*

$$L_s^{\alpha,\beta}(\varphi) := \left\{ (x, t) \in \mathbb{R}_+^{n+1} : |R_{\alpha,\beta}\varphi(x, t^{\alpha/\beta})| > s \right\}$$

and

$$M_s^{\alpha,\beta}(\varphi) := \left\{ y \in \mathbb{R}^n : \sup_{|y-x|<t} |R_{\alpha,\beta}\varphi(x, t^{\alpha/\beta})| > s \right\}.$$

Then the following four statements are true.

- (i) For any natural number k

$$\mu \left(L_s^{\alpha,\beta}(\varphi) \cap T(B(0, k)) \right) \leq \mu \left(T \left(M_s^{\alpha,\beta}(\varphi) \cap B(0, k) \right) \right).$$

- (ii) For any natural number k ,

$$\text{Cap}_{\mathbb{R}^n}^{\gamma,p} \left(M_s^{\alpha,\beta}(\varphi) \cap B(0, k) \right) \geq c_p^\gamma \left(\mu, \mu \left(T \left(M_s^{\alpha,\beta}(\varphi) \cap B(0, k) \right) \right) \right).$$

- (iii) There exists a constant $\theta_{n,\alpha} > 0$ such that

$$\sup_{|y-x|<t} |R_{\alpha,\beta}\varphi(y, t^{\alpha/\beta})| \leq \theta_{n,\alpha} \mathcal{M}\varphi(x), \quad x \in \mathbb{R}^n,$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator:

$$\mathcal{M}\varphi(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |\varphi(y)| dy, \quad x \in \mathbb{R}^n.$$

- (iv) Let O be a bounded open set contained in $\text{Int}(\{x \in \mathbb{R}^n : \varphi(x) \geq 1\})$. There exists a constant $\eta_{n,\alpha} > 0$ such that if $(x, t) \in T(O)$, then $R_{\alpha,\beta}(|\varphi|)(x, t^{\alpha/\beta}) \geq \eta_{n,\alpha}$.

Proof. (i) Since $\sup_{|y-x|<t} |R_{\alpha,\beta}\varphi(x, t^{\alpha/\beta})|$ is lower semicontinuous on $\mathbb{R}^n$, we can see that $M_s^{\alpha,\beta}(\varphi)$ is an open subset of $\mathbb{R}^n$ and

$$\begin{cases} L_s^{\alpha,\beta}(\varphi) \subseteq T(M_s^{\alpha,\beta}(\varphi)); \\ \mu(L_s^{\alpha,\beta}(\varphi)) \leq \mu(T(M_s^{\alpha,\beta}(\varphi))). \end{cases}$$

Then

$$\mu(L_s^{\alpha,\beta}(\varphi) \cap T(B(0, k))) \leq \mu(T(M_s^{\alpha,\beta}(\varphi) \cap T(B(0, k)))) = \mu(T(M_s^{\alpha,\beta}(\varphi) \cap B(0, k))).$$

(ii) It follows from the definition of $c_p^\gamma(\mu; t)$.

(iii) Let $\Phi_t(x) := t^{-n}\Phi(x/t)$ with $\Phi(x) = (|x| + 1)^{-n-\alpha}$. By (2.2), it holds

$$\sup_{|y-x|<t} |R_{\alpha,\beta}\varphi(y, t^{\alpha/\beta})| \leq C_1 \sup_{|y-x|<t} |\Phi_t * \varphi(y)|.$$

Since Φ is radial, bounded, decreasing and integrable on $\mathbb{R}^n$, it follows from [19, page 57, Proposition] that

$$\sup_{|y-x|<t} |\Phi_t * \varphi(y)| \leq C_2 \mathcal{M}\varphi(x)$$

and thus for a constant $\theta_{n,\alpha}$,

$$\sup_{|y-x|<t} |R_{\alpha,\beta}\varphi(y, t^{\alpha/\beta})| \leq \theta_{n,\alpha} \mathcal{M}\varphi(x).$$

(iv) For any $(x, t) \in T(O)$, we have

$$B(x, t) \subseteq O \subseteq \text{Int} \left(\left\{ x : \varphi(x) > 1 \right\} \right).$$

It can be deduced from (i) of Proposition 2.2 that there exist σ and C such that

$$\inf \left\{ G_{t^{\alpha/\beta}}(x) : |x| < \sigma t \right\} \geq Ct^{-n}.$$

Then

$$G_{t^{\alpha/\beta}} * |\varphi|(x, t) \geq Ct^{-n} \int_{B(x, \sigma t) \cap \text{Int}(\{x: \varphi(x) \geq 1\})} |\varphi|(y) dy.$$

If $\sigma > 1$, then

$$B(x, \sigma t) \cap \text{Int}(\{x: \varphi(x) \geq 1\}) \supseteq B(x, t) \cap \text{Int}(\{x: \varphi(x) \geq 1\}) = B(x, t).$$

If $\sigma \leq 1$, then

$$B(x, \sigma t) \cap \text{Int}(\{x: \varphi(x) \geq 1\}) = B(x, \sigma t).$$

Thus $G_{t^{\alpha/\beta}} * |\varphi|(x, t) \geq \eta_{n,\alpha}$ for some constant $\eta_{n,\alpha} > 0$.

□

Lemma 2.6. If $\alpha > n$, $\beta \in (0, 1)$, $\gamma \in (0, n)$ and $(x, t) \in \mathbb{R}_+^{n+1}$, then

$$\int_{\mathbb{R}^n} \frac{G_{t^{\alpha/\beta}}(y)}{|y-x|^{n-\gamma}} dy \lesssim (t+|x|)^{\gamma-n}.$$

Proof. Note that $G_t(x) \sim \frac{t^\beta}{(|x|+t^{\beta/\alpha})^{n+\alpha}}$. Define

$$J(x, t) = \int_{\mathbb{R}^n} \frac{t^\alpha |y-x|^{\gamma-n}}{(|x|+t)^{n+\alpha}} dy.$$

Via the change of variables: $x \rightarrow tx$ & $y \rightarrow ty$, it is sufficient to show that

$$J(x, 1) \lesssim (1+|x|)^{\gamma-n}.$$

Since $J(0, 1) \lesssim 1$, we may assume that $|x| > 0$. Write $J(x, 1) \lesssim I_1(x) + I_2(x)$, where

$$\begin{cases} I_1(x) := \int_{B(x, |x|/2)} \frac{|y-x|^{\gamma-n}}{(|x|+1)^{n+\alpha}} dy; \\ I_2(x) := \int_{\mathbb{R}^n \setminus B(x, |x|/2)} \frac{|y-x|^{\gamma-n}}{(|x|+1)^{n+\alpha}} dy. \end{cases}$$

Since $|x-y| \leq |x|/2$ implies that $|y| \simeq |x|$, we have

$$\begin{aligned} I_1(x) &\lesssim (1+|x|)^{-(n+\alpha)} \int_{B(x, |x|/2)} |y-x|^{\gamma-n} dy \\ &\lesssim (1+|x|^2)^{-(n+\alpha)/2} \int_0^{|x|/2} s^{\gamma-1} ds \\ &\lesssim (1+|x|^2)^{-(n+\alpha)/2} |x|^\gamma \\ &\lesssim (1+|x|)^{\gamma-n}. \end{aligned}$$

If $|x-y| > |x|/2$, then

$$I_2(x) \lesssim |x|^{\gamma-n} \int_{\mathbb{R}^n \setminus B(x, |x|/2)} \frac{1}{(|y|+1)^{(n+\alpha)}} dy.$$

For $|x-y| > |x|/2$, it holds $|y| < 3|x-y|$ and

$$I_2(x) \lesssim \int_{\mathbb{R}^n \setminus B(x, |x|/2)} \frac{1}{(|y|+1)^{n+\alpha} |y|^{n-\gamma}} dy \lesssim 1.$$

So, $I_2(x) \lesssim (1+|x|)^{\gamma-n}$ and $J(x, 1) \lesssim (1+|x|)^{\gamma-n}$. □

The following result provides the capacity strong estimates for $\text{Cap}_{\mathbb{R}^n}^{(\gamma, p)}(\cdot)$. For the proofs, we refer the readers to [4, 22] and the references therein.

Lemma 2.7. Let $\gamma \in (0, n)$ and $p \in [1, n/\gamma]$.

(i) For $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_0^\infty \text{Cap}_{\mathbb{R}^n}^{(\gamma, p)}(\{x \in \mathbb{R}^n : |f(x)| \geq s\}) ds^p \lesssim \|\varphi\|_{\dot{W}^{\gamma, p}(\mathbb{R}^n)}^p.$$

(ii) For $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_0^\infty \text{Cap}_{\mathbb{R}^n}^{(\gamma, p)}(\{x \in \mathbb{R}^n : |\mathcal{M}\varphi(x)| \geq s\}) ds^p \lesssim \|\varphi\|_{\dot{W}^{\gamma, p}(\mathbb{R}^n)}^p.$$

For handling the endpoint case $p = n\gamma$, we need the following Riesz potentials on $\mathbb{R}^{2n}$, see Adam-Xiao [4] and Adam [2]. For $\gamma \in (0, 2n)$,

$$I_\gamma^{(2n)} \star f(z) := \int_{\mathbb{R}^{2n}} |x-y|^{\gamma-2n} f(y) dy, \quad z \in \mathbb{R}^{2n}.$$

For $\gamma \in (0, 2n)$, the space $\dot{\mathcal{L}}_\gamma^p(\mathbb{R}^{2n})$ is defined as the set

$$\left\{ f : f = I_\gamma^{(2n)} \star \varphi, \varphi \in L^p(\mathbb{R}^{2n}) \right\}$$

with the norm $\|f\|_{\dot{\mathcal{L}}_\gamma^p(\mathbb{R}^{2n})} := \|I_\gamma^{(2n)} \star \varphi\|_{\dot{\mathcal{L}}_\gamma^p(\mathbb{R}^{2n})} = \|\varphi\|_{L^p(\mathbb{R}^{2n})}$. Formally, we write $\dot{\mathcal{L}}_\gamma^p(\mathbb{R}^{2n}) = I_\gamma^{(2n)} \star L^p(\mathbb{R}^{2n})$.

The following result is a particular case of [2, Theorem 5.2] or [4, Theorem A].

Lemma 2.8. Let $\gamma \in (0, n)$. There are a linear extension operator

$$\mathcal{E} : \dot{W}^{\gamma, n/\gamma}(\mathbb{R}^n) \longrightarrow \dot{\mathcal{L}}_{2\gamma}^{n/\gamma}(\mathbb{R}^{2n}),$$

and a linear restriction operator

$$\mathcal{R} : \dot{\mathcal{L}}_{2\gamma}^{n/\gamma}(\mathbb{R}^{2n}) \longrightarrow \dot{W}^{\gamma, n/\gamma}(\mathbb{R}^n)$$

such that $\mathcal{R}\mathcal{E}$ is the identity. Moreover,

- (i) For $\varphi \in \dot{W}^{\gamma, n/\gamma}(\mathbb{R}^n)$, $\|\mathcal{E}\varphi\|_{\dot{\mathcal{L}}_{2\gamma}^{n/\gamma}(\mathbb{R}^{2n})} \lesssim \|\varphi\|_{\dot{W}^{\gamma, n/\gamma}(\mathbb{R}^n)}$.
- (ii) For $g \in \dot{\mathcal{L}}_{2\gamma}^{n/\gamma}(\mathbb{R}^{2n})$, $\|\mathcal{R}g\|_{\dot{W}^{\gamma, n/\gamma}(\mathbb{R}^n)} \lesssim \|g\|_{\dot{\mathcal{L}}_{2\gamma}^{n/\gamma}(\mathbb{R}^{2n})}$.

3 $L^q(\mathbb{R}_+^{n+1}, \mu)$ -extension of $\dot{W}^{\gamma, p}(\mathbb{R}^n)$

In this section, we will show that the embedding (1.7) can be characterized by conditions in terms of fractional capacities. For $0 < p, q < \infty$ and a nonnegative Radon measure μ on $\mathbb{R}_+^{n+1}$, $L^{q, p}(\mathbb{R}_+^{n+1}, \mu)$ and $L^q(\mathbb{R}_+^{n+1}, \mu)$ denote the Lorentz space and the Lebesgue space of all functions on $\mathbb{R}_+^{n+1}$, respectively, for which

$$\|g\|_{L^{q, p}(\mathbb{R}_+^{n+1}, \mu)} := \left\{ \int_0^\infty \left(\mu \left(\left\{ (x, t) \in \mathbb{R}_+^{n+1} : |g(x, t)| > s \right\} \right) \right)^{p/q} ds^p \right\}^{1/p} < \infty$$

and

$$\|g\|_{L^q(\mathbb{R}_+^{n+1}, \mu)} := \left(\int_{\mathbb{R}_+^{n+1}} |g(x, t)|^q d\mu \right)^{1/q} < \infty,$$

respectively. Moreover, we denote by $L^{q, \infty}(\mathbb{R}_+^{n+1}, \mu)$ the set of all μ -measurable functions g on $\mathbb{R}_+^{n+1}$ with

$$\|g\|_{L^{q, \infty}(\mathbb{R}_+^{n+1}, \mu)} := \sup_{s>0} s \left(\mu \left(\left\{ (x, t) \in \mathbb{R}_+^{n+1} : |g(x, t)| > s \right\} \right) \right)^{1/q} < \infty.$$

3.1 $L^q(\mathbb{R}_+^{n+1}, \mu)$ -extension of $\dot{W}^{\gamma, p}(\mathbb{R}^n)$ when $p \leq q$

Theorem 3.1. Let $\alpha > n$, $\beta \in (0, 1)$. Let $\gamma \in (0, n)$ when $1 \leq p \leq n/\gamma$, $p \leq q < \infty$ and $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$. Then the following statements are equivalent:

- (i) $\|R_{\alpha, \beta} \varphi(x, t^{\alpha/\beta})\|_{L^{q, p}(\mathbb{R}_+^{n+1}, \mu)} \lesssim \|\varphi\|_{\dot{W}^{\gamma, p}(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{\gamma, p}(\mathbb{R}^n);$
- (ii) $\|R_{\alpha, \beta} \varphi(x, t^{\alpha/\beta})\|_{L^q(\mathbb{R}_+^{n+1}, \mu)} \lesssim \|\varphi\|_{\dot{W}^{\gamma, p}(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{\gamma, p}(\mathbb{R}^n);$
- (iii) $\|R_{\alpha, \beta} \varphi(x, t^{\alpha/\beta})\|_{L^{q, \infty}(\mathbb{R}_+^{n+1}, \mu)} \lesssim \|\varphi\|_{\dot{W}^{\gamma, p}(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{\gamma, p}(\mathbb{R}^n);$
- (iv) $\sup_{t>0} \frac{t^{p/q}}{c_p^\gamma(\mu; t)} < \infty;$
- (v) $(\mu(T(O)))^{p/q} \lesssim \text{Cap}_{\mathbb{R}^n}^{\gamma, p}(O)$ holds for any bounded open set $O \subseteq \mathbb{R}^n$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) can be deduced from

$$\left(s^q \mu(L_s^{\alpha, \beta}(\varphi)) \right)^{p/q} \leq \left(q \int_0^\infty \mu(L_s^{\alpha, \beta}(\varphi)) s^{q-1} ds \right)^{p/q} \leq \int_0^\infty \left(\mu(L_s^{\alpha, \beta}(\varphi)) \right)^{p/q} ds^p$$

since

$$q \mu(L_r^{\alpha, \beta}(\varphi)) r^{q-1} \leq \frac{d}{dr} \left(\int_0^r \left(\mu(L_s^{\alpha, \beta}(\varphi)) \right)^{p/q} ds^p \right)^{q/p}.$$

Now, we prove (iii) $\Rightarrow$ (v) $\Rightarrow$ (i). If (iii) is true, i.e.,

$$K_{p,q}(\mu) = \sup_{\varphi \in C_0^\infty(\mathbb{R}^n) \text{ and } \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} > 0} \frac{\sup_{s>0} s \left(\mu \left(\left\{ (x, t) \in \mathbb{R}_+^{n+1} : |R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})| > s \right\} \right) \right)^{1/q}}{\|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}} < \infty,$$

then Lemma 2.5 implies

$$(\mu(T(O)))^{1/q} \lesssim K_{p,q}(\mu) \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} \quad (3.1)$$

for any $f \in C_0^\infty(\mathbb{R}^n)$ and any open set $O \subseteq \text{Int}(\{x \in \mathbb{R}^n : f(x) \geq 1\})$. Thus (v) holds. For (v) $\Rightarrow$ (i), denote

$$Q_{p,q}(\mu) := \sup \left\{ \frac{(\mu(T(O)))^{p/q}}{\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(O)} : \text{bounded open set } O \subseteq \mathbb{R}^n \right\} < \infty.$$

Lemmas 2.5 & 2.7 imply that

$$\begin{aligned} & \int_0^\infty \left(\mu \left(L_s^{\alpha,\beta}(\varphi) \cap T(B(0, k)) \right) \right)^{p/q} ds^p \\ & \leq \int_0^\infty \left(\mu \left(T \left(M_s^{\alpha,\beta}(\varphi) \cap B(0, k) \right) \right) \right)^{p/q} ds^p \\ & \leq \int_0^\infty \left(\mu \left(T \left(\left\{ x \in \mathbb{R}^n : \theta_{n,\alpha} \mathcal{M}(\varphi)(x) > s \right\} \cap B(0, k) \right) \right) \right)^{p/q} ds^p \\ & \leq Q_{p,q}(\mu) \int_0^\infty \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)} \left(\left\{ x \in \mathbb{R}^n : \theta_{n,\alpha} \mathcal{M}(\varphi)(x) > s \right\} \cap B(0, k) \right) ds^p \\ & \leq Q_{p,q}(\mu) \int_0^\infty \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)} \left(\left\{ x \in \mathbb{R}^n : \theta_{n,\alpha} \mathcal{M}(\varphi)(x) > s \right\} \right) ds^p \\ & \leq Q_{p,q}(\mu) \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}^p \end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. Letting $k \rightarrow \infty$ reaches (i).

