Research Article

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Centered Hardy-Littlewood maximal function on product manifolds

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Abstract: Let X be the direct product of X_i where X_i is smooth manifold for $1 \leq i \leq k$. As is known, if every X_i satisfies the doubling volume condition, then the centered Hardy-Littlewood maximal function M on X is weak (1,1) bounded. In this paper, we consider the product manifold X where at least one X_i does not satisfy the doubling volume condition. To be precise, we first investigate the mapping properties of M when X_1 has exponential volume growth and X_2 satisfies the doubling condition. Next, we consider the product space of two weighted hyperbolic spaces $X_1 = (\mathbb{H}^{n+1}, d, y^{\alpha-n-1} dy dx)$ and $X_2 = (\mathbb{H}^{n+1}, d, y^{\beta-n-1} dy dx)$ which both have exponential volume growth. The mapping properties of M are discussed for every $\alpha, \beta \neq \frac{n}{2}$. Furthermore, let $X = X_1 \times X_2 \times \cdots \times X_k$ where $X_i = (\mathbb{H}^{n_i+1}, y^{\alpha_i-n_i-1} dy dx)$ for $1 \leq i \leq k$. Under the condition $\alpha_i > \frac{n_i}{2}$, we also obtained the mapping properties of M .

Keywords: Maximal function; Harmonic analysis on non-homogeneous spaces

MSC: Primary 42B25; Secondary 43A80

1 Introduction

Let S be a smooth manifold and L a second order differential operator. Furthermore, we assume that there is a σ -finite measure μ on the Borel sets of S such that L is self-adjoint on $L^2(\mu)$. Then there is a canonical distance d associated with the operator L . Denote by $B(x, r)$ the ball centered at x with radius $r > 0$ and by $V(x, r)$ the volume of $B(x, r)$ with respect to μ . The centered Hardy-Littlewood maximal function is defined by:

$$Mf(x) = \sup_{r>0} \frac{1}{V(x, r)} \int_{B(x, r)} |f(y)| d\mu(y), \quad \forall f \in L^1_{loc}(S, d, \mu).$$

As is well known, the maximal function plays an important role in harmonic analysis and is closely related to the theory of singular integral operators, square functions([23]).

Now we recall some known facts. The classical way to study the mapping properties of M is by various covering lemmas. If the manifold S supports the Besicovitch covering lemma, one can prove M is weak (1,1) bounded. However, the Besicovitch covering lemma is not easy to verify. Meanwhile it does not hold on some common manifolds. For instance, counterexamples are constructed on Heisenberg groups equipped with Carnot-Carathéodory or Korányi metric (see [9, 21]), and also on Symmetric spaces of noncompact type (see [4]).

We say (S, d, μ) satisfies the doubling volume condition if there exists a constant $C > 0$ such that

$$V(x, 2r) \leq CV(x, r), \quad \forall r > 0, x \in S. \quad (1.1)$$

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According to (1.1), there exist $v, C' > 0$ such that

$$\left(\frac{r}{R}\right)^v \leq C' \frac{V(x, r)}{V(x, R)}, \quad \forall 0 < r < R, x \in S.$$

If S satisfies the doubling volume condition, Vitali covering lemma holds on S . Then it is well known that M is weak (1,1) bounded.

On the other hand, we say a noncompact manifold (S, d, μ) has the exponential volume growth if there exist constants $c_1, c_2, C_1, C_2, C > 0$ such that

$$C_1 e^{c_1(s-r)} \leq \frac{V(x, s)}{V(x, r)} \leq C_2 e^{c_2(s-r)}, \quad \forall 1 \leq s \leq r, x \in S, \quad (1.2)$$

and

$$V(x, 2r) \leq CV(x, r), \quad \forall 0 < r \leq 1, x \in S. \quad (1.3)$$

If S has the exponential volume growth, M is not in general weak (1,1) bounded. See, for example the counterexamples in Strömberg [24] and Li [14, 15]. For more results about the maximal functions on manifolds with exponential volume growth, we refer the readers to [5, 7, 8, 13–17, 24] and references therein.

Recently, various topics about the maximal functions such as dimensional free estimates, multisublinear maximal functions and applications to function spaces have been extensively studied. See, for example [1, 2, 6, 10, 18, 20, 25, 26] and references therein.

One natural way to construct new manifolds is taking product of given manifolds. Let (X_i, d_i, μ_i) be a smooth manifold where $1 \leq i \leq k$ and k is a positive integer. For $1 \leq i \leq k$, set $x_i \in X_i$. Denote by $B_i(x_i, r)$ the ball centered at x_i with radius $r > 0$ and by $V_i(x_i, r)$ the volume of $B_i(x_i, r)$ with respect to μ_i . Now we consider the direct product space (X, d, μ) where

$$X = \prod_{1 \leq i \leq k} X_i, \quad d = \sqrt{d_1^2 + d_2^2 + \cdots + d_k^2}, \quad \mu = \mu_1 \otimes \cdots \otimes \mu_k.$$

For every point $\mathcal{X} = (x_1, x_2, \dots, x_k) \in X$, denote by $B(\mathcal{X}, r)$ the ball centered at $\mathcal{X}$ with radius $r > 0$ and by $V(\mathcal{X}, r)$ the volume of $B(\mathcal{X}, r)$ with respect to μ . The centered maximal functions on X_i and X are denoted by M_i and M respectively.

The main purpose of this paper is to study that the mapping properties of the centered maximal functions on the product manifolds will to what extent be influenced by that on submanifolds.

Note first that when every X_i ($1 \leq i \leq k$) satisfies (1.1), so does X . Therefore the centered maximal function M is of weak (1,1) type. Thus it is natural to ask about the mapping properties of M when at least one X_i does not satisfy the doubling volume condition.

To begin with, we consider the product spaces of two manifolds.

Theorem 1.1. *Suppose that X_1 has the exponential volume growth and X_2 satisfies the doubling volume condition. Then we have:*

- (i) *Let $1 < p \leq q \leq \infty$. If M_1 is bounded from $L^p(X_1)$ to $L^{p,q}(X_1)$, then M is also bounded from $L^p(X)$ to $L^{p,q}(X)$.*
- (ii) *Let $p > 1$. If M_1 is unbounded on $L^p(X_1)$, then M is unbounded on $L^p(X)$ either.*

Remark 1.2. Since X_1 has the exponential volume growth, we need first to estimate the volume of balls on X . Then inspired by the proof for the strong maximal operators ([23]), we control M by $M_1 M_2$. Finally, it turns out that the mapping properties of M are almost determined by that on exponential volume growing submanifold.

Next we will study the product spaces of two weighted hyperbolic spaces. Let us begin with some notations and known results. The real hyperbolic space $\mathbb{H}^{n+1}$ for $n \geq 1$ can be interpreted as $\mathbb{R}^+ \times \mathbb{R}^n$ with the Riemannian metric $ds^2 = \frac{dy^2 + dx^2}{y^2}$. The induced distance is

$$d((y, x), (y', x')) = \operatorname{arccosh} \frac{y^2 + y'^2 + |x - x'|^2}{2yy'}, \quad \forall (y, x), (y', x') \in \mathbb{R}^+ \times \mathbb{R}^n.$$

And the induced measure is $d\mu(y, x) = y^{-n-1} dy dx$ with dx the Lebesgue measure on $\mathbb{R}^n$.

Let $\alpha \in \mathbb{R}$. The weighed hyperbolic space of dimension $n + 1$ with the parameter α , $\mathbb{H}^{(n+1, \alpha)} := (\mathbb{H}^{n+1}, d, y^{\alpha-n-1} dy dx)$, namely $\mathbb{R}^+ \times \mathbb{R}^n$ endowed with the distance d and the measure $y^{\alpha-n-1} dy dx$. As is known, $\mathbb{H}^{(n+1, \alpha)}$ is of exponential growth. See for example [11, 14] for more details.

By [14, 16], the mapping properties of M on $\mathbb{H}^{(n+1, \alpha)}$ where $\alpha \in \mathbb{R}$ can be summarized as follows:

1. If $\alpha \leq 0$ or $\alpha > n$, then M is weak (1,1) bounded;
2. If $\frac{n}{2} \leq \alpha \leq n$, then M is unbounded on L^p for $1 \leq p < \infty$;
3. If $0 < \alpha < \frac{n}{2}$, let $\tilde{p} = \frac{n-\alpha}{n-2\alpha}$. Then M is unbounded on L^p for $1 \leq p < \tilde{p}$ and bounded on L^p for $p > \tilde{p}$.

Let $X_1 = \mathbb{H}^{(n+1, \alpha)}$, $X_2 = \mathbb{H}^{(n+1, \beta)}$. The mapping properties of the centered maximal function on the product space will be investigated for every $\alpha, \beta \neq \frac{n}{2}$. Without loss of generality, we assume $\alpha \geq \beta$ and the following cases will be investigated.

