Research Article

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Infinitely many non-radial solutions for a Choquard equation

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Abstract: In this article, we consider the non-linear Choquard equation

\[-\Delta u + V(|x|)u = \left( \int_{\mathbb{R}} \frac{|u(y)|^2}{|x-y|} \, dy \right) u \text{ in } \mathbb{R}^3,\]

where $V(r)$ is a positive bounded function. Under some proper assumptions on $V(r)$, we are able to establish the existence of infinitely many non-radial solutions.

Keywords: Choquard equation, Hardy-Littlewood-Sobolev inequality, infinitely many non-radial solutions

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1 Introduction and main results

In the past two decades, many authors have devoted to the study of existence, multiplicity, and properties of the solutions of the non-linear Choquard equation (1.1),

\[-\Delta u + V(x)u = (|x|^{-\mu} + |u|^2)u, \text{ in } \mathbb{R}^N.\]  

In a early paper [1], Lieb proved that the ground state $U$ of the equation

\[-\Delta u + u = (|x|^{-1} + |u|^2)u \text{ in } \mathbb{R}^3,\]

is radial and unique up to translations. While Lions [2] showed the existence of a sequence of radially symmetric solutions via variational methods. In [3,4], the authors proved, if $u$ is a ground state of equation (1.2), then $u$ is either positive or negative and there exist $x_0 \in \mathbb{R}^3$ and a monotone function $\nu \in C^\infty(0,\infty)$ such that for every $x \in \mathbb{R}^3$, $u(x) = \nu(|x - x_0|)$. Without loss of generality, we can suppose $U > 0$ and $x_0 = 0$, that is, $U(x) = U(|x|)$. For the non-degeneracy of the ground states, we may see [5–8]. Chen [9] proved that the ground state solution $U$ is non-degenerate, i.e., the kernel of the linearized equation

\[-\Delta \phi + (|x|^{-\mu} + |U|^2)\phi - 2|x|^{-\mu} (U\phi)U = 0\]

and can be spanned by $\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \frac{\partial U}{\partial x_3}$. Moreover,

\[|U| \leq Ce^{-\tau r},\]

where $\tau$ is an arbitrary number in $(0, 1)$, $r = |x|$, and $C$ is a positive constant depending on $\tau$. By this fact, we also have

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for some $C, \beta > 0$. For more background and recent literature of the non-linear Choquard equation, we may turn to [1–4, 7, 8, 10–14] and references therein.

The aim of the present article is to consider the following non-linear Choquard equation:

$$\begin{cases}
-\Delta u + V(|x|)u = \left( \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy \right) u \text{ in } \mathbb{R}^3, \\
u \in H^1(\mathbb{R}^3),
\end{cases}$$

where potential $V(x)$ satisfies the following assumptions:

(V) There are constants $a > 0$, $m \geq 3$, $\theta > 0$, and $V_0 > 0$ such that

$$V(r) = V_0 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right),$$

as $r \to +\infty$. (Without loss of generality, we may assume that $V_0 = 1$).

To apply variational methods, we introduce the energy functional associated with equation (1.4) by

$$J(u) = \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(|x|)|u|^2 \right) dx - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dx dy.$$

The Hardy-Littlewood-Sobolev inequality implies that $J$ is well defined on $H^1(\mathbb{R}^3)$ and belongs to $C^1$. And so $u$ is a weak solution of (1.4) if and only if $u$ is a critical point of the functional $J$.

The main result of this article is to establish the existence of infinitely many non-radial solution for (1.4) under assumption (V). The result says that

**Theorem 1.1.** Suppose that assumption (V) holds. Then equation (1.4) has infinitely many non-radial solutions.

To prove the main results, we will adopt the idea introduced by Wei and Yan in [15] to use the unique ground state $U$ of equation (1.2) to build up the approximate solutions for (1.4) with large number of bumps near the infinity. As in [15], let

$$z_j = \left( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \ldots, k,$$

and

$$r \in S_k = \left[ \left( \frac{m+1}{2\beta \pi} - \delta \right) k \ln k, \left( \frac{m+1}{2\beta \pi} + \delta \right) k \ln k \right],$$

where $m$ is the constant in the expansion for $V$, $\delta > 0$ is a small constant, and $\beta$ is given in (1.3). We denote

$$W_k(x) = \sum_{j=1}^{k} U_{z_j}(x), \quad j = 1, \ldots, k,$$

where $U_{z_j}(x) = U(x - z_j)$. Set $x = (x', x''), x' \in \mathbb{R}^2$, and $x'' \in \mathbb{R}$. Define

