Research Article

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Global attractors of the degenerate fractional Kirchhoff wave equation with structural damping or strong damping

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Abstract: This article deals with the degenerate fractional Kirchhoff wave equation with structural damping or strong damping. The well-posedness and the existence of global attractor in the natural energy space by virtue of the Faedo-Galerkin method and energy estimates are proved. It is worth mentioning that the results of this article cover the case of possible degeneration (or even negativity) of the stiffness coefficient. Moreover, under further suitable assumptions, the fractal dimension of the global attractor is shown to be infinite by using $\mathbb{Z}_2$ index theory.

Keywords: fractional Kirchhoff wave equation, degenerate, structural damping, strong damping, well-posedness, global attractor, fractal dimension

MSC 2020: 37L30, 37L15, 35B40, 35B41

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d$ is a positive integer) be a bounded domain with smooth boundary $\partial \Omega$, we consider the following fractional Kirchhoff wave equation with damping:

$$
\begin{cases}
\begin{align*}
&u_{tt} + \phi(\|\Delta^{a/2} u\|^2)(\Delta^{a/2} u + (-\Delta)^{a} u_t + f(u)) = g(x), \quad x \in \Omega, \quad t > 0, \\
&u = 0, \quad x \in \partial \Omega, \quad t > 0, \\
&u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{align*}
\end{cases}
$$

(1.1)

where the parameter $\alpha$ satisfies $1/2 \leq \alpha \leq 1$, $\Delta$ is the Laplace operator, and the fractional Laplace operator $(-\Delta)^{\beta}$ ($\beta = \alpha$ or $\beta = \frac{\alpha}{2}$) is defined as

$$
(-\Delta)^{\beta} u = \sum_{j=1}^{\infty} \lambda_j^{\beta} (u, e_j) e_j,
$$

where $\{\lambda_j\}_{j=1}^{\infty}$ is the set of eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary, $e_j$ is the corresponding eigenfunction to $\lambda_j$ such that

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j
\end{cases}$$

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(see (1.17) below), \(\phi\) and \(f\) are nonlinear scalar functions to be specified later, \(g \in L^2(\Omega)\), and \((\cdot, \cdot)\) and \(||\cdot||\) stand for the inner product and norm of \(L^2(\Omega)\), respectively.

Problem (1.1) models several interesting phenomena studied in mathematical physics. One-dimensional model (1.1) with no damping term \((-\Delta)^s u\) and source term \(f(u)\) was introduced by Kirchhoff [1] to describe the transversal vibrations of a stretched string, and the model reads as follows:

\[
\rho \phi u_{tt} = \left[ a_0 + \frac{EH}{2L} \int_0^L (u_x)^2 dx \right] u_{xx} + g,
\]

where \(L\) is the length of the string, \(0 < x < L\) is the space coordinate, \(t \geq 0\) is the time, \(u = u(x, t)\) denotes the transverse displacement of the point \(x\) at the instant \(t\), \(E\) is related to the intrinsic properties of the string (such as Young’s modulus, the string cross-sectional area, and some of the other physical quantities), \(\rho\) is the mass density, \(h\) is the cross-section area, \(a_0\) is the initial axial tension, and \(g\) is the external force. In this model, Kirchhoff used the integral term \(\int_0^L (u_x)^2 dx\) to present the average change in tension along the vibrating string taking into account the change of the string’s length. Moreover, such models can be used in tension modulations for the sound synthesis and the control practice of mechanical systems, see, for example, [2] and [3]. Further details and physical models described by Kirchhoff’s classical theory can be found in [4–10]. Since the previous study [1], the fitness and asymptotic properties of this type of models with different types of linear dissipation have been extensively studied, we refer to [11,12] for asymptotic stability, [12–18] for global existence and decay rate estimation, [12,16,19–22] for blow-up of solutions, [12,23,24] for global attractors, and [12,25,26] for steady-state solutions.

System (1.1) is said to be non-degenerate if \(\phi\) satisfies the strict hyperbolicity condition \(\phi(s) \geq a\) for some constant \(a > 0\), while degenerate if \(\phi\) just satisfies the degenerate hyperbolicity condition \(\phi(s) \geq 0\) for all \(s \geq 0\). Moreover, the term \((-\Delta)^s u\) is called strong, structural, or weak damping if \(a = 1\), \(a \in (0, 1)\), or \(a = 0\), and the strong damping models the flow of viscoelastic fluids as well as in the theory of heat conduction, see [27–30]; the structural damping model waves propagate through a lossy media, for example, fractal rock layers, human tissues, and different biomedical materials, see [31,32]; the weak damping models various oscillatory processes in many areas of modern mathematical physics including electrodynamics, quantum mechanics, nonlinear elasticity, see [33–35]. Moreover, partial differential equations with fractional Laplace operator \((-\Delta)^s\), \(0 < a < 1\) has gained great attention. This type of operator arises in many different applications, such as continuum mechanics, phase transition phenomena, population dynamics, and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes [36–38]. We refer the readers to [39–43] for some books on this topic, [41,44] for the fractional Sobolev spaces, [45–50] for fractional Laplace and \(p\)-Laplace equations, [19,25,51–54] for fractional Kirchhoff equations, and [26,55,56] for fractional Choquard equations.

Now we review some related studies. First we recall the following Kirchhoff wave model with damping

\[
\begin{cases}
\phi u_{tt} - (\|\nabla u\|^2)\Delta u + \sigma(\|\nabla u\|^2)\Delta^\theta u + f(u) = g(x), & x \in \Omega, \ t > 0, \\
u = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \Omega, \\
u_0(x) = \Phi_0(x), & x \in \Omega, \end{cases} \tag{1.2}
\]

where \(\theta \in [0, 1]\) is a constant. Well-posedness and global attractor for the aforementioned problem have been studied intensively in the past years, we refer to [6,13,16,57–61] and references therein, but most of the papers dealt with the case \(\theta = 0\) or \(\theta = 1\), \(\sigma(s) \equiv \sigma_0 > 0\) is a constant, and \(\phi(s) = \phi_0 + \phi_1 s^\gamma\), where \(\gamma > 0\) and \(\phi_1 > 0\) are constants, and \(\phi_0 \geq 0\) (degenerate and non-degenerate case) for well-posedness, \(\phi_0 > 0\) (non-degenerate case) for global attractor. For \(\theta \in (0, 1)\), problem (1.2) was studied in [63] with \(\sigma(s) \equiv \sigma_0 > 0\) is a constant and \(f(u) \equiv 0\). The main result in that article states the existence of weak solutions for \(\theta \in (0, 1/2)\) with initial data \(u_0 \in H^{max[1/4, \theta]/2}(\Omega)\) and \(u_0 \in L^2(\Omega)\) and their uniqueness for \(\theta \in [1/2, 1]\). For \(\theta < 1\) and \(f(u) \not\equiv 0\), recently, Li and Yang [64] considered (1.2) for non-degenerate case for \(\theta \in [1/2, 1]\), and they proved the well-posedness and longtime dynamics such as optimal global attractor, optimal exponential attractor, and upper semi-continuous of the global attractors with respect to \(\theta\).
For degenerate case, Chueshov [65] considered problem (1.2) with \( \theta = 1 \), i.e.,

\[
\begin{cases}
    u_t - \phi(\|u\|^2) \Delta u - \sigma(\|u\|^2) \Delta u + f(u) = g(x), & x \in \Omega, \ t > 0, \\
    u = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) = u_0(x), & u_t(x, 0) = u_t(x), \ x \in \Omega
\end{cases}
\]