	$n < \alpha$	$n/2 < \alpha \leq n$	$0 < \alpha < n/2$	$\alpha \leq 0$
$n < \beta$	Case 1			
$n/2 < \beta \leq n$	Case 2	Case 5		
$0 < \beta < n/2$	Case 3	Case 6	Case 8	
$\beta \leq 0$	Case 4	Case 7	Case 9	Case 10

The results read as follows:

Theorem 1.3. Suppose that $\alpha, \beta \in \mathbb{R}$ and $\alpha, \beta \neq \frac{n}{2}$. Let X be the direct product of $X_1 = \mathbb{H}^{(n+1, \alpha)}$ and $X_2 = \mathbb{H}^{(n+1, \beta)}$. Then we have:

- (i) M is weak (1,1) bounded in the following cases: Case 1, Case 4 and Case 10 with $\alpha^2 + \beta^2 \neq 0$. In particular, if $\alpha = \beta = 0$, M is bounded on L^p for $1 < p \leq \infty$.
- (ii) M is unbounded on L^p for $1 \leq p < \infty$ in the following cases: Case 5 and Case 2 with $\alpha^2 - \alpha + \beta^2 - \beta \leq 0$.
- (iii) In Cases 3, 7, 9 and Case 2 with $\alpha^2 - n\alpha + \beta^2 - n\beta > 0$, there exists $p_0 > 0$ determined by n, α, β such that when $p_0 < 1$, M is weak (1,1) bounded; when $1 \leq p_0$, M is bounded on L^p for $p_0 < p \leq \infty$ and unbounded for $1 \leq p < p_0$.
- (iv) In Cases 6 and 8, there exists $p_0 > 1$ such that M is bounded on L^p for $p_0 < p \leq \infty$ and unbounded for $1 \leq p < p_0$.

Remark 1.4. Since there is no doubling volume condition in X , it fails to use the covering lemmas to prove the mapping properties of M . Instead, in the spirit of the arguments in [14, 24], we use an integral operator to control the maximal functions. Moreover, we have proved some convolution inequalities on X which are of independent interest; see, for example [3, 19].

Finally we will study the product spaces of several weighted hyperbolic spaces.

Theorem 1.5. Let X be the direct product of $X_i = \mathbb{H}^{(n_i+1, \alpha_i)}$ for $1 \leq i \leq k$. Let $\alpha_i > \frac{n_i}{2}$ for $1 \leq i \leq k$. Then we have:

- (i) If $\sum_i \alpha_i(\alpha_i - n_i) > 0$, set $p_0 = \frac{\sum_i (n_i - \alpha_i)^2}{2 \sum_i \alpha_i(\alpha_i - n_i)}$. When $0 < p_0 < 1$, M is weak (1,1) bounded. When $1 \leq p_0$, M is bounded on L^p for $p_0 < p \leq \infty$ and unbounded for $1 \leq p < p_0$.
- (ii) If $\sum_i \alpha_i(\alpha_i - n_i) < 0$, M is unbounded for $1 \leq p < \infty$.

Remark 1.6. It is worth noting that our argument still works without the assumption $\alpha_i > \frac{n_i}{2}$ for $1 \leq i \leq k$. However, in that cases, we can only show that there exist two numbers $0 < p_1 \leq p_2$ such that M is bounded on L^p for $p_2 < p \leq \infty$. Moreover M is unbounded for $1 \leq p < p_1$ if $p_1 > 1$ and unbounded for $p = 1$ if $p_1 \leq 1$.

The paper is organized as follows. Section 2 is devoted to the volume estimates on the product manifolds. Among others, we give detailed estimates of the volume on product of weighted hyperbolic spaces which is of independent interest. We will prove Theorem 1.1 in Section 3. The proof of Theorem 1.3 and Theorem 1.5 will be given in Section 4 and Section 5, respectively.

Now we introduce some notations. If f and g are two functions, we say $f \sim g$ if and only if there exists a constant $c > 0$ such that $c^{-1}f \leq g \leq cf$. We say $f \lesssim g$ if and only if there exists a constant $c > 0$ such that

$f \leq cg$. Given a real number l , denote by $[l]$ the integer part of l . The constants $C, c > 0$ may change from line to line.

2 Volume Estimates

In this section, we give volume estimates of balls in the product spaces X mentioned above.

Set $X = X_1 \times X_2$ where X_1 has the exponential volume growth and X_2 satisfies the doubling volume condition. We have the following volume estimates of the balls in X .

Proposition 2.1. Suppose that X, X_1, X_2 are defined as above, then the following holds:

$$V((x_1, x_2), r) \sim \begin{cases} V_1(x_1, r)V_2(x_2, r), & 0 < r < 1 \\ V_1(x_1, r)V_2(x_2, \sqrt{r}), & r \geq 1, \end{cases} \quad \forall (x_1, x_2) \in X_1 \times X_2.$$

Proof. For all $(x_1, x_2), (y_1, y_2) \in X, r > 0$, we have

$$d_1^2(x_1, y_1) + d_2^2(x_2, y_2) \leq r^2 \Leftrightarrow \begin{cases} 0 \leq d_1(x_1, y_1) \leq r \\ d_2(x_2, y_2) \leq \sqrt{r^2 - d_1^2(x_1, y_1)}. \end{cases}$$

Case I. $0 < r < 1$.

By the local doubling properties of X_1 , we have $V((x_1, x_2), r) \sim V_1(x_1, r)V_2(x_2, r)$.

Case II. $r \geq 1$.

To begin with, we obtain

$$\begin{aligned} V((x_1, x_2), r) &= \int_{d_1 \leq r} V_2(x_2, \sqrt{r^2 - d_1^2}) d\mu_1 \\ &\sim V_2(x_2, \sqrt{r}) \int_{d_1 \leq r} \frac{V_2(x_2, \sqrt{r}\sqrt{r - d_1})}{V_2(x_2, \sqrt{r})} d\mu_1. \end{aligned}$$

Denote $I = \int_{d_1 \leq r} \frac{V_2(x_2, \sqrt{r}\sqrt{r - d_1})}{V_2(x_2, \sqrt{r})} d\mu_1$ and we have

$$\begin{aligned} I &\geq \int_{d_1 \leq r-1/2} \frac{V_2(x_2, \sqrt{r}\sqrt{r - d_1})}{V_2(x_2, \sqrt{r})} d\mu_1 \\ &\gtrsim V_1(x_1, r-1/2) \sim V_1(x_1, r), \end{aligned}$$

where we have used (1.1) and (1.2).

On the other hand,

$$\begin{aligned} I &\leq \sum_{j=1}^{[r]} \int_{r-j \leq d_1 \leq r-j+1} \frac{V_2(x_2, \sqrt{r}\sqrt{r - d_1})}{V_2(x_2, \sqrt{r})} d\mu_1 + \int_{0 < d_1 < 1} \frac{V_2(x_2, \sqrt{r}\sqrt{r - d_1})}{V_2(x_2, \sqrt{r})} d\mu_1 \\ &\leq V_1(x_1, r) \left[\sum_{j=1}^{[r]} \frac{V_1(x_1, r-j+1)}{V_1(x_1, r)} \frac{V_2(x_2, \sqrt{r}\sqrt{j})}{V_2(x_2, \sqrt{r})} + \frac{V_1(x_1, 1)}{V_1(x_1, r)} \frac{V_2(x_2, r)}{V_2(x_2, \sqrt{r})} \right] \\ &\lesssim V_1(x_1, r) \left[\sum_{j=1}^{[r]} e^{-c_2 j^{v/2}} + e^{-c_2 r^{v/2}} \right] \lesssim V_1(x_1, r), \end{aligned}$$

where v is the doubling constant and c_2 is the constant in (1.2). Thus we have finished the proof. $\square$

Next we consider the product space of several weighted hyperbolic spaces.

Proposition 2.2. Let X be the direct product of $X_i = \mathbb{H}^{(n_i+1, \alpha_i)}$ for $1 \leq i \leq k$. Set $\alpha_i^* = \max\{\alpha_i, n_i - \alpha_i\}$ and suppose $\alpha_i \neq \frac{n_i}{2}$. Then we have

$$V(\mathcal{Y}, r) \sim \begin{cases} a_1^{\alpha_1} \cdots a_k^{\alpha_k} r^{n_1 + \cdots + n_k + k}, & 0 < r < 1 \\ a_1^{\alpha_1} \cdots a_k^{\alpha_k} r^{\frac{k-1}{2}} e^{r\sqrt{\alpha_1^{*2} + \cdots + \alpha_k^{*2}}}, & r \geq 1, \end{cases}$$

for all $\mathcal{Y} = (Y_1, \dots, Y_k) \in X$ where $Y_i = (a_i, b_i) \in X_i$.

Before proving the proposition, we need the following lemma.

