Research Article

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Refined second boundary behavior of the unique strictly convex solution to a singular Monge-Ampère equation

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Abstract: In this paper, we establish the second boundary behavior of the unique strictly convex solution to a singular Dirichlet problem for the Monge-Ampère equation

$$\det(D^2u) = b(x)g(-u), \quad u < 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded, smooth and strictly convex domain in $\mathbb{R}^N$ ($N \geq 2$), $b \in C^\infty(\Omega)$ is positive and may be singular (including critical singular) or vanish on the boundary, $g \in C^1((0, \infty), (0, \infty))$ is decreasing on $(0, \infty)$ with $\lim_{t \rightarrow 0^+} g(t) = \infty$ and g is normalized regularly varying at zero with index $-\gamma$ ($\gamma > 1$). Our results reveal the refined influence of the highest and the lowest values of the $(N - 1)$ -th curvature on the second boundary behavior of the unique strictly convex solution to the problem.

Keywords: Monge-Ampère equations; Strictly convex solution; Singular boundary value problem; The second boundary behavior

MSC: 35B40; 35J25; 35J60; 35J75; 35J96

1 Introduction and main results

This presentation is to establish the second boundary behavior of the unique strictly convex solution to a singular Dirichlet problem for the Monge-Ampère equation

$$\det(D^2u) = b(x)g(-u), \quad u < 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \quad (1.1)$$

where Ω is a bounded, smooth and strictly convex domain in $\mathbb{R}^N$ ($N \geq 2$), and

$$D^2u(x) = \left(\frac{\partial^2 u(x)}{\partial x_i \partial x_j} \right)_{N \times N}$$

denotes the Hessian of u and D^2u is the so called Monge-Ampère operator. The nonlinearity g satisfies

(g₁) $g \in C^1((0, \infty), (0, \infty))$ is decreasing on $(0, \infty)$ and $\lim_{t \rightarrow 0^+} g(t) = \infty$;

(g₂) there exist a constant $\gamma > 1$ and some function $f \in C^1(0, a_1) \cap C[0, a_1)$ for a sufficiently small constant $a_1 > 0$ such that

$$\frac{-tg'(t)}{g(t)} := \gamma + f(t) \text{ with } \lim_{t \rightarrow 0^+} f(t) = 0,$$

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i.e.,

$$g(t) = c_0 t^{-\gamma} \exp \left(\int_t^{a_1} \frac{f(s)}{s} ds \right), \quad c_0 = g(a_1) a_1^\gamma,$$

where the function f satisfies

(S₁) $f \equiv 0$ on $(0, a_1]$ (or (S₂) $f(t) \neq 0, \forall t \in (0, a]$ for some $a \leq a_1$).

If (S₂) holds in (g₂), then we suppose

(g₃) there exists $\theta \geq 0$ such that

$$\lim_{t \rightarrow 0^+} \frac{tf'(t)}{f(t)} = \theta \geq 0.$$

If $\theta = 0$ in (g₃), then we further suppose

(g₄) there exist $\beta \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}$ such that

$$\lim_{t \rightarrow 0^+} (-\ln t)^\beta f(t) = \sigma.$$

The weight b satisfies

(b₁) $b \in C^\infty(\Omega)$ is positive in Ω

and one of the following two conditions

(b₂) there exist $k \in \Lambda, B_0 \in \mathbb{R}$ and $\mu \in \mathbb{R}^+$ such that

$$b(x) = k^{N+1}(d(x))(1 + B_0(d(x))^\mu + o((d(x))^\mu)), \quad d(x) \rightarrow 0,$$

where $d(x) := \text{dist}(x, \partial\Omega)$, $x \in \Omega$, Λ denotes the set of all of positive monotonic functions in $C^1(0, \delta_0) \cap L(0, \delta_0)$ which satisfy

$$\lim_{t \rightarrow 0^+} \frac{d}{dt} \left(\frac{K(t)}{k(t)} \right) = D_k \geq 0, \quad K(t) = \int_0^t k(s) ds$$

and

(b₃) there exist $\tilde{L} \in \mathcal{L}, B_0 \in \mathbb{R}$ and $\mu \in \mathbb{R}^+$ such that

$$b(x) = (d(x))^{-(N+1)} \tilde{L}^N(d(x))(1 + B_0(d(x))^\mu + o((d(x))^\mu)), \quad d(x) \rightarrow 0,$$

where $\mathcal{L}$ denotes the set of all of positive functions defined on $(0, t_0]$ by

$$\tilde{L}(t) := c \exp \left(\int_t^{t_0} \frac{y(s)}{s} ds \right), \quad t \in (0, t_0],$$

where $c \in \mathbb{R}^+, y \in C(0, t_0]$ and $\lim_{t \rightarrow 0^+} y(t) = 0$.

The set Λ in (b₂) was first introduced by Cîrstea and Rădulescu [6]-[8] for non-decreasing functions and by Mohammed [33] for non-increasing functions to study the exact boundary behavior and uniqueness of boundary blow-up elliptic problems. When b satisfies (b₂) with $D_k > 0$, we see by Lemma 3.1 (iii)-(iv) that b may be singular on the boundary with the index

$$\frac{(1 - D_k)(N + 1)}{D_k} > -(N + 1).$$

The condition (b₃) implies that b is critical singular with the index $-(N + 1)$.

Problem (1.1) has a wide range of applications in Riemannian geometry and optical physics and one important geometric application is to structure a Riemannian metric in Ω that is invariant under projective transformations. When $g(t) = t^{-(N+2)}, t > 0$ and $b \equiv 1$ in Ω , Nirenberg [38], Loewner and Nirenberg [31] for $N = 2$, Cheng and Yau [5] for $N \geq 2$ studied the existence and uniqueness of solutions to problem (1.1). In particular, Cheng and Yau [5] showed that if Ω is convex and bounded but not necessarily strictly convex then problem (1.1) possesses a unique solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ which is negative in Ω . When $g(t) = t^{-\gamma} (t > 0)$ with $\gamma > 1$

and $b \in C^\infty(\bar{\Omega})$ with $b(x) > 0$ for all $x \in \Omega$, Lazer and McKenna [27] proved the existence and uniqueness of solutions to problem (1.1). Moreover, they also obtained the following global estimate

$$c_1(d(x))^{\frac{N+1}{N+\gamma}} \leq u(x) \leq c_2(d(x))^{\frac{N+1}{N+\gamma}}, \quad x \in \Omega.$$

When b satisfies (b_1) and $g : (0, \infty) \rightarrow (0, \infty)$ is a non-increasing, smooth function, Mohammed [34] showed that problem (1.1) has a strictly convex solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ if and only if the problem

$$\det(D^2u) = b(x) \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega \quad (1.2)$$

has a strictly convex solution $v \in C^\infty(\Omega) \cap C(\bar{\Omega})$, where b may be singular or may vanish on $\partial\Omega$. In particular, the author showed that

- (i₁) if $b \in C^\infty(\bar{\Omega})$ is positive in $\bar{\Omega}$, then problem (1.1) has a strictly convex solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$;
- (i₂) if $\lim_{t \rightarrow 0^+} g(t) = \infty$, then problem (1.1) has a unique strictly convex solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ and the solution u satisfies

$$c_1\phi(d(x)) \leq -u(x) \leq c_2\phi(d(x)) \text{ and } |\nabla u(x)| \leq c_2 \frac{\phi(d(x))}{d(x)} \text{ near } \partial\Omega,$$

where c_1, c_2 are positive constants and ϕ is uniquely determined by

$$\int_0^{\phi(t)} (G(s))^{-(N+1)} ds = t, \quad G(t) = \int_t^{\hat{t}} g(s) ds, \quad t \in (0, \hat{t}), \quad \hat{t} \in (0, \infty];$$

- (i₃) if $b(x) \leq C(d(x))^{\delta-N-1}$ for some positive constants δ and C , then problem (1.2) has a strictly convex solution;
 - (i₄) if $b(x) \geq C(d(x))^{-(N+1)}$ for some positive constant C , then problem (1.2) has no strictly convex solution.
- Later, Yang and Chang [47] extended the above results (i₃)-(i₄) to the following cases:
- (i₅) if $b(x) \leq C(d(x))^{-(N+1)}(-\ln d(x))^{-q}$ near $\partial\Omega$ for some $q > N$ and $C > 0$, then problem (1.2) has a strictly convex solution;
 - (i₆) if $b(x) \geq C(d(x))^{-(N+1)}(-\ln d(x))^{-N}$ near $\partial\Omega$ for some $C > 0$, then problem (1.2) has no strictly convex solution.

Let $\mathcal{P} \in C^1(0, \infty)$ satisfy $\mathcal{P}'(t) < 0$ and $\lim_{t \rightarrow 0^+} \mathcal{P}(t) = \infty$ and define $\mathfrak{P}(t) = \int_t^1 \mathcal{P}(s) ds$. Recently, under the hypothesis of (b_1) , Zhang and Du [49] obtain the following results:

- (i₇) if $b(x) \leq \mathcal{P}(d(x))$ near $\partial\Omega$ and $\int_0^1 (\mathfrak{P}(s))^{1/N} ds < \infty$, then problem (1.2) has a strictly convex solution;
- (i₈) if $b(x) \geq \mathcal{P}(d(x))$ and $\int_0^1 (\mathfrak{P}(s))^{1/N} ds = \infty$, then problem (1.2) has no strictly convex solution.

The above facts imply that problem (1.2) has a strictly convex solution if b satisfies (b_1) and

$$b(x) \leq Ck^{N+1}(d(x)) \text{ near } \partial\Omega \text{ or } b(x) \leq C(d(x))^{-(N+1)}\tilde{L}^N(d(x)) \text{ near } \partial\Omega,$$

where C is a positive constant, $k \in \Lambda$ in (b_2) and $\tilde{L} \in \mathcal{L}$ in (b_3) with

$$\int_0^t \frac{\tilde{L}(s)}{s} ds < \infty. \quad (1.3)$$

In [28], Li and Ma studied the existence and the first boundary behavior of the strictly convex solutions to problem (1.1) by using regularity theory and sub-supersolution method. In particular, when $b \in C^3(\Omega)$ is positive in Ω and satisfies

- (b₀₁) there exist $k \in \Lambda$ and positive constants b_1 and b_2 such that

$$b_1 := \liminf_{d(x) \rightarrow 0} \frac{b(x)}{k^{N+1}(d(x))} \leq \limsup_{d(x) \rightarrow 0} \frac{b(x)}{k^{N+1}(d(x))} =: b_2,$$

g satisfies (g_1) and

(g₀₁)

$$\lim_{t \rightarrow 0^+} (G(t))^{1/(N+1)} \int_0^t (G(s))^{-1/(N+1)} ds = D_g, \quad G(t) = \int_t^{a_1} g(s) ds \text{ for some } a_1 > 0,$$

they showed that the unique strictly convex solution u to problem (1.1) satisfies

$$1 \leq \liminf_{d(x) \rightarrow 0} \frac{-u(x)}{\psi(\vartheta_1 K(d(x)))} \leq \limsup_{d(x) \rightarrow 0} \frac{-u(x)}{\psi(\vartheta_2 K(d(x)))} \leq 1,$$

where ψ is uniquely determined by

$$\int_0^{\psi(t)} ((N+1)G(s))^{-1/(N+1)} ds = t, \quad (1.4)$$

$$\vartheta_1 = \left(\frac{b_1}{\hat{m}_-(1 - D_g^{-1}(1 - D_k))} \right)^{1/(N+1)} \text{ and } \vartheta_2 = \left(\frac{b_2}{\hat{m}_+(1 - D_g^{-1}(1 - D_k))} \right)^{1/(N+1)}$$

with

$$\hat{m}_- := \max_{\bar{x} \in \partial\Omega} \omega_{N-1}(\bar{x}) \text{ and } \hat{m}_+ := \min_{\bar{x} \in \partial\Omega} \omega_{N-1}(\bar{x}), \quad (1.5)$$

where

$$\omega_{N-1}(\bar{x}) = \prod_{i=1}^{N-1} \kappa_i(\bar{x})$$

denotes the $(N-1)$ -th curvature at $\bar{x}$ and $\kappa_1(\bar{x}), \dots, \kappa_{N-1}(\bar{x})$ denote the principal curvatures of $\partial\Omega$ at $\bar{x}$. In [51], Zhang showed that if b satisfies (b₁) and (b₀₁), g satisfies (g₁) and

(g₀₂)

$$\lim_{t \rightarrow 0^+} ((g(t))^{1/N})' \int_0^t (g(s))^{-1/N} ds = -C_g,$$

and $ND_k + (1+N)C_g > 1+N$, then the unique strictly convex solution u to problem (1.1) satisfies

$$\vartheta_3^{1-C_g} := \liminf_{d(x) \rightarrow 0} \frac{-u(x)}{\phi_g((K(d(x)))^{(N+1)/N})} \leq \limsup_{d(x) \rightarrow 0} \frac{-u(x)}{\phi_g((K(d(x)))^{(N+1)/N})} =: \vartheta_4^{1-C_g},$$

where ϕ_g is uniquely determined by

$$\int_0^{\phi_g(t)} (Ng(s))^{-1/N} ds = t, \quad t > 0,$$

$$\vartheta_3 = \left(\left(\frac{N}{N+1} \right)^N \frac{b_1}{\hat{m}_-((1+N)C_g + ND_k - 1 - N)} \right)^{1/N}$$

and

$$\vartheta_4 = \left(\left(\frac{N}{N+1} \right)^N \frac{b_2}{\hat{m}_+((1+N)C_g + ND_k - 1 - N)} \right)^{1/N}.$$

Especially, if (b₀₁) is replaced by the following condition

(b₀₂) there exist $\tilde{L} \in \mathcal{L}$ with (1.3) and positive constants b_1 and b_2 such that

$$b_1 := \liminf_{d(x) \rightarrow 0} \frac{b(x)}{(d(x))^{-(N+1)} \tilde{L}^N(d(x))} \leq \limsup_{d(x) \rightarrow 0} \frac{b(x)}{(d(x))^{-(N+1)} \tilde{L}^N(d(x))} =: b_2,$$

Zhang [51] showed that the unique strictly convex solution u to problem (1.1) satisfies

$$\vartheta_5^{1-C_g} \leq \liminf_{d(x) \rightarrow 0} \frac{-u(x)}{\phi_g \left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)} \leq \limsup_{d(x) \rightarrow 0} \frac{-u(x)}{\phi_g \left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)} \leq \vartheta_6^{1-C_g},$$

where

$$\vartheta_5 = \left(\frac{b_1}{\hat{m}_- N} \right)^{1/N} \text{ and } \vartheta_6 = \left(\frac{b_2}{\hat{m}_+ N} \right)^{1/N}.$$

Then, Sun and Feng [43] and Li and Ma [29] generalized the above boundary behavior results to the case of the following Hessian equation for $i = 1, \dots, N$

$$S_i(D^2 u) = b(x)g(-u), \quad u < 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

where

$$S_i(D^2 u) = S_i(\lambda_1, \dots, \lambda_N) = \sum_{1 \leq j_1 < \dots < j_i \leq N} \lambda_{j_1} \cdots \lambda_{j_i}$$

and $\lambda_1, \dots, \lambda_N$ are the eigenvalues of $D^2 u$. Furthermore $S_0(\lambda) \equiv 1$ for $\lambda \in \mathbb{R}^N$. Espically, Li and Ma [29] also studied the existence and uniqueness of viscosity solution to the problem. For related insights on the existence, regularity and asymptotic behavior of solutions to the Monge-Ampère equations, please refer to [4], [11], [19], [21]-[25], [30], [35]-[37], [44]-[45] and the references therein. When the Monge-Ampère operator ($\det(D^2 u)$) is replaced by the Laplace operator (Δ), many papers have been dedicated to resolving existence, uniqueness and asymptotic behavior issues for solutions, please refer to [1]-[2], [10], [12]-[17], [26], [39], [46], [48], [50] and the references therein.

In this paper, by making a complete and detailed analysis to some indexes in various cases, we establish the exact second boundary behavior of the unique strictly convex solution to problem (1.1), which is quite different from the first behavior of this solution. For all we know, in literature there aren't articles on the second boundary behavior of the strictly convex solution to problem (1.1).

To our aims, we define the following subclasses of Λ and $\mathcal{L}$ as follows:

$$\begin{aligned} \Lambda_1 &:= \left\{ k \in \Lambda : \lim_{t \rightarrow 0^+} t^{-1} \left[\frac{d}{dt} \left(\frac{K(t)}{k(t)} \right) - D_k \right] = E_{1,k} \right\}; \\ \Lambda_{2,\beta} &:= \left\{ k \in \Lambda : \lim_{t \rightarrow 0^+} (-\ln t)^\beta \left[\frac{d}{dt} \left(\frac{K(t)}{k(t)} \right) - D_k \right] = E_{2,k} \right\}; \\ \mathcal{L}_\beta &:= \{ \tilde{L} \in \mathcal{L} : \lim_{t \rightarrow 0^+} (-\ln t)^\beta y(t) = E_3 \}, \end{aligned}$$

where β is a positive constant and the relation between $\tilde{L}$ and y is given in **(b₃)**. Our results are summarized as follows and $\hat{m}_\pm$ (given in Theorems 1.1-1.3) are defined by (1.5).