(1.3)

with the assumptions that

\((\phi\sigma_{(65)})\) The damping \( \sigma \) and the stiffness \( \phi \) are \( C^1 \) functions on the semi-axis \( \mathbb{R}_+ = [0, \infty) \). Moreover, \( \sigma(s) > 0 \) for all \( \mathbb{R}_+ \) and there exist \( \eta_0 \geq 0 \) and \( c_2 \geq 0 \) such that

\[
\int_0^s (\phi(\xi) \xi^2 + \eta_0 \sigma(\xi)) d\xi \to \infty \text{ as } s \to \infty,
\]

(1.4)

and

\[
s\phi(s) + c_1 \int_0^s \sigma(\xi) d\xi \geq -c_2 \text{ for } s \in \mathbb{R}_+.
\]

(1.5)

\((f_{(65)})\) \( f \) is a \( C^1(\mathbb{R}_+) \) function such that \( f(0) = 0 \) (without loss of generality),

\[
\mu_s = \liminf_{|s| \to \infty} [s^{-1} f(s)] > -\infty,
\]

(1.6)

and the following properties are valid:

(i) if \( d = 1 \), then \( f \) is arbitrary;
(ii) if \( d = 2 \), then there exist constants \( C > 0 \) and \( p \geq 1 \) such that

\[
|f'(s)| \leq C(1 + |s|^{p-1});
\]

(1.7)

(iii) if \( d \geq 3 \), then there exist constants \( C > 0 \) and \( p \in \left[ 1, \frac{d+2}{d-2} \right] \) such that

\[
|f'(s)| \leq C(1 + |s|^{p-1}),
\]

(1.8)

or else there exist constants \( C_i > 0 \) \( (i = 0, 1, 2) \) and \( p \in \left( \frac{d+2}{d-2}, \frac{d+4}{d-4} \right) \) such that

\[
C_0 |u|^{p-1} - C_1 \leq f'(u) \leq C_2(1 + |u|^{p-1}),
\]

(1.9)

where \( s_* = (s + |s|)/2 \).

**Remark 1.** As said in [65, Remark 1.2], the stiffness \( \phi \) satisfying \( (\phi\sigma_{(65)}) \) could be degenerate or even negative. A standard example is \( \phi(s) = \phi_0 + \phi_1 s^\gamma \) with \( \phi_0 \in \mathbb{R}, \ \phi_1 > 0, \) and \( \gamma \geq 1 \), then (1.4) and (1.5) hold with \( \eta_0 = c_0 = 0 \). However, we can also take \( \phi(s) \) with finite support, or even \( \phi(s) \equiv \text{const} \leq 0 \). In this case, we need additional hypotheses concerning the behavior of \( \sigma(s) \) as \( s \to \infty \).

With the assumptions \( (\phi\sigma_{(65)}) \) and \( (f_{(65)}) \), the author got the following results:

\((65_{\text{exp}})\) Existence and uniqueness of weak solutions \( u \in C(0, T; H^2_0(\Omega) \cap L^{p+1}(\Omega)) \) with \( u_t \in C(0, T; L^2(\Omega)) \);

\((65_{\text{gda}})\) Existence of a global attractor with finite fractal dimension in \( (H^2_0(\Omega) \cap L^{p+1}(\Omega)) \times L^2(\Omega) \) endowed with a partially strong topology with additional assumption that

\[
\phi(s) > 0, \ \forall s \geq 0,
\]

(1.10)

i.e., the system is non-degenerate.

In [66], the authors showed that the method used in study the asymptotically smooth in [65] can only be used when (1.10) is true: however, for the degenerate case, the method cannot apply (see [66, page 3] for details). To study the degenerate case, Ma et al. [66] considered problem (1.3) with \( \sigma(x) \equiv \sigma_0 > 0 \) as a constant, i.e.,

\[
\begin{cases}
    u_t - \phi(\|u\|^2) \Delta u - \sigma_0 \Delta u_t + f(u) = g(x), & x \in \Omega, \ t > 0, \\
    u = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) = u_0(x), & u_t(x, 0) = u_t(x), \ x \in \Omega
\end{cases}
\]

(1.11)
with assumption that

\( (\phi_{f(66)}) \) the function \( \phi \in C^1(\mathbb{R}_+) \) possessing the following properties:

(i) \( \phi(s) \geq \min\{L_1 s^\gamma, L_2 s\} \), where \( \gamma \geq 0, L_1, L_2 > 0 \) are constants;

(ii) \( \mu_\phi \lambda_I + \mu_f > 0 \), where \( \mu_\phi = \liminf_{s \to \infty} \phi(s) > 0 \) and \( \mu_f \) is defined in (1.6).

\( (f_{(66)}) \) \( f \) satisfies \( \phi_{f(69)} \) but for the case (iii) (i.e., \( d \geq 3 \)), just assumed that \( p \in \left[1, \frac{d+2}{d-2}\right] \) and (1.8) holds.

By using the method of Condition (C) (see [67]) and \( \mathbb{Z}_2 \) index (see [68, Section 2.5]), the authors got the following results:

\( (66)_{6a} \) Existence of a global attractor in \( H_0^2(\Omega) \times L^2(\Omega) \);

\( (66)_6 \) The fractal dimension of the global attractor is infinite with \( g(x) \equiv 0 \) and suitable additional assumptions on \( \phi \) and \( f \).

In view of papers [65] and [66], for the existence of global attractor, we have the following concerns:

(P1) First, by comparing the assumptions \( \phi_{(66)} \) and \( \phi_{\sigma(65)} \) with \( \sigma(x) \equiv a_0 > 0 \) is a constant, we know that there are much more restrictions on \( \phi \). Can we weaken those restrictions make it closer to the assumption \( \phi_{\sigma(65)} \)?

(P2) Second, when \( d \geq 3 \), in assumption \( \phi_{(66)} \), \( f \) satisfying (1.8) with \( p \in \left[1, \frac{d+2}{d-2}\right] \) was assumed, the main reason is that when verifying Condition (C) the embedding \( H^1(\Omega) \to L^{p+4}(\Omega) \) needs to be compact. Can we weaken this assumption?