Lemma 2.3. Let f be a continuous function on $[0, \infty)$ and $\mathbb{H}^{(n+1, \alpha)}$ be a weighted hyperbolic space with $\alpha \in \mathbb{R}$. Then we have

$$\int_{1 < d(Y_0, Y) < C} f(d(Y_0, Y)) d\mu \sim \int_1^C f(t) V(Y_0, t) dt, \quad \forall Y_0 \in \mathbb{H}^{n+1}, C > 1.$$

Proof. By the Coarea formula, we have

$$\int_{1 < d(Y_0, Y) < C} f(d(Y_0, Y)) d\mu = \int_1^C f(t) \int_{d(Y_0, Y)=t} dS dt,$$

where $dS = y^\alpha d\sigma$ and $d\sigma$ is the standard surface measure on the hyperbolic space. Use the Coarea formula again and we obtain

$$V(Y_0, r) = \int_{d(Y_0, Y) < r} d\mu = \int_0^r \int_{d(Y_0, Y)=t} dS dt.$$

Set $S(Y_0, r) = \int_{d(Y_0, Y)=r} dS$ and it follows $S(Y_0, r) = \frac{d}{dr} V(Y_0, r)$. Hence it is sufficient to prove $S(Y_0, r) \sim V(Y_0, r)$ for $r \geq 1$ and $Y_0 = (a_0, b_0) \in \mathbb{H}^{n+1}$.

When $\alpha \geq \frac{n}{2}$, we have

$$\begin{aligned} V(Y_0, r) &= C_n \int_{a_0 e^{-r}}^{a_0 e^r} y^{\alpha-n-1} (2a_0 y \cosh r - a_0^2 - y^2)^{n/2} dy \\ &= C_n a_0^\alpha e^{ar} \int_{e^{-2r}}^1 s^{\alpha-n-1} (1-s)^{n/2} (s - e^{-2r})^{n/2} ds \\ &\triangleq C_n a_0^\alpha e^{ar} g(r). \end{aligned}$$

Thus $S(Y_0, r) = \alpha C_n a_0^\alpha e^{ar} g(r) + C_n a_0^\alpha e^{ar} g'(r)$. Since

$$\begin{aligned} g'(r) &= n e^{-2r} \int_{e^{-2r}}^1 s^{\alpha-n-1} (1-s)^{n/2} (s - e^{-2r})^{n/2-1} ds \\ &\lesssim e^{-2r} \int_{e^{-2r}}^1 s^{\alpha-n-1} s^{n/2-1} ds \lesssim 1, \end{aligned}$$

then this inequality together with the fact $g'(r) > 0$ implies $S(Y_0, r) \sim V(Y_0, r)$ for $\alpha \geq \frac{n}{2}$.

When $\alpha < \frac{n}{2}$, we have

$$V(Y_0, r) = C_n \int_{a_0 e^{-r}}^{a_0 e^r} y^{\alpha-n-1} (2a_0 y \cosh r - a_0^2 - y^2)^{n/2} dy$$

$$\begin{aligned}
&= C_n a_0^\alpha e^{(n-\alpha)r} \int_1^{e^{2r}} s^{\alpha-n-1} (s-1)^{n/2} (1-e^{-2r}s)^{n/2} ds \\
&\triangleq C_n a_0^\alpha e^{(n-\alpha)r} h(r).
\end{aligned}$$

Thus $S(Y_0, r) = (n - \alpha) C_n a_0^\alpha e^{(n-\alpha)r} h(r) + C_n a_0^\alpha e^{(n-\alpha)r} h'(r)$. In fact,

$$\begin{aligned}
h'(r) &= n e^{-2r} \int_1^{e^{2r}} s^{\alpha-n} (s-1)^{n/2} (1-e^{-2r}s)^{n/2-1} ds \\
&\lesssim e^{-2r} \int_1^{e^{2r}} s^{\alpha-n} s^{n/2} ds \lesssim 1.
\end{aligned}$$

Thus we have proved $S(Y_0, r) \sim V(Y_0, r)$ for $\alpha < \frac{n}{2}$ and the lemma follows. $\square$

Proof of Proposition 2.2.: We will prove it by induction. When $k = 1$, the volume estimates can be found in [14, Proposition 2.1]. For $k \geq 2$, we have

$$V(\mathcal{Y}, r) = \int_{d_k < r} V((Y_1, \dots, Y_{k-1}), \sqrt{r^2 - d_k^2}) d\mu_k.$$

Then by induction we obtain for $0 < r < 1$

$$\begin{aligned}
V(\mathcal{Y}, r) &\sim a_1^{\alpha_1} \dots a_{k-1}^{\alpha_{k-1}} r^{n_1 + \dots + n_{k-1} + k - 1} \int_{d_k < r} 1 d\mu_k \\
&\sim a_1^{\alpha_1} \dots a_k^{\alpha_k} r^{n_1 + \dots + n_k + k}.
\end{aligned}$$

Now we prove the result for $r \geq C^*$ where $C^* > 1$ is a constant to be determined later. Note first that

$$\begin{aligned}
V(\mathcal{Y}, r) &= \int_{d_k < r} V((Y_1, \dots, Y_{k-1}), \sqrt{r^2 - d_k^2}) d\mu_k \\
&= \int_{d_k \leq 1} + \int_{1 < d_k < \sqrt{r^2 - 1}} + \int_{\sqrt{r^2 - 1} \leq d_k < r} \\
&\triangleq J_1 + J_2 + J_3.
\end{aligned}$$

For J_1 , we obtain

$$\begin{aligned}
J_1 &\lesssim a_1^{\alpha_1} \dots a_{k-1}^{\alpha_{k-1}} r^{\frac{k-2}{2}} e^{r\sqrt{\alpha_1^{*2} + \dots + \alpha_{k-1}^{*2}}} a_k^{\alpha_k} \\
&\lesssim a_1^{\alpha_1} \dots a_k^{\alpha_k} r^{\frac{k-1}{2}} e^{r\sqrt{\alpha_1^{*2} + \dots + \alpha_k^{*2}}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
J_3 &\lesssim a_1^{\alpha_1} \dots a_{k-1}^{\alpha_{k-1}} a_k^{\alpha_k} e^{r\alpha_k^*} \\
&\lesssim a_1^{\alpha_1} \dots a_k^{\alpha_k} r^{\frac{k-1}{2}} e^{r\sqrt{\alpha_1^{*2} + \dots + \alpha_k^{*2}}}.
\end{aligned}$$

Hence it is sufficient to prove $J_2 \sim a_1^{\alpha_1} \dots a_k^{\alpha_k} r^{\frac{k-1}{2}} e^{r\sqrt{\alpha_1^{*2} + \dots + \alpha_k^{*2}}}$ for $r \geq C^*$.

In fact, by Lemma 2.3 we have

$$J_2 = \int_{1 < d_k < \sqrt{r^2 - 1}} V((Y_1, \dots, Y_{k-1}), \sqrt{r^2 - d_k^2}) d\mu_k$$

$$\begin{aligned}
& \sim \int_1^{\sqrt{r^2-1}} V((Y_1, \dots, Y_{k-1}), \sqrt{r^2-t^2}) V_k(Y_k, t) dt \\
& \sim a_1^{\alpha_1} \dots a_k^{\alpha_k} r^{\frac{k}{2}} \int_{\frac{1}{r}}^{\sqrt{1-r^2}} (1-s^2)^{\frac{k-2}{4}} e^{r\sqrt{\alpha_1^{*2}+\dots+\alpha_{k-1}^{*2}}\sqrt{1-s^2}} e^{r\alpha_k^*s} ds.
\end{aligned}$$

Set $A = \sqrt{\alpha_1^{*2} + \dots + \alpha_{k-1}^{*2}}$, $B = \alpha_k^*$ and denote

$$\begin{aligned}
I &= \int_{\frac{1}{r}}^{\sqrt{1-r^2}} (1-s^2)^{\frac{k-2}{4}} e^{r(A\sqrt{1-s^2}+Bs)} ds \\
&= \int_{\frac{1}{r}}^{\frac{B}{\sqrt{A^2+B^2}}} + \int_{\frac{B}{\sqrt{A^2+B^2}}}^{\sqrt{1-r^2}} \\
&\triangleq I_1 + I_2.
\end{aligned}$$

For I_1 , we have

$$I_1 \sim \int_{\frac{1}{r}}^{\frac{B}{\sqrt{A^2+B^2}}} e^{r(A\sqrt{1-s^2}+Bs)} ds.$$

Let $y = A\sqrt{1-s^2} + Bs$ and we have

$$\begin{aligned}
I_1 &\sim \int_{A\sqrt{1-r^2}+B/r}^{\sqrt{A^2+B^2}} e^{ry} d\left(\frac{By - A\sqrt{A^2+B^2-y^2}}{A^2+B^2}\right) \\
&\sim \int_{A\sqrt{1-r^2}+B/r}^{\sqrt{A^2+B^2}} e^{ry} \left(B + \frac{Ay}{\sqrt{A^2+B^2-y^2}}\right) dy.
\end{aligned}$$