Now, we will prove (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). If (iii) is true, let O be a bounded open subset of $\text{Int}(\{x \in \mathbb{R}^n, \varphi(x) \geq 1\})$. Then

$$t^{p/q} \lesssim (K_{p,q}(\mu))^p \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(O)$$

whenever $t \in (0, \mu(T(O)))$. So, $t^{p/q} \lesssim (K_{p,q}(\mu))^p c_p^\gamma(\mu; t)$, which means (iv) is true.

Assume that (iv) is true. By Lemmas 2.5 & 2.7, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} & \int_0^\infty \left(\mu \left(L_s^{\alpha,\beta}(\varphi) \cap T(B(0, k)) \right) \right)^{p/q} ds^p \\ & \leq \int_0^\infty \frac{\left(\mu \left(L_s^{\alpha,\beta}(\varphi) \cap T(B(0, k)) \right) \right)^{p/q}}{c_p^\gamma(\mu; \mu(L_s^{\alpha,\beta}(\varphi) \cap T(B(0, k))))} \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)} \left(M_s^{\alpha,\beta}(\varphi) \cap B(0, k) \right) ds^p \\ & \lesssim \sup_{t>0} \frac{t^{p/q}}{c_p^\gamma(\mu; t)} \int_0^\infty \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)} \left(\left\{ x \in \mathbb{R}^n : \theta_{n,\alpha} \mathcal{M}(\varphi)(x) > s \right\} \cap B(0, k) \right) ds^p \\ & \lesssim \sup_{t>0} \frac{t^{p/q}}{c_p^\gamma(\mu; t)} \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}^p, \end{aligned}$$

which gives (i) via letting $k \rightarrow \infty$. □

Theorem 3.2. Let $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$, $\alpha > n$, $\beta \in (0, 1)$, $\gamma \in (0, n)$, $1 < p < \min\{n/\gamma, q\}$, or $1 = p \leq q < \infty$. Then the following two statements are equivalent.

(i) For any $\varphi \in \dot{W}^{\gamma,p}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})|^q d\mu(x, t) \right)^{1/q} \lesssim \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}.$$

(ii)

$$\sup_{x \in \mathbb{R}^n, r > 0} \frac{(\mu(T(B(x, r))))^{p/q}}{\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(B(x, r))} < \infty.$$

Proof. It is enough to prove that (ii) implies (iii) or (v) of Theorem 3.1. When $1 = p \leq q < \infty$, if (ii) holds, then

$$\|\mu\|_{1,q} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{(\mu(T(B(x, r))))^{1/q}}{\text{Cap}_{\mathbb{R}^n}^{(\gamma,1)}(B(x, r))} < \infty.$$

Suppose that a bounded open set $O \subseteq \mathbb{R}^n$ is covered by a sequence of dyadic cubes $\{I_j\}$ in $\mathbb{R}^n$ with $\sum_j |I_j|^{(n-\gamma)/n} < \infty$. It follows from [9, Lemma 4.1] that there exists another sequence of dyadic cubes $\{J_j\}$ in $\mathbb{R}^n$ such that

$$\begin{cases} \text{Int}(J_j) \cap \text{Int}(I_k) = \emptyset, & j \neq k; \\ \bigcup_k I_k = \bigcup_j J_j; \\ \sum_j |J_j|^{1-\gamma/n} \leq \sum_k |I_k|^{1-\gamma/n}; \\ T(O) \lesssim \bigcup_j T(\text{Int}(5\sqrt{n}J_j)). \end{cases}$$

Thus,

$$\mu(T(O)) \lesssim \|\mu\|_{1,q} \sum_j |5\sqrt{n}J_j|^{q(n-\gamma)/n} \lesssim \|\mu\|_{1,q} \left(\sum_k |I_k|^{(n-\gamma)/n} \right)^q,$$

which implies

$$\mu(T(O)) \lesssim \|\mu\|_{1,q} H_\infty^{n-\gamma}(O).$$

Here $H_\infty^d(\cdot)$ is the d -dimensional Hausdorff capacity. Then, the classical result $\text{Cap}_{\mathbb{R}^n}^{(\gamma,1)}(\cdot) \sim H_\infty^d(\cdot)$ implies

$$\mu(T(O)) \lesssim \|\mu\|_{1,q} (\text{Cap}_{\mathbb{R}^n}^{(\gamma,1)}(O))^q.$$

Thus, (v) of Proposition 3.1 holds.

When $1 < p < \min\{n/\gamma, q\}$,

$$\|\mu\|_{p,q} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{(\mu(T(B(x, r))))^{p/q}}{\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(B(x, r))} < \infty.$$

Let $\varphi_0 \in \dot{W}^{\gamma,p}(\mathbb{R}^n)$ and μ_s be the restriction of μ to $L_s^{\alpha,\beta}(\dot{W}^{\gamma,p}(\mathbb{R}^n))$. For any $\varphi \in \dot{W}^{\gamma,p}(\mathbb{R}^n)$, we have

$$|\varphi(x)| \lesssim \int_{\mathbb{R}^n} \frac{|(-\Delta)^{\gamma/2} \varphi(y)|}{|x-y|^{n-\gamma}} dy, \quad x \in \mathbb{R}^n.$$

Thus, Lemma 2.6 implies

$$s\mu(L_s^{\alpha,\beta}(\varphi_0)) \lesssim \int_{L_s^{\alpha,\beta}(\varphi_0)} |R_{\alpha,\beta} \varphi_0(x, t^{\alpha/\beta})| d\mu(x, t)$$

$$\begin{aligned}
&\lesssim \int_{L_s^{\alpha,\beta}(\varphi_0)} \left| \int_{\mathbb{R}^n} G_{t^{\alpha/\beta}}(y) \varphi_0(x-y) dy \right| d\mu(x, t) \\
&\lesssim \int_{L_s^{\alpha,\beta}(\varphi_0)} \left\{ \int_{\mathbb{R}^n} G_{t^{\alpha/\beta}}(y) \left(\int_{\mathbb{R}^n} \frac{|(-\Delta)^{\gamma/2} \varphi_0(z)|}{|x-y-z|^{n-\gamma}} dz \right) dy \right\} d\mu(x, t) \\
&\lesssim \int_{L_s^{\alpha,\beta}(\varphi_0)} \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{G_{t^{\alpha/\beta}}(y) dy}{|y-(x-z)|^{n-\gamma}} \right) |(-\Delta)^{\gamma/2} \varphi_0(z)| dz \right\} d\mu_s(x, t) \\
&\lesssim \int_{\mathbb{R}_+^{n+1}} \left\{ \int_{\mathbb{R}^n} \frac{|(-\Delta)^{\gamma/2} \varphi_0(z)|}{(t+|x-z|)^{n-\gamma}} dz \right\} d\mu_s(x, t) \\
&\lesssim \int_{\mathbb{R}^n} |(-\Delta)^{\gamma/2} \varphi_0(z)| \left\{ \int_{\mathbb{R}_+^{n+1}} \frac{d\mu_s(x, t)}{(t+|x-z|)^{n-\gamma}} \right\} dz \\
&\lesssim \int_{\mathbb{R}^n} |(-\Delta)^{\gamma/2} \varphi_0(z)| \left\{ \int_0^\infty \mu_s(T(B(z, r)) r^{\gamma-n-1} dr \right\} dz \\
&\lesssim J(\lambda) + K(\lambda),
\end{aligned}$$

where

$$J(\lambda) := \int_0^\lambda \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{\gamma/2} \varphi_0(z)| \mu_s(T(B(z, r))) dz \right\} r^{\gamma-n-1} dr$$

and

$$K(\lambda) := \int_\lambda^\infty \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{\gamma/2} \varphi_0(z)| \mu_s(T(B(z, r))) dz \right\} r^{\gamma-n-1} dr.$$

Since $Cap_{\mathbb{R}^n}^{(\gamma,p)}(B(x, r)) \sim r^{n-p\gamma}$, it follows from the definition of $\|\mu\|_{p,q}$ and Hölder's inequality that for $1/p + 1/p' = 1$,

$$\mu_s(T(B(z, r))) \lesssim (\mu_s(T(B(z, r))))^{1/p'} \|\mu\|_{p,q}^{q/p^2} r^{q(n-p\gamma)/p^2}.$$

Thus we have

$$\int_{\mathbb{R}^n} \mu_s(TB(x, r)) dx \lesssim r^n \mu_s(L_s^{\alpha,\beta}(\varphi_0))$$

and

$$\begin{aligned}
J(\lambda) &\lesssim \int_0^\lambda \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{\gamma/2} \varphi_0(z)| (\mu_s(T(B(z, r))))^{1/p'} \|\mu\|_{p,q}^{q/p^2} r^{q(n-p\gamma)/p^2} dz \right\} r^{\gamma-n-1} dr \\
&\lesssim \|\varphi_0\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} \|\mu\|_{p,q}^{q/p^2} \int_0^\lambda \left\{ \int_{\mathbb{R}^n} \mu_s(T(B(z, r))) dz \right\}^{1/p'} r^{q(n-p\gamma)/p^2 + \gamma - n - 1} dr \\
&\lesssim \|\varphi_0\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} \|\mu\|_{p,q}^{q/p^2} \int_0^\lambda (r^n \mu_s(T(B(z, r))))^{1/p'} r^{q(n-p\gamma)/p^2 + \gamma - n - 1} dr \\
&\lesssim \|\varphi_0\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} \|\mu\|_{p,q}^{q/p^2} \left(\mu(L_s^{\alpha,\beta}(\varphi_0)) \right)^{1/p'} \lambda^{(q-p)(n-p\gamma)/p^2}.
\end{aligned}$$

Similarly, we can obtain

$$K(\lambda) \lesssim \|\varphi_0\|_{\dot{W}^{\gamma,p}} \mu(L_s^{\alpha,\beta}(\varphi_0)) \lambda^{\gamma-n/p}.$$

Thus,

$$\begin{aligned} s\mu(L_s^{\alpha,\beta}(\varphi_0)) &\lesssim \|\varphi_0(\cdot, t^{\alpha/\beta})\|_{\dot{W}^{\gamma,p}} \mu_s(L_s^{\alpha,\beta}(\varphi_0)) \\ &\times \left\{ \lambda^{\gamma-n/p} + \|\mu\|_{p,q}^{q/p^2} \left(\mu_s(L_s^{\alpha,\beta}(\varphi_0)) \right)^{-1/p} \lambda^{(q-p)(n-p\gamma)/p^2} \right\}. \end{aligned} \quad (3.2)$$

In (3.2), take

$$\lambda = \|\mu\|_{p,q}^{1/(p\gamma-n)} \left(\mu_s(L_s^{\alpha,\beta}(\varphi_0)) \right)^{\frac{p}{q(n-p\gamma)}}$$

such that

$$\lambda^{\gamma-n/p} = \|\mu\|_{p,q}^{q/p^2} \left(\mu_s(L_s^{\alpha,\beta}(\varphi_0)) \right)^{-1/p} \lambda^{(q-p)(n-p\gamma)/p^2}.$$

It holds

$$s \left(\mu(L_s^{\alpha,\beta}(\varphi_0)) \right)^{1/q} \lesssim \|\mu\|_{p,q}^{1/p} \|\varphi_0\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}$$

which implies (iii) of Proposition 3.1. $\square$

Let $q = np/(n - p\gamma)$ and $\mu(x, t) = \delta_{(x_0, t_0)}$ be the Dirac measure at (x_0, t_0) . If $(x_0, t_0) \notin T(B(x, r))$, then $\delta_{(x_0, t_0)}(T(B(x, r))) = 0$. If $(x_0, t_0) \in T(B(x, r))$, then $B(x_0, t_0) \subseteq B(x, r)$ and $t_0^n \leq r^n$ and so

$$\delta_{(x_0, t_0)}(T(B(x, r))) \leq \frac{r^n}{t_0^n} = \frac{r^{(n-p\gamma)q/p}}{t_0^n}.$$

Then the equivalence $Cap_{\mathbb{R}^n}^{(\gamma,p)}(B(x, r)) \sim r^{n-p\gamma}$ implies

$$\frac{\delta_{(x_0, t_0)} T(B(x, r))^{p/q}}{Cap_{\mathbb{R}^n}^{(\gamma,p)}(B(x, r))} \leq t_0^{-np/q}.$$

So,

$$\|\delta_{(x_0, t_0)}(T(B(x, r)))\|_{p,q} = t_0^{p\gamma-n}.$$

It follows from the proof of Proposition 3.2 that

$$|R_{\alpha,\beta}\varphi(x_0, t_0^{\alpha/\beta})| \lesssim \|\delta_{(x_0, t_0)}\|_{p,q} \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}.$$

Thus, we have the following decay estimate of the solution.

Corollary 3.3. *Let $\alpha > n$ and $\beta \in (0, 1)$. If $\varphi \in \dot{W}^{\gamma,p}(\mathbb{R}^n)$, then for $1 \leq p < n/\gamma$ and $\gamma \in (0, n)$,*

$$|R_{\alpha,\beta}\varphi(x_0, t_0)| \lesssim t_0^{\beta(p\gamma-n)/\alpha} \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} \quad \forall (x_0, t_0) \in \mathbb{R}_+^{n+1}.$$

Let $cap_p(S)$ denote the p -variational capacity of an arbitrary set $S \subseteq \mathbb{R}^n$ defined by

$$cap_p(S) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p dx : f \in \dot{W}^{1,p}(\mathbb{R}^n), S \subseteq \text{Int} \left(\left\{ x \in \mathbb{R}^n : f(x) \geq 1 \right\} \right) \right\},$$

where $\text{Int}(E)$ stands for the interior of a set $E \subseteq \mathbb{R}^n$; $c_p(\mu, t)$ denotes the corresponding p -variational capacity minimizing function of $t \in (0, \infty)$ associated with a nonnegative measure μ on $\mathbb{R}_+^{n+1}$:

$$c_p(\mu, t) := \inf \left\{ cap_p(O) : \text{bounded open } O \subseteq \mathbb{R}^n, \mu(T(O)) > t \right\}.$$

As an immediate corollary of Theorem 3.1, it holds

Theorem 3.4. *Let $\alpha > n$, $\beta \in (0, 1)$. Assume that $p \leq q < \infty$, $1 \leq p < n$ and $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$. The following statements are equivalent:*

(i)

$$\left(\int_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})|^q d\mu(x, t) \right)^{1/q} \lesssim \|\nabla \varphi\|_{L^p(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{1,p}(\mathbb{R}^n);$$

(ii)

$$\sup_{\lambda > 0} \lambda \left(\mu \left(\left\{ (x, t) \in \mathbb{R}_+^{n+1} : |R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})| > \lambda \right\} \right) \right)^{1/q} \lesssim \|\nabla \varphi\|_{L^p(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{\gamma,p}(\mathbb{R}^n);$$

(iii)

$$\sup_{t>0} \frac{t^{p/q}}{c_p(\mu; t)} < \infty;$$

(iv)

$$\sup \left\{ \frac{(\mu(T(O)))^{p/q}}{\text{cap}_p(O)} : \text{bounded open } O \subseteq \mathbb{R}^n \right\}.$$

3.2 $L^q(\mathbb{R}_+^{n+1}, \mu)$ -extension of $\dot{W}^{\gamma,p}(\mathbb{R}^n)$ when $p > q$