Theorem 1.1. *Let b satisfy **(b₁)**-**(b₂)** with $(\gamma + N)D_k > N + 1$, g satisfy **(g₁)**-**(g₂)**.*

(I) *When **(S₁)** holds (or **(S₂)** and **(g₃)**-**(g₄)** hold with $\theta = 0$ in **(g₃)**), we have*

(i) *If $k \in \Lambda_1$, then the unique strictly convex solution u to problem (1.1) satisfies*

$$\begin{aligned} &\xi_- \psi(K(d(x))) (1 + C_- (-\ln d(x))^{-\beta} + o((- \ln d(x))^{-\beta})) \\ &\leq -u(x) \leq \xi_+ \psi(K(d(x))) (1 + C_+ (-\ln d(x))^{-\beta} + o((- \ln d(x))^{-\beta})), \quad d(x) \rightarrow 0, \end{aligned} \quad (1.6)$$

where ψ is uniquely determined by (1.4) and

$$\xi_\pm = \left(\frac{((\gamma + N)D_k - (N + 1))\hat{m}_\pm}{\gamma - 1} \right)^{-1/(\gamma + N)}, \quad (1.7)$$

$$C_\pm = \begin{cases} -\left(\frac{(\gamma + N)D_k}{N + 1}\right)^\beta \mathfrak{C}_\pm, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta = 0, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds,} \end{cases}$$

where

$$\begin{cases} \mathfrak{C}_{\pm} = \frac{A_{\pm}}{\gamma+N}, & \text{if } A_{+} \leq 0 \text{ and } A_{-} \leq 0, \\ \mathfrak{C}_{+} = \frac{A_{+}}{\gamma+N\frac{\hat{m}_{-}}{\hat{m}_{+}}} \text{ and } \mathfrak{C}_{-} = \frac{A_{-}}{\gamma+N\frac{\hat{m}_{+}}{\hat{m}_{-}}}, & \text{if } A_{+} \geq 0 \text{ and } A_{-} \geq 0, \\ \mathfrak{C}_{+} = \frac{A_{+}}{\gamma+N\frac{\hat{m}_{-}}{\hat{m}_{+}}} \text{ and } \mathfrak{C}_{-} = \frac{A_{-}}{\gamma+N}, & \text{if } A_{+} > 0 \text{ and } A_{-} < 0, \\ \mathfrak{C}_{+} = \frac{A_{+}}{\gamma+N} \text{ and } \mathfrak{C}_{-} = \frac{A_{-}}{\gamma+N\frac{\hat{m}_{+}}{\hat{m}_{-}}}, & \text{if } A_{+} < 0 \text{ and } A_{-} > 0 \end{cases}$$

with

$$A_{\pm} = \left(C_0 + \frac{1}{\gamma+N} \ln \hat{m}_{\pm}^{-1} \right) \sigma, \quad (1.8)$$

and

$$C_0 = \frac{(N+1)(1-D_k)}{((\gamma+N)D_k - (N+1))(\gamma-1)} - (\gamma+N)^{-1} \ln \frac{(\gamma+N)D_k - (N+1)}{\gamma-1}. \quad (1.9)$$

(ii) If $k \in \Lambda_{2,\beta}$ (β is the same as the one in $(\mathbf{g}_4)$), then (1.6) still holds, where

$$C_{\pm} = \begin{cases} -\hat{C}_{\pm}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta = 0, \\ -\hat{D}_{\pm}, & \text{if } (\mathbf{S}_1) \text{ holds,} \end{cases}$$

and

$$\begin{cases} \hat{C}_{\pm} = \frac{\mathfrak{A}_{\pm} + \mathfrak{B}}{\gamma+N}, & \text{if } \mathfrak{A}_{+} + \mathfrak{B} \leq 0 \text{ and } \mathfrak{A}_{-} + \mathfrak{B} \leq 0, \\ \hat{C}_{+} = \frac{\mathfrak{A}_{+} + \mathfrak{B}}{\gamma+N\frac{\hat{m}_{-}}{\hat{m}_{+}}} \text{ and } \hat{C}_{-} = \frac{\mathfrak{A}_{-} + \mathfrak{B}}{\gamma+N\frac{\hat{m}_{+}}{\hat{m}_{-}}}, & \text{if } \mathfrak{A}_{+} + \mathfrak{B} \geq 0 \text{ and } \mathfrak{A}_{-} + \mathfrak{B} \geq 0, \\ \hat{C}_{+} = \frac{\mathfrak{A}_{+} + \mathfrak{B}}{\gamma+N\frac{\hat{m}_{-}}{\hat{m}_{+}}} \text{ and } \hat{C}_{-} = \frac{\mathfrak{A}_{-} + \mathfrak{B}}{\gamma+N}, & \text{if } \mathfrak{A}_{+} + \mathfrak{B} > 0 \text{ and } \mathfrak{A}_{-} + \mathfrak{B} < 0, \\ \hat{C}_{+} = \frac{\mathfrak{A}_{+} + \mathfrak{B}}{\gamma+N} \text{ and } \hat{C}_{-} = \frac{\mathfrak{A}_{-} + \mathfrak{B}}{\gamma+N\frac{\hat{m}_{+}}{\hat{m}_{-}}}, & \text{if } \mathfrak{A}_{+} + \mathfrak{B} < 0 \text{ and } \mathfrak{A}_{-} + \mathfrak{B} > 0, \end{cases}$$

$$\begin{cases} \hat{D}_{+} = \frac{\mathfrak{B}}{\gamma+N\frac{\hat{m}_{-}}{\hat{m}_{+}}} \text{ and } \hat{D}_{-} = \frac{\mathfrak{B}}{\gamma+N\frac{\hat{m}_{+}}{\hat{m}_{-}}}, & \text{if } \mathfrak{B} \geq 0, \\ \hat{D}_{+} = \hat{D}_{-} = \frac{\mathfrak{B}}{\gamma+N}, & \text{if } \mathfrak{B} < 0 \end{cases} \quad (1.10)$$

with

$$\mathfrak{A}_{\pm} = \left(\frac{(\gamma+N)D_k}{N+1} \right)^{\beta} A_{\pm} \text{ and } \mathfrak{B} = \frac{(\gamma+N)E_{2,k}}{(\gamma+N)D_k - (N+1)}, \quad (1.11)$$

where $A_{\pm}$ are given by (1.8).

(II) When $(\mathbf{S}_2)$ and $(\mathbf{g}_3)$ hold with $\theta > 0$ in $(\mathbf{g}_3)$ and $k \in \Lambda_{2,\beta}$, then the unique strictly convex solution u to problem (1.1) satisfies (1.6) with $C_{\pm} = -\hat{D}_{\pm}$, where $\hat{D}_{\pm}$ are given by (1.10).

Corollary 1.1. In Theorem 1.1, if Ω is a ball with radius R and center x_0 , then

(I) When $(\mathbf{S}_1)$ holds (or $(\mathbf{S}_2)$ and $(\mathbf{g}_3)$ -($\mathbf{g}_4$) hold with $\theta = 0$ in $(\mathbf{g}_3)$), we have

(i) If $k \in \Lambda_1$, then the unique strictly convex solution u to problem (1.1) satisfies

$$-u(x) = \xi_R^1 \psi(K(R-r))(1 + C_R^1(-\ln(R-r))^{\beta} + o((-\ln(R-r))^{\beta})), \quad r \rightarrow R, \quad (1.12)$$

where $r = |x - x_0|$,

$$\xi_R^1 = \left(\frac{(\gamma+N)D_k - (N+1)}{(\gamma-1)R^{N-1}} \right)^{-1/(\gamma+N)} \quad (1.13)$$

and

$$C_R^1 = \begin{cases} \hat{C}_R^1, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta = 0, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds,} \end{cases}$$

where

$$\hat{C}_R^1 = - \left(\frac{(\gamma+N)D_k}{N+1} \right)^{\beta} \frac{(C_0 + \frac{N-1}{\gamma+N} \ln R) \sigma}{\gamma+N} \quad (1.14)$$

and C_0 is given by (1.9).

(ii) If $k \in \Lambda_{2,\beta}$ (β is the same as the one in $(\mathbf{g}_4)$), then (1.12) still holds, where

$$C_R^1 = \begin{cases} \hat{C}_R^1 - \frac{E_{2,k}}{(\gamma+N)D_k-(N+1)}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta = 0, \\ -\frac{E_{2,k}}{(\gamma+N)D_k-(N+1)}, & \text{if } (\mathbf{S}_1) \text{ holds,} \end{cases}$$

where $\hat{C}_R^1$ is given by (1.14) and C_0 is given by (1.9).

(II) When $(\mathbf{S}_2)$ and $(\mathbf{g}_3)$ hold with $\theta > 0$ in $(\mathbf{g}_3)$ and $k \in \Lambda_{2,\beta}$, then the unique strictly convex solution u to problem (1.1) satisfies (1.12), where

$$C_R^1 = -\frac{E_{2,k}}{(\gamma+N)D_k-(N+1)}.$$

Theorem 1.2. Let b satisfy $(\mathbf{b}_1)$ – $(\mathbf{b}_2)$ with $\mu \in (0, 1)$, g satisfy $(\mathbf{g}_1)$ – $(\mathbf{g}_2)$ with $(1-\mu)(\gamma+N)D_k > N+1$, and if $(\mathbf{S}_2)$ holds in $(\mathbf{g}_2)$, we further suppose that $(\mathbf{g}_3)$ with $\theta > 0$ and the following hold

$$\begin{cases} \frac{\theta(N+1)}{\mu} - N > \gamma > \frac{N+1}{(1-\mu)D_k} - N, & \text{if } D_k \in (0, 1), \\ \theta(N+1) - N > \gamma > \max\left\{\frac{N+1}{(1-\mu)D_k} - N, \frac{2N+1}{N}\right\} \text{ and } \mu D_k \in (0, 1), & \text{if } D_k \in [1, \infty). \end{cases} \quad (1.15)$$

If $k \in \Lambda_1$, then the unique strictly convex solution u to problem (1.1) satisfies

$$\begin{aligned} & \xi_- \psi(K(d(x))) (1 + \tilde{C}_-(d(x))^\mu + o((d(x))^\mu)) \\ & \leq -u(x) \leq \xi_+ \psi(K(d(x))) (1 + \tilde{C}_+(d(x))^\mu + o((d(x))^\mu)), \quad d(x) \rightarrow 0, \end{aligned} \quad (1.16)$$

where ψ is uniquely determined by (1.4), $\xi_\pm$ are given by (1.7) and

$$\begin{cases} \tilde{C}_\pm = \frac{B_0(N+1)((\gamma+N)D_k-(N+1))}{\tilde{\eta} + \gamma(N+1)((\gamma+N)D_k-(N+1))}, & \text{if } B_0 \geq 0, \\ \tilde{C}_+ = \frac{B_0(N+1)((\gamma+N)D_k-(N+1))}{\tilde{\eta}^{\frac{\tilde{m}_+}{m_+}} + \gamma(N+1)((\gamma+N)D_k-(N+1))}, & \text{if } B_0 < 0, \\ \tilde{C}_- = \frac{B_0(N+1)((\gamma+N)D_k-(N+1))}{\tilde{\eta}^{\frac{\tilde{m}_-}{m_-}} + \gamma(N+1)((\gamma+N)D_k-(N+1))}, & \text{if } B_0 < 0 \end{cases}$$

with

$$\begin{aligned} \tilde{\eta} = & ((1-\mu)(\gamma+N)D_k - (N+1))(N(N+1) + \mu(N-1)(\gamma+N)D_k) \\ & + \mu(\gamma+N)D_k((N-2)(N+1) + \mu(N-1)(\gamma+N)D_k) + \mu(1-\mu)(\gamma+N)^2 D_k^2. \end{aligned}$$

Corollary 1.2. In Theorem 1.2, if Ω is a ball with radius R and center x_0 , then the unique strictly convex solution u to problem (1.1) satisfies

$$-u(x) = \xi_R^1 \psi(K(R-r)) (1 + C_R^2(R-r)^\mu + o((R-r)^\mu)), \quad r \rightarrow R,$$

where ξ_R^1 is given by (1.13), $r = |x - x_0|$ and

$$C_R^2 = \frac{B_0(N+1)((\gamma+N)D_k-(N+1))}{\tilde{\eta} + \gamma(N+1)((\gamma+N)D_k-(N+1))}.$$

Remark 1.1. In Theorem 1.2, if we replace $k \in \Lambda_1$ by

$$k \in \Lambda' := \left\{ k \in \Lambda : \lim_{t \rightarrow 0^+} t^{-\mu} \left(\frac{d}{dt} \left(\frac{K(t)}{k(t)} \right) - D_k \right) = E_{1,k} \right\}$$

and other conditions still hold, then (1.16) holds, where

$$\begin{cases} \tilde{C}_\pm = \frac{(N+1)[B_0((\gamma+N)D_k-(N+1)) - (\gamma+N)E_{1,k}]}{\tilde{\eta} + \gamma(N+1)((\gamma+N)D_k-(N+1))}, & \text{if } B_0((\gamma+N)D_k-(N+1)) \geq (\gamma+N)E_{1,k}, \\ \tilde{C}_+ = \frac{(N+1)[B_0((\gamma+N)D_k-(N+1)) - (\gamma+N)E_{1,k}]}{\tilde{\eta}^{\frac{\tilde{m}_+}{m_+}} + \gamma(N+1)((\gamma+N)D_k-(N+1))}, & \text{if } B_0((\gamma+N)D_k-(N+1)) < (\gamma+N)E_{1,k}, \\ \tilde{C}_- = \frac{(N+1)[B_0((\gamma+N)D_k-(N+1)) - (\gamma+N)E_{1,k}]}{\tilde{\eta}^{\frac{\tilde{m}_-}{m_-}} + \gamma(N+1)((\gamma+N)D_k-(N+1))}, & \text{if } B_0((\gamma+N)D_k-(N+1)) < (\gamma+N)E_{1,k}. \end{cases}$$

When b is critical singular on $\partial\Omega$, we have the following second boundary behavior.

Theorem 1.3. *Let b satisfy $(\mathbf{b}_1)$ and $(\mathbf{b}_3)$, g satisfy $(\mathbf{g}_1)$ – $(\mathbf{g}_2)$, and if $(\mathbf{S}_2)$ holds in $(\mathbf{g}_2)$, we further suppose that $(\mathbf{g}_3)$ – $(\mathbf{g}_4)$ hold with $\theta = 0$ in $(\mathbf{g}_3)$ and with $\beta \in (0, 1]$ in $(\mathbf{g}_4)$. If $\tilde{L} \in \mathcal{L}_\beta$ with*

$$E_3 < \begin{cases} 0, & \beta \in (0, 1), \\ -1, & \beta = 1, \end{cases} \quad (1.17)$$

then the unique strictly convex solution u to problem (1.1) satisfies

$$\begin{aligned} & \eta_- \psi \left(\left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right) (1 + C_-^* \mathfrak{M}(d(x)) + o(\mathfrak{M}(d(x)))) \\ & \leq -u(x) \leq \eta_+ \psi \left(\left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right) (1 + C_+^* \mathfrak{M}(d(x)) + o(\mathfrak{M}(d(x)))) , \quad d(x) \rightarrow 0, \end{aligned}$$

where ψ is uniquely determined by (1.4),

$$\begin{aligned} \eta_\pm &= \left(\left(\frac{N}{N+1} \right)^N \frac{\gamma+N}{\gamma-1} \hat{m}_\pm \right)^{-1/(\gamma+N)}, \\ \mathfrak{M}(d(x)) &= \left(-\ln \psi \left(\left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right) \right)^{-\beta}, \end{aligned}$$

and

$$C_\pm^* = \begin{cases} \mathfrak{D}_\pm, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta = 0, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds,} \end{cases}$$

where

$$\begin{cases} \mathfrak{D}_\pm = \frac{\mathfrak{X}_\pm}{\gamma+N}, & \text{if } \mathfrak{X}_+ \geq 0 \text{ and } \mathfrak{X}_- \geq 0, \\ \mathfrak{D}_+ = \frac{\mathfrak{X}_+}{\gamma+N \frac{\hat{m}_-}{\hat{m}_+}} \text{ and } \mathfrak{D}_- = \frac{\mathfrak{X}_-}{\gamma+N \frac{\hat{m}_+}{\hat{m}_-}}, & \text{if } \mathfrak{X}_+ \leq 0 \text{ and } \mathfrak{X}_- \leq 0, \\ \mathfrak{D}_+ = \frac{\mathfrak{X}_+}{\gamma+N} \text{ and } \mathfrak{D}_- = \frac{\mathfrak{X}_-}{\gamma+N \frac{\hat{m}_+}{\hat{m}_-}}, & \text{if } \mathfrak{X}_+ > 0 \text{ and } \mathfrak{X}_- < 0, \\ \mathfrak{D}_+ = \frac{\mathfrak{X}_+}{\gamma+N \frac{\hat{m}_-}{\hat{m}_+}} \text{ and } \mathfrak{D}_- = \frac{\mathfrak{X}_-}{\gamma+N}, & \text{if } \mathfrak{X}_+ < 0 \text{ and } \mathfrak{X}_- > 0 \end{cases}$$

with

$$\mathfrak{X}_\pm = (\gamma+N)^{-1} \left(\frac{N+1}{\gamma-1} + \ln \left(\left(\frac{N}{N+1} \right)^N \frac{\gamma+N}{\gamma-1} \right) + \ln \hat{m}_\pm \right) \sigma.$$

Corollary 1.3. *In Theorem 1.3, if Ω is a ball with radius R and center x_0 , then the unique strictly convex solution u to problem (1.1) satisfies*

$$-u(x) = \eta_R^1 \psi \left(\left(\int_0^{R-r} \frac{L(s)}{s} ds \right)^{\frac{N}{N+1}} \right) (1 + C_R^3 \mathfrak{M}(R-r) + o(\mathfrak{M}(R-r))), \quad r \rightarrow R,$$

where

$$\eta_R^1 = \left(\left(\frac{N}{N+1} \right)^N \frac{\gamma+N}{(\gamma-1)R^{N-1}} \right)^{-1/(\gamma+N)}, \quad r = |x - x_0|$$

and

$$C_R^3 = \frac{\left(\frac{N+1}{\gamma-1} + \ln \left(\left(\frac{N}{N+1} \right)^N \frac{\gamma+N}{\gamma-1} \right) - (N-1) \ln R \right) \sigma}{(\gamma+N)^2}.$$

Remark 1.2. In Theorem 1.3, we obtain by Lemma 3.2 (i) that if $\tilde{L} \in \mathfrak{L}_\beta$ with (1.17), then (1.3) holds. But, when $\tilde{L} \in \mathfrak{L}_\beta$ with $\beta > 1$, by a direct calculation we see that

$$\int_0^{t_0} \frac{\tilde{L}(s)}{s} ds = \infty. \quad (1.18)$$

Since $\lim_{t \rightarrow 0^+} (-\ln t)^\beta y(t) = E_3$, for any $\varepsilon > 0$ we can choose a small enough constant $\tilde{t} < \min\{t_0, 1\}$ such that

$$\frac{E_3 - \varepsilon}{(-\ln t)^\beta} \leq y(t), \quad t \in (0, \tilde{t}].$$

A simple calculation shows that

$$\exp\left(\int_t^{\tilde{t}} \frac{y(s)}{s} ds\right) \geq \exp\left(\int_t^{\tilde{t}} \frac{E_3 - \varepsilon}{s(-\ln s)^\beta} ds\right) = \exp\left(\frac{E_3 - \varepsilon}{\beta - 1} ((-\ln \tilde{t})^{1-\beta} - (-\ln t)^{1-\beta})\right).$$

Moreover, we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} \exp\left(\int_t^{\tilde{t}} \frac{E_3 - \varepsilon}{s(-\ln s)^\beta} ds\right) &= \lim_{t \rightarrow 0^+} \exp\left(\frac{E_3 - \varepsilon}{\beta - 1} ((-\ln \tilde{t})^{1-\beta} - (-\ln t)^{1-\beta})\right) \\ &= \exp\left(\frac{E_3 - \varepsilon}{\beta - 1} (-\ln \tilde{t})^{1-\beta}\right) > 0. \end{aligned} \quad (1.19)$$

It follows by the definition of $\tilde{L}$ in $(\mathbf{b}_3)$ that there exists a positive constant $\tilde{C}$ such that

$$\tilde{C} \exp\left(\int_t^{t_0} \frac{E_3 - \varepsilon}{s(-\ln s)^\beta} ds\right) \leq \tilde{L}(t), \quad t \in (0, t_0]. \quad (1.20)$$

Combining (1.19) with (1.20), we obtain (1.18) holds.

Remark 1.3. If $y(t) = \pm(-\ln t)^{-1}$, $0 < t \leq \tilde{t} < \min\{t_0, 1\}$ in $(\mathbf{b}_3)$, then we have

$$\begin{aligned} \tilde{L}(t) &= \exp\left(\int_t^{t_0} \frac{y(s)}{s} ds\right) = \exp\left(\int_t^{\tilde{t}} \frac{y(s)}{s} ds\right) \exp\left(\pm \int_{\tilde{t}}^{t_0} \frac{ds}{s(-\ln s)}\right) \\ &= \exp\left(\int_t^{t_0} \frac{y(s)}{s} ds\right) \left(\frac{\ln t}{\ln \tilde{t}}\right)^{\pm 1} \in \mathfrak{L}_\beta, \end{aligned}$$

where $\beta \in (0, 1)$ with $E_3 = 0$ or $\beta = 1$ with $E_3 = \pm 1$. It's clear that (1.18) holds here. This implies that when $\tilde{L} \in \mathfrak{L}_\beta$ with $\beta \leq 1$, (1.3) is not always true if (1.17) fails.

Remark 1.4. In Theorem 1.3, if we substitute $\tilde{L} \in \mathfrak{L}_{\beta_0}$ ($\beta_0 > \beta$) and

$$E_3 < \begin{cases} 0, & \beta_0 \in (0, 1), \\ -1, & \beta_0 = 1, \end{cases}$$

for $\tilde{L} \in \mathfrak{L}_\beta$ with (1.17), then the conclusion of Theorem 1.3 still holds.

Remark 1.5. $\tilde{L} \in \mathcal{L}$ is normalized slowly varying at zero and $\lim_{t \rightarrow 0^+} \frac{t\tilde{L}'(t)}{\tilde{L}(t)} = 0$.

The rest of the paper is organized as follows. In Section 2, we give some bases of Karamata regular variation theory. In Section 3, we show some auxiliary lemmas. The proofs of Theorems 1.1-1.3 are given in Section 4.

2 Some basic facts from Karamata regular variation theory

In this section, we introduce some preliminaries of Karamata regular variation theory which come from [3], [32], [40]–[41].