$$H_k = \left\{ u \in H^1(\mathbb{R}^3), \text{ u is even in } x_h, h = 2, 3, u(r \cos \theta, r \sin \theta, x'') = u \left( r \cos \left( \theta + \frac{2jn}{k} \right), r \sin \left( \theta + \frac{2jn}{k} \right), x'' \right) \right\}.$$
has a solution \( u_k \) of the form

\[
 u_k = W_k(x) + w_k,
\]

where \( w_k \in H_1, \ r_k \in S_k \), and as \( k \to +\infty \),

\[
 \int_{\mathbb{R}^3} (|\nabla w_k|^2 + w_k^2)dx \to 0.
\]

This article is organized as follows. In Section 2, we prove two basic estimates. In Section 3, we carry out the reduction. Then, we study the reduced finite dimensional problem and prove Theorem 1.2 in Section 4.

## 2 Preliminaries

Throughout this article we write \( |\cdot|_q \) for the \( L^q(\mathbb{R}^3) \)-norm, \( q \in [1, \infty] \), always assume that condition \((V)\) holds, and the norm of \( H^1(\mathbb{R}^3) \) is defined as follows:

\[
 \|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(|x|)|u|^2)dx.
\]

Let

\[
 \Omega_j = \left\{ x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R} : \frac{x'}{|x'|}, \frac{z'}{|z'|} \geq \cos \frac{\pi}{k}, \ j = 1, \ldots, k \right\}.
\]

Then, we have the following basic estimates:

**Lemma 2.1.** For any \( x \in \Omega_1 \) and \( \eta \in (0, 1) \), there is a constant \( C > 0 \), such that

\[
 \sum_{i=2}^{k} U_{2i}(x) \leq Ce^{-\beta m_1^2}e^{-\beta(1-\eta)|x-z|}.
\]

**Proof.** For any \( x \in \Omega_1 \), we have \( |x - z| \geq |x - z| \). If \( |x - z| \geq \frac{1}{2}|z_i - z_i| \), then for any \( x \in \Omega_1 \),

\[
 |x - z| \geq \frac{1}{2}|z_i - z_i|.
\]

If \( |x - z| \leq \frac{1}{2}|z_i - z_i| \), then for any \( x \in \Omega_1 \),

\[
 |x - z| \geq |z_i - z_i| - |x - z| \geq \frac{1}{2}|z_i - z_i|.
\]

So, for any \( x \in \Omega_1 \), we have

\[
 U_{2i}(x) \leq Ce^{-\beta|x-z|} \leq Ce^{-\beta|x-z|} = Ce^{-\beta|x-z|}e^{-\beta(1-\eta)|x-z|} \leq Ce^{-\beta|x-z|}e^{-\beta(1-\eta)|x-z|}.
\]

Thus,

\[
 \sum_{i=2}^{k} U_{2i}(x) \leq Ce^{-\beta(1-\eta)|x-z|} \sum_{i=2}^{k} e^{-\beta|x-z|} \leq Ce^{-\beta(1-\eta)|x-z|} \sum_{i=2}^{k} e^{-\beta|x-z|} \sin \frac{\pi}{2} \leq Ce^{-\beta(1-\eta)|x-z|}e^{-\beta m_1^2}. \quad \square
\]

**Lemma 2.2.** For any \( q \geq 1 \), there is a constant \( \sigma > 0 \), such that
\[ W_i^q(x) = U_i^q(x) + O \left( \frac{1}{r^q} e^{-\frac{1}{2} |r(x-z)|} \right), \quad \forall x \in \Omega_i. \] (2.2)

**Proof.** In view of the symmetry, we only estimate the function \( W_i^q(x) \) in \( \Omega_i \). Notice that

\[ W_i^q(x) = U_i^q(x) + O \left( U_i^{q-1}(x) \sum_{j=1}^k U_j(x) + \left( \sum_{j=1}^k U_j(x) \right)^q \right). \]

By Lemma 2.1, we have

\[ |x-z| \geq \frac{1}{2} |z_1-z_2|, \quad \forall x \in \Omega_i. \] (2.3)

Thus for any \( \alpha > 0 \), using (2.3), we know

\[ \sum_{i=1}^k U_i^q(x) \leq C \sum_{i=1}^k e^{-q\alpha |x-z_i|} \leq C \sum_{i=1}^k e^{-\frac{q}{k^q} |x-z_i|} \leq Ce^{-\frac{q}{k^q}}, \quad \forall x \in \Omega_i. \]