Since $A\sqrt{1-r^2} + B/r > A/2$ when $r \geq 2$, then $\frac{Ay}{\sqrt{A^2+B^2-y^2}}$ has a lower bound in the integrating interval. Thus we obtain

$$\begin{aligned}
I_1 &\sim \int_{A\sqrt{1-r^2}+B/r}^{\sqrt{A^2+B^2}} \frac{Aye^{ry}}{\sqrt{A^2+B^2-y^2}} dy \\
&\sim \int_{A\sqrt{1-r^2}+B/r}^{\sqrt{A^2+B^2}} \frac{e^{ry}}{\sqrt{\sqrt{A^2+B^2}-y}} dy,
\end{aligned}$$

where we have used the fact that $\frac{y}{\sqrt{\sqrt{A^2+B^2}+y}} \sim 1$ in the integrating interval. Denote $\tilde{A} = A\sqrt{1-r^2} + B/r$ and set $y = t\sqrt{A^2+B^2}$. Then we have

$$\begin{aligned}
I_1 &\sim \int_{\frac{\tilde{A}}{\sqrt{A^2+B^2}}}^1 (1-t)^{-1/2} e^{r\sqrt{A^2+B^2}t} dt \\
&\sim \int_0^{1-\frac{\tilde{A}}{\sqrt{A^2+B^2}}} s^{-1/2} e^{r\sqrt{A^2+B^2}(1-s)} ds
\end{aligned}$$

$$\begin{aligned} & \sim e^{r\sqrt{A^2+B^2}} r^{-1/2} \int_0^{(\sqrt{A^2+B^2}-\tilde{A})r} x^{-1/2} e^{-x} dx \\ & \sim e^{r\sqrt{A^2+B^2}} r^{-1/2}. \end{aligned}$$

Now we deal with I_2 . Note first that

$$I_2 \leq \int_{\frac{B}{\sqrt{A^2+B^2}}}^{\sqrt{1-r^2}} e^{r(A\sqrt{1-s^2}+Bs)} ds.$$

Let $y = A\sqrt{1-s^2} + Bs$. Then we have

$$\begin{aligned} I_2 & \lesssim \int_{A/r+B\sqrt{1-r^2}}^{\sqrt{A^2+B^2}} e^{ry} \left(\frac{Ay}{\sqrt{A^2+B^2-y^2}} - B \right) dy \\ & \lesssim \int_{A/r+B\sqrt{1-r^2}}^{\sqrt{A^2+B^2}} e^{ry} (A^2+B^2-y^2)^{-1/2} dy \\ & \lesssim \int_{A/r+B\sqrt{1-r^2}}^{\sqrt{A^2+B^2}} e^{ry} (\sqrt{A^2+B^2}-y)^{-1/2} dy. \end{aligned}$$

Then we have $I_2 \lesssim e^{r\sqrt{A^2+B^2}} r^{-1/2}$. Combining the estimates of I_1 and I_2 gives the desired estimate of J_2 . Now set

$$C^* = \max \left\{ 2, \sqrt{1 + \left(\frac{A}{B} \right)^2}, \sqrt{1 + \left(\frac{B}{A} \right)^2} \right\}.$$

Then we have proved

$$V(y, r) \sim \begin{cases} a_1^{\alpha_1} \dots a_k^{\alpha_k} r^{n_1 + \dots + n_k + k}, & 0 < r < 1 \\ a_1^{\alpha_1} \dots a_k^{\alpha_k} r^{\frac{k-1}{2}} e^{r\sqrt{\alpha_1^2 + \dots + \alpha_k^2}}, & r \geq C^*, \end{cases}$$

Note that the argument in proving the volume estimates for $0 < r < 1$ still works for $0 < r < C^*$. Moreover when $1 \leq r \leq C^*$, we have

$$a_1^{\alpha_1} \dots a_k^{\alpha_k} r^{n_1 + \dots + n_k + k} \sim a_1^{\alpha_1} \dots a_k^{\alpha_k} r^{\frac{k-1}{2}} e^{r\sqrt{\alpha_1^2 + \dots + \alpha_k^2}}.$$

Thus we have finished the proof. $\square$

Note that Lemma 2.3 plays an important role in our proof of Proposition 2.2. Although the results in Lemma 2.3 are stated for integral over the weighted hyperbolic spaces, they can be generalized to the integral over the product space X .

Corollary 2.4. Let f be a continuous function on $[0, \infty)$ and X be defined as in Proposition 2.2. Then we have

$$\int_{1 < d(y_0, y) < C} f(d(y_0, y)) d\mu \sim \int_1^C f(t) V(y_0, t) dt, \quad \forall y_0 \in X, C > 1.$$

Proof. As above, it is sufficient to show $S(y_0, r) \sim V(y_0, r)$ for $\forall r > 1, y_0 \in X$ where $S(y_0, r) = \frac{d}{dr} V(y_0, r)$. For the sake of brevity, in the following proof, we will show $S(y, r) \sim V(y, r)$ for $\forall r > 1, y \in X$.

By Proposition 2.2 and Lemma 2.3, we know that

$$V(y, r) = \int_0^r V((Y_1, \dots, Y_{k-1}), \sqrt{r^2 - t^2}) S(Y_k, t) dt.$$

Then we have

$$S(\mathcal{Y}, r) = \int_0^r S((Y_1, \dots, Y_{k-1}), \sqrt{r^2 - t^2}) \frac{r}{\sqrt{r^2 - t^2}} S(Y_k, t) dt.$$

Note that by this expression and Lemma 2.3, we can show inductively that

$$S(\mathcal{Y}, r) \lesssim a_1^{\alpha_1} \cdots a_k^{\alpha_k}, \quad \forall \mathcal{Y} \in X, 0 < r \leq 1.$$

We will prove $S(\mathcal{Y}, r) \sim V(\mathcal{Y}, r)$ for $r > 1$ by induction. When $k = 1$, it is just the Lemma 2.3. The proof is similar to the proof of Proposition 2.2, so we omit some details here. First we will prove the result for $r > C^{**}$ where $C^{**} > 1$ is a constant to be determined later. First we have

$$\begin{aligned} S(\mathcal{Y}, r) &= \int_0^1 + \int_1^{\sqrt{r^2-1}} + \int_{\sqrt{r^2-1}}^r \\ &\triangleq S_1 + S_2 + S_3. \end{aligned}$$

For S_1 , we have

$$\begin{aligned} S_1 &\lesssim a_1^{\alpha_1} \cdots a_k^{\alpha_k} r^{\frac{k-2}{2}} e^{r\sqrt{\alpha_1^{*2} + \cdots + \alpha_{k-1}^{*2}}} \int_0^1 \frac{r}{\sqrt{r^2 - t^2}} dt \\ &\lesssim V(\mathcal{Y}, r). \end{aligned}$$

For S_3 , we obtain

$$\begin{aligned} S_3 &\lesssim a_1^{\alpha_1} \cdots a_k^{\alpha_k} e^{r\alpha_k^*} \int_{\sqrt{r^2-1}}^r \frac{r}{\sqrt{r^2 - t^2}} dt \\ &\lesssim a_1^{\alpha_1} \cdots a_k^{\alpha_k} e^{r\alpha_k^*} (1 + \sqrt{1 - r^{-2}})^{-\frac{1}{2}} \\ &\lesssim V(\mathcal{Y}, r). \end{aligned}$$

Now we are ready to deal with S_2 . By changing variable, we have

$$S_2 \sim a_1^{\alpha_1} \cdots a_k^{\alpha_k} r^{\frac{k}{2}} \int_{\frac{1}{r}}^{\sqrt{1-r^{-2}}} (1-s^2)^{\frac{k-4}{4}} e^{r(A\sqrt{1-s^2}+Bs)} ds,$$

where $A = \sqrt{\alpha_1^{*2} + \cdots + \alpha_{k-1}^{*2}}$, $B = \alpha_k^*$. Following the method in the proof of Proposition 2.2, we have

$$\int_{\frac{1}{r}}^{\frac{B}{\sqrt{A^2+B^2}}} (1-s^2)^{\frac{k-4}{4}} e^{r(A\sqrt{1-s^2}+Bs)} ds \sim e^{r\sqrt{A^2+B^2}} r^{-1/2}.$$

On the other hand, we have for $k \geq 2$

$$\begin{aligned} &\int_{\frac{B}{\sqrt{A^2+B^2}}}^{\sqrt{1-r^{-2}}} (1-s^2)^{\frac{k-4}{4}} e^{r(A\sqrt{1-s^2}+Bs)} ds \\ &\lesssim \int_{\frac{B}{\sqrt{A^2+B^2}}}^{\sqrt{1-r^{-2}}} (1-s^2)^{-\frac{1}{2}} e^{r(A\sqrt{1-s^2}+Bs)} ds \end{aligned}$$

$$\lesssim e^{r\sqrt{A^2+B^2}} r^{-1/2}.$$

Set $C^{**} = \max \left\{ 2, \sqrt{1 + (\frac{A}{B})^2}, \sqrt{1 + (\frac{B}{A})^2} \right\}$. Then we have proved $S(y, r) \sim V(y, r)$ for $r > C^{**}$. Note that for $1 < r \leq C^{**}$, both $S(y, r)$ and $V(y, r)$ are positive and finite. Then we have finished the proof. $\square$

3 Proof of Theorem 1.1

The positive part of Theorem 1.1 follows from the following lemma.