Theorem 3.5. Let $\alpha > n$, $\beta \in (0, 1)$, $\gamma \in (0, n)$, $0 < q < p$, $1 < p \leq n/\gamma$ and $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$. Then the following statements are equivalent:

(i)

$$\left(\int_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})|^q d\mu(x, t) \right)^{1/q} \lesssim \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{\gamma,p}(\mathbb{R}^n);$$

(ii)

$$\int_0^\infty \left(\frac{t^{p/q}}{c_p^\gamma(\mu; t)} \right)^{q/(p-q)} \frac{dt}{t} < \infty. \quad (3.3)$$

Proof. (ii) $\Rightarrow$ (i). If (ii) holds, then

$$I_{p,q}(\mu) := \int_0^\infty \left(\frac{t^{p/q}}{c_p^\gamma(\mu; t)} \right)^{q/(p-q)} \frac{dt}{t} < \infty.$$

For each $\varphi \in C_0^\infty(\mathbb{R}^n)$, each $j = 0, \pm 1, \pm 2, \dots$, and each natural number k , Lemma 2.5 (iii) implies that

$$\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)} \left(M_{2^j}^{\alpha,\beta}(\varphi) \cap (B(0, k)) \right) \leq \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)} \left(\left\{ x \in \mathbb{R}^n : \theta_{n,\alpha} \mathcal{M} \varphi(x) > 2^j \right\} \cap B(0, k) \right).$$

Define

$$\begin{cases} \mu_{j,k}(\varphi) := \mu \left(T(M_{2^j}^{\alpha,\beta}(\varphi) \cap (B(0, k))) \right); \\ S_{p,q,k}(\mu; \varphi) := \sum_{j=-\infty}^\infty \frac{(\mu_{j,k}(\varphi) - \mu_{j+1,k}(\varphi))^{p/(p-q)}}{(\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(M_{2^j}^{\alpha,\beta}(\varphi) \cap (B(0, k))))^{q/(p-q)}}. \end{cases}$$

It follows from (ii) of Lemma 2.5 that

$$\begin{aligned} (S_{p,q,k}(\mu; \varphi))^{(p-q)/p} &\lesssim \left(\sum_{j=-\infty}^\infty \frac{\mu_{j,k}^{p/(p-q)}(\varphi) - \mu_{j+1,k}^{p/(p-q)}(\varphi)}{(c_p^\gamma(\mu; \mu_{j,k}(\varphi)))^{q/(p-q)}} \right)^{(p-q)/p} \\ &\lesssim \left(\int_0^\infty \frac{1}{(c_p^\gamma(\mu; s))^{q/(p-q)}} ds^{p/(p-q)} \right)^{(p-q)/p} \simeq (I_{p,q}(\mu))^{(p-q)/p}. \end{aligned}$$

On the other hand, By (ii) of Lemma 2.7 and (ii)-(iii) of Lemma 2.5, we can apply Hölder's inequality to deduce that

$$\begin{aligned}
 & \int_{T(B(0,k))} |R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})|^q d\mu(x, t) \\
 &= \int_0^\infty \mu \left(L_s^{\alpha,\beta}(\varphi) \cap T(B(0, k)) \right) ds^q \\
 &\lesssim \sum_{j=-\infty}^\infty 2^{jq} \left(\mu_{j,k}(\varphi) - \mu_{j+1,k}(\varphi) \right) \\
 &\lesssim (S_{p,q,k}(\mu; \varphi))^{1-q/p} \left(\sum_{j=-\infty}^\infty 2^{jp} \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(M_{2^j(\varphi)}^{\alpha,\beta} \cap (B(0, k))) \right)^{q/p} \\
 &\lesssim (S_{p,q,k}(\mu; \varphi))^{1-q/p} \left(\int_0^\infty \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)} \left(\{x \in \mathbb{R}^N : \theta_{n,\alpha} \mathcal{M}(\varphi)(x) > 2^j\} \cap (B(0, k)) \right) ds^p \right)^{q/p} \\
 &\lesssim (S_{p,q,k}(\mu; \varphi))^{1-q/p} \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}^q.
 \end{aligned}$$

So, we get

$$\left(\int_{T(B(0,k))} |R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})|^q d\mu(x, t) \right)^{1/q} \lesssim (I_{p,q}(\mu))^{(p-q)/pq} \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}.$$

Letting $k \rightarrow \infty$ derives (1.7).

(i) $\Rightarrow$ (ii). If (i) holds, then

$$C_{p,q}(\mu) := \sup_{\varphi \in C_0^\infty(\mathbb{R}^n) \text{ \& } \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} > 0} \frac{1}{\|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}} \left(\int_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})|^q d\mu(x, t) \right)^{1/q} < \infty.$$

For each $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)} > 0$, there holds

$$\left(\int_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta} \varphi(x, t^{\alpha/\beta})|^q d\mu(x, t) \right)^{1/q} \leq C_{p,q}(\mu) \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)},$$

which indicates that

$$\sup_{s>0} s \left(\mu(L_s^{\alpha,\beta}(\varphi)) \right)^{1/q} \lesssim C_{p,q}(\mu) \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}.$$

This, together with (iv) of Lemma 2.5, implies that for fixed $\varphi \in C_0^\infty(\mathbb{R}^n)$ and any bounded open set $O \subseteq \text{Int}(\{x \in \mathbb{R}^n : \varphi(x) \geq 1\})$.

$$\mu(T(O)) \lesssim C_{p,q}^q(\mu) \|\varphi\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}^q.$$

The definition of $c_p^\gamma(\mu; t)$ implies that $c_p^\gamma(\mu; t) > 0$. For $t \in (0, \infty)$ and every j , there exists a bounded open set $O_j \subseteq \mathbb{R}^n$ such that

$$\begin{cases} \text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(O_j) \leq 2c_p^\gamma(\mu; 2^j); \\ \mu(T(O_j)) > 2^j. \end{cases}$$

Below we divide the rest of the proof into two cases.

Case 1: $p \in (1, n/\gamma)$. Since

$$\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(S) \simeq \inf \left\{ \|g\|_{L^p(\mathbb{R}^n)}^p : g \in C_0^\infty(\mathbb{R}^n), g \geq 0, S \subseteq \text{Int}(\{x \in \mathbb{R}^n : I_\gamma * g(x) \geq 1\}) \right\},$$

there exists $g_j \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} I_\gamma \star g_j(x) \geq 1, & x \in O_j; \\ \|g_j\|_{L^p(\mathbb{R}^n)}^p \leq 2Cap_{\mathbb{R}^n}^{(\gamma,p)}(O_j) \leq 4c_p^\gamma(\mu; 2^j). \end{cases}$$

For the integers i, k with $i < k$, define

$$g_{i,k} = \sup_{i \leq j \leq k} \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{1/(p-q)} g_j.$$

Then $g_{i,k} \in L^p(\mathbb{R}^n)$ with

$$\|g_{i,k}\|_{L^p(\mathbb{R}^n)}^p \lesssim \sum_{j=i}^k \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{p/(p-q)} c_p^\gamma(\mu; 2^j).$$

It is easy to see that for $i \leq j \leq k$, if $x \in O_j$, then

$$I_\gamma \star g_{i,k}(x) \geq \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{1/(p-q)}.$$

Set

$$u_{i,k}(x, t) = R_{\alpha,\beta}(I_\gamma \star g_{i,k})(x, t^{\alpha/\beta}).$$

We can deduce from (iv) of Lemma 2.5 that there exists a constant $\eta_{n,\alpha}$ such that for $(x, t) \in T(O_j)$,

$$|u_{i,k}(x, t)| \geq \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{1/(p-q)} \eta_{n,\alpha}.$$

Thus, with $s = \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{1/(p-q)} \frac{\eta_{n,\alpha}}{2}$,

$$2^j \leq \mu(T(O_j)) \leq \mu(L_s^{\alpha,\beta}(I_\gamma \star g_{i,k}(x))).$$

Equivalently, if $s > 0$ such that $\mu(L_s^{\alpha,\beta}(I_\gamma \star g_{i,k}(x))) \leq s$, then

$$s > \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{1/(p-q)} \frac{\eta_{n,\alpha}}{2}.$$

This indicates that

$$\begin{aligned} (C_{p,q}(\mu) \|g_{i,k}\|_{L^p(\mathbb{R}^n)})^q &\gtrsim \int_{\mathbb{R}_+^{n+1}} |u_{i,k}(x, t^{\alpha/\beta})|^q d\mu(x, t) \\ &\simeq \int_0^\infty \left(\inf \left\{ s : \mu(L_s^{\alpha,\beta}(I_\gamma \star g_{i,k}(x))) \leq s \right\} \right)^q ds \\ &\gtrsim \sum_{j=i}^k \left(\inf \left\{ s : \mu(L_s^{\alpha,\beta}(I_\gamma \star g_{i,k}(x))) \leq 2^j \right\} \right)^q 2^j \\ &\gtrsim \sum_{j=i}^k 2^j \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{q/(p-q)} \\ &\gtrsim \left(\sum_{j=i}^k \frac{2^{jp/(p-q)}}{c_p^\gamma(\mu; 2^j)^{q/(p-q)}} \right)^{(p-q)/p} \|g_{i,k}\|_{L^p(\mathbb{R}^n)}^q. \end{aligned}$$

Thus,

$$\sum_{j=i}^k \frac{2^{jp/(p-q)}}{c_p^\gamma(\mu; 2^j)^{q/(p-q)}} \lesssim (C_{p,q}(\mu))^{pq/(p-q)}.$$

Case 2: $p = n/\gamma$. The definition of $\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(\cdot)$ implies that there exists a positive function $\varphi_j \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_j \geq 1$ on O_j and

$$\|\varphi_j\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}^p \leq 2\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(O_j) \leq 4c_p^\gamma(\mu; 2^j).$$

By Lemma 2.8, there exists $g_i(\cdot, \cdot) \in L^p(\mathbb{R}^{2n})$ such that

$$\varphi_j(x) = I_{2\gamma}^{(2n)} \star g_j(x, 0) = \mathcal{R}\mathcal{E}\varphi_j(x)$$

and

$$\|I_{2\gamma}^{(2n)} \star g_j\|_{\dot{L}_{2\gamma}^p(\mathbb{R}^{2n})} = \|\mathcal{E}\varphi_j\|_{\dot{L}_{2\gamma}^p(\mathbb{R}^{2n})} \lesssim \|\varphi_j\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}.$$

We can define $g_{i,k}$ similar to the previous case. It is easy to show that $g_{i,k} \in L^p(\mathbb{R}^{2n})$ and $I_{2\gamma}^{(2n)} \star g_{i,k} \in \dot{L}_{2\gamma}^p(\mathbb{R}^{2n})$. Then Lemma 2.8 implies

$$\begin{aligned} \|\mathcal{R}I_{2\gamma}^{(2n)} \star g_{i,k}\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}^p &\lesssim \sum_{j=i}^k \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{p/(p-q)} \|I_{2\gamma}^{(2n)} \star g_{i,k}\|_{\dot{L}^p(\mathbb{R}^{2n})}^p \\ &\lesssim \sum_{j=i}^k \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{p/(p-q)} \|\varphi_j\|_{\dot{W}^{\gamma,p}(\mathbb{R}^n)}^p \\ &\lesssim \sum_{j=i}^k \left(\frac{2^j}{c_p^\gamma(\mu; 2^j)} \right)^{p/(p-q)} c_p^\gamma(\mu; 2^j). \end{aligned}$$

Then consider $\mathcal{R}I_{2\gamma}^{(2n)} \star g_{i,k}$. Similar to the previous case, we can get

$$\sum_{j=i}^k \frac{2^{jp/(p-q)}}{c_p^\gamma(\mu; 2^j)^{q/(p-q)}} \lesssim (C_{p,q}(\mu))^{pq/(p-q)}.$$

Letting $i, k \rightarrow \infty$, we reach

$$\int_0^\infty \left(\frac{t^{p/q}}{c_p^\gamma(\mu; t)} \right)^{q/(p-q)} dt \lesssim \sum_{-\infty}^\infty \frac{2^{jp/(p-q)}}{(c_p^\gamma(\mu; 2^j))^{q/(p-q)}} \lesssim (C_{p,q}(\mu))^{pq/(p-q)},$$

which implies (ii). □

Similar to Theorem 3.4, letting $\gamma = 1$ in Theorem 3.5, we can obtain

Theorem 3.6. Let $\alpha > n$, $\beta \in (0, 1)$. Assume that $0 < q < p$, $1 \leq p < n$ and $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$. The following states are equivalent:

(i)

$$\left(\int_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta}\varphi(x, t^{\alpha/\beta})|^q d\mu(x, t) \right)^{1/q} \lesssim \|\nabla\varphi\|_{L^p(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{1,p}(\mathbb{R}^n);$$

(ii)

$$\int_0^\infty \left(\frac{t^{p/q}}{c_p(\mu; t)} \right)^{q/(p-q)} \frac{dt}{t} < \infty.$$

When $q < p = 1$, we can establish the following necessary conditions for the embedding (1.7).

Proposition 3.7. Let $\gamma \in (0, 1)$, $0 < q < p = 1$ and $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$. Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv):

(i) $\|R_{\alpha,\beta}\varphi(x, t^{\alpha/\beta})\|_{L^q(\mathbb{R}_+^{n+1}, \mu)} \lesssim \|\varphi\|_{\dot{W}^{\gamma,1}(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{\gamma,1}(\mathbb{R}^n);$

(ii) $\|R_{\alpha,\beta}\varphi(x, t^{\alpha/\beta})\|_{L^{q,\infty}(\mathbb{R}_+^{n+1}, \mu)} \lesssim \|\varphi\|_{\dot{W}^{\gamma,1}(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{\gamma,1}(\mathbb{R}^n);$

- (iii) $\sup \left\{ \frac{(\mu(T(O)))^{p/q}}{\text{Cap}_{\mathbb{R}^n}^{(\gamma,p)}(O)} : \text{bounded open } O \subseteq \mathbb{R}^n \right\} < \infty;$
 (iv) $\|R_{\alpha,\beta}\varphi(x, t^{\alpha/\beta})\|_{L^{q,1}(\mathbb{R}_+^{n+1}, \mu)} \lesssim \|\varphi\|_{\dot{W}^{\gamma,1}(\mathbb{R}^n)} \quad \forall \varphi \in \dot{W}^{\gamma,1}(\mathbb{R}^n).$

Proof. The proofs of (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are similar to those of (ii) $\Rightarrow$ (iii) $\Rightarrow$ Theorem 3.1 (v). The implication (iii) $\Rightarrow$ (iv) follows from the estimate

$$\begin{aligned} \mu(L_s^{\alpha,\beta}(\varphi)) &\leq \mu(T(M_s^{\alpha,\beta}(\varphi))) \\ &\leq \mu(T(\{x \in \mathbb{R}^n : \theta_{n,\alpha}\mathcal{M}(\varphi)(x) > s\})) \\ &\lesssim \left(\text{Cap}_{\mathbb{R}^n}^{(\gamma,1)}(\{x \in \mathbb{R}^n : \theta_{n,\alpha}\mathcal{M}(\varphi)(x) > s\}) \right)^q, \end{aligned}$$

which is a consequence of Lemma 2.5 and Theorem 3.1 (v). $\square$

4 L^p -capacity for space-time fractional heat equations

To establish the adjoint formulation of $C_p^{(\alpha,\beta)}(\cdot)$, we need to find out the adjoint operator $R_{\alpha,\beta}^*$ of the operator $R_{\alpha,\beta}$. Note that

$$\begin{aligned} \iint_{\mathbb{R}_+^{n+1}} R_{\alpha,\beta}(f)(x, t)g(x, t)dxdt &= \iint_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}^n} G_t(x-y)f(y)dy \right) g(x, t)dxdt \\ &= \int_{\mathbb{R}^n} f(y) \left(\iint_{\mathbb{R}_+^{n+1}} G_t(x-y)g(x, t)dxdt \right) dy \end{aligned}$$

holds for all $f, g \in C_0^\infty(\mathbb{R}_+^{n+1})$. So $R_{\alpha,\beta}^*$ is defined via

$$R_{\alpha,\beta}^*(g)(x) := \iint_{\mathbb{R}_+^{n+1}} G_t(x-y)g(x, t)dxdt, \quad g \in C_0^\infty(\mathbb{R}_+^{n+1}).$$

For a Borel measure μ with compact support in $\mathbb{R}_+^{n+1}$,

$$R_{\alpha,\beta}^*\mu(x) := \iint_{\mathbb{R}_+^{n+1}} G_t(z-x)d\mu(z, t).$$

Proposition 4.1. Given $p \in (1, \infty)$ and a compact subset K of $\mathbb{R}_+^{n+1}$, let $p' = p/(p-1)$ and $\mathcal{M}_+(K)$ be the class of nonnegative Radon measures supported by K .