Definition 2.1. A positive continuous function g defined on $(0, a_0]$, for some $a_0 > 0$, is called **regularly varying at zero** with index p , denoted by $g \in RVZ_p$, if for each $\xi > 0$ and some $p \in \mathbb{R}$,

$$\lim_{t \rightarrow 0^+} \frac{g(\xi t)}{g(t)} = \xi^p. \quad (2.1)$$

In particular, when $p = 0$, g is called **slowly varying at zero**.

Clearly, if $g \in RVZ_p$, then $L(t) := g(t)/t^p$ is slowly varying at zero.

Proposition 2.2. (Uniform Convergence Theorem). If $g \in RVZ_p$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$, where $0 < c_1 < c_2 < a_0$.

Proposition 2.3. (Representation Theorem). A function L is slowly varying at zero if and only if it may be written in the form

$$L(t) = l(t) \exp \left(\int_t^{a_0} \frac{y(s)}{s} ds \right), \quad t \leq a_0,$$

where the functions l and y are continuous and for $t \rightarrow 0^+$, $y(t) \rightarrow 0$ and $l(t) \rightarrow c_0$ with $c_0 > 0$.

Definition 2.4. The function

$$L(t) = c_0 \exp \left(\int_t^{a_0} \frac{y(s)}{s} ds \right), \quad t \leq a_0,$$

is called **normalized** slowly varying at zero and

$$h(t) = t^p L(t), \quad t \geq a_0$$

is called **normalized** regularly varying at zero with index p (written as $f \in NRVZ_p$).

A function $h \in C^1(0, a_0]$ for some $a_0 > 0$ belongs to $NRVZ_p$ if and only if

$$\lim_{t \rightarrow 0^+} \frac{th'(t)}{h(t)} = p.$$

Proposition 2.5. Let functions L, L_1 be slowly varying at zero, then

- (i) $L^p, p \in \mathbb{R}, L_1 \cdot L$ and $L_1 \circ L$ satisfying $\lim_{t \rightarrow 0^+} L(t) = 0$, are also slowly varying at zero;
- (ii) for any $p > 0$,

$$t^p L(t) \rightarrow 0 \text{ and } t^{-p} L(t) \rightarrow \infty \text{ as } t \rightarrow 0^+;$$

- (iii) for any $p \in \mathbb{R}$,

$$\ln L(t)/\ln t \rightarrow 0 \text{ and } \ln(t^p L(t))/\ln t = p \text{ as } t \rightarrow 0^+.$$

Proposition 2.6. Let $g_1 \in (N)RVZ_{p_1}$ and $g_2 \in (N)RVZ_{p_2}$, then $g_1 \cdot g_2 \in (N)RVZ_{p_1+p_2}$.

Proposition 2.7. Let $g_1 \in (N)RVZ_{p_1}, g_2 \in (N)RVZ_{p_2}$ and $\lim_{t \rightarrow 0^+} g_2(t) = 0$, then $g_1 \circ g_2 \in (N)RVZ_{p_1 p_2}$.

Proposition 2.8. (Asymptotic Behavior). Let L is a slowly varying function at zero, then for $a_1 > 0$,

- (i) $\int_t^{a_1} s^p L(s) ds \sim -(1+p)^{-1} t^{1+p} L(t)$, for $p < -1, t \rightarrow 0^+$;
- (ii) $\int_0^t s^p L(s) ds \sim (1+p)^{-1} t^{1+p} L(t)$, for $p > -1, t \rightarrow 0^+$.

3 Auxiliary results

In this section, we show some useful results, which are necessary in the proofs of our results.

Lemma 3.1. (Lemma 2.9 in [51]). Let $k \in \Lambda$, then

- (i) when k is non-decreasing, $D_k \in [0, 1]$; and when k is non-increasing, $D_k \in [1, \infty)$;
- (ii) $\lim_{t \rightarrow 0^+} \frac{K(t)}{k(t)} = 0$, $\lim_{t \rightarrow 0^+} \frac{K(t)}{tk(t)} = D_k$ and $\lim_{t \rightarrow 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - D_k$;
- (iii) when $D_k > 0$ and $D_k \neq 1$, $k \in \text{NRVZ}_{(1-D_k)/D_k}$;
- (iv) when $D_k = 1$, k is normalized slowly varying at zero.

Lemma 3.2. Let $\tilde{L} \in \mathfrak{L}_\beta$ and (1.17) hold, then

- (i) $\int_0^t \frac{\tilde{L}(s)}{s} ds < \infty$;
- (ii) $\lim_{t \rightarrow 0^+} (-\ln t)^\beta \frac{\tilde{L}(s)}{\int_0^t \frac{\tilde{L}(s)}{s} ds} = \begin{cases} -E_3, & \beta \in (0, 1), \\ -(E_3 + 1), & \beta = 1; \end{cases}$
- (iii) $\lim_{t \rightarrow 0^+} (-\ln t)^\beta \frac{\tilde{L}'(t)t}{\tilde{L}(t)} = E_3$.

Proof. (i) By

$$\lim_{t \rightarrow 0^+} (-\ln t)^\beta y(t) = E_3 < \begin{cases} 0, & \beta \in (0, 1), \\ -1, & \beta = 1, \end{cases}$$

we see that for any $0 < \varepsilon < -\frac{E_3 + \tau}{2}$ with

$$\tau = \begin{cases} 0, & \beta \in (0, 1), \\ 1, & \beta = 1, \end{cases}$$

there exists a small enough positive constant $t_* \in (0, 1)$ such that

$$y(t) \leq \frac{E_3 + \varepsilon}{(-\ln t)^\beta}, \quad t \in (0, t_*] \text{ with } E_3 + \varepsilon < \begin{cases} 0, & \beta \in (0, 1), \\ -1, & \beta = 1. \end{cases}$$

A straightforward calculation shows that for any $t \in (0, t_*]$,

$$\begin{aligned} \exp \left(\int_t^{t_*} \frac{y(s)}{s} ds \right) &\leq \exp \left(\int_t^{t_*} \frac{E_3 + \varepsilon}{s(-\ln s)^\beta} ds \right) \\ &= \begin{cases} \exp \left(\frac{E_3 + \varepsilon}{1 - \beta} ((-\ln t)^{1-\beta} - (-\ln t_*)^{1-\beta}) \right), & \beta \in (0, 1), \\ \left(\frac{\ln t}{\ln t_*} \right)^{E_3 + \varepsilon}, & \beta = 1. \end{cases} \end{aligned}$$

So, we see that there exists a large constant $C > 0$ such that for any $t \in (0, t_*]$,

$$\tilde{L}(t) \leq \begin{cases} C \exp \left(\frac{E_3 + \varepsilon}{1 - \beta} (-\ln t)^{1-\beta} \right), & \beta \in (0, 1), \\ C(-\ln t)^{E_3 + \varepsilon}, & \beta = 1. \end{cases} \quad (3.1)$$

Case 1. If $\beta \in (0, 1)$, it is clear that that

$$\exp \left(-\frac{E_3 + \varepsilon}{1 - \beta} (-\ln t)^{1-\beta} \right) = \sum_{n=0}^{\infty} \frac{\left(-\frac{E_3 + \varepsilon}{1 - \beta} \right)^n (-\ln t)^{n(1-\beta)}}{n!}.$$

This together with (3.1) implies that we can choose a positive integer $n_* > (1 - \beta)^{-1}$ such that

$$\begin{aligned} C \exp \left(\frac{E_3 + \varepsilon}{1 - \beta} (-\ln t)^{1-\beta} \right) &= C \left(\exp \left(-\frac{E_3 + \varepsilon}{1 - \beta} (-\ln t)^{1-\beta} \right) \right)^{-1} \\ &= C \left(\sum_{n=0}^{\infty} \frac{\left(-\frac{E_3 + \varepsilon}{1 - \beta} \right)^n (-\ln t)^{n(1-\beta)}}{n!} \right)^{-1} \\ &\leq C \left(\left(-\frac{E_3 + \varepsilon}{1 - \beta} \right)^{n_*} (-\ln t)^{n_*(1-\beta)} \right)^{-1} \\ &= C_1 (-\ln t)^{-n_*(1-\beta)}, \quad t \in (0, t_*], \end{aligned} \quad (3.2)$$

where $C_1 = C \left(-\frac{E_3 + \varepsilon}{1 - \beta} \right)^{-n_*}$. So, we have

$$\begin{aligned} \int_0^{t_*} \frac{\tilde{L}(s)}{s} ds &\leq C \int_0^{t_*} \frac{\exp \left(\frac{E_3 + \varepsilon}{1 - \beta} (-\ln s)^{1-\beta} \right)}{s} ds \\ &\leq C_1 \int_0^{t_*} \frac{ds}{s (-\ln s)^{n_*(1-\beta)}} = \frac{C_1 (-\ln t_*)^{1-n_*(1-\beta)}}{n_*(1-\beta) - 1} < \infty. \end{aligned}$$

Case 2. If $\beta = 1$, then by (3.1) we obtain

$$\int_0^{t_*} \frac{\tilde{L}(s)}{s} ds \leq \int_0^{t_*} \frac{C(-\ln s)^{E_3 + \varepsilon}}{s} ds = -\frac{C(-\ln t_*)^{1+E_3 + \varepsilon}}{1 + E_3 + \varepsilon} < \infty.$$

Combining Cases 1-2, we obtain (i) holds.

(ii) It follows by (3.1)-(3.2) that $\lim_{t \rightarrow 0^+} (-\ln t)^\beta \tilde{L}(t) = 0$. By the l'Hospital's rule, we obtain

$$\lim_{t \rightarrow 0^+} (-\ln t)^\beta \frac{\tilde{L}(s)}{\int_0^t \frac{\tilde{L}(s)}{s} ds} = \lim_{t \rightarrow 0^+} \left(-(-\ln t)^\beta y(t) - \beta(-\ln t)^{\beta-1} \right) = \begin{cases} -E_3, & \beta \in (0, 1), \\ -(E_3 + 1), & \beta = 1. \end{cases}$$

(iii) A direct calculation shows that

$$\lim_{t \rightarrow 0^+} (-\ln t)^\beta \frac{\tilde{L}'(t)t}{\tilde{L}(t)} = \lim_{t \rightarrow 0^+} (-\ln t)^\beta y(t) = E_3.$$

□

Define

$$\varphi(t) := \int_0^t ((N+1)G(s))^{-1/(N+1)} ds, \quad t > 0, \quad (3.3)$$

where φ is the inverse of solution ψ to (1.4).

Lemma 3.3. Let g satisfy (g_1) -(g_2), then

(i) $\lim_{t \rightarrow 0^+} \frac{tg'(t)}{g(t)} = -\gamma$;

(ii) $G(t) \sim \frac{t^{1-\gamma}}{\gamma-1} \hat{L}_0(t)$, $t \rightarrow 0^+$;

(iii) $\varphi(t) \sim \left(\frac{\gamma-1}{N+1} \right)^{1/(N+1)} \frac{N+1}{\gamma+N} t^{\frac{\gamma+N}{N+1}} (\hat{L}_0(t))^{-1/(N+1)}$, $t \rightarrow 0^+$, where

$$\hat{L}_0(t) = c_0 \exp \left(\int_t^{a_1} \frac{f(s)}{s} ds \right), \quad c_0 = g(a_1) a_1^\gamma;$$

(iv) $\lim_{t \rightarrow 0^+} \frac{\varphi(t)}{t\varphi'(t)} = \lim_{t \rightarrow 0^+} \frac{\varphi(t)((N+1)G(t))^{1/(N+1)}}{t} = \frac{N+1}{\gamma+N}$, i.e., $\varphi \in NRVZ_{\frac{\gamma+N}{N+1}}$;

$$(v) \lim_{t \rightarrow 0^+} \frac{((N+1)G(t))^{N/(N+1)}}{g(t)\varphi(t)} = \frac{\gamma+N}{\gamma-1}.$$

Proof. (i) It follows by (g_2) that (i) holds.

(ii)-(iii) By (g_2) we see that

$$g(t) = t^{-\gamma} \hat{L}_0(t), \quad t \in (0, a_1]. \quad (3.4)$$

It follows by Proposition 2.8 (i) that (ii) holds. Furthermore, we have

$$((N+1)G(t))^{-1/(N+1)} \sim \left(\frac{\gamma-1}{N+1} \right)^{1/(N+1)} t^{(\gamma-1)/(N+1)} (\hat{L}_0(t))^{-1/(N+1)}, \quad t \rightarrow 0^+.$$

It follows by Proposition 2.8 (ii) that (iii) holds.

(iv)-(v) (3.3) implies that

$$\varphi'(t) = ((N+1)G(t))^{-1/(N+1)}, \quad t > 0. \quad (3.5)$$

By (3.4)-(3.5) and (i)-(ii), we obtain (iv)-(v) hold. $\square$

Lemma 3.4. Suppose that g satisfies (g_1) -(g_2). In particular, if (S_2) holds in (g_2) , we suppose that (g_3) holds. And if $\theta = 0$ in (g_3) , we further suppose that (g_4) holds. Then,

(i)

$$\lim_{t \rightarrow 0^+} v(t) \left(\frac{tg'(t)}{g(t)} + \gamma \right) = \chi_1,$$

where

$$\chi_1 = \begin{cases} -\sigma, & \text{if } (S_2) \text{ and } (g_3)\text{-(}g_4) \text{ hold with } \theta = 0 \text{ in } (g_3), \quad v(t) = (-\ln t)^\beta, \\ 0, & \text{if } (S_1) \text{ holds (or } (S_2) \text{ and } (g_3) \text{ hold with } \theta > 0), \quad v(t) = (-\ln t)^\beta, \\ 0, & \text{if } (S_1) \text{ holds (or } (S_2) \text{ and } (g_3) \text{ hold with } \theta(N+1) > (\gamma+N)\rho), \\ & v(t) = (\varphi(t))^{-\rho} \text{ with } \rho \in (0, 1]; \end{cases}$$

(ii)

$$\lim_{t \rightarrow 0^+} v(t) \left(\frac{G(t)}{tg(t)} - \frac{1}{\gamma-1} \right) = \chi_2,$$

where

$$\chi_2 = \begin{cases} -\frac{\sigma}{(\gamma-1)^2}, & \text{if } (S_2) \text{ and } (g_3)\text{-(}g_4) \text{ hold with } \theta = 0 \text{ in } (g_3), \quad v(t) = (-\ln t)^\beta, \\ 0, & \text{if } (S_1) \text{ holds (or } (S_2) \text{ and } (g_3) \text{ hold with } \theta > 0), \quad v(t) = (-\ln t)^\beta, \\ 0, & \text{if } (S_1) \text{ holds and } \gamma > \frac{(1+\rho)N+1}{N+1-\rho} \text{ in } (g_2), \quad v(t) = (\varphi(t))^{-\rho}, \quad \rho \in (0, 1], \\ 0, & \text{if } (S_2) \text{ and } (g_3) \text{ hold with } \frac{\theta(N+1)}{\rho} - N > \gamma > \frac{(1+\rho)N+1}{N+1-\rho}, \\ & v(t) = (\varphi(t))^{-\rho} \text{ with } \rho \in (0, 1]; \end{cases}$$

(iii)

$$\lim_{t \rightarrow 0^+} v(t) \left(\frac{((N+1)G(t))^{N/(N+1)}}{g(t)\varphi(t)} - \frac{\gamma+N}{\gamma-1} \right) = \chi_3,$$

where

$$\chi_3 = \begin{cases} -\frac{(N+1)\sigma}{(\gamma-1)^2}, & \text{if } (S_2) \text{ and } (g_3)\text{-(}g_4) \text{ hold with } \theta = 0 \text{ in } (g_3), \quad v(t) = (-\ln t)^\beta, \\ 0, & \text{if } (S_1) \text{ holds (or } (S_2) \text{ and } (g_3) \text{ hold with } \theta > 0), \quad v(t) = (-\ln t)^\beta, \\ 0, & \text{if } (S_1) \text{ holds and } \gamma > \frac{(1+\rho)N+1}{N+1-\rho} \text{ in } (g_2), \quad v(t) = (\varphi(t))^{-\rho}, \quad \rho \in (0, 1], \\ 0, & \text{if } (S_2) \text{ and } (g_3) \text{ hold with } \frac{\theta(N+1)}{\rho} - N > \gamma > \frac{(1+\rho)N+1}{N+1-\rho}, \\ & v(t) = (\varphi(t))^{-\rho} \text{ with } \rho \in (0, 1]; \end{cases}$$

(iv) let ξ be a positive constant, then

$$\lim_{t \rightarrow 0^+} v(t) \left(\frac{g(\xi t)}{\xi^N g(t)} - \xi^{-(\gamma+N)} \right) = \chi_4,$$

where

$$\chi_4 = \begin{cases} -\frac{\sigma \ln \xi}{\xi^{\gamma+N}}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3)\text{--}(\mathbf{g}_4) \text{ hold with } \theta = 0 \text{ in } (\mathbf{g}_3), v(t) = (-\ln t)^\beta, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds (or } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta > 0), v(t) = (-\ln t)^\beta, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds (or } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta(N+1) > (\gamma+N)\rho), \\ & v(t) = (\varphi(t))^{-\rho} \text{ with } \rho \in (0, 1]. \end{cases}$$

Proof. (i) When $(\mathbf{S}_2)$ and $(\mathbf{g}_3)\text{--}(\mathbf{g}_4)$ hold with $\theta = 0$ in $(\mathbf{g}_3)$, we have

$$\lim_{t \rightarrow 0^+} (-\ln t)^\beta \left(\frac{tg'(t)}{g(t)} + \gamma \right) = -\lim_{t \rightarrow 0^+} (-\ln t)^\beta f(t) = -\sigma. \quad (3.6)$$

When $(\mathbf{S}_1)$ holds, we have

$$\frac{tg'(t)}{g(t)} = -\gamma, \quad t \in (0, a_1].$$

This implies that for $v(t) = (-\ln t)^\beta$ or $v(t) = (\varphi(t))^{-\rho}$, we obtain

$$\lim_{t \rightarrow 0^+} v(t) \left(\frac{tg'(t)}{g(t)} + \gamma \right) = 0. \quad (3.7)$$

When $(\mathbf{S}_2)$ and $(\mathbf{g}_3)\text{--}(\mathbf{g}_4)$ hold with $\theta > 0$, we conclude by Proposition 2.5 (ii) that

$$\lim_{t \rightarrow 0^+} (-\ln t)^\beta \left(\frac{tg'(t)}{g(t)} + \gamma \right) = -\lim_{t \rightarrow 0^+} (-\ln t)^\beta f(t) = -\lim_{t \rightarrow 0^+} t^\theta (-\ln t)^\beta L(t) = 0, \quad (3.8)$$

where $L \in \text{NRVZ}_0$. Moreover, it follows by Lemma 3.3 (iv) and Proposition 2.7 that

$$1/\varphi^\rho \in \text{NRVZ}_{-\frac{(\gamma+N)\rho}{N+1}}. \quad (3.9)$$

When $(\mathbf{g}_3)$ holds with $\theta(N+1) > (\gamma+N)\rho$, we conclude by Proposition 2.6 and Proposition 2.5 (ii) that

$$\lim_{t \rightarrow 0^+} (\varphi(t))^{-\rho} \left(\frac{tg'(t)}{g(t)} + \gamma \right) = -\lim_{t \rightarrow 0^+} (\varphi(t))^{-\rho} f(t) = -\lim_{t \rightarrow 0^+} t^{\theta - \frac{(\gamma+N)\rho}{N+1}} L(t) = 0, \quad (3.10)$$

where $L \in \text{NRVZ}_0$. So, (i) follows by (3.6)–(3.10).