In view of the aforementioned introductions, in this article, we consider problem (1.3) with \( -\Delta \) replacing by \( (-\Delta)^a \) and \( \sigma(s) \equiv 1 \), i.e., problem (1.1). We mainly deal with the problems (P1) and (P2) listed above:

- First, for the existence of weak solutions, we make the following assumptions on \( \phi \) and \( f \), which is the counterpart of the assumptions in [65] with \( -\Delta \) replaced by \( (-\Delta)^a \) and \( \sigma(s) \equiv 1 \):

\[
(\phi_{\text{ass1}}) \phi \text{ is a } C^1(\mathbb{R}_+) \text{ function. Besides, for every } s \in \mathbb{R}_+, \text{ there exist } \eta_0 \geq 0 \text{ and } c_i \geq 0 \text{ (i = 1, 2)} \text{ such that }
\int_0^s \phi(\xi)d\xi + \eta_0 s \to \infty \text{ as } s \to \infty \tag{1.12}
\]

and

\[
\phi(s)s + cs \geq -c_2 \text{ for } s \in \mathbb{R}_+. \tag{1.13}
\]

\( (f_{\text{ass1}}) \) \( f \) is a \( C^1(\mathbb{R}_+) \) function such that \( f(0) = 0 \) (without loss of generality) and (1.6) holds, and the following properties are valid:

- (f-1) if \( d < 2a \), then \( f \) is arbitrary;

- (f-2) if \( d = 2a \), then there exist constants \( C > 0 \) and \( p \geq 1 \) such that

\[
|f'(s)| \leq C(1 + |s|^{p-1}); \tag{1.14}
\]

- (f-3) if \( d > 2a \), then there exist constants \( C > 0 \) and \( p \in \left[1, \frac{d+2a}{d-2a}\right] \) such that

\[
|f'(s)| \leq C(1 + |s|^{p-1}). \tag{1.15}
\]

- Second, for uniqueness of weak solutions, we add the following further assumptions on \( \phi \):

\( (\phi_{\text{ass2}}) \phi \) is nonnegative and \( \int_{a_1}^{a_2} \phi(s)ds > 0 \) for any \( 0 \leq a_1 < a_2 < \infty \).

- Third, to obtain the existence of global attractor, we make the following additional assumptions on the function \( \phi \):

\( (\phi_{\text{ass3}}) \lim_{r \to \infty} \phi(r) = \infty \) or \( \lim_{r \to \infty} \phi(r) = \zeta \) and

\[
\zeta > \max\left\{\mu_f - 2\lambda_1^{-a}, \frac{1}{2}[\mu_f - 2\lambda_1^{-a} + \eta\lambda_1^{-a} + \eta]\right\},
\]
where \( \mu_f \) is the constant given in (1.6), \( \lambda_i \) is the first eigenvalue of \(-\Delta\) with homogeneous Dirichlet boundary and

\[
\eta = \frac{\lambda_i^{2\alpha}}{2(\mu_f - 2) + \lambda_i^{2\alpha}} \in (0, 1).
\]

(1.16)

**Remark 2.** When \( \alpha = 1 \), comparing the aforementioned assumptions with the assumptions in [66], i.e., the assumptions \((\phi_{(66)})\) and \((f_{(66)})\), our assumption is much more weaker, especially, the assumption (ii) of \((\phi_{(66)})\) is eliminated and \( p \) can be taken equal to \( \frac{d+2}{d-2} \) for \( d \geq 3 \) in \((f_{(66)})\). So problems (P1) and (P2) are partially solved, and the results of this article can be regarded as a generalization of the results obtained in [65] and [66].

At the end of this section, we introduce some notations, which will be used in this article.

- Let \( X \) and \( Y \) be two Banach spaces, the notation \( X \hookrightarrow Y \) means \( X \) is continuously embedding in \( Y \), and the notation \( X \hookrightarrow\hookrightarrow Y \) means \( X \) is compactly embedding in \( Y \). The norm of a general Banach space \( X \) is denoted by \( \| \cdot \|_X \).
- Let \( X \) be a Banach space, we denote by \( X' \) the dual space of \( X \).
- Let \( X \) be a Banach space with norm \( \| \cdot \|_X \), \( |a, b| \subset \mathbb{R} \), and \( m = 0, 1, 2, \ldots \). We denote by \( C^m(a, b; X) \equiv C^m([a, b]; X) \) the space of \( m \)-differentiable (in the norm topology) functions on \([a, b]\) with values in \( X \). If \([a, b]\) is a finite interval, then \( C^m(a, b; X) \) equipped with the norm

\[
\|u\|_{C^m(a, b; X)} = \max\{\|u^{(k)}(t)\|_X : t \in [a, b],\ k = 0, 1, \ldots, m\}
\]

becomes a Banach space. Here \( u^{(k)}(t) = \partial_t^k u(t) \) is the strong derivative of \( u \) of order \( k \). We denote by \( C^m((a, b); X) \) the space of functions \( u : (a, b) \to X \) such that \( u \in C^m([a, b]; X) \) for any \( b' \in (a, b) \). A similar meaning has the notation \( C^m((a, b]; X) \) and \( C^m((a, b); X) \). We also use the notation \( C_0^m(a, b; X) \) for the space of the functions on \([a, b]\) that are continuous with respect to weak topology on \( X \). \( L^p(a, b; X) \), \( 1 \leq p \leq \infty \) are classical \( L^p \) spaces defined as sets of \((\text{classes of almost everywhere equal})\) strong Bochner-measurable functions \( f(t) \) with values in \( X \) such that \( \|f(\cdot)\|_X \in L^p(a, b; \mathbb{R}) \). Each \( L^p(a, b; X) \) is a Banach space with the norm

\[
\|f\|_{L^p(a, b; X)} = \left\{ \int_a^b \|f(t)\|_X^p dt \right\}^{1/p}, \quad \text{if } 1 \leq p < \infty,
\]

\[= \text{esssup} \|f(t)\|_X : t \in [a, b]), \quad \text{if } p = \infty.\]

- For brevity, we use the following abbreviations:

\[
L^p = L^p(\Omega), \quad H^s = H^s(\Omega), \quad H = L^2, \quad V_t = H^0_t, \quad V_{-1} = H^{-1}, \quad \|\cdot\| = \|\cdot\|_{L^2}, \quad \|\cdot\|_p = \|\cdot\|_{L^p}
\]

with \( p \geq 1 \) and \( s \in \mathbb{R} \), \( H^0 = H, \ H^s \) is the \( L^2 \)-based Sobolev spaces for \( s > 0 \), \( H^0 \) are the completion of \( C_0^\infty(\Omega) \) in \( H^s \) for \( s > 0 \), and \( H^{-s} = (H^s)^\prime \) for \( s > 0 \).

\( (\cdot, \cdot) \) is used for the notation of the \( H \)-inner product, and it is also used for the notation of duality pairing between dual spaces.

\( C_\cdot \) stands for positive constants depending on the quantities appearing in the bottom right corner.