- (i) $C_p^{(\alpha,\beta)}(K) = \sup \left\{ \|\mu\|_1^p : \mu \in \mathcal{M}_+(K) \text{ \& } \|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)} \leq 1 \right\}.$
 (ii) There exists a $\mu_K \in \mathcal{M}_+(K)$ such that

$$\begin{aligned} \mu_K(K) &= \int_{\mathbb{R}^n} \left(R_{\alpha,\beta}^*\mu_K(x) \right)^{p'} dx \\ &= \iint_{\mathbb{R}_+^{n+1}} R_{\alpha,\beta} \left(R_{\alpha,\beta}^*\mu_K \right)^{p'} dx \\ &= C_p^{(\alpha,\beta)}(K). \end{aligned}$$

Proof. (i) We set

$$\widetilde{C_p^{(\alpha,\beta)}}(K) = \sup \left\{ \|\mu\|_1^p : \mu \in \mathcal{M}_+(K) \text{ \& } \|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)} \leq 1 \right\}.$$

Since $\|\mu\|_1 = \mu(K)$, for any $f \geq 0$ & $R_{\alpha,\beta}f \geq 1_K$,

$$\begin{aligned}\|\mu\|_1 &\leq \iint_K R_{\alpha,\beta}f(x,t)d\mu(x,t) \\ &\leq \iint_{\mathbb{R}_+^{n+1}} R_{\alpha,\beta}f(x,t)d\mu(x,t) \\ &\leq \int_{\mathbb{R}^n} f(x)R_{\alpha,\beta}^*\mu(x)dx \\ &\leq \|f\|_{L^p(\mathbb{R}^n)}\|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

Because $\|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)} \leq 1$, then $\|\mu\|_1 \leq \|f\|_{L^p(\mathbb{R}^n)}$ and

$$\|\mu\|_1^p \leq \inf \left\{ \|f\|_{L^p(\mathbb{R}^n)}^p : f \geq 0 \text{ & } R_{\alpha,\beta}f \geq 1_K \right\},$$

which means that $\widetilde{C_p^{(\alpha,\beta)}}(K) \leq C_p^{(\alpha,\beta)}(K)$.

Define

$$\begin{cases} \mathcal{X} := \left\{ \mu : \mu \in \mathcal{M}_+(K) \text{ & } \mu(K) = 1 \right\}; \\ \mathcal{Y} := \left\{ f : 0 < f \in L^p(\mathbb{R}^n) \text{ & } \|f\|_{L^p(\mathbb{R}^n)} \leq 1 \right\}; \\ \mathcal{Z} := \left\{ f : 0 \leq f \in L^p(\mathbb{R}^n) \text{ & } R_{\alpha,\beta}f \geq 1_K \right\}; \\ E(\mu, f) := \int_{\mathbb{R}^n} (R_{\alpha,\beta}^*\mu)(x)f(x)dx = \iint_{\mathbb{R}_+^{n+1}} R_{\alpha,\beta}f(x,t)d\mu(x,t). \end{cases}$$

By [3, Theorem 2.4.1], $\sup_{f \in \mathcal{Y}} \min_{\mu \in \mathcal{X}} E(\mu, f) = \min_{\mu \in \mathcal{X}} \sup_{f \in \mathcal{Y}} E(\mu, f)$. We can get

$$\begin{aligned}\min_{\mu \in \mathcal{M}_+(K)} \frac{\|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)}}{\mu(K)} &= \min_{\mu \in \mathcal{M}_+(K)} \frac{\sup_{f \in L^p(\mathbb{R}^n) \text{ & } \|f\|_{L^p(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} f(x)R_{\alpha,\beta}^*\mu(x)dx}{\mu(K)} \\ &\leq \min_{\mu \in \mathcal{M}_+(K)} \sup_{f \in \mathcal{Y}} \frac{\int_{\mathbb{R}^n} f(x)R_{\alpha,\beta}^*\mu(x)dx}{\|f\|_{L^p(\mathbb{R}^n)}\mu(K)} \\ &= \sup_{f \in \mathcal{Y}} \min_{\mu \in \mathcal{X}} \frac{1}{\|f\|_{L^p(\mathbb{R}^n)}} \left\{ \int_K R_{\alpha,\beta}f(x,t)d\mu(x,t) \right\} \\ &= \sup_{f \in \mathcal{Y}} \frac{1}{\|f\|_{L^p(\mathbb{R}^n)}} \left(\min_{(x,t) \in K} R_{\alpha,\beta}f(x,t) \right) \min_{\mu \in \mathcal{X}} \mu(K) \\ &\leq \sup_{0 < f \in L^p(\mathbb{R}^n)} \frac{(\min_{(x,t) \in K} R_{\alpha,\beta}f(x,t))}{\|f\|_{L^p(\mathbb{R}^n)}} \\ &\leq \frac{1}{\inf_{0 \leq f \in L^p \text{ & } \min R_{\alpha,\beta}f=1} \|f\|_{L^p}} = \left(C_p^{(\alpha,\beta)}(K) \right)^{-1/p}.\end{aligned}$$

Then for $\|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)} \leq 1$, it holds

$$\left(C_p^{(\alpha,\beta)}(K) \right)^{-1/p} = \min_{\mu \in \mathcal{M}_+(K)} \frac{1}{\mu(K)} \|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)}.$$

Recall that

$$\widetilde{C_p^{(\alpha,\beta)}}(K) = \sup \left\{ \|\mu\|_1^p : \mu \in \mathcal{M}_+(K) \text{ & } \|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)} \leq 1 \right\}.$$

For any $\mu \in \mathcal{M}_+(K)$, take $\mu_1 := \|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)}^{-1}\mu$, which means that $\|R_{\alpha,\beta}^*\mu_1\|_{L^{p'}(\mathbb{R}^n)} = 1$.

$$\left[\widetilde{C_p^{(\alpha,\beta)}}(K) \right]^{1/p} \geq \sup \left\{ \mu \in \mathcal{M}_+(K) : \frac{\|\mu\|_1}{\|R_{\alpha,\beta}^*\mu\|_{L^{p'}(\mathbb{R}^n)}} \right\}$$

$$= \sup \left\{ \|\mu_1\|_1, \mu_1 \in \mathcal{M}_+(K) \right\},$$

which implies that

$$\min_{\mu \in \mathcal{M}_+(K)} \left\{ \frac{\|R_{\alpha,\beta}^* \mu\|_{L^{p'}(\mathbb{R}^n)}}{\mu(K)} \right\} = \min_{\mu \in \mathcal{M}_+(K)} \left\{ \frac{\|R_{\alpha,\beta}^* \mu\|_{L^{p'}(\mathbb{R}^n)}}{\|\mu\|_1} \right\} \geq \left(\widetilde{C_p^{(\alpha,\beta)}}(K) \right)^{-1/p}.$$

This gives $\left(C_p^{(\alpha,\beta)}(K) \right)^{-1/p} \geq \left(\widetilde{C_p^{(\alpha,\beta)}}(K) \right)^{-1/p}$. The proof of (i) is completed.

Next, we prove (ii). According to (i), we select a sequence $\{\mu_j\} \subset \mathcal{M}_+(K)$ such that

$$\begin{cases} \lim_{j \rightarrow \infty} (\mu_j(K))^p = C_p^{(\alpha,\beta)}(K); \\ \|R_{\alpha,\beta}^* \mu_j\|_{L^{p'}(\mathbb{R}^n)} \leq 1. \end{cases}$$

A direct computation implies that

$$C_p^{(\alpha,\beta)}(K) = \sup \left\{ \|\mu\|_1^p : \mu \in \mathcal{M}_+(K) \text{ \& } \|R_{\alpha,\beta}^* \mu\|_{L^{p'}(\mathbb{R}^n)} = 1 \right\}.$$

Then, using the fact $\|R_{\alpha,\beta}^* \mu_j\|_{L^{p'}(\mathbb{R}^n)} = 1$, we get

$$\left| \int_{\mathbb{R}_+^{n+1}} R_{\alpha,\beta} f(x, t) d\mu_j(x, t) \right| = \left| \int_{\mathbb{R}^n} f(x) R_{\alpha,\beta}^* \mu_j(x) dx \right| \leq \|R_{\alpha,\beta}^* \mu_j\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p} \leq \|f\|_{L^p}.$$

There exists $\mu \in \mathcal{M}_+(K)$ such that μ_j weak $\star$ convergence to μ . Hence $\mu^p(K) = C_p^{(\alpha,\beta)}(K)$. Since the Lebesgue space $L^p(\mathbb{R}^n)$ is uniformly convex for $1 < p < \infty$, noting that the set

$$\left\{ f \in C_0^\infty(\mathbb{R}^n) : f \geq 0 \text{ on } \mathbb{R}^n \text{ and } f \geq 1 \right\}$$

is a convex set. Following the procedure of [3, Theorem 2.3.10], we can prove that there exists a unique function denoted by f_K such that

$$\begin{cases} f_K \in L^p(\mathbb{R}^n); \\ C_p^{(\alpha,\beta)}(\{(x, t) \in K : R_{\alpha,\beta} f_K < 1\}) = 0; \\ \|f_K\|_{L^p(\mathbb{R}^n)}^p = C_p^{(\alpha,\beta)}(K). \end{cases} \quad (4.1)$$

Then we have

$$\begin{aligned} (C_p^{(\alpha,\beta)}(K))^{1/p} = \mu(K) &\leq \int_{\mathbb{R}_+^{n+1}} R_{\alpha,\beta} f(x, t) d\mu(x, t) \\ &\leq \int_{\mathbb{R}^n} f_K(x) R_{\alpha,\beta}^* \mu(x) dx \\ &\leq \|f_K\|_{L^p(\mathbb{R}^n)} \|R_{\alpha,\beta}^* \mu\|_{L^{p'}(\mathbb{R}^n)} \\ &= (C_p^{(\alpha,\beta)}(K))^{1/p} \|R_{\alpha,\beta}^* \mu\|_{L^{p'}(\mathbb{R}^n)}, \end{aligned} \quad (4.2)$$

which implies that $\|R_{\alpha,\beta}^* \mu\|_{L^{p'}(\mathbb{R}^n)} \geq 1$. On the other hand, following the procedure of [3, Proposition 2.3.2, (b)], we can prove that $\mu \mapsto P_\alpha^* \mu(x)$ is lower semi-continuous on $M_+(\mathbb{R}_+^{n+1})$ in the weak $\star$ topology, i.e.,

$$R_{\alpha,\beta}^* \mu(x) \leq \liminf_{j \rightarrow \infty} R_{\alpha,\beta}^* \mu_j(x),$$

which, together with $\|R_{\alpha,\beta}^* \mu_j\|_{L^{p'}(\mathbb{R}^n)} \leq 1$, gives $\|R_{\alpha,\beta}^* \mu\|_{L^{p'}(\mathbb{R}^n)} \leq 1$. Hence $\|R_{\alpha,\beta}^* \mu\|_{L^{p'}(\mathbb{R}^n)} = 1$. Taking $\mu_K = (C_p^{(\alpha,\beta)}(K))^{1/p'} \mu$ yields

$$\mu_K(K) = \int_K (C_p^{(\alpha,\beta)}(K))^{1/p'} d\mu = (C_p^{(\alpha,\beta)}(K))^{1/p'} \mu(K)$$

$$= (C_p^{(\alpha, \beta)}(K))^{1/p'} (C_p^{(\alpha, \beta)}(K))^{1/p} = C_p^{(\alpha, \beta)}(K).$$

On the other hand,

$$\int_{\mathbb{R}^n} (R_{\alpha, \beta}^* \mu_K(x))^{p'} dx = \|R_{\alpha, \beta}^* \mu_K\|_{L^{p'}(\mathbb{R}^n)}^{p'} = (C_p^{(\alpha, \beta)}(K)) \|R_{\alpha, \beta}^* \mu\|_{L^{p'}(\mathbb{R}^n)}^{p'} = C_p^{(\alpha, \beta)}(K).$$

This indicates that

$$\mu_K(K) = \int_{\mathbb{R}^n} (R_{\alpha, \beta}^* \mu_K(x))^{p'} dx = C_p^{(\alpha, \beta)}(K). \quad (4.3)$$

Let f_K be the function mentioned above. Then

$$\mu_K(\{(x, t) \in K : P_\alpha f_K(x, t) \leq 1\}) = 0.$$

By Hölder's inequality, we can get

$$\begin{aligned} C_p^{(\alpha, \beta)}(K) &= \mu_K(K) \leq \int_K R_{\alpha, \beta} f_K d\mu_K \\ &\leq \int_{\mathbb{R}^n} f_K(x) R_{\alpha, \beta}^* \mu_K(x) dx \\ &\leq \|f_K\|_{L^p(\mathbb{R}^n)} \|R_{\alpha, \beta}^* \mu_K\|_{L^{p'}(\mathbb{R}^n)} \\ &= (C_p^{(\alpha, \beta)}(K))^{1/p} (C_p^{(\alpha, \beta)}(K))^{1/p'} = C_p^{(\alpha, \beta)}(K). \end{aligned} \quad (4.4)$$

It follows from (4.4) that

$$C_p^{(\alpha, \beta)}(K) = \int_{\mathbb{R}^n} f_K R_{\alpha, \beta}^* \mu_K dx.$$

Hence

$$\int_{\mathbb{R}^n} f_K R_{\alpha, \beta}^* \mu_K dx = \int_{\mathbb{R}^n} (R_{\alpha, \beta}^* \mu_K)^{p'} dx = C_p^{(\alpha, \beta)}(K).$$

The above identity implies that

$$C_{\mathbb{R}_+^{n+1}}^{\alpha, p}(K) = \int_{\mathbb{R}_+^{n+1}} R_{\alpha, \beta} f_K d\mu_K = \int_{\mathbb{R}_+^{n+1}} R_{\alpha, \beta} (R_{\alpha, \beta}^* \mu_K)^{p'-1} d\mu_K,$$

which completes the proof of (ii). □

Below we investigate some basic properties of $C_p^{(\alpha, \beta)}(\cdot)$.

Proposition 4.2.

- (i) $C_p^{(\alpha, \beta)}(\emptyset) = 0$;
- (ii) If $K_1 \subseteq K_2 \subset \mathbb{R}_+^{n+1}$, then $C_p^{(\alpha, \beta)}(K_1) \leq C_p^{(\alpha, \beta)}(K_2)$;
- (iii) For any sequence $\{K_j\}_{j=1}^\infty$ of subsets of $\mathbb{R}_+^{n+1}$,

$$C_p^{(\alpha, \beta)}\left(\bigcup_{j=1}^\infty K_j\right) \leq \sum_{j=1}^\infty C_p^{(\alpha, \beta)}(K_j);$$

- (iv) For any $K \subset \mathbb{R}_+^{n+1}$ and any $x_0 \in \mathbb{R}^n$, $C_p^{(\alpha, \beta)}(K + (0, x_0)) = C_p^{(\alpha, \beta)}(K)$.

Proof. (i) By the definition of $C_p^{(\alpha, \beta)}$, if $K = \emptyset$, then $1_K \equiv 0$. Take $f = 0$. Then $0 \leq C_p^{(\alpha, \beta)}(K) \leq \|f\|_{L^p(\mathbb{R}^n)}^p = 0$, i.e., $C_p^{(\alpha, \beta)}(K) = 0$.

(ii) Let $K_1 \subseteq K_2 \subset \mathbb{R}_+^{n+1}$. For any fixed $(x, t) \in \mathbb{R}_+^{n+1}$, $1_{K_1}(x, t) \leq 1_{K_2}(x, t)$ due to

$$\begin{cases} 1_{K_1}(x, t) = 1_{K_2}(x, t), & (x, t) \in K_1; \\ 1_{K_1}(x, t) = 0 < 1 = 1_{K_2}(x, t), & (x, t) \notin K_1 \text{ \& } (x, t) \in K_2; \\ 1_{K_1}(x, t) = 1_{K_2}(x, t) = 0, & (x, t) \notin K_2. \end{cases}$$

This means that if $f \geq 0$ such that $R_{\alpha, \beta} f \geq 1_{K_2}$, then $R_{\alpha, \beta} f \geq 1_{K_1}$, equivalently,

$$\left\{ \|f\|_{L^p(\mathbb{R}^n)}^p : f \geq 0 \text{ \& } R_{\alpha, \beta}(f) \geq 1_{K_2} \right\} \subseteq \left\{ \|f\|_{L^p(\mathbb{R}^n)}^p : f \geq 0 \text{ \& } R_{\alpha, \beta}(f) \geq 1_{K_1} \right\}.$$

By the definition of $C_p^{(\alpha, \beta)}$, we can get $C_p^{(\alpha, \beta)}(K_1) \leq C_p^{(\alpha, \beta)}(K_2)$.