Lemma 3.1. *Suppose that X_1 has the exponential volume growth and X_2 is noncompact satisfying the doubling volume condition. Then we have*

$$Mf(x_1, x_2) \lesssim M_1 M_2 f(x_1, x_2), \quad \forall (x_1, x_2) \in X_1 \times X_2.$$

Proof of Lemma 3.1:

Without loss of generality, we suppose $f \geq 0$.

To begin with, let

$$T_1 f(x_1, x_2) = \sup_{0 < r < 1} \frac{1}{V((x_1, x_2), r)} \int_{B((x_1, x_2), r)} f(y_1, y_2) d\mu_1 d\mu_2,$$

$$T_2 f(x_1, x_2) = \sup_{r \geq 1} \frac{1}{V((x_1, x_2), r)} \int_{B((x_1, x_2), r)} f(y_1, y_2) d\mu_1 d\mu_2.$$

And we have

$$Mf(x_1, x_2) \leq T_1 f(x_1, x_2) + T_2 f(x_1, x_2).$$

According to the volume estimates, we have

$$T_1 f(x_1, x_2) \lesssim \sup_{0 < r < 1} \frac{1}{V_1(x_1, r)} \int_{B_1(x_1, r)} M_2 f(y_1, x_2) \frac{V_2(x_2, \sqrt{r^2 - d_1^2})}{V_2(x_2, r)} d\mu_1$$

$$\lesssim M_1 M_2 f(x_1, x_2).$$

Now we deal with T_2 . For every $r \geq 1$, we have

$$\begin{aligned} & \frac{1}{V((x_1, x_2), r)} \int_{B((x_1, x_2), r)} f(y_1, y_2) d\mu_1 d\mu_2 \\ & \lesssim \frac{1}{V_1(x_1, r)} \int_{B_1(x_1, r)} M_2 f(y_1, x_2) \frac{V_2(x_2, \sqrt{r^2 - d_1^2})}{V_2(x_2, \sqrt{r})} d\mu_1 \\ & \lesssim \frac{1}{V_1(x_1, r)} \left[\int_{d_1 \leq r-1} M_2 f(y_1, x_2) \frac{V_2(x_2, \sqrt{r^2 - d_1^2})}{V_2(x_2, \sqrt{r})} d\mu_1 + \int_{r-1 < d_1 < r} M_2 f(y_1, x_2) d\mu_1 \right] \\ & \lesssim \frac{1}{V_1(x_1, r)} \int_{B_1(x_1, r)} M_2 f(y_1, x_2) (r - d_1)^{\frac{\nu}{2}} d\mu_1 + M_1 M_2 f(x_1, x_2), \end{aligned}$$

where ν is the doubling constant. It remains to show the first term in the last inequality can be controlled by $M_1 M_2 f$.

In fact, we have

$$\begin{aligned} & \frac{1}{V_1(x_1, r)} \int_{B_1(x_1, r)} M_2 f(y_1, x_2) (r - d_1)^{\frac{v}{2}} d\mu_1 \\ & \lesssim \sum_{j=1}^{[r]} \frac{V_1(x_1, r-j+1)}{V_1(x_1, r)} j^{\frac{v}{2}} \frac{1}{V_1(x_1, r-j+1)} \int_{B_1(x_1, r-j+1)} M_2 f(y_1, x_2) d\mu_1 \\ & \quad + \frac{V_1(x_1, 1)}{V_1(x_1, r)} \frac{r^{\frac{v}{2}}}{V_1(x_1, 1)} \int_{B_1(x_1, 1)} M_2 f(y_1, x_2) d\mu_1 \\ & \lesssim M_1 M_2 f(x_1, x_2) \left[\sum_{j=1}^{[r]} j^{\frac{v}{2}} e^{-c(j-1)} + r^{\frac{v}{2}} e^{-cr} \right] \lesssim M_1 M_2 f(x_1, x_2). \end{aligned}$$

Thus we have proved the lemma. $\square$

Proof of the positive part of Theorem 1.1:

According to the definition of Lorentz space (see [22]), we have for $1 < p \leq q < \infty$

$$\|M_1 M_2 f\|_{L^{p,q}(X)}^p = c \left(\int_0^\infty t^{q-1} \mu_1 \times \mu_2 \{ (x_1, x_2) \in X_1 \times X_2 : M_1 M_2 f(x_1, x_2) \geq t \}^{\frac{q}{p}} dt \right)^{\frac{p}{q}}.$$

When $q = \infty$ and $1 < p < \infty$, we have

$$\|M_1 M_2 f\|_{L^{p,\infty}(X)}^p = \sup_{t>0} t^p \mu_1 \times \mu_2 \{ (x_1, x_2) \in X_1 \times X_2 : M_1 M_2 f(x_1, x_2) \geq t \}.$$

Since M_1 is bounded from $L^p(X_1)$ to $L^{p,q}(X_1)$, by the Minkowski inequality, we have for $1 < p \leq q < \infty$

$$\begin{aligned} \|M_1 M_2 f\|_{L^{p,q}(X)}^p & \lesssim \int_{X_2} \left(\int_0^\infty t^{q-1} \mu_1 \{ x_1 \in X_1 : M_1 M_2 f(x_1, x_2) \geq t \}^{\frac{q}{p}} dt \right)^{\frac{p}{q}} d\mu_2 \\ & \lesssim \int_{X_2} \int_{X_1} |M_2 f|^p d\mu_1 d\mu_2 \\ & \lesssim \|f\|_{L^p(X)}^p. \end{aligned}$$

Note that we have used the fact M_2 is bounded from $L^p(X_2)$ to $L^p(X_2)$ for $1 < p \leq \infty$. When $q = \infty$, with a slight modification, the above argument still works. $\square$

Proof of negative part of Theorem 1.1:

Since M_1 is unbounded, for any $\delta > 0$, there exists a function $f_1 \geq 0 \in L^p(X_1)$ such that

$$\frac{\|M_1 f_1\|_{L^p}^p}{\|f_1\|_{L^p}^p} > \delta.$$

Set

$$\begin{aligned} \widetilde{M}_1 f(x_1) &= \sup_{0 < r \leq 1} \frac{1}{V_1(x_1, r)} \int_{B(x_1, r)} |f(y_1)| d\mu_1, \quad \forall f \in L^1_{loc}(X_1, d\mu_1), \\ M_1^* f(x_1) &= \sup_{1 < r} \frac{1}{V_1(x_1, r)} \int_{B(x_1, r)} |f(y_1)| d\mu_1, \quad \forall f \in L^1_{loc}(X_1, d\mu_1). \end{aligned}$$

Thanks to (1.3), $\widetilde{M}_1$ is bounded from $L^p(X_1)$ to $L^p(X_1)$ for $p > 1$. Denote by $\widetilde{A}$ the operator norm of $\widetilde{M}_1$ on $L^p(X_1)$. Then we obtain

$$\frac{\|M_1^* f_1\|_{L^p}^p}{\|f_1\|_{L^p}^p} > 2^{-p} (\delta - (\widetilde{A})^p).$$

Fix a point $P_2 \in X_2$ and set $E = X_1 \times B_2(P_2, \sqrt{R})$ where $R > 1$. Now consider the function $f(y_1, y_2) = f_1(y_1)\chi_{B_2(P_2, 2\sqrt{R})}$ and for $(x_1, x_2) \in E$, the following holds,

$$\begin{aligned} & Mf(x_1, x_2) \\ & \gtrsim \sup_{\frac{1+\sqrt{5}}{2} \leq r \leq R} \frac{1}{V_1(x_1, r)V_2(x_2, \sqrt{r})} \int_{B((x_1, x_2), r)} f_1(y_1)\chi_{B_2(P_2, 2\sqrt{R})}(y_2) d\mu_1 d\mu_2 \\ & \gtrsim \sup_{\frac{1+\sqrt{5}}{2} \leq r \leq R} \frac{1}{V_1(x_1, r)} \int_{B_1(x_1, \sqrt{r^2-r})} f_1(y_1) \frac{\mu_2(B_2(x_2, \sqrt{r}) \cap B_2(P_2, 2\sqrt{R}))}{V_2(x_2, \sqrt{r})} d\mu_1. \end{aligned}$$

Since $x_2 \in B_2(P_2, \sqrt{R})$, we have $B_2(x_2, \sqrt{r}) \cap B_2(P_2, 2\sqrt{R}) = B_2(x_2, \sqrt{r})$.

$$\begin{aligned} Mf(x_1, x_2) & \gtrsim \sup_{\frac{1+\sqrt{5}}{2} \leq r \leq R} \frac{1}{V_1(x_1, r)} \int_{B_1(x_1, \sqrt{r^2-r})} f_1(y_1) d\mu_1 \\ & \gtrsim \sup_{\frac{1+\sqrt{5}}{2} \leq r \leq R} \frac{1}{V_1(x_1, \sqrt{r^2-r})} \int_{B_1(x_1, \sqrt{r^2-r})} f_1(y_1) d\mu_1. \end{aligned}$$

We have used (1.2) in the second inequality. Letting $R \rightarrow \infty$, by the Fatou theorem we have

$$\frac{\|Mf\|_{L^p(E)}^p}{\|f\|_{L^p(X_1 \times X_2)}^p} \gtrsim \frac{\|M_1^* f_1\|_{L^p(X_1)}^p}{\|f_1\|_{L^p(X_1)}^p} > 2^{-p}(\delta - (\tilde{A})^p).$$