Obviously, \( V_t \hookrightarrow H \hookrightarrow V_{-1}. \) Define the operator \( A : V_t \to V_{-1} \) by \((A\phi, \psi) = (\nabla\phi, \nabla\psi)\) for all \( \phi, \psi \in V_t \). Then \( A \) is a closed positive self-adjoint operator in \( H \) with domain

\[
V_2 = D(A) = \{ \phi \in H : A\phi \in H \} = H^2 \cap H^0.
\]
Let \( \{e_j\}_{j=1}^{\infty} \subset C_0^\infty(\Omega) \) be a complete orthonormal family of \( H \), which is made of eigenfunctions of \( A \):

\[
\begin{align*}
A e_j &= \lambda_j e_j, & j &= 1, 2, \ldots, \\
(e_i, e_j) &= \delta_{i,j} = \begin{cases} 
1 & \text{if } i = j; \\
0 & \text{if } i \neq j,
\end{cases} \\
0 < \lambda_1 \leq \lambda_2 \leq \cdots, & \lambda_j \to \infty & \text{as } j \to \infty.
\end{align*}
\]

(1.17)

Following [69, Section 2.2.1], we can define the powers \( A^k \) of \( A \) for all \( k \in \mathbb{R} \):

- For every \( k > 0 \), \( A^k \) is an unbounded self-adjoint operator in \( H \) with a dense domain

\[
V_{2k} = D(A^k) = \{ \phi \in H : A^k \phi \in H \},
\]

and

\[
A^k \phi = \sum_{j=1}^{\infty} \lambda_j^k (\phi, e_j) e_j.
\]

The operator \( A^k \) is strictly positive and injective. The space \( V_k \) is endowed with the scalar product and the norm

\[
\begin{align*}
(\phi, \psi)_{V_k} &= (A^k u, A^k v) = \sum_{j=1}^{\infty} \lambda_j^k (\phi, e_j)(\psi, e_j), \\
\|\phi\|_{V_k} &= \left\| A^k \phi \right\| = \left[ \sum_{j=1}^{\infty} \lambda_j^k (\phi, e_j)^2 \right]^\frac{1}{2},
\end{align*}
\]

(1.18)

which makes \( V_k \) a Hilbert space and \( A^k \) is an isomorphism from \( V_{2k} \) onto \( H \). Obviously,

\[
\|\phi\|_{V_k} = \|A\phi\| = \|A^k \phi\|, \quad \|\phi\|_{V_k} = \|A^k \phi\| = \|\nabla \phi\|.
\]

- For every \( k > 0 \), we define \( V_k \) as the dual of \( V_k \) and can extend \( A^k \) as an isomorphism from \( H \) onto \( V_{2k} \). Alternatively, \( V_k \) can be endowed with the scalar product and the norm in (1.18) where \( k \) is replaced by \(-k\).

- We obtain, finally, an increasing family of spaces \( V_k, k \in \mathbb{R} \), and

\[
\begin{align*}
V_{k_1} &\hookrightarrow V_{k_2} \text{ for all } k_1, k_2 \in \mathbb{R} \text{ and } k_1 \geq k_2, \\
V_{k_1} &\hookrightarrow V_{k_2} \text{ for all } k_1, k_2 \in \mathbb{R} \text{ and } k_1 > k_2.
\end{align*}
\]

Each space is dense in the following one, the injection is continuous, and \( A^{k-k_2} \) is an isomorphism of \( V_{2k_1} \) into \( V_{2k_2} \) for all \( k_1, k_2 \in \mathbb{R} \) satisfying \( k_1 > k_2 \).

By using the aforementioned notations, we define the following fundamental Hilbert space:

\[
\begin{align*}
\mathcal{H} &= V_0 \times H, \\
((\phi_1, \psi_1), (\phi_2, \psi_2))_{\mathcal{H}} &= (\phi_1, \phi_2)_{V_0} + (\psi_1, \psi_2), \\
\|(\phi, \psi)\|_{\mathcal{H}} &= \sqrt{((\phi, \psi), (\phi, \psi))_{\mathcal{H}}} = \sqrt{\|\phi\|_{V_0}^2 + \|\psi\|_H^2}.
\end{align*}
\]

(1.19)

By using the notations introduced above, problem (1.1) can be equivalently rewritten as the following abstract form:

\[
\begin{align*}
\frac{d}{dt} u - \phi(\|u\|_{V_0}^2) A^ku + A^k u_t + f(u) &= g, \\
u(0) &= u_0, \quad u_t(0) = u_1.
\end{align*}
\]

(1.20)

The remaining of this article is organized as follows. In Section 2, we study the well-posedness and the additional regularity of the weak solution. In Section 3, we discuss the existence of global attractors. In Section 4, we investigate the fractal dimension of the global attractor.
2 Well-posedness

In this section, we study the well-posedness of solutions to problem (1.1), and the following lemmas are used frequently.

**Lemma 1.** [70] Let $T \in (0, \infty)$, $q \in [1, \infty)$, and $r \in (1, \infty)$. Assume $X$, $Z$, and $Y$ as three Banach spaces and satisfy $X \hookrightarrow Z \hookrightarrow Y$. Then
\[ W_1 \hookrightarrow L^q(0, T; Z) \text{ and } W_2 \hookrightarrow C([0, T]; Z), \]
where
\[ W_1 = \{ u \in L^q(0, T; X) : u_t \in L^1(0, T; Y) \} \text{ and } W_2 = \{ u \in L^\infty(0, T; X) : u_t \in L^1(0, T; Y) \}. \]

**Lemma 2.** [71, Chapter 3, Lemma 8.1] Assume $X, Y$ as two Banach spaces and satisfy $X \hookrightarrow Y$. If $X$ is reflexive and $\psi \in L^\infty(0, T; X) \cap C_0(0, T; Y)$, then $\psi \in C_0(0, T; X)$.

**Lemma 3.** [72, Lemma 2.4] Let $\{ x_{n+1}^\infty \} \subset \mathbb{R}$ be a bounded sequence and $f \in C(\mathbb{R})$ be a monotone increasing function. Then
\[ f(\liminf_{n \to \infty} x_n) \leq \liminf_{n \to \infty} f(x_n). \] (2.1)

In this article, we mainly concern the weak solutions of problem (1.20), which are defined as follows:

**Definition 1.** Assume $(\phi_{ass})$ and $(f_{ass})$ hold. Let $T > 0$, $0 < \frac{1}{2} \leq \alpha < 1$, $(u_0, u_1) \in \mathcal{H}$, and $g \in H$. A function $u$ such that
\[ u \in L^\infty(0, T; V_a), \quad u_t \in L^\infty(0, T; H) \cap L^1(0, T; V_a), \quad u_{tt} \in L^2(0, T; V_a) \] (2.2)
is a weak solution to problem (1.20) on the time interval $[0, T]$, with initial data $u_0$ and $u_1$, if
\[ (u_{tt}, \psi) + (A^\alpha u_t, \psi) + \phi(u L^2) (A^\alpha u, \psi) + (f(u), \psi) = (g, \psi) \] (2.3)
holds every $\psi \in V_a$, almost every $t \in [0, T]$, $u(0) = u_0$, and $u_t(0) = u_1$.