(iii) Let $\epsilon > 0$. Take $f_j \geq 0$ such that $P_{\alpha} f_j \geq 1$ on K_j and

$$\int_{\mathbb{R}^n} |f_j(x)|^p dx \leq C_p^{(\alpha, \beta)}(K_j) + \epsilon/2^j.$$

Let $f = \sup_{j \in \mathbb{N}_+} f_j$. For any $(x, t) \in \cup_{j=1}^{\infty} K_j$, there exists a j_0 such that $(x, t) \in K_{j_0}$ and $P_{\alpha} f_{j_0}(x, t) \geq 1$. Hence $P_{\alpha} f(x, t) \geq 1$ on $\cup_{j=1}^{\infty} K_j$. On the other hand,

$$\|f\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |f(x)|^p dx \leq \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |f_j(x)|^p dx = \sum_{j=1}^{\infty} C_p^{(\alpha, \beta)}(K_j) + \epsilon,$$

which indicates

$$C_p^{(\alpha, \beta)}\left(\bigcup_{j=1}^{\infty} K_j\right) \leq \sum_{j=1}^{\infty} C_p^{(\alpha, \beta)}(K_j).$$

(iv) Define $f_{x_0}(x) = f(x - x_0)$. Then $\|f\|_{L^p(\mathbb{R}^n)} = \|f_{x_0}\|_{L^p(\mathbb{R}^n)}$. If $(x, t) \in K + (0, x_0)$, then $(t, x - x_0) \in K$, and vice versa. Take $f \geq 0$ and $R_{\alpha, \beta} f \geq 1_K$. We have

$$\begin{aligned} R_{\alpha, \beta} f_{x_0}(x, t) &= \int_{\mathbb{R}^n} G_t(x - y) f_{x_0}(y, t) dy \\ &= \int_{\mathbb{R}^n} G_t(x - y) f(y - x_0) dy \\ &= \int_{\mathbb{R}^n} G_t(x - x_0 - (y - x_0)) f(y - x_0) dy \\ &= R_{\alpha, \beta} f(x - x_0), \end{aligned}$$

which means that $R_{\alpha, \beta} f_{x_0}(x, t) \geq 1_{K+(0, x_0)}$ is equivalent to $R_{\alpha, \beta} f(x - x_0, t) \geq 1_K$. We have $C_p^{(\alpha, \beta)}(K + (0, x_0)) = C_p^{(\alpha, \beta)}(K)$. $\square$

Proposition 4.3. For $t_0 > 0$, $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, set the (α, β) -parabolic ball centered at (x_0, t_0) with the radius r_0 as

$$B_{r_0}^{(\alpha, \beta)}(x_0, t_0) = \left\{ (x, t) \in \mathbb{R}_+^{n+1}, r_0^\alpha < t - t_0 < 2r_0^\alpha, |x - x_0| < r_0^\beta/2 \right\}.$$

(i) If $1 \leq p < \infty$, then

$$C_p^{(\alpha, \beta)}\left(B_{r_0}^{(\alpha, \beta)}(0, 0)\right) = r_0^{\beta n} C_p^{(\alpha, \beta)}\left(B_1^{(\alpha, \beta)}(0, 0)\right), \quad \forall r_0 > 0.$$

(ii) For $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ with $t_0 \lesssim r_0^\alpha$,

$$r_0^{\beta n} \leq C_p^{(\alpha, \beta)}\left(B_{r_0}^{(\alpha, \beta)}(t_0, x_0)\right) \leq \left(\frac{(t_0 + r_0^\alpha)^{n\beta/\alpha}}{r_0^{\beta n}}\right)^p r_0^{\beta n}.$$

Specially, for $t_0 \lesssim r_0^\alpha$,

$$C_p^{(\alpha, \beta)}\left(B_{r_0}^{(\alpha, \beta)}(t_0, x_0)\right) \sim r_0^{\beta n}.$$

Proof. (i) If $f \geq 0$ obeys $R_{\alpha,\beta}f \geq 1_{B_{r_0}^{(\alpha,\beta)}(0,0)}$, then for $(x, t) \in B_{r_0}^{(\alpha,\beta)}(0, 0)$, we can see that $(x, t) \in B_{r_0}^{(\alpha,\beta)}(0, 0)$ is equivalent to $(x/r_0^\beta, t/r_0^\alpha) \in B_1^{(\alpha,\beta)}(0, 0)$. Let $t/r_0^\alpha = s$ and $x/r_0^\beta = y$. If $f \geq 0$ and $R_{\alpha,\beta}f(x, t) \geq 1$, $\forall (x, t) \in B_{r_0}^{(\alpha,\beta)}(0, 0)$, we can get

$$\begin{aligned} R_{\alpha,\beta}f(x, t) &= \int_{\mathbb{R}^n} G_t(x - z)f(z)dz \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty K_{\alpha,(t/u)^\beta}(x - z)g_\beta(u)du \right) f(z)dz \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty K_{\alpha,(r_0^\alpha s/u)^\beta}(x - r_0^\beta v)g_\beta(u)du \right) f(r_0^\beta v)r_0^{\beta n}dv. \end{aligned}$$

By the Fourier transform, we can see that, letting $r_0^\beta \xi = \eta$,

$$\begin{aligned} K_{\alpha,(r_0^\alpha s/u)^\beta}(x - r_0^\beta v) &= \int_{\mathbb{R}^n} \exp\left(-\frac{r_0^{\alpha\beta}s^\beta}{u^\beta}|\xi|^\beta\right) \exp(i(x - r_0^\beta v)\xi)d\xi \\ &= \int_{\mathbb{R}^n} \exp(i(r_0^{-\beta}x - v)\eta) \exp(-|\eta|^\alpha(s/u)^\beta)r_0^{-\beta n}d\eta \\ &= r_0^{-\beta n}K_{\alpha,(s/u)^\beta}(r_0^{-\beta}x - v). \end{aligned}$$

The above identities show that

$$\begin{aligned} R_{\alpha,\beta}f(x, t) &= \int_{\mathbb{R}^n} \left(\int_0^\infty K_{\alpha,(s/u)^\beta}(r_0^{-\beta}x - v)g_\beta(u)du \right) r_0^{-\beta}f(r_0^\beta v)r_0^{\beta n}dv \\ &= \int_{\mathbb{R}^n} G_s(r_0^{-\beta}x - v)f_{r_0}(v)dv \\ &= R_{\alpha,\beta}f_{r_0}(r_0^{-\beta}x, s) \\ &= R_{\alpha,\beta}f_{r_0}(s, y), \end{aligned}$$

which indicates that $R_{\alpha,\beta}f(x, t) \geq 1 \forall (x, t) \in B_{r_0}^{(\alpha,\beta)}(0, 0)$ is equivalent to $R_{\alpha,\beta}f(y, s) \geq 1, \forall (y, s) \in B_1^{(\alpha,\beta)}(0, 0)$. Finally, we obtain

$$\begin{aligned} C_p^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(0, 0)) &= \inf \left\{ \|f\|_{L^p(\mathbb{R}^n)}^p : f \geq 0 \text{ \& } R_{\alpha,\beta}f \geq 1_{B_{r_0}^{(\alpha,\beta)}(0,0)} \right\} \\ &= r_0^{\beta n} \inf \left\{ \|f_{r_0}\|_{L^p(\mathbb{R}^n)}^p : f_{r_0} \geq 0 \text{ \& } R_{\alpha,\beta}f_{r_0} \geq 1_{B_1^{(\alpha,\beta)}(0,0)} \right\} \\ &\geq r_0^{\beta n} \inf \left\{ \|g\|_{L^p(\mathbb{R}^n)}^p : g \geq 0 \text{ \& } R_{\alpha,\beta}g \geq 1_{B_1^{(\alpha,\beta)}(0,0)} \right\} \\ &= r_0^{\beta n} C_p^{(\alpha,\beta)}(B_1^{(\alpha,\beta)}(0, 0)). \end{aligned}$$

Conversely, we can also get, changing the order of $C_p^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(0, 0))$ and $C_p^{(\alpha,\beta)}(B_1^{(\alpha,\beta)}(0, 0))$,

$$r_0^{-\beta n} C_p^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(0, 0)) \leq C_p^{(\alpha,\beta)}(B_1^{(\alpha,\beta)}(0, 0)),$$

which means (i) holds.

(ii) For $p \in [1, \infty)$, choose $\tilde{p}, \tilde{q}$ such that

$$\begin{cases} 1 \leq p < \tilde{p} < \frac{np}{n-\min\{n,\alpha\}} = \infty; \\ 1/\tilde{q} = \frac{n\beta}{\alpha}(1/p - 1/\tilde{p}). \end{cases}$$

If $0 \leq f \in L^p(\mathbb{R}^n)$ and $R_{\alpha,\beta}f(x, t) \geq 1$ for $(x, t) \in B_{r_0}^{(\alpha,\beta)}(x_0, t_0) \subset \mathbb{R}_+^{n+1}$, then

$$\left\{ \int_{r_0^\alpha < t-t_0 < 2r_0^\alpha} \left(\int_{|x-x_0| < r_0^\beta/2} |R_{\alpha,\beta}f(x, t)|^{\tilde{p}} dx \right)^{\tilde{q}/\tilde{p}} dt \right\}^{1/\tilde{q}} \\ \geq \left\{ \int_{r_0^\alpha < t-t_0 < 2r_0^\alpha} \left(\int_{|x-x_0| < r_0^\beta/2} 1 dx \right)^{\tilde{q}/\tilde{p}} dt \right\}^{1/\tilde{q}} \geq r_0^{\beta n/\tilde{p} + \alpha/\tilde{q}}.$$

On the other hand, by Lemma 2.3,

$$\left\{ \int_{r_0^\alpha < t-t_0 < 2r_0^\alpha} \left(\int_{|x-x_0| < r_0^\beta/2} |R_{\alpha,\beta}f(x, t)|^{\tilde{p}} dx \right)^{\tilde{q}/\tilde{p}} dt \right\}^{1/\tilde{q}} \lesssim \|R_{\alpha,\beta}f\|_{L^{\tilde{q}}((t_0+r_0^\alpha, t_0+2r_0^\alpha); L^{\tilde{p}})} \\ \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Since $1/\tilde{q} = \beta n(1/p - 1/\tilde{p})/\alpha$, the above estimates indicate that $r_0^{\beta n} \leq \|f\|_{L^p(\mathbb{R}^n)}^p$.

For the converse, choose $f = 1_{\{x \in \mathbb{R}^n: |x-x_0| < r_0^\beta/2\}}$. Then

$$R_{\alpha,\beta}f(x, t) = \int_{\mathbb{R}^n} G_t(x-y)f(y)dy \\ = \int_{\{y \in \mathbb{R}^n: |y-x_0| < r_0^\beta/2\}} G_t(x-y)f(y)dy.$$

If $(x, t) \in B_{r_0}^{(\alpha,\beta)}(x_0, t_0)$, then $|x-x_0| < r_0^\beta/2$. Since $t_0 + r_0^\alpha < t < t_0 + 2r_0^\alpha$,

$$\frac{t^\beta}{(t^\beta/\alpha + |x-y|)^{n+\alpha}} \geq \frac{1}{t^{\beta n/\alpha}} \geq \frac{1}{(t_0 + r_0^\alpha)^{n\beta/\alpha}},$$

which gives

$$R_{\alpha,\beta}(f)(x, t) \geq \int_{\{y \in \mathbb{R}^n: |y-x_0| < r_0^\beta/2\}} \frac{t^\beta}{(t^\beta/\alpha + |x-y|)^{n+\alpha}} dy \geq \frac{r_0^{n\beta}}{(t_0 + r_0^\alpha)^{n\beta/\alpha}},$$

equivalently, $R_{\alpha,\beta}((t_0 + r_0^\alpha)^{n\beta/\alpha} r_0^{-\beta n} f) \geq 1$. Under the assumption $t_0 \leq r_0^\alpha$, we obtain

$$C_p^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(x_0, t_0)) \lesssim \left\| \left((t_0 + r_0^\alpha)^{n\beta/\alpha} r_0^{-\beta n} \right) f(x) \right\|_{L^p(\mathbb{R}^n)}^p = \left((t_0 + r_0^\alpha)^{n\beta/\alpha} r_0^{-\beta n} \right)^p r_0^{\beta n}.$$

Specially, for $t_0 \leq r_0^\alpha$, $C_p^{(\alpha,\beta)}(B_{r_0}^{(\alpha,\beta)}(x_0, t_0)) \sim r_0^{\beta n}$. This completes the proof of Proposition 4.3. $\square$

We investigate the capacity strong and weak type inequalities. As some preliminaries, we prove several further properties of the capacity $C_p^{(\alpha,\beta)}(\cdot)$. At first, the capacity $C_p^{(\alpha,\beta)}(\cdot)$ is an outer capacity, i.e.,

Proposition 4.4. For any $E \subset \mathbb{R}_+^{n+1}$,

$$C_p^{(\alpha,\beta)}(E) = \inf \left\{ C_p^{(\alpha,\beta)}(O) : O \supset E, O \text{ open} \right\}.$$

Proof. Without loss of generality, we assume that $C_p^{(\alpha,\beta)}(E) < \infty$. By (ii) of Proposition 4.2,

$$C_p^{(\alpha,\beta)}(E) \leq \inf \left\{ C_p^{(\alpha,\beta)}(O) : O \supset E, O \text{ open} \right\}.$$

For $\epsilon \in (0, 1)$, there exists a measurable, nonnegative function f such that $R_{\alpha,\beta}f \geq 1$ on E and

$$\int_{\mathbb{R}^n} |f(x)|^p dx \leq C_p^{(\alpha,\beta)}(E) + \epsilon.$$

Since $R_{\alpha,\beta}f$ is lower semi-continuous, then the set $O_\epsilon := \{(x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha,\beta}f(x, t) > 1 - \epsilon\}$ is an open set. On the other hand, $E \subset O_\epsilon$. This implies that

$$C_p^{(\alpha,\beta)}(O_\epsilon) \leq \frac{1}{(1-\epsilon)^p} \int_{\mathbb{R}^n} |f(x)|^p dx < \frac{1}{(1-\epsilon)^p} (C_p^{(\alpha,\beta)}(E) + \epsilon).$$

The arbitrariness of ϵ indicates that $C_p^{(\alpha,\beta)}(E) \geq \inf \left\{ C_p^{(\alpha,\beta)}(O) : O \supset E, O \text{ open} \right\}$. □

An immediate corollary of Proposition 4.4 is the following result.

Corollary 4.5. *If $\{K_j\}_{j=1}^\infty$ is a decreasing sequence of compact sets, then*

$$C_p^{(\alpha,\beta)}(\cap_{j=1}^\infty K_j) = \lim_{j \rightarrow \infty} C_p^{(\alpha,\beta)}(K_j).$$

Proposition 4.6. *Let $1 < p < \infty$. If $\{E_j\}_{j=1}^\infty$ is an increasing sequence of arbitrary subsets of $\mathbb{R}^n$, then $C_p^{(\alpha,\beta)}(\cup_{j=1}^\infty E_j) = \lim_{j \rightarrow \infty} C_p^{(\alpha,\beta)}(E_j)$.*

Proof. Since $\{E_j\}_{j=1}^\infty$ is increasing, then $C_p^{(\alpha,\beta)}(\cup_{j=1}^\infty E_j) \geq \lim_{j \rightarrow \infty} C_p^{(\alpha,\beta)}(E_j)$.

Conversely, without loss of generality, we assume that $\lim_{j \rightarrow \infty} C_p^{(\alpha,\beta)}(E_j)$ is finite. For each j , let f_{E_j} be the unique function such that $f_{E_j} \geq 1$ on E_j and $\|f_{E_j}\|_{L^p}^p = C_p^{(\alpha,\beta)}(E_j)$. Then for $i < j$, it holds that $R_{\alpha,\beta}f_{E_j} \geq 1$ on E_i and further, $R_{\alpha,\beta}((f_{E_i} + f_{E_j})/2) \geq 1$ on E_i , which means that

$$\int_{\mathbb{R}^n} ((f_{E_i} + f_{E_j})/2)^p dx \geq C_p^{(\alpha,\beta)}(E_i).$$

It follows from [3, Corollary 1.3.3] that the sequence $\{f_{E_j}\}_{j=1}^\infty$ converges strongly to a function f satisfying

$$\|f\|_{L^p}^p = \lim_{j \rightarrow \infty} C_p^{(\alpha,\beta)}(E_j).$$

Similar to [3, Proposition 2.3.12], it can be deduced that $R_{\alpha,\beta}f \geq 1$ on $\cup_{j=1}^\infty E_j$, except possibly on a countable union of sets with $C_p^{(\alpha,\beta)}(\cdot)$ zero. Hence

$$\lim_{j \rightarrow \infty} C_p^{(\alpha,\beta)}(E_j) \geq \int_{\mathbb{R}^n} |f(x)|^p dx \geq C_p^{(\alpha,\beta)}(\cup_{j=1}^\infty E_j).$$

□

As a corollary of Proposition 4.6, we can get

Corollary 4.7. *Let O be an open subset of $\mathbb{R}_+^{n+1}$. Then*

$$C_p^{(\alpha,\beta)}(O) = \sup \left\{ C_p^{(\alpha,\beta)}(K) : \text{compact } K \subset O \right\}.$$

The strong type inequality corresponding to $C_p^{(\alpha,\beta)}(\cdot)$ is as follows.