Therefore,

\[ U_i^{q-1}(x) \sum_{j=1}^k U_j(x) \leq U_i^{q-1}(x) \sum_{j=1}^k U_j(x) \leq C e^{-\frac{q}{k^q}}, \quad \forall x \in \Omega_i, \]

and

\[ \left( \sum_{i=1}^k U_i(x) \right)^q \leq U_i^q(x) \left( \sum_{i=1}^k U_j(x) \right)^q \leq C e^{-\frac{q}{k^q}}, \quad \forall x \in \Omega_i. \]

Consequently, (2.2) follows. \( \square \)

### 3 The reduction argument

Let \( T_j = \frac{\partial U_j}{\partial t}, \ j = 1, \ldots, k \), where \( z_j \) is given in (1.5). We denote by

\[ E = \left\{ u \in H_0 : \int \int_{R^3 \times R^3} \frac{|U_i(x)|^2 T_j(y) u(y)}{|x-y|} dxdy + 2 \int \int_{R^3 \times R^3} \frac{U_i(x) u(x) T_j(y)}{|x-y|} dxdy = 0, \quad j = 1, \ldots, k \right\}. \]

Applying Lemma 2.2, there exists a bounded linear operator \( L \) from \( E \) to \( E \) such that

\[ \langle Lv_1, v_2 \rangle = \int \int_{R^3} (V_1 V_2 + V(|x|) v_1 v_2) dx - \int \int_{R^3} \frac{|W(x)|^2 v_1(y) v_2(y)}{|x-y|} dxdy \]

\[ - 2 \int \int_{R^3} \frac{W(x) v_1(x) W(y) v_2(y)}{|x-y|} dxdy, \quad v_1, v_2 \in E. \]

Thus, we have

**Lemma 3.1.** There is a constant \( C' > 0 \), independent of \( k \), such that for any \( r \in S_k \),

\[ \|Lv\| \leq C'\|v\|, \quad v \in E. \]

Next, we show that \( L \) is invertible in \( E \).
Lemma 3.2. There is a constant $C' > 0$, independent of $k$, such that for any $r \in S_k$, 
\[ \|Lv\| \geq C' \|v\|, \quad v \in E. \]

Proof. Suppose to the contrary that there are $k \to +\infty$, $\eta \in S_k$, and $v_k \in E$, with 
\[ \|Lv_k\| = o(1) \|v_k\|. \]

Then
\[ \langle Lv_k, \psi \rangle = o(1) \|v_k\| \|\psi\|, \quad \forall \psi \in E. \quad (3.1) \]

We may assume that $\|v_k\|^2 = k$.

By symmetry, we see from (3.1),
\[ \int_{\Omega_1} (\nabla v_k \nabla \psi + V(|x|)v_k \psi) \, dx - \int_{\Omega_1} \frac{|W(x)|^2 v_k(y)\psi(y)}{|x - y|} \, dy - 2 \int_{\Omega_1} \frac{W(x)v_k(x)W(y)\psi(y)}{|x - y|} \, dy \]
\[ = \frac{1}{k} \langle Lv_k, \psi \rangle = o\left(\frac{1}{\sqrt{k}} \right) \|v_k\|, \quad \forall \psi \in E. \quad (3.2) \]

In particular,
\[ \int_{\Omega_1} (|\nabla v_k|^2 + V(|x|)|v_k|^2) \, dx - \int_{\Omega_1} \frac{|W(x)|^2 v_k(y)^2}{|x - y|} \, dy - 2 \int_{\Omega_1} \frac{W(x)v_k(x)W(y)v_k(y)}{|x - y|} \, dy = o(1) \]
and
\[ \int_{\Omega_1} (|\nabla v_k|^2 + V(|x|)|v_k|^2) \, dx = 1. \quad (3.3) \]

Let $\varphi_k(x) = v_k(x - z_i)$. Then for any $R > 0$, since $|z_i - z_i| = r \sin \frac{\pi}{k} \geq \frac{m}{4} \ln k$, we see that $B_R(z_i) \subset \Omega_1$. As a result, from (3.3), we find that for any $R > 0$,
\[ \int_{B_R(z_i)} (|\nabla \varphi_k|^2 + V(|x|)|\varphi_k|^2) \, dx \leq 1. \]