(ii) By $(\mathbf{g}_2)$, we have

$$-tg'(t) = \gamma g(t) + g(t)f(t), \quad t \in (0, a_1].$$

Integrating it from t to a_1 and integration by parts, we obtain

$$tg(t) = (\gamma - 1)G(t) + \int_t^{a_1} g(s)f(s)ds + c, \quad t \in (0, a_1],$$

i.e.,

$$\frac{G(t)}{tg(t)} - \frac{1}{\gamma - 1} = -\frac{1}{\gamma - 1} \frac{\int_t^{a_1} g(s)f(s)ds}{tg(t)} - \frac{c}{(\gamma - 1)g(t)t}, \quad t \in (0, a_1], \quad (3.11)$$

where c is a constant. The condition $(\mathbf{g}_2)$ implies that $g \in \text{NRVZ}_{1-\gamma}$ with $\gamma > 1$. Moreover, by $t \mapsto (-\ln t)^\beta \in \text{NRVZ}_0$ and (3.9), we see that

$$v \in \begin{cases} \text{NRVZ}_0, & \text{if } v(t) = (-\ln t)^\beta, \\ \text{NRVZ}_{-\frac{(\gamma+N)\rho}{N+1}}, & \text{if } v(t) = (\varphi(t))^{-\rho}. \end{cases}$$

We conclude by Proposition 2.7 and Proposition 2.6 that

$$t \mapsto tg(t)(v(t))^{-1} \in \begin{cases} \text{NRVZ}_{1-\gamma}, & \text{if } v(t) = (-\ln t)^\beta, \\ \text{NRVZ}_{\frac{(1-\gamma)(N+1)+(\gamma+N)\rho}{N+1}}, & \text{if } v(t) = (\varphi(t))^{-\rho}. \end{cases}$$

So, we can take $L \in NRVZ_0$ such that

$$tg(t)(v(t))^{-1} = t^{\zeta}L(t)$$

with

$$\zeta = \begin{cases} 1 - \gamma, & \text{if } v(t) = (-\ln t)^{\beta}, \\ \frac{(1-\gamma)(N+1)+(\gamma+N)\rho}{N+1}, & \text{if } v(t) = (\varphi(t))^{-\rho}. \end{cases}$$

It's clear that

$$\begin{cases} \zeta < 0, & \text{if } \gamma > 1 \text{ in } (\mathbf{g}_2) \text{ and } v(t) = (-\ln t)^{\beta}, \\ \zeta < 0, & \text{if } \gamma > \frac{(1+\rho)N+1}{N+1-\rho} \text{ in } (\mathbf{g}_2) \text{ and } v(t) = (\varphi(t))^{-\rho}. \end{cases}$$

By Proposition 2.5 (ii), we have

$$\lim_{t \rightarrow 0^+} tg(t)(v(t))^{-1} = \infty,$$

i.e.,

$$\lim_{t \rightarrow 0^+} -\frac{c}{(\gamma-1)tg(t)(v(t))^{-1}} = 0. \quad (3.12)$$

Combining with (3.6)-(3.10), by the l'Hospital's rule, we can obtain

$$\begin{aligned} & \lim_{t \rightarrow 0^+} v(t) \frac{-\int_t^{a_1} g(s)f(s)ds}{(\gamma-1)tg(t)} \\ &= \lim_{t \rightarrow 0^+} \frac{\int_{a_1}^t g(s)f(s)ds}{(\gamma-1)tg(t)(v(t))^{-1}} \\ &= \lim_{t \rightarrow 0^+} \frac{1}{\gamma-1} \frac{g(t)f(t)}{g(t)(v(t))^{-1} + tg'(t)(v(t))^{-1} + tg(t)[(v(t))^{-1}]'} \\ &= \lim_{t \rightarrow 0^+} \frac{1}{\gamma-1} \frac{v(t)f(t)}{1 + \frac{tg'(t)}{g(t)} + \frac{t[(v(t))^{-1}]'}{(v(t))^{-1}}} \\ &= \begin{cases} \lim_{t \rightarrow 0^+} \frac{(-\ln s)^{\beta}f(s)}{-(\gamma-1)^2}, & \text{if } \gamma > 1 \text{ in } (\mathbf{g}_2) \text{ and } v(t) = (-\ln t)^{\beta} \\ \lim_{t \rightarrow 0^+} \frac{(N+1)(\varphi(t))^{-\rho}f(t)}{(\gamma-1)((1-\gamma)(N+1)+(\gamma+N)\rho)}, & \text{if } \gamma > \frac{(1+\rho)N+1}{N+1-\rho} \text{ in } (\mathbf{g}_2) \text{ and } v(t) = (\varphi(t))^{-\rho} \end{cases} \\ &= \chi_2. \end{aligned} \quad (3.13)$$

(3.11)-(3.13) imply that

$$\begin{aligned} & \lim_{t \rightarrow 0^+} v(t) \left(\frac{G(t)}{tg(t)} - \frac{1}{\gamma-1} \right) \\ &= -\lim_{t \rightarrow 0^+} \frac{\int_t^{a_1} g(s)f(s)ds}{(\gamma-1)tg(t)(v(t))^{-1}} - \lim_{t \rightarrow 0^+} \frac{c}{(\gamma-1)tg(t)(v(t))^{-1}} = \chi_2. \end{aligned}$$

(iii) By Lemma 3.3 (iv), Proposition 2.7 and Proposition 2.6, we have

$$t \mapsto (v(t))^{-1}\varphi(t) \in \begin{cases} NRVZ_{\frac{(\gamma+N)\rho}{N+1}}, & \text{if } v(t) = (-\ln t)^{\beta}, \\ NRVZ_{\frac{(\gamma+N)(\rho+1)}{N+1}}, & \text{if } v(t) = (\varphi(t))^{-\rho}. \end{cases}$$

This implies that

$$(v(t))^{-1}\varphi(t) \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

Moreover, it follows by Lemma 3.3 (v) that

$$\frac{((N+1)G(t))^{1/(N+1)}}{g(t)} - \frac{\gamma+N}{\gamma-1}\varphi(t) \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

If $v(t) = (-\ln t)^\beta$, then we have by the l'Hospital's rule, (3.5) and (i)-(ii) that

$$\begin{aligned}
 & \lim_{t \rightarrow 0^+} v(t) \left(\frac{((N+1)G(t))^{N/(N+1)}}{g(t)\varphi(t)} - \frac{\gamma+N}{\gamma-1} \right) \\
 &= \lim_{t \rightarrow 0^+} \frac{\frac{((N+1)G(t))^{N/(N+1)}}{g(t)} - \frac{\gamma+N}{\gamma-1} \varphi(t)}{(-\ln t)^{-\beta} \varphi(t)} \\
 &= \lim_{t \rightarrow 0^+} \frac{\frac{-N((N+1)G(t))^{-1/(N+1)} g^2(t) - ((N+1)G(t))^{N/(N+1)} g'(t)}{g^2(t)} - \frac{\gamma+N}{\gamma-1} \varphi'(t)}{(-\ln t)^{-\beta} \varphi'(t) + \beta(-\ln t)^{-\beta-1} \varphi(t)t^{-1}} \\
 &= \lim_{t \rightarrow 0^+} -(N+1)(-\ln t)^\beta \left(\frac{G(t)}{tg(t)} \frac{tg'(t)}{g(t)} + \frac{\gamma}{\gamma-1} \right) \\
 &= \lim_{t \rightarrow 0^+} -(N+1)(-\ln t)^\beta \left[\left(\frac{G(t)}{tg(t)} - \frac{1}{\gamma-1} \right) \left(\frac{tg'(t)}{g(t)} + \gamma \right) - \gamma \left(\frac{G(t)}{tg(t)} - \frac{1}{\gamma-1} \right) \right. \\
 &\quad \left. + \frac{1}{\gamma-1} \left(\frac{tg'(t)}{g(t)} + \gamma \right) \right] = \begin{cases} -\frac{(N+1)\sigma}{(\gamma-1)^2}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3)\text{-(}\mathbf{g}_4\text{) hold with } \theta = 0 \text{ in } (\mathbf{g}_3), \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds (or } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta > 0). \end{cases}
 \end{aligned} \tag{3.14}$$

If $v(t) = (\varphi(t))^{-\rho}$, then we have by the l'Hospital's rule, (3.5) and (i)-(ii) that

$$\begin{aligned}
 & \lim_{t \rightarrow 0^+} v(t) \left(\frac{((N+1)G(t))^{N/(N+1)}}{g(t)\varphi(t)} - \frac{\gamma+N}{\gamma-1} \right) \\
 &= \lim_{t \rightarrow 0^+} \frac{\frac{((N+1)G(t))^{N/(N+1)}}{g(t)} - \frac{\gamma+N}{\gamma-1} \varphi(t)}{(\varphi(t))^{\rho+1}} \\
 &= \lim_{t \rightarrow 0^+} \frac{\frac{-N((N+1)G(t))^{-1/(N+1)} g^2(t) - ((N+1)G(t))^{N/(N+1)} g'(t)}{g^2(t)} - \frac{\gamma+N}{\gamma-1} \varphi'(t)}{(\rho+1)(\varphi(t))^\rho \varphi'(t)} \\
 &= \lim_{t \rightarrow 0^+} -\frac{N+1}{\rho+1} (\varphi(t))^{-\rho} \left(\frac{G(t)}{tg(t)} \frac{tg'(t)}{g(t)} + \frac{\gamma}{\gamma-1} \right) \\
 &= \lim_{t \rightarrow 0^+} \frac{N+1}{\rho+1} (\varphi(t))^{-\rho} \left[-\left(\frac{G(t)}{tg(t)} - \frac{1}{\gamma-1} \right) \left(\frac{tg'(t)}{g(t)} + \gamma \right) + \gamma \left(\frac{G(t)}{tg(t)} - \frac{1}{\gamma-1} \right) \right. \\
 &\quad \left. - \frac{1}{\gamma-1} \left(\frac{tg'(t)}{g(t)} + \gamma \right) \right] = \begin{cases} 0, & \text{if } (\mathbf{S}_1) \text{ holds and } \gamma > \frac{(1+\rho)N+1}{N+1-\rho} \text{ in } (\mathbf{g}_2), \\ 0, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \frac{\theta(N+1)}{\rho} - N > \gamma > \frac{(1+\rho)N+1}{N+1-\rho}. \end{cases}
 \end{aligned} \tag{3.15}$$

(3.14)-(3.15) imply that (iii) holds.

(iv) When $(\mathbf{S}_1)$ holds in $(\mathbf{g}_2)$, we have

$$\frac{g(\xi t)}{\xi^N g(t)} = \xi^{-(\gamma+N)}, \quad t \in (0, a_1].$$

In this case, for $v(t) = (-\ln t)^\beta$ or $v(t) = (\varphi(t))^{-\rho}$, we obtain

$$\lim_{t \rightarrow 0^+} v(t) \left(\frac{g(\xi t)}{\xi^N g(t)} - \xi^{-(\gamma+N)} \right) = 0. \tag{3.16}$$

Now, we investigate the case that $(\mathbf{S}_2)$ holds in $(\mathbf{g}_2)$. If $\xi = 1$, then the result is obvious. If $\xi \neq 1$, then we have

$$\frac{g(\xi t)}{\xi^N g(t)} - \xi^{-(\gamma+N)} = \xi^{-(\gamma+N)} \left(\exp \left(\int_{\xi t}^t \frac{f(s)}{s} ds \right) - 1 \right).$$

It follows by Proposition 2.2 that

$$\lim_{t \rightarrow 0^+} \frac{f(st)}{s} = 0 \text{ and } \lim_{t \rightarrow 0^+} \frac{f(st)}{sf(t)} = s^{\theta-1}$$

uniformly with respect to $s \in [1, \xi]$ or $s \in [\xi, 1]$. So, we have

$$\lim_{t \rightarrow 0^+} \int_{\xi t}^t \frac{f(\tau)}{\tau} d\tau \stackrel{\tau := ts}{=} \lim_{t \rightarrow 0^+} \int_{\xi}^1 \frac{f(ts)}{s} ds = 0$$

and

$$\lim_{t \rightarrow 0^+} \int_{\xi}^1 \frac{f(st)}{sf(t)} ds = \int_{\xi}^1 s^{\theta-1} ds = \begin{cases} -\ln \xi, & \text{if } (\mathbf{g}_3) \text{ holds with } \theta = 0, \\ \frac{1-\xi^{\theta}}{\theta}, & \text{if } (\mathbf{g}_3) \text{ holds with } \theta > 0. \end{cases}$$

Since $e^r - 1 \sim r$ as $r \rightarrow 0^+$, we conclude by (3.6), (3.8)-(3.10) that

$$\begin{aligned} & \lim_{t \rightarrow 0^+} v(t) \left(\frac{g(\xi t)}{\xi^N g(t)} - \xi^{-(\gamma+N)} \right) \\ &= \lim_{t \rightarrow 0^+} \xi^{-(\gamma+N)} v(t) f(t) \int_{\xi}^1 \frac{f(st)}{sf(t)} ds \\ &= \begin{cases} -\frac{\sigma \ln \xi}{\xi^{\gamma+N}}, & \text{if } (\mathbf{g}_3)\text{-(}\mathbf{g}_4\text{) hold with } \theta = 0 \text{ in } (\mathbf{g}_3), v(t) = (-\ln t)^{\beta}, \\ 0, & \text{if } (\mathbf{g}_3) \text{ hold with } \theta > 0, v(t) = (-\ln t)^{\beta}, \\ 0, & \text{if } (\mathbf{g}_3) \text{ hold with } \theta(N+1) > (\gamma+N)\rho, v(t) = (\varphi(t))^{-\rho}. \end{cases} \end{aligned}$$

This, combined with (3.16), shows that (iv) holds. $\square$

Lemma 3.5. (Lemma 2.3 in [48]) Let $\tilde{L} \in \mathcal{L}$, then

$$\lim_{t \rightarrow 0^+} \frac{\tilde{L}(t)}{\int_t^{t_0} \frac{\tilde{L}(s)}{s} ds} = 0.$$

If further $\int_0^{t_0} \frac{\tilde{L}(s)}{s} ds < \infty$, then

$$\lim_{t \rightarrow 0^+} \frac{\tilde{L}(t)}{\int_0^t \frac{\tilde{L}(s)}{s} ds} = 0.$$

Lemma 3.6. Suppose that g satisfies $(\mathbf{g}_1)$ -($\mathbf{g}_2$). In particular, if $(\mathbf{S}_2)$ holds in $(\mathbf{g}_2)$, we suppose that $(\mathbf{g}_3)$ holds, and if $\theta = 0$ in $(\mathbf{g}_3)$, we further suppose that $(\mathbf{g}_4)$ holds. ψ is uniquely determined by (1.4). Then,

(i) $\psi'(s) = ((N+1)G(\psi(t)))^{\frac{1}{N+1}}$, $\psi(t) > 0$, $\psi(0) = 0$ and

$$-\psi''(t) = ((N+1)G(\psi(t)))^{\frac{1-N}{N+1}} g(\psi(t)), \quad t \in (0, a_1);$$

(ii) $-(\psi'(t))^{N-1} \psi''(t) = g(\psi(t))$;

(iii) $\lim_{t \rightarrow 0^+} \frac{t\psi'(t)}{\psi(t)} = \frac{N+1}{\gamma+N}$, i.e., $\psi \in \text{NRVZ}_{\frac{N+1}{\gamma+N}}$;

(iv) $\lim_{t \rightarrow 0^+} \frac{\psi'(t)}{t\psi''(t)} = -\frac{\gamma+N}{\gamma-1}$, i.e., $\psi' \in \text{NRVZ}_{-\frac{\gamma-1}{\gamma+N}}$;

(v) if $k \in \Lambda$, then $\lim_{t \rightarrow 0^+} \frac{\ln t}{\ln \psi(K(t))} = \frac{(\gamma+N)D_k}{N+1}$;

(vi) if $\tilde{L} \in \mathcal{L}$ and (1.3) holds, then $\lim_{t \rightarrow 0^+} \frac{\ln \psi \left(\left(\int_0^t \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right)}{\ln t} = 0$;

(vii)

$$\begin{aligned} & \lim_{t \rightarrow 0^+} (-\ln \psi(t))^{\beta} \left(\frac{\psi'(t)}{t\psi''(t)} + \frac{\gamma+N}{\gamma-1} \right) \\ &= \begin{cases} \frac{(N+1)\sigma}{(\gamma-1)^2}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3)\text{-(}\mathbf{g}_4\text{) hold with } \theta = 0 \text{ in } (\mathbf{g}_3), \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds (or } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta > 0); \end{cases} \end{aligned}$$

(viii) let $\rho \in (0, 1]$, then

$$\begin{aligned} & \lim_{t \rightarrow 0^+} t^{-\rho} \left(\frac{\psi'(t)}{t\psi''(t)} + \frac{\gamma + N}{\gamma - 1} \right) \\ &= \begin{cases} 0, & \text{if } (\mathbf{S}_1) \text{ holds and } \gamma > \frac{(1+\rho)N+1}{N+1-\rho} \text{ in } (\mathbf{g}_2), \\ 0, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \frac{\theta(N+1)}{\rho} - N > \gamma > \frac{(1+\rho)N+1}{N+1-\rho}; \end{cases} \end{aligned}$$

(ix) let ξ be a positive constant, then

$$\begin{aligned} & \lim_{t \rightarrow 0^+} (-\ln \psi(t))^\beta \left(\frac{g(\xi\psi(t))}{\xi^N g(\psi(t))} - \xi^{-(\gamma+N)} \right) \\ &= \begin{cases} -\sigma \xi^{-(\gamma+N)} \ln \xi, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3)\text{-(g}_4) \text{ hold with } \theta = 0 \text{ in } (\mathbf{g}_3), \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds (or } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta > 0); \end{cases} \end{aligned}$$

(x) let $(\mathbf{S}_1)$ hold (or $(\mathbf{S}_2)$ and $(\mathbf{g}_3)$ hold with $\theta(N+1) > (\gamma+N)\rho$), ξ be a positive constant and $\rho \in (0, 1]$, then

$$\lim_{t \rightarrow 0^+} t^{-\rho} \left(\frac{g(\xi\psi(t))}{\xi^N g(\psi(t))} - \xi^{-(\gamma+N)} \right) = 0.$$

Proof. (i)-(ii) By the definition of ψ , we obtain that (i)-(ii) hold.

(iii)-(iv) It follows by (i) and Lemma 3.3 (iv)-(v) that

$$\lim_{t \rightarrow 0^+} \frac{t\psi'(t)}{\psi(t)} = \lim_{t \rightarrow 0^+} \frac{t((N+1)G(\psi(t)))^{\frac{1}{N+1}}}{\psi(t)} \stackrel{z:=\psi(t)}{=} \lim_{z \rightarrow 0^+} \frac{\varphi(z)((N+1)G(z))^{\frac{1}{N+1}}}{z} = \frac{N+1}{\gamma+N}$$

and

$$\lim_{t \rightarrow 0^+} \frac{\psi'(t)}{t\psi''(t)} = - \lim_{z \rightarrow 0^+} \frac{((N+1)G(z))^{\frac{N}{N+1}}}{g(z)\varphi(z)} = -\frac{\gamma+N}{\gamma-1}.$$

(v) We conclude by (iii), Lemma 3.1 (ii) and Lemma 2.5 (iii) that (v) holds.

(vi) By using Lemma 3.5, we have

$$\lim_{t \rightarrow 0^+} \frac{t \left(\left(\int_0^t \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right)'}{\left(\int_0^t \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}}} = \frac{N}{N+1} \lim_{t \rightarrow 0^+} \frac{\tilde{L}(t)}{\int_0^t \frac{\tilde{L}(s)}{s} ds} = 0.$$

This implies that

$$t \mapsto \left(\int_0^t \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \in NRVZ_0. \quad (3.17)$$

So, by (3.17), (iii) and Proposition 2.7 we further obtain

$$t \mapsto \psi \left(\left(\int_0^t \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right) \in NRVZ_0. \quad (3.18)$$

We have by Proposition 2.5 (iii) that (vi) holds.