**Remark 3.**

(i) By (2.2) and assumptions $(\phi_{ass})$ and $(f_{ass})$, it is easy to prove that every term in (2.3) is meaningful except the term item $(f(u), \psi)$. Next we show $(f(u), \psi)$ makes sense. Note (A.4) and (A.7) in the Appendix, we obtain from $u \in L^\infty(0, T; V_a)$ that there exists a constant $C_{u(t), u_{tt}}$
\[ \| f(u(t)) \|_Y \leq C_{u(t), u_{tt}} \text{ for a.e. } t \in [0, T], \]
where
\[ Y \in \begin{cases} [1, \infty), & \text{if } d < 2\alpha, \\ [1, \infty), & \text{if } d = 2\alpha, \\ \left[1, \frac{2d}{p(d-2\alpha)}\right], & \text{if } d > 2\alpha. \end{cases} \] (4.4)

Note $p \in \left[1, \frac{d+2\alpha}{d-2\alpha}\right]$ for $d > 2\alpha$ (see assumption $(f_{ass})$), we obtain $V_a \hookrightarrow L^\frac{2d}{d+2\alpha}(\Omega)$ for $d > 2\alpha$. So it follows from the above analysis that
\[ (f(u(t)), \psi) \leq \begin{cases} \| f(u) \|_Y \| \psi \|, & \text{if } d = 1, 2, \\ \| f(u) \|_Y \| \psi \|_\frac{2d}{d+2\alpha}, & \text{if } d = 3, 4, 5, \ldots, \\ \leq C_{u(t), u_{tt}} \| \psi \|_{V_a} \text{ for a.e. } t \in [0, T]. \end{cases} \]
(ii) For any $\theta \in [0, \alpha]$ and $y \in (0, a]$, we have $V_a \hookrightarrow V_0 \hookrightarrow H \hookrightarrow V_y \hookrightarrow V_a$. So it follows from Lemma 1 and (2.2) that $u \in C([0, T]; V_0)$ and $u_t \in C([0, T]; V_y)$, then $u(0) = u|_{x=0}$ and $u_t(0) = u_t|_{x=0}$ make sense.

The main result of this section is the following theorem.

**Theorem 1.** Let assumptions (\(\phi_{ass1}\)) and (\(f_{ass1}\)) be in force, $\frac{1}{2} \leq \alpha \leq 1$, $g \in H$, and $(u_0, u_1) \in \mathcal{H}$ satisfying $\|(u_0, u_1)\|_H \leq R$, where $R > 0$ is a constant. Then for every $T > 0$, problem (1.20) has a weak solution $u$ on $[0, T]$ such that $u \in L^\infty(0, T; V_0) \cap L^2(0, T; V_0) \cap C_a([0, T]; V_0)$, $u_t \in L^\infty(0, T; H) \cap L^2(0, T; V_a) \cap C_a([0, T]; H)$, $(u_0, u_1) \in L^2(0, T; V_0)$, and (2.3) holds. Moreover, there exists a constant $C_{R,T}$ such that

$$\|u\|_{L^2(0, T; V_0)} + \|u_t\|_{L^2(0, T; V_0)} + \|u_t\|_H + \|u_t\|_{L^2(0, T; V_a)} \leq C_{R,T} \tag{2.5}$$

For every $a \in (0, T)$, $u \in L^\infty(a, T; V_0)$, $u_t \in L^\infty(a, T; V_a)$, $u_{\alpha} \in L^\infty(a, T; V_\alpha)$, $u_{\alpha t} \in L^2(a, T; V_\alpha \cap L^2(0, T; H)$, and there exists a constant $C_{R,T}$ such that

$$\|u\|_{L^\infty(a, T; V_\alpha)} + \|u_t\|_{L^\infty(a, T; V_\alpha)} + \|u_{\alpha t}\|_{L^2(a, T; V_\alpha)} \leq C_{R,T} \tag{2.6}$$

and it holds

$$E(u(t), u_t(t)) + \int_s^t \|u_t(\tau)\|_{V_0}^2 d\tau = E(u(s), u_t(s)), \quad \forall 0 < s < t \leq T, \tag{2.7}$$

$$E(u(t), u_\alpha(t)) + \int_0^t \|u_\alpha(\tau)\|_{V_\alpha}^2 d\tau \leq E(u_0, u_\alpha), \quad \forall 0 < t \leq T, \tag{2.8}$$

where

$$E(u(t), u_\alpha(t)) = \frac{1}{2} \|u_\alpha(t)\|^2 + \Phi(\|u_\alpha(t)\|_{V_\alpha}) + \int_\Omega F(u(t))dx + \int_\Omega g\alpha(t)dx. \tag{2.9}$$

Here,

$$\Phi(s) = \int_0^s \phi(\xi)d\xi \text{ and } F(s) = \int_0^s f(\xi)d\xi. \tag{2.10}$$

In addition,

(i) if $\phi$ is nonnegative the “$\leq$” in (2.8) can be changing to “$=$”, which, together with (2.7), implies the following energy equality:

$$E(u(t), u_t(t)) + \int_s^t \|u_t(\tau)\|_{V_0}^2 d\tau = E(u(s), u_t(s)), \quad \forall 0 \leq s < t \leq T; \tag{2.11}$$

(ii) if the assumption (\(\phi_{ass2}\)) holds, the function $t \mapsto (u(t), u_\alpha(t))$ is (strongly) continuous in $\mathcal{H}$; moreover, if $u(t)$ and $w(t)$ are two weak solutions with initial data $(u_0, u_1)$ and $(w_0, w_1)$, respectively, such that $\|(u_0, u_1)\|_H \leq R$ and $\|(w_0, w_1)\|_H \leq R$, where $R > 0$ is a constant, then for every $t > 0$ and $\varepsilon \in (0, 1/2]$, there exists a constant $C_{\varepsilon, R, T} > 0$ such that $z(t) = u(t) - w(t)$ satisfies the relation

$$\frac{1 - \varepsilon}{2} \|z(t)\|_{V_\alpha}^2 + \frac{\varepsilon(1 - \varepsilon)}{4} \|z(t)\|_{V_\alpha}^4 + \left(1 - \frac{1}{\varepsilon}\right) \int_0^t \|z\|^2 d\tau \leq C_{\varepsilon, R, T}(\|z(0)\|_{V_\alpha}^2 + \|z(0)\|_{V_\alpha}^4), \quad t \in [0, T]. \tag{2.12}$$