Lemma 4.8. Let $p \in (1, \infty)$. Then

$$\int_0^\infty C_p^{(\alpha, \beta)} \left(\left\{ (x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha, \beta} f(x, t) \geq \lambda \right\} \right) d\lambda^p \lesssim \|f\|_{L^p(\mathbb{R}^n)}^p, \quad \forall f \in L_+^p(\mathbb{R}^n).$$

Here and henceforth, the symbol $L_+^p(\mathbb{R}^n)$ stands for the class of all nonnegative functions in $L^p(\mathbb{R}^n)$ and $d\lambda^p = p\lambda^{p-1}d\lambda$.

Proof. Since $C_0^\infty(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$, we are about to verify this inequality for any nonnegative $C_0^\infty(\mathbb{R}^n)$ -function. Assume that $f \in C_0^\infty(\mathbb{R}^n)$. For $r > 0$, let

$$E_j := \left\{ (x, t) \in \overline{B_r(0, 0)} : R_{\alpha, \beta} f(x, t) \geq 2^j \right\}.$$

Note that E_j depends on r . Since $f \in C_0^\infty(\mathbb{R}^n)$, $R_{\alpha, \beta} f(x, t)$ is continuous. Thus, E_j is the intersection of $\overline{B_r(0, 0)}$ and a close set which is equivalent to a compact set in $\mathbb{R}^n$.

Let μ_j stand for the measure corresponding to E_j such that

$$\mu_j(E_j) = \int_{\mathbb{R}^n} (R_{\alpha, \beta}^* \mu_j(x))^{p'} dx = \int_{\mathbb{R}_+^{n+1}} R_{\alpha, \beta} (R_{\alpha, \beta}^* \mu_j)^{p'-1} d\mu_j = C_p^{(\alpha, \beta)}(E_j).$$

Let $S := \sum_{j=-\infty}^\infty 2^{jp} \mu_j(E_j)$ and $T := \left\| \sum_{j=-\infty}^\infty 2^{j(p-1)} (R_{\alpha, \beta}^* \mu_j) \right\|_{L^{p'}}^{p'}$. Because $f \in C_0^\infty(\mathbb{R}^n)$, there exists a positive integer j_0 such that E_j are empty sets for $j > j_0$, i.e., $\mu_j(E_j) = C_p^{(\alpha, \beta)}(E_j) = 0$, $j > j_0$. Hence $\sum_{j=1}^\infty 2^{jp} \mu_j(E_j) < \infty$. On the other hand,

$$\begin{aligned} \sum_{j=-\infty}^{-1} 2^{jp} \mu_j(E_j) &= \sum_{j=-\infty}^{-1} 2^{jp} \sum_{j=-\infty}^{-1} 2^{jp} C_p^{(\alpha, \beta)}(E_j) \\ &\leq \sum_{j=-\infty}^{-1} 2^{jp} \sum_{j=-\infty}^{-1} 2^{jp} C_p^{(\alpha, \beta)}(B_r(0, 0)) \\ &\lesssim r^n \sum_{j=-\infty}^{-1} 2^{jp} \sum_{j=-\infty}^{-1} 2^{jp}, \end{aligned}$$

which gives $S < \infty$. It follows from Hölder's inequality that

$$S \leq \sum_{j=-\infty}^\infty 2^{j(p-1)} \int_{\mathbb{R}_+^{n+1}} R_{\alpha, \beta} f d\mu_j \leq \int_{\mathbb{R}^n} f \left(\sum_{j=-\infty}^\infty 2^{j(p-1)} (R_{\alpha, \beta}^* \mu_j) \right) dx \leq T^{1/p'} \|f\|_{L^p}. \quad (4.5)$$

Below we prove that

$$T \lesssim S, \quad (4.6)$$

The proof of (4.6) is divided into two cases.

Case 1: $2 \leq p < \infty$. For $k = 0, \pm 1, \pm 2, \dots$, let

$$\begin{cases} \sigma_k(x) = \sum_{j=k}^\infty 2^{j(p-1)} R_{\alpha, \beta}^* \mu_j(x); \\ \sigma(x) = \sum_{j=-\infty}^\infty 2^{j(p-1)} R_{\alpha, \beta}^* \mu_j(x). \end{cases}$$

Since $f \in C_0^\infty(\mathbb{R}^n)$, the sets E_j are empty for sufficiently large j , i.e., there exists a positive integer j_0 such that $C_p^{(\alpha, \beta)}(E_j) = 0$, $j > j_0$. Then by (ii) of Proposition 4.1,

$$\|\sigma_k\|_{L^{p'}} = \left\| \sum_{j=k}^\infty 2^{j(p-1)} R_{\alpha, \beta}^* \mu_j \right\|_{L^{p'}} \leq \sum_{j=k}^\infty 2^{j(p-1)} \|R_{\alpha, \beta}^* \mu_j\|_{L^{p'}} \simeq \sum_{j=k}^{j_0} 2^{j(p-1)} (C_p^{(\alpha, \beta)}(E_j))^{1/p'} < \infty,$$

which implies that for any k , $\sigma_k \in L^{p'}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, the sequence $\{\sigma_k(x)\}$ is increasing as $k \rightarrow -\infty$ since $R_{\alpha,\beta}^* \mu_j(x) \geq 0$. If there exists an upper bound for $\{\sigma_k(x)\}$, we know that $\sigma_k(x)$ converges to the series $\sum_{j=-\infty}^{\infty} 2^{j(p-1)} R_{\alpha,\beta}^* \mu_j(x)$. If the sequence $\{\sigma_k\}$ is unbounded, then σ_k tends to ∞ as $k \rightarrow -\infty$. Without loss of generality, formally, we write $\lim_{k \rightarrow -\infty} \sigma_k(x) = \sigma(x)$. Then by the mean value theorem, noticing that $\sigma_k(x) \geq \sigma_{k+1}(x)$, we have

$$\begin{aligned} \sigma(x)^{p'} &= \left(\sum_{j=-\infty}^{\infty} 2^{j(p-1)} R_{\alpha,\beta}^* \mu_j(x) \right)^{p'} = p' \sum_{k=-\infty}^{\infty} (\sigma_k(x))^{p'-1} \\ &= \lim_{n \rightarrow -\infty} \sigma_n(x)^{p'} = \lim_{n \rightarrow -\infty} \sum_{k=n}^{\infty} (\sigma_k(x)^{p'} - \sigma_{k+1}(x)^{p'}) \\ &\leq p' (\sigma_k(x))^{p'-1} (\sigma_k(x) - \sigma_{k+1}(x)) \\ &= p' \sum_{k=-\infty}^{\infty} (\sigma_k(x))^{p'-1} 2^{k(p-1)} R_{\alpha,\beta}^* \mu_k(x). \end{aligned} \quad (4.7)$$

We use Cauchy-Schwartz's inequality to obtain

$$\begin{aligned} T &= p' \int_{\mathbb{R}^n} \sum_{k=-\infty}^{\infty} \sigma_k^{p'-1}(x) 2^{k(p-1)} R_{\alpha,\beta}^* \mu_k(x) dx \\ &\lesssim p' \int_{\mathbb{R}^n} \left(\sum_{k=-\infty}^{\infty} (\sigma_k(x)) 2^{k \frac{1}{(p-1)} \frac{1}{p'-1}} (R_{\alpha,\beta}^* \mu_k)^{p'-1} \right)^{p'-1} \\ &\quad \times \left(\sum_{k=-\infty}^{\infty} 2^{k(p-1) \frac{p(p-2)}{(p-1)^2} \frac{1}{2-p'}} (R_{\alpha,\beta}^* \mu_k(x))^{p'} \right)^{2-p'} dx \\ &= p' \int_{\mathbb{R}^n} \left(\sum_{k=-\infty}^{\infty} \sigma_k(x) 2^k (R_{\alpha,\beta}^* \mu_k)^{p'-1} \right)^{p'-1} \left(\sum_{k=-\infty}^{\infty} 2^{kp} (R_{\alpha,\beta}^* \mu_k(x))^{p'} \right)^{2-p'} dx, \end{aligned}$$

which, together with Hölder's inequality, indicates that $T \lesssim p' T_1^{2-p'} T_2^{p'-1}$, where

$$\begin{cases} T_1 := \int_{\mathbb{R}^n} \left(\sum_{k=-\infty}^{\infty} 2^{kp} (R_{\alpha,\beta}^* \mu_k(x))^{p'} \right) dx; \\ T_2 := \int_{\mathbb{R}^n} \left(\sum_{k=-\infty}^{\infty} \sigma_k(x) 2^k (R_{\alpha,\beta}^* \mu_k)^{p'-1} \right) dx. \end{cases}$$

For T_1 , we have

$$\begin{aligned} T_1 &= \sum_{k=-\infty}^{\infty} 2^{kp} \int_{\mathbb{R}^n} (R_{\alpha,\beta}^* \mu_k(x))^{p'} dx = \sum_{k=-\infty}^{\infty} 2^{kp} C_p^{(\alpha,\beta)}(E_k) \\ &\lesssim \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} C_p^{(\alpha,\beta)} \left(\left\{ (x,t) \in \overline{B_r(0,0)} : R_{\alpha,\beta} f(x,t) \geq 2^k \right\} \right) d\lambda^p \\ &\lesssim \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^k} C_p^{(\alpha,\beta)} \left(\left\{ (x,t) \in \overline{B_r(0,0)} : R_{\alpha,\beta} f(x,t) \geq \lambda \right\} \right) d\lambda^p \\ &\lesssim \int_0^{\infty} C_p^{(\alpha,\beta)} \left(\left\{ (x,t) \in \overline{B_r(0,0)} : R_{\alpha,\beta} f(x,t) \geq \lambda \right\} \right) d\lambda^p \lesssim S. \end{aligned}$$

For T_2 , we can get

$$\begin{aligned} T_2 &= \sum_{k=-\infty}^{\infty} 2^k \int_{\mathbb{R}^n} \left(\sum_{j \geq k} 2^{j(p-1)} R_{\alpha, \beta}^* \mu_j(x) \right) (R_{\alpha, \beta}^* \mu_k)^{p'-1} dx \\ &= \sum_{k=-\infty}^{\infty} \sum_{j \geq k} 2^{k+j(p-1)} \int_{\mathbb{R}^n} R_{\alpha, \beta}^* \mu_j(x) (R_{\alpha, \beta}^* \mu_k)^{p'-1} dx \\ &= \sum_{k=-\infty}^{\infty} \sum_{j \geq k} 2^{k+j(p-1)} \int_{\mathbb{R}_+^{n+1}} R_{\alpha, \beta} (R_{\alpha, \beta}^* \mu_k)^{p'-1} d\mu_j(x). \end{aligned}$$

Note that E_k is compact subsets. Let f_{E_k} be the function satisfying (4.1). Suppose that $R_{\alpha, \beta} f_{E_k}(x_0, t_0) > 1$, by the lower-semicontinuity, $R_{\alpha, \beta} f_{E_k} \geq 1 + \delta > 1$ on some neighborhood U of (x_0, t_0) . On the other hand, denote by Ω_{E_k} the set $\{(x, t) \in E_k : R_{\alpha, \beta} f_{E_k} < 1\}$ and let F be any compact subset of Ω_{E_k} . For any $0 \leq f \in L^p(\mathbb{R}^n)$ such that $R_{\alpha, \beta} f \geq 1$ on F

$$\mu_k(F) \leq \int_{\mathbb{R}_+^{n+1}} R_{\alpha, \beta} f d\mu_k = \int_{\mathbb{R}^n} f R_{\alpha, \beta}^* \mu_k dx \leq \|R_{\alpha, \beta}^* \mu_k\|_{L^{p'}} \|f\|_{L^p},$$

which, together with (4.3), gives

$$\mu_k(F) \leq \|R_{\alpha, \beta}^* \mu_k\|_{L^{p'}} C_p^{(\alpha, \beta)}(F) \leq \|R_{\alpha, \beta}^* \mu_k\|_{L^{p'}} C_p^{(\alpha, \beta)}(\Omega_{E_k}) = 0.$$

Hence, $\mu_k(\Omega_{E_k}) = 0$, i.e., $R_{\alpha, \beta} f_{E_k} \geq 1$ on E_k for μ_k a.e. We can deduce that

$$\begin{aligned} C_p^{(\alpha, \beta)}(E_k) &= \int_{\mathbb{R}_+^{n+1}} R_{\alpha, \beta} (R_{\alpha, \beta}^* \mu_k)^{p'-1} d\mu_k = \int_{\mathbb{R}_+^{n+1}} R_{\alpha, \beta} f_{E_k} d\mu_k \\ &\geq (1 + \delta) \mu_k(U) + \mu_k(E_k \setminus U) \\ &\geq \delta \mu_k(U) + \mu_k(E_k), \end{aligned}$$

which gives $\mu_k(U) = 0$, i.e., $(x_0, t_0) \notin \text{supp } \mu_k$. Equivalently, for all $(x, t) \in \text{supp } \mu_k$,

$$R_{\alpha, \beta} (R_{\alpha, \beta}^* \mu_k)^{p'-1} = R_{\alpha, \beta} f_{E_k} \leq 1.$$

Since for $j \geq k$, $E_j \subset E_k$ and we obtain

$$\begin{aligned} T_2 &\lesssim \sum_{k=-\infty}^{\infty} \sum_{j \geq k} 2^{k+j(p-1)} C_p^{(\alpha, \beta)}(E_j) \\ &= \sum_{j=-\infty}^{\infty} 2^{j(p-1)} C_p^{(\alpha, \beta)}(E_j) \sum_{k=-\infty}^j 2^k \\ &\lesssim \sum_{j=-\infty}^{\infty} 2^{jp} C_p^{(\alpha, \beta)}(E_j) \lesssim S. \end{aligned}$$

The estimates for T_1 and T_2 yields (4.6). It can be deduced from (4.5) & (4.6) that

$$\left\| \sum_{j=-\infty}^{\infty} 2^{j(p-1)} (R_{\alpha, \beta}^* \mu_j) \right\|_{L^{p'}}^{p'} \lesssim \|f\|_{L^p}^p,$$

i.e., $\sigma \in L^{p'}(\mathbb{R}^n)$ and the series $\sum_{j=-\infty}^{\infty} 2^{j(p-1)} R_{\alpha, \beta}^* \mu_j(x) < \infty$ a.e. $x \in \mathbb{R}^n$.