Since the constants in the above inequalities are independent of δ , we have proved the desired results. $\square$

4 Proof of theorem 1.3

We need some preparation before the proof. Given a weighted hyperbolic space $\mathbb{H}^{(n+1, \alpha)}$, define the following group operation for any $Y = (a, b)$, $Y' = (a', b') \in \mathbb{H}^{n+1}$,

$$(a, b) \cdot (a', b') = (aa', b + ab').$$

It is easy to verify $E = (1, 0)$ is the identity element and $(a^{-1}, -ba^{-1})$ is the inverse element of (a, b) . For any two points $Y, Y' \in \mathbb{H}^{n+1}$, we have $d(Y, Y') = d(E, Y^{-1} \cdot Y')$. The left and right Haar measures are given by

$$d\lambda(y, x) = y^{-n-1} dy dx \quad \text{and} \quad d\rho(y, x) = y^{-1} dy dx,$$

respectively. Thus it is not a unimodular group. For the measure $d\mu = y^{\alpha-n-1} dy dx$, we have

$$d\mu(Y \cdot Z) = a^\alpha d\mu(Z) \quad \text{and} \quad d\mu(Z \cdot Y) = a^{\alpha-n} d\mu(Z),$$

where $Y = (a, b)$, $Z = (y, x) \in \mathbb{H}^{n+1}$.

Let X be the direct product group of $X_i = \mathbb{H}^{(n_i+1, \alpha_i)}$ where $1 \leq i \leq k$. The identity element in X is denoted by $\mathcal{E}$ and the inverse element of $\mathcal{Y}$ is denoted by $\mathcal{Y}^{-1}$. Then we still have $d(\mathcal{Y}, \mathcal{Y}') = d(\mathcal{E}, \mathcal{Y}^{-1} \cdot \mathcal{Y}')$.

In this section, we consider the product space X of two weighted hyperbolic spaces $X_1 = \mathbb{H}^{(n+1, \alpha)}$, $X_2 = \mathbb{H}^{(n+1, \beta)}$. To begin with, we consider the positive parts of Theorem 1.3. Since X has the local doubling properties, the operator $\tilde{M}$ defined by

$$\tilde{M}f(\mathcal{Y}) = \sup_{0 < r \leq 1} \frac{1}{V(\mathcal{Y}, r)} \int_{B(\mathcal{Y}, r)} |f(\mathcal{Y}')| d\mu,$$

is weak (1,1) bounded. We only need to prove the results for the following operator

$$M^*f(\mathcal{Y}) = \sup_{r > 1} \frac{1}{V(\mathcal{Y}, r)} \int_{B(\mathcal{Y}, r)} |f(\mathcal{Y}')| d\mu.$$

Without loss of generality, suppose $f \geq 0$ and it is sufficient to consider the following operator

$$Tf(y) = \int_X f(y') V^{-1}(y, d(y, y') + 1) d\mu(y').$$

Since $V^{-1}(y, d(y, y') + 1) = V^{-1}(\mathcal{E}, d(\mathcal{E}, y^{-1} \cdot y') + 1) a_1^{-\alpha} a_2^{-\beta}$, we have

$$Tf(y) = \int_X f(y') V^{-1}(\mathcal{E}, d(\mathcal{E}, y^{-1} \cdot y') + 1) a_1^{-\alpha} a_2^{-\beta} d\mu(y').$$

Set $z = y^{-1} \cdot y'$ and change the variables :

$$\begin{aligned} Tf(y) &= \int_X f(y \cdot z) V^{-1}(\mathcal{E}, d(\mathcal{E}, z) + 1) a_1^{-\alpha} a_2^{-\beta} d\mu(y \cdot z) \\ &= \int_X f(y \cdot z) V^{-1}(\mathcal{E}, d(\mathcal{E}, z) + 1) d\mu(z), \end{aligned}$$

where we have used the fact $d\mu(y \cdot z) = a_1^{\alpha} a_2^{\beta} d\mu(z)$.

Denote by $z = ((y_1, x_1), (y_2, x_2))$ where $(y_1, x_1), (y_2, x_2) \in \mathbb{H}^{n+1}$. By the Minkowski inequality, we obtain

$$\begin{aligned} \|Tf\|_{L^p(X)} &\leq \int_X V^{-1}(\mathcal{E}, d(\mathcal{E}, z) + 1) \left[\int_X f(y \cdot z)^p d\mu(y) \right]^{\frac{1}{p}} d\mu(z) \\ &\leq \int_X V^{-1}(\mathcal{E}, d(\mathcal{E}, z) + 1) y_1^{\frac{n-\alpha}{p}} y_2^{\frac{n-\beta}{p}} d\mu(z) \|f\|_{L^p(X)}. \end{aligned}$$

Set $d\tilde{\mu}(z) = y_1^{\frac{n-\alpha}{p}} y_2^{\frac{n-\beta}{p}} d\mu(z)$ and we obtain

$$\int_X V^{-1}(\mathcal{E}, d(\mathcal{E}, z) + 1) y_1^{\frac{n-\alpha}{p}} y_2^{\frac{n-\beta}{p}} d\mu(z) \leq \sum_{l \geq 0} \frac{\tilde{V}(\mathcal{E}, l+1)}{V(\mathcal{E}, l+1)},$$

where $\tilde{V}(\mathcal{E}, l+1)$ is the volume of ball with respect to measure $d\tilde{\mu}$. Set

$$\Omega = \max\{\alpha, n - \alpha\}^2 + \max\{\beta, n - \beta\}^2,$$

$$\Sigma = \frac{1}{p^2} \left\{ \max\{n + \alpha(p-1), (n - \alpha)(p-1)\}^2 + \max\{n + \beta(p-1), (n - \beta)(p-1)\}^2 \right\}.$$

According to the volume estimates, we have

$$\sum_{l \geq 0} \frac{\tilde{V}(\mathcal{E}, l+1)}{V(\mathcal{E}, l+1)} \lesssim \sum_{l \geq 0} e^{l(\sqrt{\Sigma} - \sqrt{\Omega})}. \quad (4.1)$$

Thus the positive results hold whenever $\Sigma < \Omega$.

Now we consider the negative results. Let χ_B be the characteristic function of $B(\mathcal{E}, 1)$. Thus we have

$$M(\chi_B)(y) \geq C V^{-1}(y, d(y, \mathcal{E}) + 1).$$

Now we consider the L^p norm of the right side. Set

$$\begin{aligned} I &= \int_X V^{-p}(y, d(y, \mathcal{E}) + 1) d\mu(y) \\ &= \int_X V^{-p}(\mathcal{E}, d(\mathcal{E}, y) + 1) d\hat{\mu}(y) \end{aligned}$$

where $d\hat{\mu} = a_1^{-\alpha p} a_2^{-\beta p} d\mu$. By Corollary 2.4, we have

$$\begin{aligned} I &\geq \int_{d(\mathcal{Y}, \mathcal{E}) \geq 1} V^{-p}(\mathcal{E}, d(\mathcal{Y}, \mathcal{E}) + 1) d\hat{\mu}(\mathcal{Y}) \\ &\gtrsim \int_1^\infty V^{-p}(\mathcal{E}, t + 1) \widehat{V}(\mathcal{E}, t) dt, \end{aligned}$$

where $\widehat{V}(\mathcal{E}, t)$ is the volume of ball $B(\mathcal{E}, t)$ with respect to the measure $d\hat{\mu}$. Set

$$\Pi = \frac{1}{p^2} \left\{ \max\{\alpha(1-p), n + \alpha(p-1)\}^2 + \max\{\beta(1-p), n + \beta(p-1)\}^2 \right\}.$$

Then we have

$$I \gtrsim \int_1^\infty (t+1)^{-\frac{p-1}{2}} e^{tp(\sqrt{\Pi}-\sqrt{\Omega})} dt. \quad (4.2)$$

The right side is infinite whenever $\Pi > \Omega$ or $\Pi = \Omega$, $1 \leq p \leq 3$.

Next we will discuss the mapping properties of the centered maximal function in several different cases. Note that Cases 1, 2, 5 are special cases of the Theorem 1.5 which will be discussed later.

Without loss of generality we assume $\alpha \geq \beta$. As mentioned above, the negative condition is $\Pi \geq \Omega$ and the positive condition is $\Sigma < \Omega$. It is easy to check that

$$\Pi = \Sigma \Leftrightarrow \frac{n-\alpha}{p} + \alpha \geq (n-\alpha) \left(1 - \frac{1}{p}\right) \quad \text{and} \quad \frac{n-\beta}{p} + \beta \geq (n-\beta) \left(1 - \frac{1}{p}\right).$$

For $p \geq 1$ we have

$$\frac{n-\beta}{p} + \beta \geq (n-\beta) \left(1 - \frac{1}{p}\right), \Leftrightarrow \begin{cases} 1 \leq p & \beta \geq n/2, \\ 1 \leq p \leq \frac{n-\beta}{n/2-\beta} & \beta < n/2. \end{cases}$$

Note also that $\frac{n-\beta}{n/2-\beta} \leq \frac{n-\alpha}{n/2-\alpha}$ when $\beta \leq \alpha < n/2$. Since in this section only the cases with $\beta < n/2$ are studied, we can assume $1 \leq p \leq \bar{p} = \frac{n-\beta}{n/2-\beta}$ and it follows that $\Pi = \Sigma$ under the assumption. On the other hand, consider the equation $\Sigma = \Omega$. In fact, in the cases we will study the equation which always has two distinct roots with different sign. Denote by p_0 the positive root. To get positive results, it is sufficient to prove $p_0 < \bar{p}$ in each case below.