**Proof.** For every $\eta > 0$ we introduce the following energy-type function $E^\eta(u_0, u_1) \in \mathcal{H}$:

$$E^\eta(u_0, u_1) = \|u_1\|^2 + \Phi(\|u_0\|_{V_\alpha}) + \eta\|u_0\|_{V_\alpha}^2 - \alpha(\eta) + \|u_0\|^2 \tag{2.13}$$
with

\[ a(\eta) = \inf_{\psi \in \mathbb{R}} \Phi(\psi) + \eta \psi. \]  

(2.14)

Let \( \eta_0 \) be the nonnegative constant given in (\( \phi_{\text{ass}} \)), then it follows from (1.12) that the infimum \( \inf_{\psi \geq 0} \Phi(\psi) + \eta_0 \psi \) exists, which is denoted by \( q \), i.e.,

\[ q = \inf_{\psi \geq 0} \Phi(\psi) + \eta_0 \psi. \]  

(2.15)

Then,

\[ a(\eta) \geq q, \quad \forall \eta \geq \eta_0. \]  

(2.16)

So it follows

\[ E_t(u_0, u_t) = \|u_t\|^2 + [\Phi(\|u_0\|_{H^{s+1}}^2) + \eta\|u_0\|_{H^s}^2 - a(\eta)] + \|u_0\| \leq \|u_t\|^2 + [\Phi(\|u_0\|_{H^{s+1}}^2) + \eta\|u_0\|_{H^s}^2 - q] + \|u_0\|^2 < \infty, \quad \forall \eta \geq \eta_0. \]

For all \( \nu \in \mathbb{R} \), and \( \eta > 0 \), let

\[ W^{\theta, \nu}(u(t), u_t(t)) = E(u(t), u_t(t)) + \eta \left[ (u(t), u_t(t)) + \frac{1}{2} \|u_t(t)\|_{H^s}^2 \right] + \nu \|u(t)\|^2. \]  

(2.17)

For every \( \eta \geq \eta_0 \), we can find \( \nu \geq 0 \), which depends on \( \eta \) and \( \mu_f \), a positive constant \( a_1 \), and a monotonic positive function \( G(s) \) such that

\[ \frac{1}{4} E_t(u_0, u_t) - a_1 \leq W^{\theta, \nu}(u_0, u_t) \leq \frac{3}{4} E_t(u_0, u_t) + G(\|u_0\|_{H^s}), \quad \forall (u_0, u_t) \in \mathcal{H}, \]  

(2.18)

where \( \mu_f \) is the constant defined in (1.6). We refer to the Appendix for the proof of (2.18).

In the following, we divide the proof into several steps. We denote by \( C \) a positive constants independent of \( m \), which may change from line to line.

**Step 1: Existence of weak solutions.** Let \( \{e_i\}_{i=1}^{\infty} \) be defined in Section 2, we seek for the approximate solutions of the form

\[ u_m(t) = \sum_{i=1}^{m} g_{im}(t)e_i, \quad m = 1, 2, \ldots, \]  

(2.19)

which satisfy, for \( i = 1, 2, \ldots, m \),

\[ \begin{cases} 
(u_{m}, e_i) + (A^s u_m, e_i) + \phi(\|u_m\|_{H^{s+1}}^2)(A^{s} u_m, e_i) + (f(u_m), e_i) = (g, e_i), \quad t > 0, \\
u_{m}(0) = \nu_{m0} = \sum_{i=1}^{m} \nu_{im} e_i, u_{m0}(0) = u_{m1} = \sum_{i=1}^{m} \eta_{im} e_i, 
\end{cases} \]  

(2.20)

i.e.,

\[ \begin{cases} 
g_{im}' + \lambda_{i}^{g} g_{im}' + \lambda_{i}^{g} \left( \sum_{j=1}^{m} \lambda_{j}^{g} g_{jm} \right) g_{im} + \left( \frac{m}{\sum_{j=1}^{m} \lambda_{j}^{g} g_{jm}} \right) f \left( \sum_{j=1}^{m} g_{jm}(t)e_j \right) e_i = (g, e_i), \quad t > 0, \\
g_{im}(0) = \epsilon_{im}, \quad g_{im}'(0) = \eta_{im}, 
\end{cases} \]  

where \( \epsilon_{im} = (u_0, e_i), \eta_{im} = (u_t, e_i) \). In view of \( g \in \mathcal{H} \), and the assumptions (\( \phi_{\text{ass}} \)) and (\( f_{\text{ass1}} \)), by standard theory of ordinary differential equations, \( g_{im} (i = 1, 2, \ldots, m) \) exist. Then, \( u_m(x, t) \) satisfies

\[ \begin{cases} 
u_{m}(t) + A^s u_m + \phi(\|u_m\|_{H^{s+1}}^2)A^{s} u_m + f(u_m) = g_{m}, \quad t > 0, \\
u_{m}(0) = \nu_{m0}, \quad u_{m0}(0) = u_{m1}, 
\end{cases} \]  

(2.21)
where $g_m = \sum_{i=1}^m (g, e_i)e_i$. Since $(u_0, u_i) \in \mathcal{H}$, it is easy to see

$$(u_{m0}, u_{mi}) \to (u_0, u_i)$$ in $\mathcal{H}$ as $m \to \infty$. \hfill (2.22)

Then, in view of $\| (u_0, u_i) \|_H \leq R$, we can assume that

$$|(u_{m0}, u_{mi})|_H \leq C_R, \ m = 1, 2, \ldots$$ \hfill (2.23)

for some positive constant $C_R$ depending only on $R$.

By multiplying both sides of (2.21) by $u_{mi}$ and integrating over $\Omega$, we obtain

$$\frac{d}{dt} E(u_m(t), u_m(t)) + \| u_m(t) \|^2_{\mathcal{V}_g} = 0,$$ \hfill (2.24)

where $E(\cdot, \cdot)$ is defined in (2.9).

By multiplying both sides of (2.21) by $u_m$ and integrating over $\Omega$, we obtain

$$\frac{d}{dt} \left( u_m(t) + \frac{1}{2} \| u_m \|^2_{\mathcal{V}_g} \right) \leq \| u_m \|^2 - \phi (\| u_m \|^2_{\mathcal{V}_g}) - (f(u_m), u_m) + (g, u_m).$$

Since

$$-\phi (\| u_m \|^2_{\mathcal{V}_g}) \| u_m \|^2 \leq c_1 \| u_m \|^2_{\mathcal{V}_g} + c_2 \ (\text{by (1.3)}),$$

$$-(f(u_m), u_m) \leq \int_{\Omega} [(1 - \mu_t) u_m + \bar{c} u_m] dx \leq \left( \frac{3}{2} - \mu_t \right) \| u_m \|^2 + \frac{C^2}{2} |\Omega| \ (\text{by (A.1) in the appendix}),$$

$$(g, u_m) \leq \| g \| \| u_m \| \leq \frac{1}{2} \| g \|^2 + \frac{1}{2} \| u_m \|^2 \ (\text{by g} \in H),$$

we obtain

$$\frac{d}{dt} \left( u_m(t) + \frac{1}{2} \| u_m \|^2_{\mathcal{V}_g} \right) \leq \| u_m \|^2 + C (\| u_m \|^2_{\mathcal{V}_g} + \| u_m \|^2 + 1).$$ \hfill (2.25)