Case 2: $1 < p < 2$. For $k = 0, \pm 1, \pm 2, \dots$, let

$$\begin{cases} \sigma_k(x) = \sum_{j=-\infty}^k 2^{j(p-1)} R_{\alpha, \beta}^* \mu_j(x); \\ \sigma(x) = \sum_{j=-\infty}^{\infty} 2^{j(p-1)} R_{\alpha, \beta}^* \mu_j(x). \end{cases}$$

Similarly, because $\sigma_k(x) \geq 0$ for $x \in \mathbb{R}^n$, we can also write $\lim_{k \rightarrow \infty} \sigma_k(x) = \sigma(x)$ formally. Below, similar to *Case 1*, we will prove (4.6) and $\sigma \in L^{p'}(\mathbb{R}^n)$. Following the procedure of (4.7), we can deduce that

$$(\sigma(x))^{p'} = \left(\sum_{j=-\infty}^{\infty} 2^{j(p-1)} R_{\alpha,\beta}^* \mu_j(x) \right)^{p'} \leq p' \sum_{k=-\infty}^{\infty} (\sigma_k(x))^{p'-1} 2^{k(p-1)} R_{\alpha,\beta}^* \mu_k(x).$$

We get

$$\begin{aligned} T &= p' \sum_{k=-\infty}^{\infty} 2^{k(p-1)} \int_{\mathbb{R}^n} P_{\alpha}^* \mu_k(x) \left(\sum_{j=-\infty}^k 2^{j(p-1)} R_{\alpha,\beta}^* \mu_j(x) \right)^{p'-1} dx \\ &= p' \sum_{k=-\infty}^{\infty} 2^{k(p-1)} \int_{\mathbb{R}^n} \left(\sum_{j=-\infty}^k 2^{j(p-1)} \left(R_{\alpha,\beta}^* \mu_k(x) \right)^{1/(p'-1)} R_{\alpha,\beta}^* \mu_j(x) \right)^{p'-1} dx \\ &= p' \sum_{k=-\infty}^{\infty} 2^{k(p-1)} \left\| \sum_{j=-\infty}^k 2^{j(p-1)} \left(R_{\alpha,\beta}^* \mu_k(x) \right)^{1/(p'-1)} R_{\alpha,\beta}^* \mu_j(x) \right\|_{L^{p'-1}}^{p'-1} \\ &\lesssim p' \sum_{k=-\infty}^{\infty} 2^{k(p-1)} \left\{ \sum_{j=-\infty}^k 2^{j(p-1)} \left\| \left(R_{\alpha,\beta}^* \mu_k(x) \right)^{1/(p'-1)} R_{\alpha,\beta}^* \mu_j(x) \right\|_{L^{p'-1}} \right\}^{p'-1}. \end{aligned}$$

Also, for $j \leq k$, if $x \in E_k$ then $x \in E_j$. Similarly, we can deduce that $R_{\alpha,\beta} \left(R_{\alpha,\beta}^* \mu_j(x) \right)^{p'-1} (x) \leq 1$, $x \in E_k$. This indicates that

$$\int_{\mathbb{R}^n} \left(R_{\alpha,\beta}^* \mu_k(x) \right) \left(R_{\alpha,\beta}^* \mu_j(x) \right)^{p'-1} dx = \int_{\mathbb{R}_+^{n+1}} R_{\alpha,\beta} \left(R_{\alpha,\beta}^* \mu_j(x) \right)^{p'-1} d\mu_k \leq \mu_k(E_k) = C_p^{(\alpha,\beta)}(E_k).$$

We obtain

$$\begin{aligned} T &\lesssim \sum_{k=-\infty}^{\infty} 2^{k(p-1)} \left\{ \sum_{j=-\infty}^k 2^{j(p-1)} \left(\int_{\mathbb{R}^n} R_{\alpha,\beta}^* \mu_k(x) \left(R_{\alpha,\beta}^* \mu_j(x) \right)^{p'-1} dx \right)^{1/(p'-1)} \right\}^{p'-1} \\ &\lesssim \sum_{k=-\infty}^{\infty} 2^{k(p-1)} C_p^{(\alpha,\beta)}(E_k) \left(\sum_{j=-\infty}^k 2^{j(p-1)} \right)^{p'-1} \\ &\lesssim \sum_{k=-\infty}^{\infty} 2^{kp} C_p^{(\alpha,\beta)}(E_k) \lesssim S. \end{aligned}$$

which gives (4.6) for $1 < p < 2$.

It follows from (4.5) & (4.6) that $S \lesssim \|f\|_{L^p} T^{1/p'} \lesssim \|f\|_{L^p} S^{1/p'}$. Then we can get $S \lesssim \|f\|_{L^p}^p$ and consequently, $T < \infty$. Then $\sigma \in L^{p'}(\mathbb{R}^n)$ for $1 < p < 2$. By the fact that $f \in C_0^\infty(\mathbb{R}^n)$, there exists an integer j_0 such that

$$\left\| \sum_{j=0}^{\infty} 2^{j(p-1)} R_{\alpha,\beta}^* \mu_j \right\|_{L^{p'}} \leq \sum_{j=0}^{\infty} 2^{j(p-1)} \left\| R_{\alpha,\beta}^* \mu_j \right\|_{L^{p'}} \simeq \sum_{j=0}^{j_0} 2^{j(p-1)} (C_p^{(\alpha,\beta)}(E_j))^{1/p'} < \infty.$$

This indicates that $\sum_{j=-\infty}^{-1} 2^{j(p-1)} R_{\alpha,\beta}^* \mu_j \in L^{p'}(\mathbb{R}^n)$. For σ_k , if $k \leq -1$, because $R_{\alpha,\beta}^* \mu_j(x) \geq 0$, then

$$\|\sigma_k\|_{L^{p'}} \leq \left\| \sum_{j=-\infty}^{-1} 2^{j(p-1)} R_{\alpha,\beta}^* \mu_j \right\|_{L^{p'}}$$

and $\sigma_k \in L^{p'}(\mathbb{R}^n)$. If $k \geq 0$, we can also use the identity

$$\sigma_k = \sum_{j=-\infty}^{-1} 2^{j(p-1)} R_{\alpha,\beta}^* \mu_j + \sum_{j=0}^{j_0} 2^{j(p-1)} R_{\alpha,\beta}^* \mu_j,$$

to get $\sigma_k \in L^p(\mathbb{R}^n)$. Further, we obtain that

$$\begin{aligned} & \int_0^\infty C_p^{(\alpha,\beta)} \left(\left\{ (x, t) \in \overline{B_r(0, 0)} : R_{\alpha,\beta} f(x, t) \geq \lambda \right\} \right) d\lambda^p \\ &= \sum_{j=-\infty}^\infty \int_{2^j}^{2^{j+1}} C_p^{(\alpha,\beta)} \left(\left\{ (x, t) \in \overline{B_r(0, 0)} : R_{\alpha,\beta} f(x, t) \geq \lambda \right\} \right) d\lambda^p \\ &\lesssim \sum_{j=-\infty}^\infty C_p^{(\alpha,\beta)}(E_j) \int_{2^j}^{2^{j+1}} d\lambda^p \\ &\lesssim \sum_{j=-\infty}^\infty 2^{pj} \mu_j(E_j) \lesssim \|f\|_{L^p}^p. \end{aligned}$$

Hence we can deduce from Proposition 4.6 and Fauto's lemma that

$$\begin{aligned} & \int_0^\infty C_p^{(\alpha,\beta)} \left(\left\{ (x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha,\beta} f(x, t) > \lambda \right\} \right) d\lambda^p \\ &= \int_0^\infty C_p^{(\alpha,\beta)} \left(\bigcup_{r=1}^\infty \left\{ (x, t) \in B_r(0, 0) : R_{\alpha,\beta} f(x, t) > \lambda \right\} \right) d\lambda^p \\ &= \int_0^\infty \lim_{r \rightarrow \infty} C_p^{(\alpha,\beta)} \left(\left\{ (x, t) \in B_r(0, 0) : R_{\alpha,\beta} f(x, t) > \lambda \right\} \right) d\lambda^p \\ &\leq \lim_{r \rightarrow \infty} \int_0^\infty C_p^{(\alpha,\beta)} \left(\left\{ (x, t) \in \overline{B_r(0, 0)} : R_{\alpha,\beta} f(x, t) \geq \lambda \right\} \right) d\lambda^p \lesssim \|f\|_{L^p}^p. \end{aligned}$$

□

By the above capacity strong type inequality, it is easy to get the following capacity weak type inequality

$$\lambda^p C_p^{(\alpha,\beta)} \left(\left\{ (x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha,\beta} f(x, t) \geq \lambda \right\} \right) \leq \|f\|_{L^p(\mathbb{R}^n)}^p, \quad \forall f \in L_+^p(\mathbb{R}^n).$$

5 $L^q(\mathbb{R}_+^{n+1}, \mu)$ -extension of $L^p(\mathbb{R}^n)$

Let $1 < p \leq q < \infty$ and $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$. For $\lambda > 0$, define

$$\kappa(\mu; \lambda) := \inf \left\{ C_p^{(\alpha,\beta)}(K) : \text{compact } K \subset \mathbb{R}_+^{n+1}, \mu(K) \geq \lambda \right\}.$$

Theorem 5.1. Let $1 < p \leq q < \infty$ and $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$.

(i) The extension $R_{\alpha,\beta} : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}_+^{n+1})$ is bounded if and only if

$$\sup_{\lambda \in (0, \infty)} \frac{\lambda^{p/q}}{\kappa(\mu; \lambda)} < \infty;$$

(ii) If $1 < p < q < \infty$, then $\lambda^{p/q} \lesssim \kappa(\mu; \lambda) \forall \lambda \in (0, \infty)$ can be replaced by $\mu(B_r^{(\alpha, \beta)}(x_0, t_0)) \lesssim r^{\beta q n/p} \forall (r, t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$ with $t_0 \lesssim r^\alpha$.

Proof. (i). Suppose that $R_{\alpha, \beta}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}_+^{n+1}, \mu)$. For a compact set $K \subset \mathbb{R}_+^{n+1}$, denote by $\mu|_K$ the restriction of μ to the set K . Then

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) R_{\alpha, \beta}^* \mu|_K dx &= \iint_{\mathbb{R}_+^{n+1}} R_{\alpha, \beta} f(x, t) d\mu|_K(x, t) \\ &\leq \left\{ \iint_{\mathbb{R}_+^{n+1}} (R_{\alpha, \beta} f(x, t))^q d\mu|_K(x, t) \right\}^{1/q} \left\{ \iint_{\mathbb{R}_+^{n+1}} d\mu|_K(x, t) \right\}^{1-1/q} \\ &= \|R_{\alpha, \beta} f\|_{L^q(\mathbb{R}_+^{n+1}, \mu)} (\mu(K))^{1-1/q} \\ &\lesssim \|f\|_{L^p(\mathbb{R}^n)} (\mu(K))^{1-1/q}, \end{aligned}$$

which implies that $\|R_{\alpha, \beta}^* \mu|_K\|_{L^{p'}(\mathbb{R}^n)} \lesssim (\mu(K))^{1/q'}$. Letting

$$E_\lambda(f) := \{(x, t) \in \mathbb{R}_+^{n+1} : |R_{\alpha, \beta} f(x, t)| > \lambda\},$$

we have

$$\begin{aligned} \lambda \mu(E_\lambda(f)) &= \lambda \int_{E_\lambda(f)} 1 d\mu(x, t) \\ &\leq \int_{E_\lambda(f)} |R_{\alpha, \beta} f(x, t)| d\mu \\ &= \int_{\mathbb{R}_+^{n+1}} |R_{\alpha, \beta} f(x, t)| d\mu|_{E_\lambda(f)} \\ &\leq \|R_{\alpha, \beta} f\|_{L^q(\mathbb{R}_+^{n+1}, \mu)} (\mu(E_\lambda))^{1/q'} \\ &\leq \|f\|_{L^p(\mathbb{R}^n)} (\mu(E_\lambda))^{1/q'}, \end{aligned}$$

which implies that $\lambda(\mu(E_\lambda(f)))^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ and, hence, $\sup_{\lambda \in (0, \infty)} \lambda^q \mu(E_\lambda(f)) \lesssim \|f\|_{L^p(\mathbb{R}^n)}^q$. Choose a function $f \in L^p(\mathbb{R}^n)$ such that $R_{\alpha, \beta} f \geq 1$ on a given compact set $K \subset \mathbb{R}_+^{n+1}$, that is, $K \subset E_1(f)$. It follows from the above estimate that

$$(\mu(K))^{1/q} \leq (\mu(E_1(f)))^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

and

$$(\mu(K))^{p/q} \lesssim \inf \left\{ \|f\|_{L^p(\mathbb{R}^n)}^p : f \geq 0 \text{ \& } R_{\alpha, \beta} f \geq 1 \right\} \leq C_p^{(\alpha, \beta)}(K).$$

If K is compact and $\mu(K) \geq \lambda$, then

$$\lambda^{1/q} \leq (\mu(K))^{1/q} \leq (C_p^{(\alpha, \beta)}(K))^{1/p}.$$

This means that $\lambda^{p/q} \leq C_p^{(\alpha, \beta)}(K)$, equivalently,

$$\lambda^{p/q} \leq \inf \left\{ C_p^{(\alpha, \beta)}(K) : K \text{ is compact and } \mu(K) \geq \lambda \right\},$$

that is, $\lambda^{p/q} \lesssim \kappa(\mu, \lambda)$.

Conversely, we assume that $\sup_{\lambda \in (0, \infty)} \lambda^{p/q} / \kappa(\mu, \lambda) < \infty$. We know that for any compact set K , $(\mu(K))^{1/q} \lesssim (C_p^{(\alpha, \beta)}(K))^{1/p}$. We have proved that

$$\int_0^\infty C_p^{(\alpha, \beta)} \left(\left\{ (x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha, \beta} f(x, t) \geq \lambda \right\} \right) d\lambda^p \lesssim \|f\|_{L^p(\mathbb{R}^n)}^p. \quad (5.1)$$

For any $\tau > 0$, we can get

$$\begin{aligned}
 \tau^p C_p^{(\alpha, \beta)}(E_\tau) &= \tau^p C_p^{(\alpha, \beta)}\left(\left\{(x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha, \beta} f(x, t) \geq \tau\right\}\right) \\
 &= \int_0^\tau C_p^{(\alpha, \beta)}\left(\left\{(x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha, \beta} f(x, t) \geq \lambda\right\}\right) d\lambda^p \\
 &\lesssim \int_0^\tau C_p^{(\alpha, \beta)}\left(\left\{(x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha, \beta} f(x, t) \geq \lambda\right\}\right) d\lambda^p \\
 &\lesssim \int_0^\infty C_p^{(\alpha, \beta)}\left(\left\{(x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha, \beta} f(x, t) \geq \lambda\right\}\right) d\lambda^p \\
 &\lesssim \|f\|_{L^p(\mathbb{R}^n)}^p.
 \end{aligned} \tag{5.2}$$

We can use (5.1) & (5.2) to obtain

$$\begin{aligned}
 \iint_{\mathbb{R}_+^{n+1}} |R_{\alpha, \beta} f(x, t)|^q d\mu(x, t) &= \int_0^\infty \mu(E_\lambda) d\lambda^q \\
 &\leq \int_0^\infty \left(C_p^{(\alpha, \beta)}(E_\lambda)\right)^{q/p} d\lambda^q \\
 &= q \int_0^\infty \left(C_p^{(\alpha, \beta)}(E_\lambda)\right)^{q/p-1} \left(C_p^{(\alpha, \beta)}(E_\lambda)\right) \lambda^{q-1} d\lambda \\
 &\lesssim \int_0^\infty \left(\lambda^{-p} \|f\|_{L^p(\mathbb{R}^n)}^p\right)^{q/p-1} \left(C_p^{(\alpha, \beta)}(E_\lambda)\right) \lambda^{q-1} d\lambda \\
 &\lesssim \|f\|_{L^p(\mathbb{R}^n)}^{q-p} \int_0^\infty C_p^{(\alpha, \beta)}(E_\lambda) d\lambda^p \\
 &\lesssim \|f\|_{L^p(\mathbb{R}^n)}^q,
 \end{aligned}$$

which completes the proof of (i).

(ii). If $\lambda^{p/q} \leq \kappa(\mu, \lambda)$, then for any set $K \subset \mathbb{R}_+^{n+1}$, $(\mu(K))^{1/q} \leq (C_p^{(\alpha, \beta)}(K))^{1/p}$. Let $K = B_r^{(\alpha, \beta)}(t_0, x_0)$. Then

$$\left(\mu(B_r^{(\alpha, \beta)}(x_0, t_0))\right)^{1/q} \leq \left(C_p^{(\alpha, \beta)}(B_r^{(\alpha, \beta)}(x_0, t_0))\right)^{1/p}.$$

By Proposition 4.3,

$$r^{n\beta} \lesssim C_p^{(\alpha, \beta)}(B_r^{(\alpha, \beta)}(x_0, t_0)) \lesssim \left((t_0 + r^\alpha)^{n\beta/\alpha} r_0^{-\beta n}\right)^p r^{\beta n}.$$

Then for $t_0 \leq r^\alpha$, $C_p^{(\alpha, \beta)}(B_r^{(\alpha, \beta)}(x_0, t_0)) \lesssim r^{n\beta}$ and $(\mu(B_r^{(\alpha, \beta)}(x_0, t_0)))^{1/q} \lesssim r^{n\beta/p}$.