(vii)-(viii) It follows by (i) and Lemma 3.4 (iii) that

$$\begin{aligned} & \lim_{t \rightarrow 0^+} (-\ln \psi(t))^\beta \left(\frac{\psi'(t)}{t\psi''(t)} + \frac{\gamma+N}{\gamma-1} \right) \\ &= \lim_{t \rightarrow 0^+} (-\ln \psi(t))^\beta \left(-\frac{((N+1)G(\psi(t)))^{\frac{N}{N+1}}}{g(\psi(t))t} + \frac{\gamma+N}{\gamma-1} \right) \\ &\stackrel{z:=\psi(t)}{=} \lim_{z \rightarrow 0^+} (-\ln z)^\beta \left(-\frac{((N+1)G(z))^{\frac{N}{N+1}}}{g(z)\varphi(z)} + \frac{\gamma+N}{\gamma-1} \right) \\ &= \begin{cases} \frac{(N+1)\sigma}{(\gamma-1)^2}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3)\text{-(g}_4) \text{ hold with } \theta = 0 \text{ in } (\mathbf{g}_3), \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds (or } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta > 0), \end{cases} \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{-\rho} \left(\frac{\psi'(t)}{t\psi''(t)} + \frac{\gamma + N}{\gamma - 1} \right) &= \lim_{t \rightarrow 0^+} t^{-\rho} \left(- \frac{((N+1)G(\psi(t)))^{\frac{N}{N+1}}}{g(\psi(t))t} + \frac{\gamma + N}{\gamma - 1} \right) \\ &\stackrel{z:=\psi(t)}{=} \lim_{z \rightarrow 0^+} (\varphi(z))^{-\rho} \left(- \frac{((N+1)G(z))^{\frac{N}{N+1}}}{g(z)\varphi(z)} + \frac{\gamma + N}{\gamma - 1} \right) \\ &= \begin{cases} 0, & \text{if } (\mathbf{S}_1) \text{ holds and } \gamma > \frac{(1+\rho)N+1}{N+1-\rho} \text{ in } (\mathbf{g}_2), \\ 0, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \frac{\theta(N+1)}{\rho} - N > \gamma > \frac{(1+\rho)N+1}{N+1-\rho}. \end{cases} \end{aligned}$$

(ix)-(x) It follows by Lemma 3.4 (iv) that

$$\begin{aligned} &\lim_{t \rightarrow 0^+} (-\ln \psi(t))^\beta \left(\frac{g(\xi\psi(t))}{\xi^N g(\psi(t))} - \xi^{-(\gamma+N)} \right) \\ &\stackrel{z:=\psi(t)}{=} \lim_{z \rightarrow 0^+} (-\ln z)^\beta \left(\frac{g(\xi z)}{\xi^N g(z)} - \xi^{-(\gamma+N)} \right) \\ &= \begin{cases} -\sigma \xi^{-(\gamma+N)} \ln \xi, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3)\text{-(}\mathbf{g}_4) \text{ hold with } \theta = 0 \text{ in } (\mathbf{g}_3), \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds (or } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta > 0). \end{cases} \end{aligned}$$

Moreover, if $(\mathbf{S}_1)$ holds or $(\mathbf{S}_2)$ and $(\mathbf{g}_3)$ hold with $\theta(N+1) > (\gamma+N)\rho$, then by Lemma 3.4 (iv), we have

$$\lim_{t \rightarrow 0^+} t^{-\rho} \left(\frac{g(\xi\psi(t))}{\xi^N g(\psi(t))} - \xi^{-(\gamma+N)} \right) \stackrel{z:=\psi(t)}{=} \lim_{z \rightarrow 0^+} (\varphi(z))^{-\rho} \left(\frac{g(\xi z)}{\xi^N g(z)} - \xi^{-(\gamma+N)} \right) = 0.$$

□

4 The Second Boundary Behavior

In this section, we prove Theorems 1.1-1.3. We first introduce some lemmas as follows.

Lemma 4.1. (Lemma 2.1 in [27]) Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$, and let $u, v \in C(\bar{\Omega}) \cap C^2(\Omega)$. Suppose $h(x, t)$ is defined for $x \in \Omega$ and t in some interval containing the ranges of u and v . If the following hold:

- (i) h is strictly increasing in t for all $x \in \Omega$,
- (ii) the matrix D^2v is positive definite in Ω ,
- (iii) $\det(D^2v) \geq h(x, v)$ and $\det(D^2u) \leq h(x, u)$, $x \in \Omega$,
- (iv) $u \geq v$ on $\partial\Omega$,

then, we have $u \geq v$ in Ω .

For any $\delta > 0$, let

$$\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}.$$

Since Ω is C^m -smooth for $m \geq 2$, we can always take $\delta_1 > 0$ such that (see Lemmas 14.16 and 14.17 in [18])

$$d \in C^m(\Omega_{\delta_1}) \text{ and } |\nabla d(x)| = 1, \quad x \in \Omega_{\delta_1}.$$

Let $\bar{x} \in \partial\Omega$ be the projection of the point $x \in \Omega_{\delta_1}$ to $\partial\Omega$, and $\kappa_i(\bar{x})$ ($i = 1, \dots, N-1$) be the principal curvatures of $\partial\Omega$ at $\bar{x}$, then

$$D^2(d(x)) = \text{diag} \left(\frac{-\kappa_1(\bar{x})}{1 - d(x)\kappa_1(\bar{x})}, \dots, \frac{-\kappa_{N-1}(\bar{x})}{1 - d(x)\kappa_{N-1}(\bar{x})}, 0 \right).$$

Lemma 4.2. (See the proof of Proposition 2.4 in [27], Proposition 2.1 and Corollary 2.3 in [9]) Let h be a C^2 -function on $(0, \delta_1)$, then

$$\det(D^2 h(d(x))) = (-h'(d(x)))^{N-1} h''(d(x)) \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})}, \quad x \in \Omega_{\delta_1}.$$

Lemma 4.3. (Corollary 2.3 in [20]) Let h be a C^2 -function on $(0, \delta_1)$ and Ω be a bounded domain with $\partial\Omega \in C^m$ for $m \geq 2$, then, for $i = 1, \dots, N$,

$$S_i(D^2(h(d(x)))) = (-h'(d(x)))^i S_i(\epsilon_1, \dots, \epsilon_{N-1}) + (-h'(d(x)))^{i-1} h''(d(x)) S_{i-1}(\epsilon_1, \dots, \epsilon_{N-1}), \quad x \in \Omega_{\delta_1}$$

where

$$\epsilon_j = \frac{\kappa_j(\bar{x})}{1 - \kappa_j(\bar{x})d(x)}, \quad j = 1, \dots, N-1.$$

Definition 4.4. (Definition 1.1 in [42]) Let $i \in \{1, \dots, N\}$ and Ω be an open bounded subset of $\mathbb{R}^N$; a function $u \in C^2(\Omega)$ is (strictly) i -convex if $S_l(D^2 u)(\cdot) \geq 0$ in Ω for $l = 1, \dots, i$. In particular, if $i = N$, then we say that u is (strictly) convex in Ω .

4.1 Proof of Theorem 1.1

Next, we prove Theorem 1.1 and we first show some preliminaries as follows.

Fix $\varepsilon > 0$ and let

$$w_{\pm}(d(x)) = \xi_{\pm} \psi(K(d(x))) (1 + (C_{\pm} \pm \varepsilon)(-\ln d(x))^{-\beta}), \quad x \in \Omega_{\delta_1},$$

where $\xi_{\pm}$ and $C_{\pm}$ are in Theorem 1.1. By the Lagrange's mean value theorem, we obtain that there exist $\lambda_{\pm} \in (0, 1)$ and

$$\Theta_{\pm}(d(x)) = \xi_{\pm} \psi(K(d(x))) (1 + \lambda_{\pm}(C_{\pm} \pm \varepsilon)(-\ln d(x))^{-\beta}) \quad (4.1)$$

such that for any $x \in \Omega_{\delta_1}$

$$g(w_{\pm}(d(x))) = g(\xi_{\pm} \psi(K(d(x)))) + \xi_{\pm} \psi(K(d(x))) g'(\Theta_{\pm}(d(x))) (C_{\pm} \pm \varepsilon)(-\ln d(x))^{-\beta}.$$

Since $g \in \text{NRVZ}_{-\gamma}$, we have by Proposition 2.2 that

$$\lim_{d(x) \rightarrow 0} \frac{g(\xi_{\pm} \psi(K(d(x))))}{g(\Theta_{\pm}(d(x)))} = \lim_{d(x) \rightarrow 0} \frac{g'(\xi_{\pm} \psi(K(d(x))))}{g'(\Theta_{\pm}(d(x)))} = 1. \quad (4.2)$$

Moreover, by the definitions of $\xi_{\pm}$ and $C_{\pm}$, we can take a sufficiently small positive constant still denoted by δ_1 such that

$$w_{+}(d(x)) \geq w_{-}(d(x)), \quad x \in \Omega_{\delta_1}. \quad (4.3)$$

Proof. Our proof is divided into two steps and the outline of the proof is given as below.

- In Step 1, for fixed $\varepsilon > 0$ and $x \in \Omega_{\delta_1}$, we first give some functions $I_{1\pm}(d(x))$, $I_{2\pm}(d(x))$ and $I_{3\pm}(d(x))$ (which are corresponding to $\varepsilon > 0$), and then by detailed calculation we will show that there exists a sufficiently small positive constant $\delta_{\varepsilon} < \delta_1$ such that

$$I_{1+}(d(x)) + I_{2+}(d(x)) + I_{3+}(d(x)) > 0, \quad x \in \Omega_{\delta_{\varepsilon}} \quad (4.4)$$

and

$$I_{1-}(d(x)) + I_{2-}(d(x)) + I_{3-}(d(x)) < 0, \quad x \in \Omega_{\delta_{\varepsilon}}. \quad (4.5)$$

- In Step 2, we will define two functions $\underline{u}_{\varepsilon}$, $\bar{u}_{\varepsilon}$ in $\Omega_{\delta_{\varepsilon}}$ and show they are sub- and super-solutions of Eq. (1.1) in $\Omega_{\delta_{\varepsilon}}$ by (4.4) and (4.5), respectively. In particular, we will show $\underline{u}_{\varepsilon}$ is strictly convex in $\Omega_{\delta_{\varepsilon}}$. Finally, we will establish the second boundary behavior of the unique strictly convex solution to problem (1.1) by using Lemma 4.1.

By the above analysis, we see that how to get (4.4)-(4.5) is the key of the research.

Step 1. We first define functions $I_{1\pm}(d(x))$, $I_{2\pm}(d(x))$ and $I_{3\pm}(d(x))$ as follows.

For fixed $\varepsilon > 0$ and $\forall x \in \Omega_{\delta_1}$, we define

$$\begin{aligned}
 I_{1\pm}(d(x)) &= (-\ln d(x))^\beta \left[\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \hat{m}_\pm \right. \\
 &\quad \left. - \frac{g(\xi_\pm \psi(K(d(x))))}{\xi_\pm^N g(\psi(K(d(x))))} \right]; \\
 I_{2\pm}(d(x)) &= C_\pm \left[N \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \mathcal{A}_\pm \prod_{i=1}^{N-1} (1 - d(x)\kappa_i(\bar{x}))^{-1} \right. \\
 &\quad \left. - \frac{\xi_\pm \psi(K(d(x)))g'(\xi_\pm \psi(K(d(x))))}{g(\xi_\pm \psi(K(d(x))))} \frac{g'(\Theta_\pm(d(x)))}{g'(\xi_\pm \psi(K(d(x))))} \frac{g(\xi_\pm \psi(K(d(x))))}{\xi_\pm^N g(\psi(K(d(x))))} \right] \\
 &\quad \pm \varepsilon \left[N \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \hat{m}_\pm \prod_{i=1}^{N-1} (1 - d(x)\kappa_i(\bar{x}))^{-1} \right. \\
 &\quad \left. - \frac{\xi_\pm \psi(K(d(x)))g'(\xi_\pm \psi(K(d(x))))}{g(\xi_\pm \psi(K(d(x))))} \frac{g'(\Theta_\pm(d(x)))}{g'(\xi_\pm \psi(K(d(x))))} \frac{g(\xi_\pm \psi(K(d(x))))}{\xi_\pm^N g(\psi(K(d(x))))} \right]; \\
 I_{3\pm}(d(x)) &= (-\ln d(x))^\beta \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \\
 &\quad \times \frac{\hat{m}_\pm \sum_{i=1}^{N-1} C_{N-1}^i (-1)^{i+1} (m_\pm d(x))^i}{(1 - d(x)m_\pm)^{N-1}} + \frac{\zeta_\pm(d(x))}{(-\ln d(x))^{-\beta}} \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \\
 &\quad - (B_0 \pm \varepsilon) \frac{g(\xi_\pm \psi(K(d(x))))}{\xi_\pm^N g(\psi(K(d(x))))} (d(x))^\mu (-\ln d(x))^\beta \\
 &\quad - (B_0 \pm \varepsilon)(C_\pm \pm \varepsilon) \frac{\xi_\pm \psi(K(d(x)))g'(\xi_\pm \psi(K(d(x))))}{g(\xi_\pm \psi(K(d(x))))} \frac{g'(\Theta_\pm(d(x)))}{g'(\xi_\pm \psi(K(d(x))))} \frac{g(\xi_\pm \psi(K(d(x))))}{\xi_\pm^N g(\psi(K(d(x))))} (d(x))^\mu,
 \end{aligned}$$

where

$$\left\{ \begin{array}{l} (\mathbf{S}_2) \text{ holds and } \theta = 0 \\ (\mathbf{S}_1) \text{ holds} \\ (\mathbf{S}_2) \text{ holds and } \theta > 0 \end{array} \right\} \left\{ \begin{array}{l} k \in \Lambda_1 \\ k \in \Lambda_{2,\beta} \\ \left\{ \begin{array}{l} k \in \Lambda_1, \\ k \in \Lambda_{2,\beta}, \end{array} \right. \end{array} \right. \left\{ \begin{array}{l} \mathcal{A}_\pm = \hat{m}_\pm, \quad \text{if } A_+ \leq 0 \text{ and } A_- \leq 0, \\ \mathcal{A}_\pm = \hat{m}_\mp, \quad \text{if } A_+ \geq 0 \text{ and } A_- \geq 0, \\ \mathcal{A}_+ = \mathcal{A}_- = \hat{m}_-, \quad \text{if } A_+ > 0 \text{ and } A_- < 0, \\ \mathcal{A}_+ = \mathcal{A}_- = \hat{m}_+, \quad \text{if } A_+ < 0 \text{ and } A_- > 0; \\ \mathcal{A}_\pm = \hat{m}_\pm, \quad \text{if } \mathfrak{A}_+ + \mathfrak{B} \leq 0 \text{ and } \mathfrak{A}_+ + \mathfrak{B} \leq 0, \\ \mathcal{A}_\pm = \hat{m}_\mp, \quad \text{if } \mathfrak{A}_+ + \mathfrak{B} \geq 0 \text{ and } \mathfrak{A}_+ + \mathfrak{B} \geq 0, \\ \mathcal{A}_+ = \mathcal{A}_- = \hat{m}_-, \quad \text{if } \mathfrak{A}_+ + \mathfrak{B} > 0 \text{ and } \mathfrak{A}_+ + \mathfrak{B} < 0, \\ \mathcal{A}_+ = \mathcal{A}_- = \hat{m}_+, \quad \text{if } \mathfrak{A}_+ + \mathfrak{B} < 0 \text{ and } \mathfrak{A}_+ + \mathfrak{B} > 0; \\ \mathcal{A}_\pm = \hat{m}_\pm; \\ \left\{ \begin{array}{l} \mathcal{A}_+ = \hat{m}_-, \quad \text{if } \mathfrak{B} \geq 0, \\ \mathcal{A}_- = \hat{m}_+, \quad \text{if } \mathfrak{B} < 0; \end{array} \right. \\ \left\{ \begin{array}{l} \mathcal{A}_+ = \hat{m}_-, \quad \text{if } \mathfrak{B} \geq 0, \\ \mathcal{A}_- = \hat{m}_+, \quad \text{if } \mathfrak{B} < 0, \end{array} \right. \end{array} \right.$$

$$m_+ := \min\{\min_{\bar{x} \in \Omega} \kappa_i(\bar{x}) : i = 1, \dots, N-1\} \text{ and } m_- := \max\{\max_{\bar{x} \in \partial\Omega} \kappa_i(\bar{x}) : i = 1, \dots, N-1\} \quad (4.6)$$

and

$$\begin{aligned}
 \zeta_\pm(d(x)) &= (C_\pm \pm \varepsilon)^2 (N-1) \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) (-\ln d(x))^{-2\beta} \\
 &\quad + 2\beta(C_\pm \pm \varepsilon) \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{k(d(x))d(x)} (-\ln d(x))^{-\beta-1} \\
 &\quad \times (1 + (C_\pm \pm \varepsilon)(N-1)(-\ln d(x))^{-\beta} + v_\pm(d(x)))
 \end{aligned}$$

$$\begin{aligned}
& -\beta(C_{\pm} \pm \varepsilon) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 (-\ln d(x))^{-\beta-1} \\
& \times (1 - (\beta + 1)(-\ln d(x))^{-1}) (1 + (C_{\pm} \pm \varepsilon)(N - 1)(-\ln d(x))^{-\beta} + v_{\pm}(d(x))) \\
& + v_{\pm}(d(x)) \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) (1 + (C_{\pm} \pm \varepsilon)(-\ln d(x))^{-\beta}),
\end{aligned}$$

where $A_{\pm}$ are given by (1.8), $\mathfrak{A}_{\pm}$ and $\mathfrak{B}$ are given by (1.11) and

$$v_{\pm}(d(x)) = \mathfrak{R}_{2\pm}(d(x)) + \sum_{i=2}^{N-1} C_{N-1}^i (\mathfrak{R}_{1\pm}(d(x)) + \mathfrak{R}_{2\pm}(d(x)))^i,$$

where

$$C_{N-1}^i = \frac{(N-1)!}{i!(N-1-i)!}, \quad \mathfrak{R}_{1\pm}(d(x)) = (C_{\pm} \pm \varepsilon)(-\ln d(x))^{-\beta}, \quad (4.7)$$

and

$$\mathfrak{R}_{2\pm}(d(x)) = (N-1)\beta(C_{\pm} \pm \varepsilon) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{k(d(x))d(x)} (-\ln d(x))^{-\beta-1}.$$

Next, we prove

$$\lim_{d(x) \rightarrow 0} (I_{1\pm}(d(x)) + I_{2\pm}(d(x)) + I_{3\pm}(d(x))) = \pm \varepsilon \left(\gamma + N \frac{\hat{m}_{\pm}}{\hat{m}_{\pm}} \right) \xi_{\pm}^{-(\gamma+N)}. \quad (4.8)$$

To prove (4.8), we calculate the limits of $I_{1\pm}(d(x))$, $I_{2\pm}(d(x))$ and $I_{3\pm}(d(x))$ as $d(x) \rightarrow 0$.

• First, by Lemma 3.1 (ii), Lemma 3.6 (v), (vii) and (ix), we obtain that

$$\lim_{d(x) \rightarrow 0} I_{1\pm}(d(x)) = \begin{cases} \varpi_{1\pm} + \varpi_{2\pm}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta = 0, k \in \Lambda_1, \\ \varpi_{1\pm} + \varpi_{2\pm} + \frac{(\gamma+N)E_{2,k}\hat{m}_{\pm}}{\gamma-1}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta = 0, k \in \Lambda_{2,\beta}, \\ \frac{\gamma+N}{\gamma-1} E_{2,k} \hat{m}_{\pm}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta > 0, k \in \Lambda_{2,\beta}, \\ \frac{\gamma+N}{\gamma-1} E_{2,k} \hat{m}_{\pm}, & \text{if } (\mathbf{S}_1) \text{ holds, } k \in \Lambda_{2,\beta}, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds, } k \in \Lambda_1, \end{cases} \quad (4.9)$$

where

$$\varpi_{1\pm} = \left(\frac{(\gamma+N)D_k}{N+1} \right)^{\beta} \frac{(N+1)(1-D_k)\sigma\hat{m}_{\pm}}{(\gamma-1)^2} \text{ and } \varpi_{2\pm} = \left(\frac{(\gamma+N)D_k}{N+1} \right)^{\beta} \frac{\sigma \ln \xi_{\pm}}{\xi_{\pm}^{\gamma+N}}.$$

• Second, by (4.2), Lemma 3.1 (ii), Lemma 3.6 (iv) and the choices of $\xi_{\pm}$, $C_{\pm}$ in Theorem 1.1, we obtain

$$\lim_{d(x) \rightarrow 0} I_{2\pm}(d(x)) = C_{\pm} \frac{[(\gamma+N)D_k - (N+1)]N}{\gamma-1} \mathcal{A}_{\pm} + \gamma \xi_{\pm}^{-(\gamma+N)} \pm \varepsilon \left(\gamma + N \frac{\hat{m}_{\pm}}{\hat{m}_{\pm}} \right) \xi_{\pm}^{-(\gamma+N)}. \quad (4.10)$$

• Third, by Lemma 3.1 (ii), Lemma 3.6 (iii)-(iv) and a straightforward calculation, we obtain

$$\lim_{d(x) \rightarrow 0} I_{3\pm}(d(x)) = 0. \quad (4.11)$$

Combining (4.9) and (4.10)-(4.11), we obtain (4.8) holds. By $(\mathbf{b}_1)$ -($\mathbf{b}_2$) and (4.8), we see that there exists a sufficiently small constant $\delta_{\varepsilon} < \delta_1$ such that

$$k^{N+1}(d(x))(1 + (B_0 - \varepsilon)(d(x))^{\mu}) \leq b(x) \leq k^{N+1}(d(x))(1 + (B_0 + \varepsilon)(d(x))^{\mu}), \quad x \in \Omega_{\delta_{\varepsilon}} \quad (4.12)$$

and (4.4)-(4.5) hold.