Case 3. $\alpha > n, 0 < \beta < \frac{n}{2}$.

To start with, M^* is bounded on L^p for p satisfying

$$\left(\frac{n-\alpha}{p} + \alpha\right)^2 + \left(\frac{n-\beta}{p} + \beta\right)^2 < \alpha^2 + (n-\beta)^2. \quad (4.3)$$

Now we consider the equation $\Sigma = \Omega$. In this case, it can be written as

$$p^2[(n-\beta)^2 - \beta^2] + p[2\beta(\beta-n) + 2\alpha(\alpha-n)] - (n-\alpha)^2 - (n-\beta)^2 = 0,$$

which has two roots. Next we will show $p_0 < \bar{p}$. Set

$$f(p) = p^2[(n-\beta)^2 - \beta^2] + p[2\beta(\beta-n) + 2\alpha(\alpha-n)] - (n-\alpha)^2 - (n-\beta)^2.$$

According to the graph of $f(p)$, we only need to show $f(\bar{p}) > 0$ which is equivalent to showing $\bar{p}$ satisfying (4.3). In fact, we have $(\frac{n-\alpha}{p} + \alpha)^2 = (\frac{n}{p} + \frac{\alpha}{p'})^2 < \alpha^2$ for every $\alpha > n, p \geq 1$ where p' is conjugate to p . Moreover, $(\frac{n-\beta}{p} + \beta)^2 = \frac{n^2}{4} < (n-\beta)^2$. Then we have shown the inequality $p_0 < \bar{p}$. As a result, if $p_0 < 1$, M^* is bounded on L^p for $1 \leq p \leq \infty$ and hence M is bounded on L^p for $1 < p \leq \infty$; if $1 \leq p_0$, M^* is bounded for $p_0 < p \leq \infty$ and M is unbounded on L^p for $1 \leq p < p_0$.

Case 4. $\alpha > n, \beta \leq 0$.

In this case, the positive condition can be interpreted as

$$\left(\frac{n-\alpha}{p} + \alpha\right)^2 + \left(\frac{n-\beta}{p} + \beta\right)^2 < \alpha^2 + (n-\beta)^2.$$

Note that the above inequality holds when $p = 1$. Thus by interpolation, we can get the desired results.

Case 6. $\frac{n}{2} < \alpha \leq n, 0 < \beta < \frac{n}{2}$.

We will show the positive root p_0 of the following equation is smaller than $\bar{p}$,

$$(n-\alpha+\alpha p)^2 + (n-\beta+\beta p)^2 = p^2[\alpha^2 + (n-\beta)^2].$$

Note that

$$\begin{aligned} & \left(\frac{n-\alpha}{\bar{p}} + \alpha\right)^2 + \left(\frac{n-\beta}{\bar{p}} + \beta\right)^2 - \alpha^2 - (n-\beta)^2 \\ &= \frac{n^2}{4} + (n-\alpha)^2 \left(\frac{n/2-\beta}{n-\beta}\right)^2 + 2\alpha(n-\alpha)\frac{n/2-\beta}{n-\beta} - (n-\beta)^2 \\ &< \frac{n^2}{4} + \frac{n^2}{4} \left(\frac{n/2-\beta}{n-\beta}\right)^2 + \frac{n^2}{2} \frac{n/2-\beta}{n-\beta} - (n-\beta)^2. \end{aligned}$$

We have used the fact $\frac{n}{2} < \alpha \leq n$ in the last step. Set

$$f(\beta) = \frac{n^2}{4} + \frac{n^2}{4} \left(\frac{n/2-\beta}{n-\beta}\right)^2 + \frac{n^2}{2} \frac{n/2-\beta}{n-\beta} - (n-\beta)^2.$$

Differentiating $f(\beta)$ gives

$$\begin{aligned} f'(\beta) &= \frac{-n^3}{8} \frac{n-2\beta}{(n-\beta)^3} - \frac{n^3}{4} \frac{1}{(n-\beta)^2} + 2(n-\beta) \\ &= \frac{1}{(n-\beta)^3} [2(n-\beta)^4 - \frac{n^3}{2}(n-\beta) + \frac{n^4}{8}]. \end{aligned}$$

Since $0 < \beta < \frac{n}{2}$, we have $f'(\beta) > 0$. In turn, we obtain

$$\left(\frac{n-\alpha}{\bar{p}} + \alpha\right)^2 + \left(\frac{n-\beta}{\bar{p}} + \beta\right)^2 - \alpha^2 - (n-\beta)^2 < f(n/2) = 0.$$

However, $p = 1$ does not satisfy the positive condition. In fact, $2n^2 > \alpha^2 + (n-\beta)^2$ and hence $1 < p_0$. As a result, M^* is bounded on L^p for $p_0 < p \leq \infty$ and M is unbounded on L^p for $1 \leq p < p_0$.

Case 7. $\frac{n}{2} < \alpha \leq n, \beta \leq 0$.

Now we will show that $\bar{p}$ satisfies the condition $\Sigma < \Omega$. In fact,

$$\begin{aligned} & \left(\frac{n-\alpha}{\bar{p}} + \alpha\right)^2 + \left(\frac{n-\beta}{\bar{p}} + \beta\right)^2 - \alpha^2 - (n-\beta)^2 \\ &< \frac{n^2}{4} + \frac{n^2}{4} \left(\frac{n/2-\beta}{n-\beta}\right)^2 + \frac{n^2}{2} \frac{n/2-\beta}{n-\beta} - (n-\beta)^2 \\ &< 0. \end{aligned}$$

We have used the fact $\frac{n}{2} < \alpha \leq n$ in the second step and $\beta \leq 0$ in the last step. In turn, $\bar{p}$ satisfies (4.1) and hence $p_0 < \bar{p}$. We can conclude that if $p_0 < 1$, M^* is bounded on L^p for $1 \leq p \leq \infty$; if $1 \leq p_0$, M^* is bounded on L^p for $p_0 < p \leq \infty$ and M is unbounded on L^p for $1 \leq p < p_0$.

Case 8. $0 < \alpha < \frac{n}{2}, 0 < \beta < \frac{n}{2}$.

It is sufficient to show $p_0 < \bar{p}$. In fact, we have

$$\left(\frac{n-\alpha}{\bar{p}} + \alpha\right)^2 + \left(\frac{n-\beta}{\bar{p}} + \beta\right)^2 - (n-\alpha)^2 - (n-\beta)^2$$

$$= \frac{n^2}{4} - (n - \beta)^2 + (n - \alpha)^2 \left(\frac{n/2 - \beta}{n - \beta} \right)^2 + 2\alpha(n - \alpha) \frac{n/2 - \beta}{n - \beta} + 2\alpha n - n^2.$$

Set

$$f(\beta) = \frac{n^2}{4} - (n - \beta)^2 + (n - \alpha)^2 \left(\frac{n/2 - \beta}{n - \beta} \right)^2 + 2\alpha(n - \alpha) \frac{n/2 - \beta}{n - \beta} + 2\alpha n - n^2.$$

Then we have

$$\begin{aligned} f'(\beta) &= 2(n - \beta) - n(n - \alpha)^2 \frac{n/2 - \beta}{(n - \beta)^3} - n\alpha(n - \alpha) \frac{1}{(n - \beta)^2} \\ &= (n - \beta)^{-3} [2(n - \beta)^4 - n^2(n - \alpha)(n - \beta) + (n^2 - \alpha n)^2 / 2]. \end{aligned}$$

Now consider the function $g(x) = 2x^4 - n^2(n - \alpha)x + (n^2 - \alpha n)^2/2$ for $n - \alpha \leq x < n$. Note that $g'(x) = 8x^3 - n^2(n - \alpha) \geq 8(n - \alpha)^3 - n^2(n - \alpha) > 0$ when $0 < \alpha < \frac{n}{2}$. Thus we have

$$g(x) \geq 2(n - \alpha)^4 - n^2(n - \alpha)^2/2 = (n - \alpha)^2 [2(n - \alpha)^2 - n^2/2] \geq 0.$$

As a result, $f'(\beta) \geq 0$ for $0 < \beta \leq \alpha$. Then we obtain

$$\begin{aligned} f(\beta) &\leq \frac{n^2}{4} - (n - \alpha)^2 + (n - \alpha)^2 \left(\frac{n/2 - \alpha}{n - \alpha} \right)^2 + 2\alpha(n - \alpha) \frac{n/2 - \alpha}{n - \alpha} + 2\alpha n - n^2 \\ &\leq -2\alpha^2 + 4\alpha n - 3n^2/2. \end{aligned}$$

According to the fact $0 < \alpha < \frac{n}{2}$, we can conclude that $f(\alpha) < 0$. Moreover, it is easy to check $2n^2 > (n - \alpha)^2 + (n - \beta)^2$ which implies $p_0 > 1$. Hence, we get the desired results.