For the reverse implication, if $(x, t) \in B_r^{(\alpha, \beta)}(x_0, t_0)$, then $|x - x_0| \leq r^\beta/2$ and $r^\alpha < t - t_0 < 2r^\alpha$. We get $t > t_0 + r^\alpha > r^\alpha$ and $|x - x_0| < r^\beta/2 \lesssim t^{\beta/\alpha}$, and hence,

$$\begin{aligned}
 G_t(x - y) &\geq \frac{t^\beta}{(t^{\beta/\alpha} + |x - x_0|)^{n+\alpha}} \\
 &= \frac{1}{t^{n\beta/\alpha}} \frac{1}{(1 + |x - x_0|/t^{\beta/\alpha})^{n+\alpha}} \\
 &\geq \frac{1}{t^{n\beta/\alpha}} \geq \frac{1}{r^{n\beta}}.
 \end{aligned}$$

This means that if $(x, t) \in B_r^{(\alpha, \beta)}(x_0, t_0)$ then $r \geq (G_t(x - x_0))^{-1/n\beta}$. We obtain

$$R_{\alpha, \beta}^* \mu|_K(x_0) = \iint_{\mathbb{R}_+^{n+1}} G_t(x_0 - x) d\mu|_K$$

$$\begin{aligned}
&= \iint_{\mathbb{R}_+^{n+1}} \left(\int_{(G_t(x_0-x))^{-1/n\beta}}^{\infty} \frac{dr}{r^{n\beta+1}} \right) d\mu|_K \\
&= \iint_{\mathbb{R}_+^{n+1}} \left(\int_0^{\infty} 1_{B_r^{(\alpha,\beta)}(x_0,t_0)} \frac{dr}{r^{n\beta+1}} \right) d\mu|_K \\
&\sim \int_0^{\infty} \mu|_K(B_r^{(\alpha,\beta)}(x_0,t_0)) \frac{dr}{r^{n\beta+1}}.
\end{aligned}$$

Minkowski's inequality gives

$$\left\| R_{\alpha,\beta}^* \mu|_K \right\|_{L^{p'}(\mathbb{R}^n)} \lesssim \int_0^{\infty} \left\| \mu|_K(B_r^{(\alpha,\beta)}(\cdot, t_0)) \right\|_{L^{p'}(\mathbb{R}^n)} \frac{dr}{r^{n\beta+1}}.$$

On the other hand,

$$\begin{aligned}
\left\| \mu|_K(B_r^{(\alpha,\beta)}(\cdot, t_0)) \right\|_{L^{p'}(\mathbb{R}^n)}^{p'} &= \int_{\mathbb{R}^n} \left(\mu|_K(B_r^{(\alpha,\beta)}(x, t_0)) \right)^{p'} dx \\
&\lesssim (\mu(K))^{p'-1} \int_{\mathbb{R}^n} \mu|_K(B_r^{(\alpha,\beta)}(x, t_0)) dx \\
&\lesssim (\mu(K))^{p'} r^{\beta n}.
\end{aligned}$$

Hence,

$$\int_{\delta}^{\infty} \left\| \mu|_K(B_r^{(\alpha,\beta)}(x_0, t_0)) \right\|_{L^{p'}(\mathbb{R}^n)} \frac{dr}{r^{n\beta+1}} \lesssim \int_{\delta}^{\infty} \mu(K) r^{n\beta/p' - n\beta - 1} dr \lesssim \mu(K) \delta^{-n\beta/p}.$$

Meanwhile, using $\mu(B_r^{(\alpha,\beta)}(x_0, t_0)) \lesssim r^{n\beta q/p}$, we have

$$\begin{aligned}
\left\| \mu|_K(B_r^{(\alpha,\beta)}(x_0, t_0)) \right\|_{L^{p'}(\mathbb{R}^n)}^{p'} &= \int_{\mathbb{R}^n} \left(\mu|_K(B_r^{(\alpha,\beta)}(x_0, t_0)) \right)^{p'} dx_0 \\
&\lesssim r^{n\beta q(p'-1)/p} \int_{\mathbb{R}^n} \mu|_K(B_r^{(\alpha,\beta)}(x_0, t_0)) dx_0 \\
&\lesssim r^{n\beta q(p'-1)/p} \mu(K) r^{n\beta} \\
&\lesssim r^{n\beta(1+q/p(p-1))} \mu(K),
\end{aligned}$$

which gives

$$\begin{aligned}
\int_0^{\delta} \left\| \mu|_K(B_r^{(\alpha,\beta)}(x_0, t_0)) \right\|_{L^{p'}(\mathbb{R}^n)} \frac{dr}{r^{n\beta+1}} &\lesssim (\mu(K))^{1/p'} \int_0^{\delta} r^{n\beta(1+q/p(p-1))/p'} \frac{dr}{r^{n\beta+1}} \\
&\lesssim (\mu(K))^{1/p'} \int_0^{\delta} r^{n\beta/p' + n\beta q/p^2 - n\beta - 1} dr \\
&\lesssim (\mu(K))^{1/p'} \delta^{n\beta(q-p)/p^2}.
\end{aligned}$$

Finally,

$$\left\| R_{\alpha,\beta}^* \mu|_K \right\|_{L^{p'}(\mathbb{R}^n)} \lesssim \mu(K) \delta^{-n\beta/p} + (\mu(K))^{1/p'} \delta^{n\beta(q-p)/p^2}.$$

Take δ such that $(\mu(K))\delta^{-n\beta/p} = (\mu(K))^{1/p'} \delta^{n\beta(q-p)/p^2}$, that is, $\delta = (\mu(K))^{p/n\beta q}$. For such a δ ,

$$\left\| R_{\alpha,\beta}^* \mu|_K \right\|_{L^{p'}(\mathbb{R}^n)} \lesssim \mu(K) \left((\mu(K))^{p/n\beta q} \right)^{-n\beta/p} \lesssim (\mu(K))^{1/q'}.$$

Let $f \in C_0^\infty(\mathbb{R}^n)$ and $E_\lambda = \{x : |R_{\alpha,\beta} * f(x, t)| \geq \lambda\}$. Then the set E_λ is compact. It follows from the above estimate that

$$\lambda \mu(E_\lambda) \leq \int_{\mathbb{R}^n} |R_{\alpha,\beta} f(x, t)| d\mu|_{E_\lambda} \leq \|f\|_{L^p} \|R_{\alpha,\beta}^* \mu|_{E_\lambda}\|_{L^{p'}} \lesssim \|f\|_{L^p} (\mu(E_\lambda))^{1/q'}. \quad (5.3)$$

For an arbitrary $f \in L^p(\mathbb{R}^n)$, via approximating f by a sequence from $C_0^\infty(\mathbb{R}^n)$ in the L^p -norm, we can prove that (5.3) holds for f . For $f \in L^p(\mathbb{R}^n)$ such that $R_{\alpha,\beta} f \geq 1$ on K . By (5.3), it holds $(\mu(K))^{1/q} \lesssim \|f\|_{L^p}$, which gives $(\mu(K))^{1/q} \leq (C_p^{(\alpha,\beta)}(K))^{1/p}$. Recall that $\mu(K) \geq \lambda$. Then taking the infimum over the compact sets K such that $\mu(K) \geq \lambda$, we get $\lambda^{p/q} \lesssim C_p^{(\alpha,\beta)}(K)$, i.e.,

$$\lambda^{p/q} \lesssim \inf \left\{ C_p^{(\alpha,\beta)}(K) : \text{compact } K \subset \mathbb{R}_+^{n+1}, \mu(K) \geq \lambda \right\} = c_{\alpha,p}(\mu; \lambda).$$

This completes the proof of Theorem 5.1. □

Theorem 5.2. Let $1 < q < p < \infty$ and $\mu \in \mathcal{M}_+(\mathbb{R}_+^{n+1})$. The extension

$$R_{\alpha,\beta} : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}_+^{n+1}, \mu)$$

is bounded if and only if

$$\int_0^\infty \left(\frac{\lambda^{p/q}}{\kappa(\mu; \lambda)} \right)^{q/(p-q)} \frac{d\lambda}{\lambda} < \infty.$$

Proof. We first assume that $R_{\alpha,\beta} : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}_+^{n+1}, \mu)$ is bounded. Then

$$\left(\iint_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta} f(x, t)|^q d\mu \right)^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

We can get

$$\begin{aligned} \sup_{\lambda>0} \lambda (\mu(E_\lambda(f)))^{1/q} &= \sup_{\lambda>0} \left\{ \lambda^q \mu(E_\lambda(f)) \right\}^{1/q} \\ &\lesssim \sup_{\lambda>0} \left\{ \lambda^q \int_{E_\lambda(f)} 1 d\mu(x, t) \right\}^{1/q} \\ &\lesssim \sup_{\lambda>0} \left\{ \int_{E_\lambda(f)} |R_{\alpha,\beta} f(x, t)|^q d\mu(x, t) \right\}^{1/q} \\ &\lesssim \left\{ \int_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta} f(x, t)|^q d\mu(x, t) \right\}^{1/q} \\ &\lesssim \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

By the definition of $\kappa(\mu, 2^j)$, for any $j \in \mathbb{Z}$, there exists K_j such that $\mu(K_j) > 2^j$ and $C_p^{(\alpha,\beta)}(K_j) \leq 2\kappa(\mu, 2^j)$. On the other hand, since

$$C_p^{(\alpha,\beta)}(K_j) = \inf \left\{ \|f\|_{L^p(\mathbb{R}^n)}^p : f \geq 0, R_{\alpha,\beta} f \geq 1_{K_j} \right\}.$$

There exists a function $f_j \in L^p(\mathbb{R}^n)$ such that $R_{\alpha,\beta}f_j \geq 1_{K_j}$ and $\|f_j\|_{L^p(\mathbb{R}^n)}^p \leq 2C_p^{(\alpha,\beta)}(K_j)$.

For the integers i, k with $i < k$, let

$$f_{i,k} := \sup_{i \leq j \leq k} \left(\frac{2^j}{\kappa(\mu; 2^j)} \right)^{1/(p-q)} f_j.$$

Then

$$\begin{aligned} \|f_{i,k}\|_{L^p(\mathbb{R}^n)}^p &\leq \sum_{j=i}^k \left(\frac{2^j}{\kappa(\mu; 2^j)} \right)^{p/(p-q)} \|f_j\|_{L^p(\mathbb{R}^n)}^p \\ &\lesssim \sum_{j=i}^k \left(\frac{2^j}{\kappa(\mu; 2^j)} \right)^{p/(p-q)} C_p^{(\alpha,\beta)}(K_j) \\ &\lesssim \sum_{j=i}^k \left(\frac{2^j}{\kappa(\mu; 2^j)} \right)^{p/(p-q)} \kappa(\mu; 2^j). \end{aligned}$$

Note that for $i \leq j \leq k$, if $(x, t) \in K_j$, then

$$\begin{aligned} |R_{\alpha,\beta}f_{i,k}(x, t)| &= \left| R_{\alpha,\beta} \left(\sup_{i \leq j \leq k} \left(\frac{2^j}{\kappa(\mu; 2^j)} \right)^{1/(p-q)} f_j(x, t) \right) \right| \\ &\geq \left| \left(\frac{2^j}{\kappa(\mu; 2^j)} \right)^{1/(p-q)} R_{\alpha,\beta}f_j(x, t) \right| \\ &\geq \left(\frac{2^j}{\kappa(\mu; 2^j)} \right)^{1/(p-q)}. \end{aligned}$$

We can see from the above estimate that

$$K_j \subset \left\{ (x, t) \in \mathbb{R}_+^{n+1} : R_{\alpha,\beta}f_{i,k}(x, t) > \left(\frac{2^j}{\kappa(\mu; 2^j)} \right)^{1/(p-q)} \right\}.$$

This means that

$$2^j < \mu(K_j) < \mu \left(E_{(2^j/\kappa(\mu; 2^j))^{1/(p-q)}}(f_{i,k}) \right).$$

We can obtain

$$\begin{aligned} \|f_{i,k}\|_{L^p(\mathbb{R}^n)}^p &\geq \iint_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta}(f_{i,k})(x, t)|^q d\mu(x, t) \\ &\simeq \int_0^\infty \left(\inf \left\{ \lambda : \mu(E_\lambda(f_{i,k})) \leq s \right\} \right)^q ds \\ &\geq \sum_{j=i}^k 2^j \left(\inf \left\{ \lambda : \mu(E_\lambda(f_{i,k})) \leq 2^j \right\} \right)^q \\ &\geq \sum_{j=i}^k 2^j \left(\frac{2^j}{\kappa(\mu; 2^j)} \right)^{q/(p-q)} \\ &\geq \left(\frac{\sum_{j=i}^k \left(2^j/\kappa(\mu; 2^j) \right)^{q/(p-q)} 2^j}{\left(\sum_{j=i}^k \left(2^j/\kappa(\mu; 2^j) \right)^{p/(p-q)} \kappa(\mu; 2^j) \right)^{q/p}} \right) \|f_{i,k}\|_{L^p(\mathbb{R}^n)}^q \\ &\simeq \left(\sum_{j=i}^k \frac{2^{jp/(p-q)}}{(\kappa(\mu; 2^j))^{q/(p-q)}} \right)^{1-q/p} \|f_{i,k}\|_{L^p(\mathbb{R}^n)}^q. \end{aligned}$$

This implies

$$\int_0^\infty \left(\lambda^{p/q} / \kappa(\mu; \lambda) \right)^{q/(p-q)} \lambda^{-1} d\lambda \lesssim \sum_{j=-\infty}^\infty \frac{2^{jp/(p-q)}}{(\kappa(\mu; 2^j))^{q/(p-q)}} \lesssim 1.$$

Conversely, let

$$I_{p,q}(\mu) = \int_0^\infty \left(\frac{\lambda^{p/q}}{\kappa(\mu; \lambda)} \right)^{q/(p-q)} \frac{d\lambda}{\lambda} < \infty.$$

Now for each integer $j = 0, \pm 1, \pm 2, \dots$, and $f \in C_0(\mathbb{R}^n)$, let

$$S_{p,q}(\mu; f) = \sum_{j=-\infty}^\infty \frac{\left\{ \mu(E_{2^j}(f)) - \mu(E_{2^{j+1}}(f)) \right\}^{p/(p-q)}}{\left(C_p^{(\alpha,\beta)}(E_{2^j}(f)) \right)^{q/(p-q)}}.$$

Using integration-by-part, Hölder's inequality and Lemma 4.8, we obtain

$$\begin{aligned} & \int_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta} f|^q d\mu \\ &= - \int_0^\infty \lambda^q d\mu(E_\lambda(f)) \\ &\lesssim \sum_{j=-\infty}^\infty \left\{ \mu(E_{2^j}(f)) - \mu(E_{2^{j+1}}(f)) \right\} 2^{jq} \\ &\lesssim (S_{p,q}(\mu; f))^{1-q/p} \left(\sum_{j=-\infty}^\infty 2^{jp} C_p^{(\alpha,\beta)}(E_{2^j}(f)) \right)^{q/p} \\ &\lesssim (S_{p,q}(\mu; f))^{1-q/p} \left(\int_0^\infty C_p^{(\alpha,\beta)} \left(\left\{ (x, t) \in \mathbb{R}_+^{n+1} : |R_{\alpha,\beta} f(x, t)| > \lambda \right\} \right) d\lambda^p \right)^{q/p} \\ &\lesssim (S_{p,q}(\mu; f))^{1-q/p} \|f\|_{L^p(\mathbb{R}^n)}^q. \end{aligned}$$

Note also that

$$\begin{aligned} & (S_{p,q}(\mu; f))^{1-q/p} \\ &= \left(\sum_{j=-\infty}^\infty \frac{\left\{ \mu(E_{2^j}(f)) - \mu(E_{2^{j+1}}(f)) \right\}^{p/(p-q)}}{\left(C_p^{(\alpha,\beta)}(E_{2^j}(f)) \right)^{q/(p-q)}} \right)^{1-q/p} \\ &= \left(\sum_{j=-\infty}^\infty \frac{\left\{ \mu(E_{2^j}(f)) - \mu(E_{2^{j+1}}(f)) \right\}^{p/(p-q)}}{\left(\kappa(\mu; \mu(E_{2^j}(f))) \right)^{q/(p-q)}} \right)^{1-q/p} \\ &= \left(\sum_{j=-\infty}^\infty \frac{\left(\nu(E_{2^j}(f)) \right)^{p/(p-q)} - \left(\mu(E_{2^{j+1}}(f)) \right)^{p/(p-q)}}{\left(\kappa(\mu; \mu(E_{2^j}(f))) \right)^{q/(p-q)}} \right)^{1-q/p} \\ &\lesssim \left(\int_0^\infty \frac{ds^{p/(p-q)}}{\left(\kappa(\mu; s) \right)^{q/(p-q)}} \right)^{1-q/p} \\ &\simeq (I_{p,q}(\mu))^{1-q/p}. \end{aligned}$$

Therefore,

$$\left(\iint_{\mathbb{R}_+^{n+1}} |R_{\alpha,\beta} f(x,t)|^q d\mu \right)^{1/q} \lesssim (I_{p,q}(\mu))^{(p-q)/pq} \|f\|_{L^p(\mathbb{R}^n)}.$$

□

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