Step 2. Let

$$\underline{u}_{\varepsilon}(x) = -\xi_{+}\psi(K(d(x)))(1 + (C_{+} + \varepsilon)(-\ln d(x))^{-\beta}), \quad x \in \Omega_{\delta_{\varepsilon}}.$$

Then

$$g(-\underline{u}_{\varepsilon}(x)) = g(\xi_{+}\psi(K(d(x)))) + \xi_{+}\psi(K(d(x)))g'(\Theta_{+}(d(x)))(C_{+} + \varepsilon)(-\ln d(x))^{-\beta}, \quad x \in \Omega_{\delta_{\varepsilon}},$$

where $\Theta_+(d(x))$ is given by (4.1). By Lemma 4.2, we have for any $x \in \Omega_{\delta_\varepsilon}$

$$\begin{aligned}
 & \det(D^2 \underline{u}_\varepsilon(x)) - b(x)g(-\underline{u}_\varepsilon(x)) \geq -\xi_+^N (\psi'(K(d(x))))^{N-1} \psi''(K(d(x))) k^{N+1}(d(x)) \\
 & \times (1 + (C_+ + \varepsilon)(N-1)(-\ln d(x))^{-\beta} + v_+(d(x))) \\
 & \times \left[\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) (1 + (C_+ + \varepsilon)(-\ln d(x))^{-\beta}) + 2\beta(C_+ + \varepsilon) \right. \\
 & \times \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{k(d(x))d(x)} (-\ln d(x))^{-\beta-1} \\
 & - \beta(C_+ + \varepsilon) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 (-\ln d(x))^{-\beta-1} \\
 & \times (1 - (\beta+1)(-\ln d(x))^{-1}) \left. \right] \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} - k^{N+1}(d(x))(1 + (B_0 + \varepsilon)(d(x))^\mu) \\
 & \times [g(\xi_+ \psi(K(d(x)))) + \xi_+ \psi(K(d(x)))g'(\Theta_+(d(x)))(C_+ + \varepsilon)(-\ln d(x))^{-\beta}] \\
 & \geq -\xi_+^N (\psi'(K(d(x))))^{N-1} \psi''(K(d(x))) k^{N+1}(d(x))(-\ln d(x))^{-\beta} \left\{ (-\ln d(x))^\beta \right. \\
 & \times \left[\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \left(\prod_{i=1}^{N-1} \frac{\hat{m}_+}{(1 - d(x)m_+)^{N-1}} - \hat{m}_+ + \hat{m}_+ \right) \right. \\
 & - \frac{g(\xi_+ \psi(K(d(x))))}{\xi_+^N g(\psi(K(d(x))))} \left. \right] + (C_+ + \varepsilon) \left[N \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \right. \\
 & \times \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} - \frac{\xi_+ \psi(K(d(x)))g'(\xi_+ \psi(K(d(x))))}{g(\xi_+ \psi(K(d(x))))} \\
 & \times \frac{g'(\Theta_+(d(x)))}{g'(\xi_+ \psi(K(d(x))))} \frac{g(\xi_+ \psi(K(d(x))))}{\xi_+^N g(\psi(K(d(x))))} \left. \right] + \frac{\zeta_+(d(x))}{(-\ln d(x))^{-\beta}} \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \\
 & - (B_0 + \varepsilon) \frac{g(\xi_+ \psi(K(d(x))))}{\xi_+^N g(\psi(K(d(x))))} (d(x))^\mu (-\ln d(x))^\beta - (B_0 + \varepsilon)(C_+ + \varepsilon) \\
 & \times \frac{\xi_+ \psi(K(d(x)))g'(\xi_+ \psi(K(d(x))))}{g(\xi_+ \psi(K(d(x))))} \frac{g'(\Theta_+(d(x)))}{g'(\xi_+ \psi(K(d(x))))} \frac{g(\xi_+ \psi(K(d(x))))}{\xi_+^N g(\psi(K(d(x))))} (d(x))^\mu \left. \right\} \\
 & \geq -\xi_+^N (\psi'(K(d(x))))^{N-1} \psi''(K(d(x))) k^{N+1}(d(x))(-\ln d(x))^{-\beta} \sum_{i=1}^3 I_{i+}(d(x)) > 0,
 \end{aligned} \tag{4.13}$$

i.e., $\underline{u}_\varepsilon$ is a subsolution of Eq. (1.1) in $\Omega_{\delta_\varepsilon}$. Moreover, it follows by Lemma 4.3 that for $i = 1, \dots, N$

$$\begin{aligned}
 S_i(D^2 \underline{u}_\varepsilon(x)) &= \xi_+^i (\psi'(K(d(x))))^i \left(1 + (C_+ + \varepsilon)(-\ln d(x))^{-\beta} + \beta(C_+ + \varepsilon) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \right. \\
 & \times \frac{K(d(x))}{k(d(x))d(x)} (-\ln d(x))^{-\beta-1} \left. \right)^i k^i(d(x)) S_i(\epsilon_1, \dots, \epsilon_{N-1}) - \xi_+^i (\psi'(K(d(x))))^{i-1} \psi''(K(d(x))) \\
 & \times k^{i+1}(d(x)) \left(1 + (C_+ + \varepsilon)(-\ln d(x))^{-\beta} + \beta(C_+ + \varepsilon) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{k(d(x))d(x)} \right. \\
 & \times (-\ln d(x))^{-\beta-1} \left. \right)^{i-1} \left[\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \right. \\
 & \times (1 + (C_+ + \varepsilon)(-\ln d(x))^{-\beta}) + 2\beta(C_+ + \varepsilon) \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{k(d(x))d(x)} (-\ln d(x))^{-\beta-1} \\
 & - \beta(C_+ + \varepsilon) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 (-\ln d(x))^{-\beta-1} \\
 & \times (1 - (\beta+1)(-\ln d(x))^{-1}) \left. \right] S_{i-1}(\epsilon_1, \dots, \epsilon_{N-1}).
 \end{aligned}$$

This implies that we can adjust the above positive constant δ_ε such that for any $x \in \Omega_{\delta_\varepsilon}$

$$S_i(D^2 \underline{u}_\varepsilon(x)) > 0 \text{ for } i = 1, \dots, N.$$

We obtain by Definition 4.4 that $D^2 \underline{u}_\varepsilon$ is positive definite in $\Omega_{\delta_\varepsilon}$.

Let

$$\bar{u}_\varepsilon(x) = -\xi_- \psi(K(d(x))) (1 + (C_- - \varepsilon)(-\ln d(x))^{-\beta}), \quad x \in \Omega_{\delta_\varepsilon}.$$

By the same calculation as (4.13), we obtain

$$\begin{aligned} & \det(D^2 \bar{u}_\varepsilon(x)) - b(x)g(-\bar{u}_\varepsilon(x)) \\ & \leq -\xi_-^N (\psi'(K(d(x))))^{N-1} \psi''(K(d(x))) k^{N+1}(d(x)) (-\ln d(x))^{-\beta} \sum_{i=1}^3 I_{i-}(d(x)) < 0, \end{aligned}$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of Eq. (1.1) in $\Omega_{\delta_\varepsilon}$.

Let u be the unique strictly convex solution to problem (1.1). Then, there exists a large constant $M > 0$ such that

$$\underline{u}_\varepsilon(x) - Md(x) \leq u(x) \leq \bar{u}_\varepsilon(x) + Md(x), \quad x \in \{x \in \Omega : d(x) = \delta_\varepsilon\}.$$

We assert that

$$u(x) \geq \underline{u}_\varepsilon(x) - Md(x), \quad x \in \Omega_{\delta_\varepsilon} \quad (4.14)$$

and

$$u(x) \leq \bar{u}_\varepsilon(x) + Md(x), \quad x \in \Omega_{\delta_\varepsilon}. \quad (4.15)$$

Since $D^2 \underline{u}_\varepsilon$ is positive definite in $\Omega_{\delta_\varepsilon}$ and $D^2(-Md(x))$ is positive semidefinite in $\Omega_{\delta_\varepsilon}$, we have by the Minkowski inequality that $D^2(\underline{u}_\varepsilon(x) - Md(x))$ is positive definite in $\Omega_{\delta_\varepsilon}$ and

$$\det(D^2(\underline{u}_\varepsilon(x) - Md(x))) \geq \det(D^2 \underline{u}_\varepsilon(x)) \geq b(x)g(-\underline{u}_\varepsilon(x)) \geq b(x)g(-\underline{u}_\varepsilon(x) + Md(x)), \quad x \in \Omega_{\delta_\varepsilon}.$$

Similarly, we have $D^2(u(x) - Md(x))$ is positive definite in $\Omega_{\delta_\varepsilon}$ and

$$\det(D^2 u(x) - Md(x)) \geq \det(D^2 u(x)) = b(x)g(-u(x)) \geq b(x)g(-u(x) + Md(x)).$$

By Lemma 4.1, we see that (4.14)-(4.15) hold. Hence, for any $x \in \Omega_{\delta_\varepsilon}$

$$\begin{aligned} C_+ + \varepsilon + \frac{Md(x)(-\ln d(x))^\beta}{\xi_+ \psi(K(d(x)))} & \geq \left(\frac{-u(x)}{\xi_+ \psi(K(d(x)))} - 1 \right) (-\ln d(x))^\beta; \\ C_- - \varepsilon - \frac{Md(x)(-\ln d(x))^\beta}{\xi_- \psi(K(d(x)))} & \leq \left(\frac{-u(x)}{\xi_- \psi(K(d(x)))} - 1 \right) (-\ln d(x))^\beta. \end{aligned}$$

Since $(\gamma + N)D_k - (N + 1) > 0$, we conclude from Lemma 3.1 (ii), Lemma 3.6 (iii), Proposition 2.7 and Proposition 2.5 (ii) that

$$\begin{aligned} C_+ + \varepsilon & \geq \limsup_{d(x) \rightarrow 0} \left(\frac{-u(x)}{\xi_+ \psi(K(d(x)))} - 1 \right) (-\ln d(x))^\beta; \\ C_- - \varepsilon & \leq \liminf_{d(x) \rightarrow 0} \left(\frac{-u(x)}{\xi_- \psi(K(d(x)))} - 1 \right) (-\ln d(x))^\beta. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, the proof is finished. $\square$

4.2 Proof of Theorem 1.2

Now, we prove Theorem 1.2. For fixed $\varepsilon > 0$, we define

$$w_\pm(d(x)) = \xi_\pm \psi(K(d(x))) (1 + (\tilde{C}_\pm \pm \varepsilon)(d(x))^\mu), \quad x \in \Omega_{\delta_1},$$

where $\xi_{\pm}$ and $\tilde{C}_{\pm}$ are given in Theorem 1.2. By the Lagrange's mean value theorem, we see that there exist $\lambda_{\pm} \in (0, 1)$ and

$$\Theta_{\pm}(d(x)) = \xi_{\pm} \psi(K(d(x))) (1 + \lambda_{\pm} (\tilde{C}_{\pm} \pm \varepsilon)(d(x))^{\mu}) \quad (4.16)$$

such that for any $x \in \Omega_{\delta_1}$

$$g(w_{\pm}(d(x))) = g(\xi_{\pm} \psi(K(d(x)))) + \xi_{\pm} \psi(K(d(x))) g'(\Theta_{\pm}(d(x))) (C_{\pm} \pm \varepsilon)(d(x))^{\mu}.$$

By Proposition 2.2, we see that (4.2) still holds. Moreover, we can adjust $\delta_1 > 0$ such that (4.3) holds here.

Proof. Similar to the proof of Theorem 1.1, the proof is divided into the following two steps.

Step 1. We first define functions $I_{1\pm}(d(x))$, $I_{2\pm}(d(x))$ and $I_{3\pm}(d(x))$ as follows.

For fixed $\varepsilon > 0$ and $\forall x \in \Omega_{\delta_1}$, we define

$$\begin{aligned} I_{1\pm}(d(x)) &= \frac{1}{(d(x))^{\mu}} \left[\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \hat{m}_{\pm} - \frac{g(\xi_{\pm} \psi(K(d(x))))}{\xi_{\pm}^N g(\psi(K(d(x))))} \right]; \\ I_{2\pm}(d(x)) &= \tilde{C}_{\pm} \left\{ \left[\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \right. \right. \\ &\quad \times \left(N + \mu(N-1) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} \right) + \frac{2\mu\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} + \mu \\ &\quad \times (\mu-1) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 \Big] \mathcal{B}_{\pm} \prod_{i=1}^{N-1} (1 - d(x)\kappa_i(\bar{x}))^{-1} \\ &\quad - \frac{\xi_{\pm} \psi(K(d(x))) g'(\xi_{\pm} \psi(K(d(x))))}{g(\xi_{\pm} \psi(K(d(x))))} \frac{g'(\Theta_{\pm}(d(x)))}{g'(\xi_{\pm} \psi(K(d(x))))} \frac{g(\xi_{\pm} \psi(K(d(x))))}{\xi_{\pm}^N g(\psi(K(d(x))))} \Big\} \\ &\quad \pm \varepsilon \left\{ \left[\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \right. \right. \\ &\quad \times \left(N + \mu(N-1) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} \right) + \frac{2\mu\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} + \mu \\ &\quad \times (\mu-1) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 \Big] \\ &\quad \times \hat{m}_{\pm} \prod_{i=1}^{N-1} (1 - d(x)\kappa_i(\bar{x}))^{-1} - \frac{\xi_{\pm} \psi(K(d(x))) g'(\xi_{\pm} \psi(K(d(x))))}{g(\xi_{\pm} \psi(K(d(x))))} \frac{g'(\Theta_{\pm}(d(x)))}{g'(\xi_{\pm} \psi(K(d(x))))} \\ &\quad \times \frac{g(\xi_{\pm} \psi(K(d(x))))}{\xi_{\pm}^N g(\psi(K(d(x))))} \Big\} - (B_0 \pm \varepsilon) \frac{g(\xi_{\pm} \psi(K(d(x))))}{\xi_{\pm}^N g(\psi(K(d(x))))}; \\ I_{3\pm}(d(x)) &= (d(x))^{-\mu} \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \\ &\quad \times \frac{\hat{m}_{\pm} \sum_{i=1}^{N-1} C_{N-1}^i (-1)^{i+1} (m_{\pm} d(x))^i}{(1 - d(x)m_{\pm})^{N-1}} - (B_0 \pm \varepsilon) (\tilde{C}_{\pm} \pm \varepsilon) \frac{\xi_{\pm} \psi(K(d(x))) g'(\xi_{\pm} \psi(K(d(x))))}{g(\xi_{\pm} \psi(K(d(x))))} \\ &\quad \times \frac{g'(\Theta_{\pm}(d(x)))}{g'(\xi_{\pm} \psi(K(d(x))))} \frac{g(\xi_{\pm} \psi(K(d(x))))}{\xi_{\pm}^N g(\psi(K(d(x))))} (d(x))^{\mu} + \frac{\tilde{C}_{\pm}(d(x))}{(d(x))^{\mu}} \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})}, \end{aligned}$$

where

$$\begin{cases} \mathcal{B}_{\pm} = \hat{m}_{\pm}, & \text{if } B_0 \geq 0, \\ \mathcal{B}_{\pm} = \hat{m}_{\mp}, & \text{if } B_0 < 0, \end{cases}$$

$m_{\pm}$ are defined by (4.6) and

$$\begin{aligned} \tilde{\zeta}_{\pm}(d(x)) &= (\tilde{C}_{\pm} \pm \varepsilon)^2 (N-1) \left(1 + \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} \right) \\ &\times \left(1 + \frac{\mu\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} \right) (d(x))^{2\mu} \\ &+ 2\mu(\tilde{C}_{\pm} \pm \varepsilon)^2 (N-1) \left(1 + \frac{\mu\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} \right) \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \\ &\times \frac{K(d(x))k'(d(x))}{k^2(d(x))} (d(x))^{2\mu} + \mu(\mu-1)(\tilde{C}_{\pm} \pm \varepsilon)^2 (N-1) \left(1 + \frac{\mu\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \right. \\ &\times \left. \frac{K(d(x))}{d(x)k(d(x))} \right) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 (d(x))^{2\mu} + \tilde{v}_{\pm}(d(x)) \\ &\times \left[\left(1 + \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} \right) (1 + (\tilde{C}_{\pm} \pm \varepsilon)(d(x))^{\mu}) \right. \\ &+ 2\mu(\tilde{C}_{\pm} \pm \varepsilon) \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} (d(x))^{\mu} \\ &\left. + \mu(\mu-1)(\tilde{C}_{\pm} \pm \varepsilon) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 (d(x))^{\mu} \right] \end{aligned}$$

with

$$\tilde{v}_{\pm}(d(x)) = \sum_{i=2}^{N-1} C_{N-1}^i (\tilde{\mathfrak{R}}_{\pm}(d(x)))^i,$$

where C_{N-1}^i is given in (4.7) and

$$\tilde{\mathfrak{R}}_{\pm}(d(x)) = (\tilde{C}_{\pm} \pm \varepsilon) \left(1 + \frac{\mu\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} \right) (d(x))^{\mu}.$$

Next, we prove

$$\lim_{d(x) \rightarrow 0} (I_{1\pm}(d(x)) + I_{2\pm}(d(x)) + I_{3\pm}(d(x))) = \pm \varepsilon \varrho_{\pm} \hat{m}_{\pm}, \quad (4.17)$$

where

$$\begin{aligned} \varrho_{\pm} &= \left[\frac{(1-\mu)(\gamma+N)D_k - (N+1)}{\gamma-1} \left(N + \mu(N-1) \frac{\gamma+N}{N+1} D_k \right) \right. \\ &+ \mu \frac{\gamma+N}{\gamma-1} D_k \left(N + \mu(N-1) \frac{\gamma+N}{N+1} D_k - 2 \right) + \mu(1-\mu) \frac{(\gamma+N)^2}{(\gamma-1)(N+1)} D_k^2 \left. \right] \frac{\hat{m}_{\pm}}{\hat{m}_{\pm}} \\ &+ (\gamma+N)D_k - (N+1). \end{aligned} \quad (4.18)$$

To prove (4.17), we calculate the limits of $I_{1\pm}(d(x))$, $I_{2\pm}(d(x))$ and $I_{3\pm}(d(x))$ as $d(x) \rightarrow 0$.

• First, we investigate the limits of $I_{1\pm}(d(x))$ as $d(x) \rightarrow 0$. As the preliminaries, we prove some limits as below.