So, we may assume that there is a $v \in H^1(\mathbb{R}^3)$, such that as $k \to +\infty$,
\[ \varphi_k \to v, \quad \text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3), \]
and
\[ \varphi_k \to v, \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3). \]

Since $\varphi_k$ is even in $x_h$, $h = 2, 3$, it is easy to see that $v$ is even in $x_h$, $h = 2, 3$. On the other hand, from
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|U_z(x)|^2 T(y) v_k(y)}{|x - y|} \, dx \, dy + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U_z(x)v_k(x)U_z(y)T(y)}{|x - y|} \, dx \, dy = 0, \]
we obtain
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|U(x)|^2 \partial U(y)\varphi_k(y)}{|x - y|} \, dx \, dy + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U(x)\varphi_k(x)U(y)\partial U(y)}{|x - y|} \, dx \, dy = 0. \]

So, $v$ satisfies
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|U(x)|^2 \partial U(y)\varphi(y)}{|x - y|} \, dx \, dy + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U(x)v(x)U(y)\partial U(y)}{|x - y|} \, dx \, dy = 0. \quad (3.4) \]
Now, we claim that $v$ satisfies

$$-\Delta v + v = \left( \int_{\mathbb{R}^3} \frac{|U(y)|^2}{|x-y|} \, dy \right) v + 2 \left( \int_{\mathbb{R}^3} \frac{U(y)v(y)}{|x-y|} \, dy \right) u \quad \text{in } \mathbb{R}^3. \quad (3.5)$$

Define

$$\overline{E} = \left\{ u : u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|U(x)|^2 \partial^2 U(y)}{\partial z^2} u(y) \, dy \right) \, dx + 2 \int_{\mathbb{R}^3} \frac{U(x)u(x)U(y)\partial^2 U(y)}{\partial z^2} \, dy = 0 \right\}.$$

For any $R > 0$, let $\psi \in C_0^\infty(B_R(0)) \cap \overline{E}$ be any function, satisfying that $\psi$ is even in $x_h$, $h = 2, 3$. Then $\psi_k = \psi(x-z_i) \in C_0^\infty(B_R(0))$. With the argument in [15] we find

$$\int_{B_1} (\nabla \psi_k \nabla + V(x)\psi_k) \, dx \to \int_{\mathbb{R}^3} (\nabla \psi + v\psi) \, dx.$$

By Lemma 2.2, we know

$$\left( \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{|W(x)|^2 \psi_k(y)\psi(y)}{|x-y|} \, dy \right) \, dx \right) \to \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|U(x)|^2 \psi(y)}{|x-y|} \, dy \right) \, dx.$$

Similarly,

$$\left( \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{|W(x)\psi_k(y)W(y)\psi(y)}{|x-y|} \, dy \right) \, dx \right) \to \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|U(x)v(x)U(y)\psi(y)}{|x-y|} \, dy \right) \, dx.$$

Thus, we have

$$\int_{\mathbb{R}^3} (\nabla \psi + v\psi) \, dx = \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|U(x)|^2 \psi(y)}{|x-y|} \, dy \right) \, dx + 2 \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{U(x)v(x)U(y)\psi(y)}{|x-y|} \, dy \right) \, dx. \quad (3.6)$$

On the other hand, since $v$ is even in $x_h$, $h = 2, 3$, (3.6) holds for any function $\psi \in C_0^\infty(\mathbb{R}^3)$, which is odd in $x_h$, $h = 2, 3$. Therefore, (3.6) holds for any $\psi \in C_0^\infty(B_R(0)) \cap \overline{E}$. By the density of $C_0^\infty(\mathbb{R}^3)$ in $H^1(\mathbb{R}^3)$, it is easy to show that

$$\int_{\mathbb{R}^3} (\nabla \psi + v\psi) \, dx - \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{|U(x)|^2 \psi(y)}{|x-y|} \, dy \right) \, dx = 2 \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \frac{U(x)v(x)U(y)\psi(y)}{|x-y|} \, dy \right) \, dx = 0, \quad \forall \psi \in \overline{E}. \quad (3.7)$$