For $\eta > 0$, it follows from (2.17), (2.24), and (2.25) that

$$\frac{d}{dt} W^{\eta, \gamma}(u_m, u_{mi}) = \frac{d}{dt} E(u_m, u_m) + \eta \frac{d}{dt} \left( u_m(t) + \frac{1}{2} \| u_m \|^2_{\mathcal{V}_g} \right) + \nu \frac{d}{dt} \| u_m \|^2$$

$$\leq \eta \| u_m \|^2 + \eta C \| u_m \|^2_{\mathcal{V}_g} + \| u_m \|^2 + 1 \ + 2\nu \| u_m, u_{mi} \|$$

$$\leq \eta \| u_m \|^2 + \eta C \| u_m \|^2_{\mathcal{V}_g} + \| u_m \|^2 + 1 \ + \nu \| u_m \|^2 + \| u_m \|^2$$

$$= (\eta + \nu) \| u_m \|^2 + \eta C \| u_m \|^2_{\mathcal{V}_g} + (\eta C + \nu) \| u_m \|^2 + \eta C.$$ \hfill (2.26)

In view of (1.12) and (2.16), for $\eta > \eta_0$ and $\theta_1 = \theta_1(\eta) = \frac{1}{\eta - \eta_0}$, we obtain

$$\theta_1(\Phi(s) + \eta s - a(\eta)) - s \geq \theta_1(\Phi(s) + \eta_0 s - \Phi(0)) \to \infty \ as \ s \to \infty.$$

Then there exists a positive constant $\theta_2 = \theta_2(\eta)$ such that

$$s \leq \theta_1(\Phi(s) + \eta s - a(\eta)) + \theta_2, \ \eta > \eta_0, \ s \in \mathbb{R}^+, \ \eta > \eta_0.$$

which implies

$$\| u_m \|^2_{\mathcal{V}_g} \leq \theta_1(\Phi(\| u_m \|^2_{\mathcal{V}_g}) + \eta \| u_m \|^2_{\mathcal{V}_g} - a(\eta)) + \theta_2, \ \eta > \eta_0.$$

Then it follows from (2.26) and (2.13) that, for $\eta > \eta_0$,

$$\frac{d}{dt} W^{\eta, \gamma}(u_m, u_{mi}) \leq \eta C \theta_1 \| u_m \|^2_{\mathcal{V}_g} + \eta \| u_m \|^2_{\mathcal{V}_g} + a(\eta) + \eta C \theta_2 + (\eta C + \nu) \| u_m \|^2 + \eta C + (\eta + \nu) \| u_m \|^2$$

$$\leq \hat{C} E^\gamma(u_m, u_{mi}) + \hat{C},$$ \hfill (2.27)
where
\[ \hat{C} = \max(\eta C_0, \eta C + \nu, \eta + \nu, \eta C_0 + \eta C). \]

Then, for every \( \eta > \eta_0 \) and \( \nu \geq \nu_0 \), we obtain from (2.18) and (2.27) that
\[ \frac{d}{dt} W^{\eta, \nu}(u_m, u_m) \leq 4\hat{C} W^{\eta, \nu}(u_m, u_m) + 4\alpha \hat{C} + \hat{C}. \]

Thus, for any \( T > 0 \), by Gronwall's inequality and (2.23), it follows
\[ W^{\eta, \nu}(u_m(t), u_m(t)) \leq e^{4\hat{C}T} [W^{\eta, \nu}(u_m(0), u_m) + (4\alpha \hat{C} + \hat{C})T], \]
which, together with (2.18), implies
\[ E^0(u_m(t), u_m(t)) \leq 4W^{\eta, \nu}(u_m(t), u_m(t)) + 4\alpha \]
\[ \leq 4e^{4\hat{C}T} [W^{\eta, \nu}(u_m(0), u_m) + (4\alpha \hat{C} + \hat{C})T] + 4\alpha, \]
for all \( \eta > \eta_0, t \in [0, T], m = 1, 2, \ldots \). So, in view of (2.13) and (2.15), by the using (2.23) and (2.28) with \( \eta = 1 + \eta_0 \), we obtain
\[ \| (u_m(t); u_m(t)) \|_{\mathcal{V}_a}^2 = \| u_m \|_{\mathcal{V}_a}^2 + \| u_m \|_{\mathcal{V}_a}^2 \]
\[ \leq E^0(u_m, u_m) + \alpha(1 + \eta_0) - (\Phi(\| u_m \|_{\mathcal{V}_a}^2) + \eta_0 \| u_m \|_{\mathcal{V}_a}^2) - \| u_m \|_{\mathcal{V}_a}^2 \]
\[ \leq E^0(u_m, u_m) + \alpha(1 + \eta_0) - \inf_{\psi \in \mathcal{V}_a} [\Phi(\psi) + \eta_0 \psi] \]
\[ = E^0(u_m, u_m) + \alpha(1 + \eta_0) - \| \delta \|_{\mathcal{V}_a} \leq C_{R,T}, \quad \forall t \in [0, T], m = 1, 2, \ldots. \]

Due to (2.9), (2.23), (2.24), (A.8), and (2.29), we obtain
\[ \int_0^T \| u_m(t) \|_{\mathcal{V}_a}^2 dt = E(u_m(0), u_m) - E(u_m(T), u_m(T)) \leq C_{R,T}, \quad m = 1, 2, \ldots. \]

Similar to Remark 3, it follows for all \( t \in [0, T], m = 1, 2, \ldots \), and \( i = 1, 2, \ldots, m \),
\[ (f(u_m(t), e_i) \leq \begin{cases} \| f(u_m) \|_{\mathcal{V}_a}^2, & \text{if } d \leq 2\alpha, \\ \| f(u_m) \|_{\mathcal{V}_a}^2 \| e_i \|_{\mathcal{V}_a}^2, & \text{if } d > 2\alpha \leq C_{R,T} \| e_i \|_{\mathcal{V}_a}. \end{cases} \]

Then the above analysis and (2.20) imply
\[ \| u_m \|_{\mathcal{V}_a} = \sup_{i=1, \ldots, m} \frac{(u_m(t); e_i)}{\| e_i \|_{\mathcal{V}_a}} \]
\[ \leq \sup_{i=1, \ldots, m} \frac{\| u_m \|_{\mathcal{V}_a}^2 \| e_i \|_{\mathcal{V}_a} + \Phi(\| u_m \|_{\mathcal{V}_a}^2) \| u_m \|_{\mathcal{V}_a} \| e_i \|_{\mathcal{V}_a} + C_{R,T} \| e_i \|_{\mathcal{V}_a} + \| g \| \| e_i \|_{\mathcal{V}_a}^2}{\| e_i \|_{\mathcal{V}_a}} \]
\[ \leq C \| u_m \|_{\mathcal{V}_a} + C_{R,T}, \quad \forall t \in [0, T], m = 1, 2, \ldots. \]