Case 1. When $D_k \in (0, 1)$, by a direct calculation, we have

$$\frac{N+1}{(1-\mu)D_k} - N > \frac{N+1}{(1-\mu)} - N > \frac{(1+\mu)N+1}{N+1-\mu}.$$

So, by (1.15), Lemma 3.1 (ii) and Lemma 3.6 (viii) and (x) with $\rho = \mu \in (0, 1)$, we obtain

$$\begin{aligned} &\lim_{d(x) \rightarrow 0^+} (d(x))^{-\mu} \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} + \frac{\gamma+N}{\gamma-1} \right) \frac{K(d(x))k'(d(x))}{k^2(d(x))} \hat{m}_{\pm} \\ &= \lim_{d(x) \rightarrow 0^+} \left(\frac{K(d(x))}{d(x)} \right)^{\mu} (K(d(x)))^{-\mu} \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} + \frac{\gamma+N}{\gamma-1} \right) \frac{K(d(x))k'(d(x))}{k^2(d(x))} \hat{m}_{\pm} \\ &= \begin{cases} 0, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \frac{\theta(N+1)}{\mu} - N > \gamma > \frac{N+1}{(1-\mu)D_k} - N, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds with } \gamma > \frac{N+1}{(1-\mu)D_k} - N \text{ in } (\mathbf{g}_2) \end{cases} \end{aligned} \quad (4.19)$$

and

$$\begin{aligned}
 & \lim_{d(x) \rightarrow 0} (d(x))^{-\mu} \left(\xi_{\pm}^{-(\gamma+N)} - \frac{g(\xi_{\pm} \psi(K(d(x))))}{\xi_{\pm}^N g(\psi(K(d(x))))} \right) \\
 &= \lim_{d(x) \rightarrow 0} \left(\frac{K(d(x))}{d(x)} \right)^{\mu} (K(d(x)))^{-\mu} \left(\xi_{\pm}^{-(\gamma+N)} - \frac{g(\xi_{\pm} \psi(K(d(x))))}{\xi_{\pm}^N g(\psi(K(d(x))))} \right) \\
 &= \begin{cases} 0, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \frac{\theta(N+1)}{\mu} - N > \gamma, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds in } (\mathbf{g}_2). \end{cases}
 \end{aligned} \quad (4.20)$$

Case 2. When $D_k \in [1, \infty)$, by (1.15), Lemma 3.1 (ii) and Lemma 3.6 (viii) and (x) with $\rho = 1$, we obtain

$$\begin{aligned}
 & \lim_{d(x) \rightarrow 0} (d(x))^{-\mu} \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} + \frac{\gamma + N}{\gamma - 1} \right) \frac{K(d(x))k'(d(x))}{k^2(d(x))} \hat{m}_{\pm} \\
 &= \lim_{d(x) \rightarrow 0} \frac{K(d(x))}{(d(x))^{\mu}} (K(d(x)))^{-1} \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} + \frac{\gamma + N}{\gamma - 1} \right) \frac{K(d(x))k'(d(x))}{k^2(d(x))} \hat{m}_{\pm} \\
 &= \begin{cases} 0, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta(N+1) - N > \gamma > \frac{2N+1}{N}, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds with } \gamma > \frac{2N+1}{N} \text{ in } (\mathbf{g}_2) \end{cases}
 \end{aligned} \quad (4.21)$$

and

$$\begin{aligned}
 & \lim_{d(x) \rightarrow 0} (d(x))^{-\mu} \left(\xi_{\pm}^{-(\gamma+N)} - \frac{g(\xi_{\pm} \psi(K(d(x))))}{\xi_{\pm}^N g(\psi(K(d(x))))} \right) \\
 &= \lim_{d(x) \rightarrow 0} \frac{K(d(x))}{(d(x))^{\mu}} (K(d(x)))^{-1} \left(\xi_{\pm}^{-(\gamma+N)} - \frac{g(\xi_{\pm} \psi(K(d(x))))}{\xi_{\pm}^N g(\psi(K(d(x))))} \right) \\
 &= \begin{cases} 0, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3) \text{ hold with } \theta(N+1) - N > \gamma, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds in } (\mathbf{g}_2). \end{cases}
 \end{aligned} \quad (4.22)$$

On the other hand, a simple calculation shows that

$$\begin{aligned}
 & - \lim_{d(x) \rightarrow 0} \frac{\gamma + N}{\gamma - 1} (d(x))^{-\mu} \left(\frac{K(d(x))k'(d(x))}{k^2(d(x))} - (1 - D_k) \right) \hat{m}_{\pm} \\
 &= - \lim_{d(x) \rightarrow 0} \frac{\gamma + N}{\gamma - 1} (d(x))^{1-\mu} (d(x))^{-1} \left(\frac{K(d(x))k'(d(x))}{k^2(d(x))} - (1 - D_k) \right) \hat{m}_{\pm} = 0
 \end{aligned} \quad (4.23)$$

We obtain by combining (4.19)-(4.20) (or (4.21)-(4.22)) and (4.23) that

$$\lim_{d(x) \rightarrow 0} I_{1\pm}(d(x)) = 0. \quad (4.24)$$

• Second, by (4.2), Lemma 3.1 (ii), Lemma 3.6 (iii)-(iv) and the choices of $\xi_{\pm}$, $\tilde{C}_{\pm}$ in Theorem 1.2, we obtain

$$\begin{aligned}
 & \lim_{d(x) \rightarrow 0} I_{2\pm}(d(x)) = \tilde{C}_{\pm} \left[\left(\frac{(1-\mu)(\gamma+N)D_k - (N+1)}{\gamma-1} \left(N + \mu(N-1) \frac{\gamma+N}{N+1} D_k \right) \right. \right. \\
 & + \mu \frac{\gamma+N}{\gamma-1} D_k \left(N + \mu(N-1) \frac{\gamma+N}{N+1} D_k - 2 \right) + \mu(1-\mu) \frac{(\gamma+N)^2}{(\gamma-1)(N+1)} D_k^2 \Big) \mathcal{B}_{\pm} \\
 & + \frac{\gamma((\gamma+N)D_k - (N+1)) \hat{m}_{\pm}}{\gamma-1} \Big] - B_0 \frac{((\gamma+N)D_k - (N+1)) \hat{m}_{\pm}}{\gamma-1} \\
 & \pm \varepsilon \varrho_{\pm} \hat{m}_{\pm} = \pm \varepsilon \varrho_{\pm} \hat{m}_{\pm}, \text{ where } \varrho_{\pm} \text{ are given by (4.18).}
 \end{aligned} \quad (4.25)$$

• Third, we conclude by Lemma 3.1 (ii), Lemma 3.6 (iii)-(iv) and a straightforward calculation that

$$\lim_{d(x) \rightarrow 0} I_{3\pm}(d(x)) = 0. \quad (4.26)$$

Combining (4.24) and (4.25)-(4.26), we obtain (4.17) holds. Moreover, it follows by (1.15) that $\varrho_{\pm} > (\gamma+N)D_k - (N+1) > 0$. By (b₁)-(b₂) and (4.17), we see that there exists a sufficiently small constant $\delta_{\varepsilon} > 0$ such that (4.12) and (4.4)-(4.5) hold here.

Step 2. Let

$$\underline{u}_\varepsilon(x) = -\xi_+ \psi(K(d(x))) (1 + (\tilde{C}_+ + \varepsilon)(d(x))^\mu), \quad x \in \Omega_{\delta_\varepsilon}.$$

Then

$$g(-\underline{u}_\varepsilon(x)) = g(\xi_+ \psi(K(d(x)))) + \xi_+ \psi(K(d(x))) g'(\Theta_+(d(x))) (\tilde{C}_+ + \varepsilon)(d(x))^\mu, \quad x \in \Omega_{\delta_\varepsilon},$$

where $\Theta_+(d(x))$ is given by (4.16). By Lemma 4.2, we have for any $x \in \Omega_{\delta_\varepsilon}$

$$\begin{aligned} & \det(D^2 \underline{u}_\varepsilon(x)) - b(x)g(-\underline{u}_\varepsilon(x)) \geq -\xi_+^N (\psi'(K(d(x))))^{N-1} \psi''(K(d(x))) k^{N+1}(d(x)) \\ & \times \left(1 + (\tilde{C}_+ + \varepsilon)(N-1)(d(x))^\mu + \mu(\tilde{C}_+ + \varepsilon)(N-1) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} (d(x))^\mu \right. \\ & \left. + \tilde{v}_+(d(x)) \right) \left[\left(1 + \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} \right) (1 + (\tilde{C}_+ + \varepsilon)(d(x))^\mu) \right. \\ & \left. + 2\mu(C_+ + \varepsilon) \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} (d(x))^\mu \right. \\ & \left. + \mu(\mu-1)(\tilde{C}_+ + \varepsilon) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 (d(x))^\mu \right] \\ & \times \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} - k^{N+1}(d(x)) (1 + (B_0 + \varepsilon)(d(x))^\mu) [g(\xi_+ \psi(K(d(x)))) \\ & + \xi_+ \psi(K(d(x))) g'(\Theta_+(d(x))) (\tilde{C}_+ + \varepsilon)(d(x))^\mu] \\ & = -\xi_+^N (\psi'(K(d(x))))^{N-1} \psi''(K(d(x))) k^{N+1}(d(x)) (d(x))^\mu \\ & \times \left\{ (d(x))^{-\mu} \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \right. \\ & \left. + (\tilde{C}_+ + \varepsilon) \left[\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \left(N + \mu(N-1) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \right. \right. \right. \\ & \left. \left. \times \frac{K(d(x))}{d(x)k(d(x))} \right) + \frac{2\mu\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} + \frac{\mu(\mu-1)\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \right. \\ & \left. \left. \times \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 \right] \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} + \frac{\tilde{\zeta}_+(d(x))}{(d(x))^\mu} \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \right\} \\ & - k^{N+1}(d(x))g(\xi_+ \psi(K(d(x)))) - (B_0 + \varepsilon)k^{N+1}(d(x))g(\xi_+ \psi(K(d(x))))(d(x))^\mu \\ & - (\tilde{C}_+ + \varepsilon)k^{N+1}(d(x))\xi_+ \psi(K(d(x)))g'(\Theta_+(d(x)))(d(x))^\mu - (B_0 + \varepsilon)(\tilde{C}_+ + \varepsilon)k^{N+1}(d(x)) \\ & \times \xi_+ \psi(K(d(x)))g'(\Theta_+(d(x)))(d(x))^{2\mu} \\ & = -\xi_+^N (\psi'(K(d(x))))^{N-1} \psi''(K(d(x))) k^{N+1}(d(x)) (d(x))^\mu \\ & \times \left\{ (d(x))^{-\mu} \left[\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \hat{m}_+ - \frac{g(\xi_+ \psi(K(d(x))))}{\xi_+^N g(\psi(K(d(x))))} \right] \right. \\ & \left. + (\tilde{C}_+ + \varepsilon) \left[\left(\left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \left(N + \mu(N-1) \right. \right. \right. \right. \\ & \left. \left. \times \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} \right) + \frac{2\mu\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} \right. \right. \\ & \left. \left. + \frac{\mu(\mu-1)\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 \right) \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \right. \\ & \left. - \frac{\xi_+ \psi(K(d(x)))g'(\xi_+ \psi(K(d(x))))}{g(\xi_+ \psi(K(d(x))))} \frac{g'(\Theta_+(d(x)))}{g'(\xi_+ \psi(K(d(x))))} \frac{g(\xi_+ \psi(K(d(x))))}{\xi_+^N g(\psi(K(d(x))))} \right] \\ & - (B_0 + \varepsilon) \frac{g(\xi_+ \psi(K(d(x))))}{\xi_+^N g(\psi(K(d(x))))} - (B_0 + \varepsilon)(\tilde{C}_+ + \varepsilon) \frac{\xi_+ \psi(K(d(x)))g'(\xi_+ \psi(K(d(x))))}{g(\xi_+ \psi(K(d(x))))} \\ & \times \frac{g'(\Theta_+(d(x)))}{g'(\xi_+ \psi(K(d(x))))} \frac{g(\xi_+ \psi(K(d(x))))}{\xi_+^N g(\psi(K(d(x))))} (d(x))^\mu \end{aligned} \quad (4.27)$$

$$\begin{aligned}
& + (d(x))^{-\mu} \left(\frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} + 1 \right) \frac{\hat{m}_+ \sum_{i=1}^{N-1} C_{N-1}^i (-1)^{i+1} (m_+ d(x))^i}{(1-d(x)m_+)^{N-1}} \\
& + \frac{\tilde{\xi}_+(d(x))}{(d(x))^\mu} \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})} \Big\} \\
& \geq -\xi_+^N (\psi'(K(d(x))))^{N-1} \psi''(K(d(x))) k^{N+1}(d(x)) (d(x))^\mu \sum_{i=1}^3 I_{i+}(d(x)) > 0,
\end{aligned}$$

i.e., $\underline{u}_\varepsilon$ is a subsolution of Eq. (1.1) in $\Omega_{\delta_\varepsilon}$. Moreover, it follows by Lemma 4.3 that for $i = 1, \dots, N$

$$\begin{aligned}
S_i(D^2 \underline{u}_\varepsilon(x)) &= \xi_+^i (\psi'(K(d(x))))^i k^i(d(x)) \left(1 + (\tilde{C}_+ + \varepsilon)(d(x))^\mu + \mu(\tilde{C}_+ + \varepsilon) \frac{K(d(x))}{d(x)k(d(x))} \right. \\
&\quad \times \left. \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} (d(x))^\mu \right)^i S_i(\epsilon_1, \dots, \epsilon_{N-1}) \\
&\quad - \xi_+^i (\psi'(K(d(x))))^{i-1} \psi''(K(d(x))) k^{i+1}(d(x)) \left(1 + (\tilde{C}_+ + \varepsilon)(d(x))^\mu + \mu(\tilde{C}_+ + \varepsilon) \right. \\
&\quad \times \left. \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} (d(x))^\mu \right)^{i-1} \left[\left(1 + \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))k'(d(x))}{k^2(d(x))} \right) \right. \\
&\quad \times \left(1 + (\tilde{C}_+ + \varepsilon)(d(x))^\mu \right) + 2\mu(\tilde{C}_+ + \varepsilon) \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \frac{K(d(x))}{d(x)k(d(x))} (d(x))^\mu \\
&\quad \left. + \mu(\mu-1)(\tilde{C}_+ + \varepsilon) \frac{\psi(K(d(x)))}{\psi'(K(d(x)))K(d(x))} \frac{\psi'(K(d(x)))}{\psi''(K(d(x)))K(d(x))} \left(\frac{K(d(x))}{d(x)k(d(x))} \right)^2 (d(x))^\mu \right] \\
&\quad \times S_{i-1}(\epsilon_1, \dots, \epsilon_{N-1}).
\end{aligned}$$

This implies that we can adjust the above positive constant δ_ε such that for any $x \in \Omega_{\delta_\varepsilon}$

$$S_i(D^2 \underline{u}_\varepsilon(x)) > 0 \text{ for } i = 1, \dots, N.$$

We obtain by Lemma 4.4 that $D^2 \underline{u}_\varepsilon$ is positive definite in $\Omega_{\delta_\varepsilon}$.

Let

$$\bar{u}_\varepsilon(x) = -\xi_- \psi(K(d(x))) (1 + (\tilde{C}_- - \varepsilon)(d(x))^\mu), \quad x \in \Omega_{\delta_\varepsilon}$$

By the same calculation as (4.27), we obtain

$$\begin{aligned}
& \det(D^2 \bar{u}_\varepsilon(x)) - b(x)g(-\bar{u}_\varepsilon(x)) \\
& \leq -\xi_-^N (\psi'(K(d(x))))^{N-1} \psi''(K(d(x))) k^{N+1}(d(x)) (d(x))^\mu \sum_{i=1}^3 I_{i-}(d(x)) < 0,
\end{aligned}$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of Eq. (1.1) in $\Omega_{\delta_\varepsilon}$.

Let u be the unique strictly convex solution to problem (1.1). Through the same argument as Theorem 1.1, we see that there exists a large constant $M > 0$ such that

$$\underline{u}_\varepsilon(x) - Md(x) \leq u(x) \leq \bar{u}_\varepsilon(x) + Md(x), \quad x \in \Omega_{\delta_\varepsilon}.$$

Hence, for any $x \in \Omega_{\delta_\varepsilon}$

$$\begin{aligned}
\tilde{C}_+ + \varepsilon + \frac{M(d(x))^{1-\mu}}{\xi_+ \psi(K(d(x)))} &\geq \left(\frac{-u(x)}{\xi_+ \psi(K(d(x)))} - 1 \right) (d(x))^{-\mu}; \\
\tilde{C}_- - \varepsilon - \frac{M(d(x))^{1-\mu}}{\xi_- \psi(K(d(x)))} &\leq \left(\frac{-u(x)}{\xi_- \psi(K(d(x)))} - 1 \right) (d(x))^{-\mu}.
\end{aligned}$$

Since $(1 - \mu)(\gamma + N)D_k - (N + 1) > 0$, we conclude from Lemma 3.1 (ii), Lemma 3.6 (iii), Proposition 2.7 and Proposition 2.5 (ii) that

$$\begin{aligned}\tilde{C}_+ + \varepsilon &\geq \limsup_{d(x) \rightarrow 0} \left(\frac{-u(x)}{\xi_+ \psi(K(d(x)))} - 1 \right) (d(x))^{-\mu}; \\ \tilde{C}_- - \varepsilon &\leq \liminf_{d(x) \rightarrow 0} \left(\frac{-u(x)}{\xi_- \psi(K(d(x)))} - 1 \right) (d(x))^{-\mu}.\end{aligned}$$

Letting $\varepsilon \rightarrow 0$, the proof is finished. $\square$

4.3 Proof of Theorem 1.3

Now, we prove Theorem 1.3. As before, for fixed $\varepsilon > 0$, we define

$$w_{\pm}(d(x)) = \eta_{\pm} \psi \left(\left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right) (1 + (C_{\pm}^* \pm \varepsilon) \mathfrak{M}(d(x))), \quad x \in \Omega_{\delta_1},$$

where $\eta_{\pm}$, $C_{\pm}^*$ and $\mathfrak{M}$ are given in Theorem 1.3. By the Lagrange's mean value theorem, it is clear that there exist $\lambda_{\pm} \in (0, 1)$ and

$$\Theta_{\pm}(d(x)) = \eta_{\pm} \psi \left(\left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right) (1 + \lambda_{\pm} (C_{\pm}^* \pm \varepsilon) \mathfrak{M}(d(x))) \quad (4.28)$$

such that for $x \in \Omega_{\delta_1}$

$$\begin{aligned}g(w_{\pm}(d(x))) &= g \left(\eta_{\pm} \psi \left(\left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right) \right) \\ &\quad + \eta_{\pm} \psi \left(\left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \right) g'(\Theta_{\pm}(d(x))) (C_{\pm}^* \pm \varepsilon) \mathfrak{M}(d(x)).\end{aligned}$$

We obtain by Proposition 2.2 that (4.2) still holds. Moreover, we can adjust δ_1 such that (4.3) holds here.

Proof. The proof is still divided into the following two steps.