But (3.7) holds for $\psi = \frac{\partial U}{\partial z_j}$, Thus, (3.7) is true for any $\psi \in H^1(\mathbb{R}^3)$. So, we have proved (3.5). Since $U$ is non-degenerate, we see $v = c \frac{\partial U}{\partial z_i}$ that because $v$ is even in $x_h$, $h = 2, 3$. From (3.4), we find

$$v = 0.$$

As a result,

$$\int_{B_R(z_i)} v^2 \, dx = o(1), \quad \forall R > 0.$$

On the other hand, it follows from Lemma 2.1 that for any small $\eta > 0$, there is a constant $C > 0$, such that

$$W_\eta(x) \leq Ce^{-(1-\eta)|x-z_i|}, \quad x \in \Omega_1. \quad (3.8)$$

Thus,

$$o(1) = \int_{\Omega_1} (|\nabla v_k|^2 + V(|x|)|v_k|^2) \, dx - \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{|W(x)|^2 |v_k(y)|^2}{|x-y|} \, dy \right) \, dx - 2 \int_{\Omega_1} \left( \int_{\mathbb{R}^3} \frac{W(x)v_k(x)W(y)v_k(y)}{|x-y|} \, dy \right) \, dx$$
\[
\geq \int_{\Omega_i} (|\nabla v_k|^2 + V(|x|)|v_k|^2)dx + o(1) + O(e^{-(1-\eta)\|v\|}) \int_{\Omega_i} |v_k|^2 dx
\]
\[
\geq \frac{1}{2} \int_{\Omega_i} (|\nabla v_k|^2 + V(|x|)|v_k|^2)dx + o(1).
\]

This is a contradiction to (3.3). \(\square\)

Let
\[
I(\varphi) = J(W_f + \varphi), \quad \varphi \in E.
\]

Expand \(I(\varphi)\) as follows:
\[
I(\varphi) = I(0) + I(\varphi) + \frac{1}{2} \langle L\varphi, \varphi \rangle + R(\varphi), \quad \varphi \in E,
\]
where
\[
I(\varphi) = \sum_{j=1}^{k} \int_{\mathbb{R}^3} (V(|x|) - 1)U_j \varphi dx - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|W(x)|^2 W(y)\varphi(y)}{|x-y|} dxdy + \sum_{j=1}^{k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|U_j(x)|^2 U_j(y)\varphi(y)}{|x-y|} dxdy
\]

and
\[
R(\varphi) = -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W(x)\varphi(x)\varphi^2(y)}{|x-y|} dxdy - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\varphi^2(x)\varphi^2(y)}{|x-y|} dxdy.
\]

In order to find a critical point \(\varphi \in E\) for \(I(\varphi)\), we need to estimate each term in the expansion.

**Lemma 3.3.** There is a constant \(C > 0\), independent of \(k\), such that for any \(\varphi \in H^1(\mathbb{R}^3)\),
\[
|R(\varphi)| \leq C(\|\varphi\|^3 + \|\varphi\|), \quad (3.9)
\]
\[
\|R'(\varphi)\| \leq C(\|\varphi\|^2 + \|\varphi\|^2), \quad (3.10)
\]
and
\[
\|R''(\varphi)\| \leq C(\|\varphi\| + \|\varphi\|^2). \quad (3.11)
\]

**Proof.** Similar to the proof of (3.1), we have that for any \(v, w \in H^1(\mathbb{R}^3)\)
\[
|R(\varphi)| \leq C(\|\varphi\|^3 + \|\varphi\|),
\]
\[
|\langle R'(\varphi), v \rangle| = \left| -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W(x)v(x)\varphi^2(y)}{|x-y|} dxdy - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W(x)\varphi(x)v(y)\varphi(y)}{|x-y|} dxdy - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi^2(x)v(y)\varphi(y)dxdy \right|
\]
\[
\leq C(\|\varphi\|^2 + \|\varphi\|^3)\|v\|,
\]
and
\[
|\langle R''(\varphi)v, w \rangle| = \left| -2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W(x)v(x)\varphi(y)w(y)}{|x-y|} dxdy - 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W(x)w(x)\varphi(y)v(y)}{|x-y|} dxdy
\]
\[
-2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\varphi^2(x)w(y)v(y)}{|x-y|} dxdy - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\varphi^2(x)v(y)w(y)}{|x-y|} dxdy - 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\varphi^2(x)v(y)w(y)}{|x-y|} dxdy \right|
\]
\[
\leq C(\|\varphi\| + \|\varphi\|^2)\|v\|\|w\|.
\]

So, (3.10) and (3.11) follow. \(\square\)
Lemma 3.4. Moreover, there is a small $\sigma > 0$, such that
\[
\|u\| \leq \frac{C}{k^{\frac{m}{2} + \sigma}}.
\]

Proof. By the symmetry of the problem,
\[
\sum_{j=1}^{k} \int_{\mathbb{R}^3} (V(|x|) - 1) U_j \varphi \, dx = k \int_{\mathbb{R}^3} (V(|x|) - 1) U_j \varphi \, dx
\]
\[
= k \int_{\mathbb{R}^3} (V(|x - z_i|) - 1) U_j(x) \varphi(x - z_i) \, dx
\]
\[
\leq kO \left( \frac{1}{r^m} \right) \|\varphi\| \leq \frac{C}{k^{\frac{m}{2} + \sigma}} \|\varphi\|
\]
because $m > 1$.