Case 9. $0 < \alpha < \frac{n}{2}, \beta \leq 0$.

In this case, we have

$$\begin{aligned} &\left(\frac{n - \alpha}{\bar{p}} + \alpha \right)^2 + \left(\frac{n - \beta}{\bar{p}} + \beta \right)^2 - (n - \alpha)^2 - (n - \beta)^2 \\ &= \frac{n^2}{4} - (n - \beta)^2 + (n - \alpha)^2 \left(\frac{n/2 - \beta}{n - \beta} \right)^2 + 2\alpha(n - \alpha) \frac{n/2 - \beta}{n - \beta} + 2\alpha n - n^2 \\ &\leq \frac{n^2}{4} - n^2 + \frac{(n - \alpha)^2}{4} + \alpha(n - \alpha) + 2\alpha n - n^2 = (-3\alpha^2 + 10\alpha n - 6n^2)/4. \end{aligned}$$

It is easy to verify that the last step is negative. Then we have proved the desired results.

Case 10. $\alpha \leq 0, \beta \leq 0$.

The condition $\Sigma < \Omega$ suggests that

$$\left(\frac{n - \alpha}{p} + \alpha \right)^2 + \left(\frac{n - \beta}{p} + \beta \right)^2 < (n - \alpha)^2 + (n - \beta)^2.$$

Thus it is sufficient to consider

$$\begin{aligned} &\left(\frac{n - \alpha}{p} + \alpha \right)^2 + \left(\frac{n - \beta}{p} + \beta \right)^2 - (n - \alpha)^2 - (n - \beta)^2 \\ &= p^{-2} [(n - \alpha)^2 + (n - \beta)^2] + 2p^{-1} [\alpha(n - \alpha) + \beta(n - \beta)] + 2n(\alpha + \beta - n) \end{aligned}$$

Set

$$f(x) = x^2 [(n - \alpha)^2 + (n - \beta)^2] + 2x [\alpha(n - \alpha) + \beta(n - \beta)] + 2n(\alpha + \beta - n).$$

Since $f(1) = 2n^2 - (n - \alpha)^2 - (n - \beta)^2 \leq 0$, then by the property of function f we obtain $f(x) < 0$ for $0 \leq x \leq 1$. Moreover, we have $f(1) = 0$ when $\alpha = \beta = 0$. Thus M^* is bounded on L^p for $1 \leq p \leq \infty$ when $\alpha^2 + \beta^2 \neq 0$. When $\alpha = \beta = 0$, M^* is bounded on L^p for $1 < p \leq \infty$.

5 Proof of theorem 1.5

Now we will prove Theorem 1.5. The method is similar to that used in proving Theorem 1.3 and we omit some details here. The positive part will be considered first.

Proof of the positive part:.

Without loss of generality, suppose $f \geq 0$ and consider the operator T defined by

$$\begin{aligned} Tf(y) &= \int_X f(y') V^{-1}(y, d(y, y') + 1) d\mu(y') \\ &= \int_X f(y') V^{-1}(\varepsilon, d(\varepsilon, y^{-1} \cdot y') + 1) a_1^{-\alpha_1} \cdots a_k^{-\alpha_k} d\mu(y'), \end{aligned}$$

where $y = (Y_1, \dots, Y_k)$, $y' = (Y'_1, \dots, Y'_k) \in X$. By changing variables we obtain

$$\begin{aligned} Tf(y) &= \int_X f(y \cdot z) V^{-1}(\varepsilon, d(\varepsilon, z) + 1) a_1^{-\alpha_1} \cdots a_k^{-\alpha_k} d\mu(y \cdot z) \\ &= \int_X f(y \cdot z) V^{-1}(\varepsilon, d(\varepsilon, z) + 1) d\mu(z), \end{aligned}$$

where $z = y^{-1} \cdot y'$ and we have used the fact $d\mu(y \cdot z) = a_1^{\alpha_1} \cdots a_k^{\alpha_k} d\mu(z)$ in the last step.

Denote by $z = (Z_1, \dots, Z_k)$ where $Z_i = (y_i, x_i) \in \mathbb{H}^{n_i+1}$. By the Minkowski inequality, we get

$$\begin{aligned} \|Tf\|_{L^p(X)} &\leq \int_X V^{-1}(\varepsilon, d(\varepsilon, z) + 1) \left[\int_X f(y \cdot z)^p d\mu(y) \right]^{\frac{1}{p}} d\mu(z) \\ &\leq \int_X V^{-1}(\varepsilon, d(\varepsilon, z) + 1) y_1^{\frac{n_1-\alpha_1}{p}} \cdots y_k^{\frac{n_k-\alpha_k}{p}} d\mu(z) \|f\|_{L^p(X)}. \end{aligned}$$

Set $d\tilde{\mu}(z) = y_1^{\frac{n_1-\alpha_1}{p}} \cdots y_k^{\frac{n_k-\alpha_k}{p}} d\mu(z)$ and we obtain

$$\int_X V^{-1}(\varepsilon, d(\varepsilon, z) + 1) y_1^{\frac{n_1-\alpha_1}{p}} \cdots y_k^{\frac{n_k-\alpha_k}{p}} d\mu(z) \leq \sum_{l \geq 0} \frac{\tilde{V}(\varepsilon, l+1)}{V(\varepsilon, l+1)},$$

where $\tilde{V}(\varepsilon, l+1)$ is the volume of ball with respect to measure $d\tilde{\mu}$. Note that

$$\frac{n_i - \alpha_i}{p} + \alpha_i - \frac{n_i}{2} = \left(\frac{1}{p} - \frac{1}{2}\right)n_i + \left(1 - \frac{1}{p}\right)\alpha_i \geq \left(\frac{1}{p} - \frac{1}{2}\right)n_i + \left(1 - \frac{1}{p}\right)\frac{n_i}{2} = \frac{n_i}{2p} > 0,$$

when $\alpha_i > \frac{n_i}{2}$. By Proposition 2.2,

$$\sum_{l \geq 0} \frac{\tilde{V}(\varepsilon, l+1)}{V(\varepsilon, l+1)} \lesssim \sum_{l \geq 0} e^{l(\sqrt{\sum (\frac{n_i-\alpha_i}{p} + \alpha_i)^2} - \sqrt{\sum \alpha_i^2})}.$$

The right side of the inequality is finite when

$$\sum_{1 \leq i \leq k} \left(\frac{n_i - \alpha_i}{p} + \alpha_i\right)^2 < \sum_{1 \leq i \leq k} \alpha_i^2. \quad (5.1)$$

It is easy to verify that the above inequality holds when $\sum_i \alpha_i(\alpha_i - n_i) > 0$ and $p > p_0$. We have proved the positive part. $\square$

Proof of the negative part:.

Let χ_B be the characteristic function of $B(\mathcal{E}, 1)$ and we have

$$M(\chi_B)(\mathcal{Y}) \geq CV^{-1}(\mathcal{Y}, d(\mathcal{Y}, \mathcal{E}) + 1).$$

Set

$$\begin{aligned} I &= \int_X V^{-p}(\mathcal{Y}, d(\mathcal{Y}, \mathcal{E}) + 1) d\mu(\mathcal{Y}) \\ &= \int_X V^{-p}(\mathcal{E}, d(\mathcal{Y}, \mathcal{E}) + 1) d\hat{\mu}(\mathcal{Y}), \end{aligned}$$

where $d\hat{\mu} = a_1^{-\alpha_1 p} \cdots a_k^{-\alpha_k p} d\mu$. By Corollary 2.4, we obtain

$$\begin{aligned} I &\gtrsim \int_1^\infty V^{-p}(\mathcal{E}, t+1) \widehat{V}(\mathcal{E}, t) dt \\ &\gtrsim \int_1^\infty (t+1)^{-(p-1)\frac{k-1}{2}} e^{t(\sqrt{\sum (n_i + \alpha_i(p-1))^2} - p\sqrt{\sum \alpha_i^2})} dt \end{aligned}$$

where $\widehat{V}(\mathcal{E}, t)$ is the volume of ball with respect to the measure $d\hat{\mu}$ and we have used the assumption $\alpha_i > \frac{n_i}{2}$ in the last inequality. The right hand side is infinite when

$$\sum_{1 \leq i \leq k} (n_i + \alpha_i(p-1))^2 > p^2 \sum_{1 \leq i \leq k} \alpha_i^2. \quad (5.2)$$

The above inequality holds when $\sum_i \alpha_i(\alpha_i - n_i) < 0$ or $\sum_i \alpha_i(\alpha_i - n_i) > 0$, $p < p_0$. Then we have proved the results. $\square$

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