Step 1. As before, for fixed $\varepsilon > 0$ and $\forall x \in \Omega_{\delta_1}$, we define

$$r(d(x)) = \left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N}{N+1}} \quad (4.29)$$

and

$$\begin{aligned}I_{1\pm}(d(x)) &= (\mathfrak{M}(d(x)))^{-1} \left\{ \left[\left(\frac{N}{N+1} \right)^N \left(\frac{N}{N+1} - \frac{1}{N+1} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \right) \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} \right. \right. \\ &\quad \left. \left. + \left(\frac{N}{N+1} \right)^N \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \right] \hat{m}_{\pm} - \frac{g(\xi_{\pm} \psi(r(d(x))))}{\xi_{\pm}^N g(\psi(r(d(x))))} \right\}; \\ I_{2\pm}(d(x)) &= C_{\pm}^* \left[\left(\frac{N}{N+1} \right)^N \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \mathcal{C}_{\pm} \prod_{i=1}^{N-1} (1 - d(x) \kappa_i(\bar{x}))^{-1} \right. \\ &\quad \left. - \frac{\eta_{\pm} \psi(r(d(x))) g'(\eta_{\pm} \psi(r(d(x))))}{g(\eta_{\pm} \psi(r(d(x))))} \frac{g'(\Theta_{\pm}(d(x)))}{g'(\eta_{\pm} \psi(r(d(x))))} \frac{g(\eta_{\pm} \psi(r(d(x))))}{\eta_{\pm}^N g(\psi(r(d(x))))} \right]\end{aligned}$$

$$\begin{aligned}
& \pm \varepsilon \left[\left(\frac{N}{N+1} \right)^N N \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \hat{m}_+ \prod_{i=1}^{N-1} (1 - d(x)\kappa_i(\bar{x}))^{-1} \right. \\
& \quad \left. - \frac{\eta_{\pm}\psi(r(d(x)))g'(\eta_{\pm}\psi(r(d(x))))}{g(\eta_{\pm}\psi(r(d(x))))} \frac{g'(\Theta_{\pm}(d(x)))}{g'(\eta_{\pm}\psi(r(d(x))))} \frac{g(\eta_{\pm}\psi(r(d(x))))}{\eta_{\pm}^N g(\psi(r(d(x))))} \right]; \\
I_{3\pm}(d(x)) &= -(B_0 \pm \varepsilon) \frac{g(\eta_{\pm}\psi(r(d(x))))}{\eta_{\pm}^N g(\psi(r(d(x))))} (d(x))^{\mu} (\mathfrak{M}(d(x)))^{-1} \\
&+ (\mathfrak{M}(d(x)))^{-1} \left[\left(\frac{N}{N+1} \right)^N \left(\frac{N}{N+1} - \frac{1}{N+1} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \right) \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} \right. \\
&+ \left. \left(\frac{N}{N+1} \right)^N \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \right] \frac{\hat{m}_+ \sum_{i=1}^{N-1} C_{N-1}^i (-1)^{i+1} (m_{\pm} d(x))^i}{(1 - d(x)m_{\pm})^{N-1}} \\
&- (B_0 \pm \varepsilon) (C_{\pm}^* \pm \varepsilon) \frac{\eta_{\pm}\psi(r(d(x)))g'(\eta_{\pm}\psi(r(d(x))))}{g(\eta_{\pm}\psi(r(d(x))))} \frac{g'(\Theta_{\pm}(d(x)))}{g'(\eta_{\pm}\psi(r(d(x))))} \frac{g(\eta_{\pm}\psi(r(d(x))))}{\eta_{\pm}^N g(\psi(r(d(x))))} (d(x))^{\mu} \\
&+ \left(\frac{N}{N+1} \right)^N \zeta_{\pm}^*(d(x)) (\mathfrak{M}(d(x)))^{-1} \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})},
\end{aligned}$$

where

$$\begin{cases} \text{if } (\mathbf{S}_2) \text{ holds and } \theta = 0, \\ \text{if } (\mathbf{S}_1) \text{ holds, } \mathcal{C}_{\pm} = \hat{m}_{\pm}, \end{cases} \quad \begin{cases} \mathcal{C}_{\pm} = \hat{m}_{\pm}, & \text{if } \mathfrak{X}_{+} \geq 0 \text{ and } \mathfrak{X}_{-} \geq 0, \\ \mathcal{C}_{\mp} = \hat{m}_{\mp}, & \text{if } \mathfrak{X}_{+} \leq 0 \text{ and } \mathfrak{X}_{-} \leq 0, \\ \mathcal{C}_{+} = \mathcal{C}_{-} = \hat{m}_{+}, & \text{if } \mathfrak{X}_{+} > 0 \text{ and } \mathfrak{X}_{-} < 0, \\ \mathcal{C}_{+} = \mathcal{C}_{-} = \hat{m}_{-}, & \text{if } \mathfrak{X}_{+} < 0 \text{ and } \mathfrak{X}_{-} > 0; \end{cases}$$

$m_{\pm}$ are defined by (4.6) and

$$\begin{aligned}
\zeta_{\pm}^* &= (C_{\pm}^* \pm \varepsilon) \left(\frac{N}{N+1} - \frac{1}{N+1} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \right) \mathfrak{M}(d(x)) \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} \\
&\times (N + (C_{\pm}^* \pm \varepsilon)(N-1)\mathfrak{M}(d(x))) + (C_{\pm}^* \pm \varepsilon)^2 (N-1) \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \\
&\times (\mathfrak{M}(d(x)))^2 + \left[\frac{\beta N}{N+1} (C_{\pm}^* \pm \varepsilon) \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \frac{\psi'(r(d(x)))r(d(x))}{\psi(r(d(x)))} \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} \right. \\
&\times (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}} + \frac{\beta N}{N+1} (C_{\pm}^* \pm \varepsilon) \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}} + \frac{\beta(\beta+1)N(C_{\pm}^* \pm \varepsilon)}{N+1} \\
&\times \frac{\psi'(r(d(x)))r(d(x))}{\psi(r(d(x)))} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} (\mathfrak{M}(d(x)))^{\frac{\beta+2}{\beta}} \\
&+ \beta(C_{\pm}^* \pm \varepsilon) \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}} \Big] \\
&\times (1 + (C_{\pm}^* \pm \varepsilon)(N-1)\mathfrak{M}(d(x)) + v_{\pm}^*(d(x))) \\
&+ v_{\pm}^*(d(x)) \left[\left(\frac{N}{N+1} - \frac{1}{N+1} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \right) (1 + (C_{\pm}^* \pm \varepsilon)\mathfrak{M}(d(x))) \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} \right. \\
&+ \left. \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) (1 + (C_{\pm}^* \pm \varepsilon)\mathfrak{M}(d(x))) \right]
\end{aligned}$$

with

$$v_{\pm}^*(d(x)) = \mathfrak{R}_{2\pm}^*(d(x)) + \sum_{i=2}^{N-1} C_{N-1}^i (\mathfrak{R}_{1\pm}^*(d(x)) + \mathfrak{R}_{2\pm}^*(d(x)))^i,$$

where C_{N-1}^i is given in (4.7) and

$$\mathfrak{R}_{1\pm}^*(d(x)) = (C_{\pm}^* \pm \varepsilon)\mathfrak{M}(d(x)) \text{ and } \mathfrak{R}_{2\pm}^*(d(x)) = \beta(C_{\pm}^* \pm \varepsilon)(\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}}.$$

Next, we prove

$$\lim_{d(x) \rightarrow 0} (I_{1\pm}(d(x)) + I_{2\pm}(d(x)) + I_{3\pm}(d(x))) = \pm \varepsilon \left(\gamma + N \frac{\hat{m}_+}{\hat{m}_\pm} \right) \eta_\pm^{-(\gamma+N)}. \quad (4.30)$$

To prove (4.30), we calculate the limits of $I_{1\pm}(d(x))$, $I_{2\pm}(d(x))$ and $I_{3\pm}(d(x))$ as $d(x) \rightarrow 0$.

• First, by Lemma 3.2 (ii)-(iii), Lemma 3.6 (iv), (vi)-(vii) and (ix), we obtain

$$\lim_{d(x) \rightarrow 0} I_{1\pm}(d(x)) = \begin{cases} -\frac{\left(\frac{N}{N+1}\right)^N (N+1)\sigma\hat{m}_\pm}{(\gamma-1)^2} + \frac{\sigma \ln \eta_\pm}{\eta_\pm^{\gamma+N}}, & \text{if } (\mathbf{S}_2) \text{ and } (\mathbf{g}_3)\text{-(}\mathbf{g}_4\text{) hold with } \theta = 0, \\ 0, & \text{if } (\mathbf{S}_1) \text{ holds in } (\mathbf{g}_2). \end{cases} \quad (4.31)$$

• Second, by (4.2), Lemma 3.6 (iv) and the choices of $\eta_\pm$ and $C_\pm^*$ in Theorem 1.3, we obtain

$$\lim_{d(x) \rightarrow 0} I_{2\pm}(d(x)) = C_\pm^* \left[\left(\frac{N}{N+1} \right)^N \frac{N(\gamma+N)}{\gamma-1} \mathcal{C}_\pm + \gamma \eta_\pm^{-(\gamma+N)} \right] \pm \varepsilon \left(\gamma + N \frac{\hat{m}_+}{\hat{m}_\pm} \right) \eta_\pm^{-(\gamma+N)}. \quad (4.32)$$

• Third, we conclude by Lemma 3.5, Lemma 3.6 (iii)-(iv) and a straightforward calculation that

$$\lim_{d(x) \rightarrow 0} I_{3\pm}(d(x)) = 0. \quad (4.33)$$

Combining (4.31)-(4.33), we obtain (4.30) holds. By $(\mathbf{b}_1)$, $(\mathbf{b}_3)$ and (4.30), we see that there exists a sufficiently small positive constant $\delta_\varepsilon < \delta_1$ such that for any $x \in \Omega_{\delta_\varepsilon}$

$$(d(x))^{-(N+1)} \tilde{L}^N(d(x)) (1 + (B_0 - \varepsilon)(d(x))^\mu) \leq b(x) \leq (d(x))^{-(N+1)} \tilde{L}^N(d(x)) (1 + (B_0 + \varepsilon)(d(x))^\mu)$$

and (4.4)-(4.5) hold here.

Step 2. Let

$$\underline{u}_\varepsilon(x) = -\eta_+ \psi(r(d(x))) (1 + (C_+^* + \varepsilon)\mathfrak{M}(d(x))), \quad x \in \Omega_{\delta_\varepsilon},$$

where $r(d(x))$ is given by (4.29). Then

$$g(-\underline{u}_\varepsilon(x)) = g(\eta_+ \psi(r(d(x)))) + \eta_+ \psi(r(d(x))) g'(\Theta_+(d(x))) (C_+^* + \varepsilon) \mathfrak{M}(d(x)), \quad x \in \Omega_{\delta_\varepsilon},$$

where $\Theta_+(d(x))$ is given by (4.28). By Lemma 4.2, we have for any $x \in \Omega_{\delta_\varepsilon}$

$$\begin{aligned} \det(D^2 \underline{u}_\varepsilon(x)) - b(x) g(-\underline{u}_\varepsilon(x)) &\geq -\eta_+^N \left(\frac{N}{N+1} \right)^N (\psi'(r(d(x))))^{N-1} \psi''(r(d(x))) (d(x))^{-(N+1)} \\ &\times \tilde{L}^N(d(x)) (1 + (C_+^* + \varepsilon)(N-1)\mathfrak{M}(d(x)) + v_+^*(d(x))) \left[\left(\frac{N}{N+1} - \frac{1}{N+1} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \right) \right. \\ &\times (1 + (C_+^* + \varepsilon)\mathfrak{M}(d(x))) \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \\ &\times (1 + (C_+^* + \varepsilon)\mathfrak{M}(d(x))) + \frac{\beta N}{N+1} (C_+^* + \varepsilon) \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \frac{\psi'(r(d(x)))r(d(x))}{\psi(r(d(x)))} \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} \\ &\times (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}} + \frac{\beta N}{N+1} (C_+^* + \varepsilon) (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}} \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \frac{\beta(\beta+1)N(C_+^* + \varepsilon)}{N+1} \\ &\times \frac{\psi'(r(d(x)))r(d(x))}{\psi(r(d(x)))} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} (\mathfrak{M}(d(x)))^{\frac{\beta+2}{\beta}} \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \beta(C_+^* + \varepsilon) \\ &\times \left. \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}} \left(\frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \right] \\ &\times \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} - (d(x))^{-(N+1)} \tilde{L}^N(d(x)) (1 + (B_0 + \varepsilon)(d(x))^\mu) \\ &\times [g(\eta_+ \psi(r(d(x)))) + \eta_+ (C_+^* + \varepsilon) g'(\Theta_+(d(x))) \psi(r(d(x))) \mathfrak{M}(d(x))] \end{aligned} \quad (4.34)$$

$$\begin{aligned}
&= -\eta_+^N (\psi'(r(d(x))))^{N-1} \psi''(r(d(x))) (d(x))^{-(N+1)} \tilde{L}^N(d(x)) \mathfrak{M}(d(x)) \left\{ (\mathfrak{M}(d(x)))^{-1} \right. \\
&\quad \times \left[\left(\left(\frac{N}{N+1} \right)^N \left(\frac{N}{N+1} - \frac{1}{N+1} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \right) \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} \right. \right. \\
&\quad \left. \left. + \left(\frac{N}{N+1} \right)^N \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \right) \hat{m}_+ - \frac{g(\eta_+ \psi(r(d(x))))}{\eta_+^N g(\psi(r(d(x))))} \right] \\
&\quad + (C_+^* + \varepsilon) \left[\left(\frac{N}{N+1} \right)^N N \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \right. \\
&\quad \left. - \frac{\eta_+ \psi(r(d(x))) g'(\eta_+ \psi(r(d(x))))}{g(\eta_+ \psi(r(d(x))))} \frac{g'(\Theta_+(d(x)))}{g'(\eta_+ \psi(r(d(x))))} \frac{g(\eta_+ \psi(r(d(x))))}{\eta_+^N g(\psi(r(d(x))))} \right] \\
&\quad - (B_0 + \varepsilon) \frac{g(\eta_+ \psi(r(d(x))))}{\eta_+^N g(\psi(r(d(x))))} (d(x))^\mu (\mathfrak{M}(d(x)))^{-1} + (\mathfrak{M}(d(x)))^{-1} \left[\left(\frac{N}{N+1} \right)^N \left(\frac{N}{N+1} - \frac{1}{N+1} \right. \right. \\
&\quad \times \left. \left. \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \right) \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \left(\frac{N}{N+1} \right)^N \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \right] \\
&\quad \times \frac{\hat{m}_+ \sum_{i=1}^{N-1} C_{N-1}^i (-1)^{i+1} (m_+ d(x))^i}{(1 - d(x)m_+)^{N-1}} \\
&\quad - (B_0 + \varepsilon)(C_+^* + \varepsilon) \frac{\eta_+ \psi(r(d(x))) g'(\eta_+ \psi(r(d(x))))}{g(\eta_+ \psi(r(d(x))))} \frac{g'(\Theta_+(d(x)))}{g'(\eta_+ \psi(r(d(x))))} \frac{g(\eta_+ \psi(r(d(x))))}{\eta_+^N g(\psi(r(d(x))))} (d(x))^\mu \\
&\quad \left. + \left(\frac{N}{N+1} \right)^N \zeta_+^*(d(x)) (\mathfrak{M}(d(x)))^{-1} \prod_{i=1}^{N-1} \frac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \right\} \\
&\geq -\eta_+^N (\psi'(r(d(x))))^{N-1} \psi''(r(d(x))) (d(x))^{-(N+1)} \tilde{L}^N(d(x)) \mathfrak{M}(d(x)) \sum_{i=1}^3 I_{i+}(d(x)) > 0,
\end{aligned}$$

i.e., $\underline{u}_\varepsilon$ is a subsolution of Eq. (1.1) in $\Omega_{\delta_\varepsilon}$. Moreover, it follows by Lemma 4.3 that for $i = 1, \dots, N$

$$\begin{aligned}
S_i(D^2 \underline{u}_\varepsilon(x)) &= \eta_+^i \left(\frac{N}{N+1} \right)^i (\psi'(r(d(x))))^i (d(x))^{-i} \tilde{L}^i(d(x)) \left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{-\frac{i}{N+1}} \\
&\quad \times (1 + (C_+^* + \varepsilon) \mathfrak{M}(d(x)) + \beta(C_+^* + \varepsilon) (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}})^i S_i(\epsilon_1, \dots, \epsilon_{N-1}) \\
&\quad - \eta_+^i \left(\frac{N}{N+1} \right)^i (\psi'(r(d(x))))^{i-1} \psi''(r(d(x))) (d(x))^{-(i+1)} \tilde{L}^i(d(x)) \left(\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds \right)^{\frac{N-i}{N+1}} \\
&\quad \times (1 + (C_+^* + \varepsilon) \mathfrak{M}(d(x)) + \beta(C_+^* + \varepsilon) (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}})^{i-1} \left[\left(\frac{N}{N+1} - \frac{1}{N+1} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \right) \right. \\
&\quad \times (1 + (C_+^* + \varepsilon) \mathfrak{M}(d(x))) \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \left(\frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \\
&\quad \times (1 + (C_+^* + \varepsilon) \mathfrak{M}(d(x))) + \frac{\beta N}{N+1} (C_+^* + \varepsilon) \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} \frac{\psi'(r(d(x)))r(d(x))}{\psi(r(d(x)))} \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} \\
&\quad \times (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}} + \frac{\beta N}{N+1} (C_+^* + \varepsilon) (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}} \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \frac{\beta(\beta+1)N(C_+^* + \varepsilon)}{N+1} \\
&\quad \times \frac{\psi'(r(d(x)))r(d(x))}{\psi(r(d(x)))} \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} (\mathfrak{M}(d(x)))^{\frac{\beta+2}{\beta}} \frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \beta(C_+^* + \varepsilon) \\
&\quad \times \left. \frac{\psi'(r(d(x)))}{\psi''(r(d(x)))r(d(x))} (\mathfrak{M}(d(x)))^{\frac{\beta+1}{\beta}} \left(\frac{\tilde{L}(d(x))}{\int_0^{d(x)} \frac{\tilde{L}(s)}{s} ds} + \frac{\tilde{L}'(d(x))d(x)}{\tilde{L}(d(x))} - 1 \right) \right] S_{i-1}(\epsilon_1, \dots, \epsilon_{N-1}).
\end{aligned}$$

This implies that we can adjust the above positive constant δ_ε such that for any $x \in \Omega_{\delta_\varepsilon}$

$$S_i(D^2 \underline{u}_\varepsilon(x)) > 0 \text{ for } i = 1, \dots, N.$$

We obtain by Lemma 4.4 that $D^2 \underline{u}_\varepsilon$ is positive definite in $\Omega_{\delta_\varepsilon}$.

Let

$$\bar{u}_\varepsilon(x) = -\eta_- \psi(r(d(x))) (1 + (C_-^* - \varepsilon) \mathfrak{M}(d(x))), \quad x \in \Omega_{\delta_\varepsilon}$$

By the same calculation as (4.34), we obtain

$$\begin{aligned} & \det(D^2 \bar{u}_\varepsilon(x)) - b(x)g(-\bar{u}_\varepsilon(x)) \\ & \leq -\eta_-^N (\psi'(r(d(x))))^{N-1} \psi''(r(d(x)))(d(x))^{-(N+1)} \tilde{L}^N(d(x)) \mathfrak{M}(d(x)) \sum_{i=1}^3 I_{i-}(d(x)) < 0, \end{aligned}$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of Eq. (1.1) in $\Omega_{\delta_\varepsilon}$.

Let u be the unique strictly convex solution to problem (1.1). Through the same argument as Theorem 1.1, we see that there exists a large constant $M > 0$ such that

$$\underline{u}_\varepsilon(x) - Md(x) \leq u(x) \leq \bar{u}_\varepsilon(x) + Md(x), \quad x \in \Omega_{\delta_\varepsilon}.$$

Hence, we have for any $x \in \Omega_{\delta_\varepsilon}$

$$\begin{aligned} C_+^* + \varepsilon + \frac{M(\mathfrak{M}(d(x)))^{-1}d(x)}{\eta_+ \psi(r(d(x)))} & \geq \left(\frac{-u(x)}{\eta_+ \psi(r(d(x)))} - 1 \right) (\mathfrak{M}(d(x)))^{-1}; \\ C_-^* - \varepsilon - \frac{M(\mathfrak{M}(d(x)))^{-1}d(x)}{\eta_- \psi(r(d(x)))} & \leq \left(\frac{-u(x)}{\eta_- \psi(r(d(x)))} - 1 \right) (\mathfrak{M}(d(x)))^{-1}. \end{aligned}$$

We conclude from (3.17)-(3.18), Proposition 2.5 (i)-(ii) that

$$\begin{aligned} C_+^* + \varepsilon & \geq \limsup_{d(x) \rightarrow 0} \left(\frac{-u(x)}{\eta_+ \psi(r(d(x)))} - 1 \right) (\mathfrak{M}(d(x)))^{-1}; \\ C_-^* - \varepsilon & \leq \liminf_{d(x) \rightarrow 0} \left(\frac{-u(x)}{\eta_- \psi(r(d(x)))} - 1 \right) (\mathfrak{M}(d(x)))^{-1}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, the proof is finished. $\square$

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