By Lemma 2.1, we have
\[
\sum_{j \neq i} U_j \leq C e^{-\beta n} \quad \forall x \in \Omega_i,
\]
and
\[
\sum_{i,j} \int_{\Omega_i} \frac{|U_j|}{|x - y|} \, dy \leq C \sum_{i,j} \int_{\Omega_i} \frac{|U_j|}{|x - y|} \, dy e^{-\frac{\beta}{|z_i - z_j|}}
\]
\[
\leq C \sum_{i,j} e^{-\frac{\beta |z_i - z_j|}{|x - y|}} \int_{\Omega_i} \frac{|U_j|}{|x - y|} \, dy
\]
\[
\leq C e^{-\beta n} \sum_{i,j} \int_{\Omega_i} |U_j| \, dy.
\]

Combining these with
\[
W_i(x) = U_i(x) + O \left( \frac{1}{r^m} e^{-\frac{\beta}{|x - z_i|}} \right), \quad \forall x \in \Omega_i,
\]
we have,
\[
\left| \sum_{j=1}^{k} \int_{\mathbb{R}^3} \frac{|U_j(x)|^2 U_j(y) \varphi(y)}{|x - y|} \, dx \right| \leq \int_{\mathbb{R}^3} \frac{|W_i(x)|^2 W_i(y) \varphi(y)}{|x - y|} \, dx
\]
\[
= \left| \sum_{j=1}^{k} \int_{\mathbb{R}^3} \frac{|U_j(x)|^2 \left( \sum_{j \neq i} U_j(y) \right) \varphi(y)}{|x - y|} \, dx \right| + 2 \sum_{j=1}^{k} \int_{\Omega_i} \frac{|U_j(x) U_j(x) W_i(y) \varphi(y)}{|x - y|} \, dx
\]
\[
= \left| \sum_{j=1}^{k} \int_{\Omega_i} \frac{|U_j(x)|^2 \left( \sum_{j \neq i} U_j(y) \right) \varphi(y)}{|x - y|} \, dx \right| + 2 \sum_{j=1}^{k} \int_{\Omega_i} \frac{|U_j(x) U_j(x) W_i(y) \varphi(y)}{|x - y|} \, dx
\]
\[
\leq k e^{-\beta n} \sum_{j=1}^{k} \int_{\Omega_i} \frac{|U_j(x)|^2 \varphi(y)}{|x - y|} \, dx + 2 k e^{-\beta n} \sum_{j=1}^{k} \int_{\Omega_i} \frac{|U_j(x)| W_i(y) \varphi(y)}{|x - y|} \, dx
\]
\[
\leq C k^2 e^{-\beta n} \|\varphi\|.
\]
Since \( m \geq 3 \), we see
\[
k^2e^{\frac{\beta m}{1}} \leq k^2e^{-\beta m(\frac{m-1}{m})\ln k} \leq \frac{C}{k^{\frac{m-1}{m+1}}},
\]
Thus, we have
\[
\|l_k\| \leq \frac{C}{k^{\frac{m-1}{m+1}}}.
\]

Proposition 3.5. There is an integer \( k_0 > 0 \), such that for each \( k \geq k_0 \), there is a \( C^1 \) map from \( S_k \) to \( H_k : \varphi = \varphi(r) \), \( r = |x| \) satisfying \( \varphi \in E \), and
\[
\left\langle \frac{\delta l(\varphi)}{\delta \varphi}, \psi \right\rangle = 0, \quad \forall \psi \in E.
\]
Moreover, there is a small \( \sigma > 0 \), such that
\[
\|\varphi\| \leq \frac{C}{k^{\frac{m-1}{m+1}}}.
\] (3.13)

Proof. Since \( l(\varphi) \) is a bounded linear functional in \( E \), we know that there is an \( l_k \in E \), such that
\[
l(\varphi) = \langle l_k, \varphi \rangle.
\]
Thus, finding a critical point for \( I(\varphi) \) is equivalent to solving
\[
l_k + L(\varphi) + R'(\varphi) = 0.
\] (3.14)
By Lemma 3.2, \( L \) is invertible. Thus, (3.14) can be rewritten as
\[
\varphi = A(\varphi) = -L^{-1}l_k - L^{-1}R'(\varphi).
\]
Let
\[
S = \left\{ \varphi : \varphi \in E, \|\varphi\| \leq \frac{1}{k^{\frac{m-1}{m}}} \right\}.
\]
So, from Lemma 3.4,
\[
\|A(\varphi)\| \leq C\|l_k\| + C(\|\varphi\|^2 + \|\varphi\|^3) \leq \frac{C}{k^{\frac{m-1}{m+1}}} + \frac{C}{k^{m-3}} + \frac{C}{k^{\frac{m-1}{2}}} \leq \frac{1}{k^{\frac{m-1}{2}}}.
\]
Thus, \( A \) maps \( S \) into \( S \).
By (3.11), we have,
\[
\|A(\varphi_1) - A(\varphi_2)\| = \|L^{-1}R'(\varphi_1) - L^{-1}R'(\varphi_2)\| \leq C(\|\varphi_1\| + \|\varphi_1\|^2 + \|\varphi_1\|^3)(\|\varphi_1\| \leq \frac{1}{2}\|\varphi_1 - \varphi_2\|.
\]
So, we have proved that \( A \) is a contraction map from \( S \) to \( S \). Therefore, the result follows from the contraction mapping theorem. \( \square \)

4 Proof of Theorem 1.2

Lemma 4.1. There is a small constant \( \sigma > 0 \), such that
\[
J(W_i) = k \left( A_k + \frac{B_k}{r^m} - kB_2e^{\frac{2m}{r}} + O \left( \frac{1}{k^{2m-2+\sigma}} \right) \right).
\]
where

\[ A_k = \left( \frac{1}{2} - \frac{k}{4} \right) \int_{R^1 \times R^1} \frac{U^4(x)U^4(y)}{|x - y|} \, dx \, dy, \]

\[ B_1 = \frac{a}{2} \int_{R^1} U^2 \, dx \]

and \( B_2 > 0 \) is a positive constant.

**Proof.** Using the symmetry,

\[
\int_{R^1} (|V| W_1^2 + W_2^2) \, dx = \sum_{j=1}^{k} \sum_{i=1}^{k} \int_{R^1 \times R^1} \frac{|U_j(x)|^2 U_j(y) U_i(y)}{|x - y|} \, dx \, dy
\]

\[
= k \int_{R^1 \times R^1} \frac{|U(x)|^2 |U(y)|^2}{|x - y|} \, dx \, dy + k \sum_{i=2}^{k} \int_{R^1 \times R^1} \frac{U_i^2(x) U_i(y) U_i(y)}{|x - y|} \, dx \, dy.
\]

It follows from Lemma 2.1 that

\[
\int_{\Omega_i} (|V(x)| - 1) W_2^2 \, dx = k \int_{\Omega_i} \left( |V(x)| - 1 \right) \left( U_{z_i} + O \left( e^{-\frac{1}{2} |z_i - z_i|} e^{-\frac{1}{2} |x - z_i|} \right) \right)^2 \, dx
\]

\[
= k \int_{\Omega_i} \left( |V(x)| - 1 \right) U_{z_i}^2 \, dx + kO \left( \int_{\Omega_i} \left| V(x) \right| - 1 \mid e^{-\frac{1}{2} |z_i - z_i|} e^{-\frac{1}{2} |x - z_i|} \, dx \right)
\]

\[
= k \left( B_1 + O \left( \frac{1}{k^{m+\theta}} \right) \right).
\]

Using Lemma 2.1 and the fact

\[
\sum_{i=2}^{k} e^{-\frac{1}{2} |z_i - z_i|} \leq C e^{-\frac{2\theta k^2}{r^2}} \leq \frac{C}{k^{m+1-r}}, \quad \text{for any } r \in S_k,
\]

we obtain that if \( r > 0 \) is small enough, for any \( x \in \Omega_i \),

\[
|W_2(x)|^2 = U_{z_i}^2(x) + 2 U_{z_i}(x) \sum_{i=2}^{k} U_{z_i}(x) + \left( \sum_{i=2}^{k} U_{z_i}(x) \right)^2 = U_{z_i}^2(x) + 2 U_{z_i}(x) \sum_{i=2}^{k} U_{z_i}(x) + O \left( \frac{1}{k^{2m+\theta}} \right).
\]

Thus, we have

\[
\int_{R^1 \times R^1} \frac{|W_1(x)|^2 |W_2(y)|^2}{|x - y|} \, dx \, dy
\]

\[
= k^2 \int_{\Omega_i \times \Omega_i} \frac{|W_1(x)|^2 |W_2(y)|^2}{|x - y|} \, dx \, dy
\]

\[
= k^2 \int_{R^1 \times R^1} \frac{U_2(x) U_2(y)}{|x - y|} \, dx \, dy + 4k^2 \sum_{i=2}^{k} \int_{R^1 \times R^1} \frac{U_i^2(x) U_i(y) U_i(y)}{|x - y|} \, dx \, dy
\]

\[
+ 4k^2 \sum_{i=2}^{k} \int_{R^1 \times R^1} \frac{U_i(x) U_i(x) U_i(y) U_i(y)}{|x - y|} \, dx \, dy + k^4 O \left( \frac{1}{k^{2m+\theta}} \right).
\]
On the other hand, we have
\[
\sum_{i=2}^{k} \sum_{j=2}^{k} \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \frac{U_{x}(x)U_{y}(x)U_{y}(y)}{|x-y|} \, dx \, dy = B_{1}^{2} \sum_{i=2}^{k} e^{-\beta_{i}|z_{i}|} + \sum_{j=2}^{k} e^{-\beta_{j}|z_{j}|} = B_{1}^{2} e^{-4\eta\eta_{0}}
\]
and
\[
\sum_{i=2}^{k} \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \frac{U_{x}^{2}(x)U_{y}(y)}{|x-y|} \, dx \, dy = B_{1}^{2} \sum_{i=2}^{k} e^{-\beta_{i}|z_{i}|} = B_{1}^{2} e^{-\frac{2m}{r}}.
\]

So,
\[
J(W_{r}) = k \left( A_{k} + \frac{B_{1}}{r^{m}} - kB_{2}e^{-\frac{2m}{r}} + O\left(\frac{1}{k^{2m-2}}\right) \right),
\]
where
\[
A_{k} = \left( \frac{1}{2} - \frac{k}{4} \right) \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \frac{U^{2}(x)U^{2}(y)}{|x-y|} \, dx \, dy,
\]
\[
B_{1} = \frac{a}{2} \int_{\mathbb{R}^{3}} U^{2}dx.
\]
and \(B_{2} > 0\) is a positive constant. \(\square\)

We are ready to prove Theorem 1.2. Let \(\varphi_{r} = \varphi(r)\) be the map obtained in Proposition 3.5. Define
\[
F(r) = J(W_{r} + \varphi_{r}), \quad \forall r \in S_{k}.
\]

With the same argument in [15,16], we can easily check that for \(k\) sufficiently large, if \(r\) is a critical point of \(F(r)\), then \(W_{r} + \varphi_{r}\) is a solution of (1.4).

**Proof of Theorem 1.2.** It follows from Lemmas 3.1 and 3.3 that
\[
\|L\varphi_{r}\| \leq C\|\varphi_{r}\|, \quad |R(\varphi)| \leq C(|\varphi|^{3} + \|\varphi\|^{4}).
\]
So, Proposition 3.5 and Lemma 4.1 give
\[
F(r) = J(W_{r}) + \frac{1}{2} \langle L\varphi_{r}, \varphi_{r} \rangle + \mathcal{R}L\varphi_{r}\| \varphi_{r}\| + \|\varphi_{r}\|^{3} + \|\varphi_{r}\|^{4})
\]
\[
= J(W_{r}) + O\left(\frac{1}{k^{2m-3}}\right).
\]

Consider
\[
\max\{F(r) : r \in S_{k}\}, \quad (4.1)
\]
where \(S_{k}\) is defined in Section 1. Since the function
\[
\frac{B_{1}}{r^{m}} - kB_{2}e^{-\frac{2m}{r}}
\]
has a maximum point
\[
\hat{r}_{k} = \left(\frac{m + 1}{2\beta\pi} + o(1)\right)k \ln k,
\]

which is an interior point of \( S_k \), it is easy to check that (4.1) is achieved by some \( r_k \), which is in the interior of \( S_k \). Thus, \( r_k \) is a critical point of \( F(r) \). As a result,

\[
W_r + \varphi_{r_k}
\]

is a solution of (1.4).

\[\square\]

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## References