In view of (2.30), integrating the above inequality from 0 to T, it follows
\[ \int_0^T \| u_m(t) \|_{\mathcal{V}_a}^2 dt \leq C_{R,T}, \quad m = 1, 2, \ldots. \]

By multiplying both sides of (2.21) by \( A^\alpha u_m \) and integrating over \( \Omega \), it follows from the definition of operator \( A \) and (2.20) that
\[ \frac{d}{dt} \left[ (u_m, A^\alpha u_m) + \frac{1}{2} \| u_m \|_{\mathcal{V}_a}^2 \right] - \| u_m \|_{\mathcal{V}_a}^2 \]
\[ \leq |\Phi(\| u_m \|_{\mathcal{V}_a}^2)\| u_m \|_{\mathcal{V}_a}^2| + |f(u_m), A^\alpha u_m| + |g, A^\alpha u_m| \]
\[ \leq |f(u_m), A^\alpha u_m| + \| g \|^2 + C_{R,T} \| u_m \|_{\mathcal{V}_a}^2, \quad \forall t \in [0, T], \quad m = 1, 2, \ldots. \]
For the term $|\langle f(u_m), A^\alpha u_m \rangle|$, in view of the definition of the operator $A$, it holds

$$
|\langle f(u_m), A^\alpha u_m \rangle| = |\langle \nabla f(u_m), \nabla A^{\alpha-1}u_m \rangle| = |\langle f'(u_m)\nabla u, \nabla A^{\alpha-1}u_m \rangle|.
$$

(2.33)

Then we have the following estimate (note $\frac{1}{2} \leq \alpha \leq 1$):

- if $d < 2\alpha$, since $f'(s) \in C(\Omega)$, $V_\alpha \hookrightarrow C(\tilde{\Omega})$, $\|u_m\|_{V_\alpha} \leq C_{\alpha,T}$ (see (2.29)), $m = 1, 2, \ldots$, it follows from Hölder’s inequality that

$$
|\langle f(u_m), A^\alpha u_m \rangle| \leq \max_{|s| \leq C_{\alpha,T}} \int_{\Omega} |A^\alpha u_m| \, dx \leq C\|u_m\|_{V_\alpha};
$$

- if $d = 2\alpha$, note $V_\alpha \hookrightarrow L^{2\alpha}(\Omega)$ (see the assumptions on $p$ in (fass1)), $\|u_m\|_{V_\alpha} \leq C_{\alpha,T}, m = 1, 2, \ldots$ (see (2.29)), we obtain from (A.4) and Hölder’s inequality that

$$
|\langle f(u_m), A^\alpha u_m \rangle| \leq C \int_{\Omega} (1 + |u_m|^p)|A^\alpha u_m| \, dx \leq C(1 + \|u_m\|_{V_\alpha}^p)\|u_m\|_{V_\alpha} \leq C_{\alpha,T} \|u_m\|_{V_\alpha}^2.
$$

- if $2\alpha < d < 2$ and $p < \frac{1 + \alpha}{1 - \alpha}$, we can choose $\varepsilon > 0$ small enough such that $p + 1 \leq \frac{2 - \varepsilon}{1 - \alpha}$, then we obtain $\|\nabla u_m\|_{\frac{2 - \varepsilon}{1 - \alpha}} \leq C\|u_m\|_{V_\alpha}$. Furthermore, we have $\|\nabla A^{\alpha-1} u_m\|_{\frac{p}{p - 1}} \leq C\|u_m\|_{V_\alpha}$. Note $V_\alpha \hookrightarrow L^{p+\varepsilon}(\Omega)$ (see the assumptions on $p$ in (fass1)), $\|u_m\|_{V_\alpha} \leq C_{\alpha,T}$ (see (2.29)), $m = 1, 2, \ldots$, by Hölder’s inequality (note $\frac{p - 1}{p + 1} + \frac{2 - \varepsilon}{p + 1} + \frac{\varepsilon}{p + 1} = 1$), we obtain from (1.15) and (2.33) that

$$
|\langle f(u_m), A^\alpha u_m \rangle| \leq C \int_{\Omega} (1 + |u_m|^p)|\nabla u_m| |\nabla A^{\alpha-1} u_m| \, dx
$$

$$
\leq C(1 + \|u_m\|_{V_\alpha}^p)\|\nabla u_m\|_{\frac{2 - \varepsilon}{1 - \alpha}} \|\nabla A^{\alpha-1} u_m\|_{\frac{p}{p - 1}} \leq C_{\alpha,T} \|u_m\|_{V_\alpha}^2;
$$

- if $2\alpha < d < 2$ and $p = \frac{1 + \alpha}{1 - \alpha}$, we can choose $0 < \varepsilon < 1$ such that $\frac{1}{1 - \alpha - \varepsilon} > 1$. Since $V_\alpha \hookrightarrow L^{2\alpha}(\Omega)$, $\|\nabla u_m\|_{\frac{2}{1 - \alpha}} \leq C\|u_m\|_{V_\alpha}$, and $\|\nabla A^{\alpha-1} u_m\|_{\frac{p}{p - 1}} \leq C\|u_m\|_{V_\alpha}$. Note $\|u_m\|_{V_\alpha} \leq C_{\alpha,T}$ (see (2.29)), $m = 1, 2, \ldots$, by Hölder’s inequality (note $\alpha + \frac{1}{1 - \alpha - \varepsilon} + \varepsilon = 1$), we obtain from (1.15) and (2.33) that

$$
|\langle f(u_m), A^\alpha u_m \rangle| \leq C \int_{\Omega} (1 + |u_m|^p)|\nabla u_m| |\nabla A^{\alpha-1} u_m| \, dx
$$

$$
= C \int_{\Omega} \left(1 + \|u_m\|_{V_\alpha}^p\right)|\nabla u_m| |\nabla A^{\alpha-1} u_m| \, dx
$$

$$
\leq C \int_{\Omega} \left(1 + \|u_m\|_{V_\alpha}^p\right)|\nabla u_m| |\nabla A^{\alpha-1} u_m| \, dx
$$

$$
\leq C(1 + \|u_m\|_{V_\alpha}^p)\|\nabla u_m\|_{\frac{2}{1 - \alpha}} \|\nabla A^{\alpha-1} u_m\|_{\frac{p}{p - 1}} \leq C_{\alpha,T} \|u_m\|_{V_\alpha}^2;
$$

- if $d > 2$, since $p \in \left[\frac{d + 2\alpha}{d - 2\alpha}, \frac{d + 2\alpha}{d - 2\alpha}\right]$ (see the assumptions on $p$ in (fass1)), it follows that $\|\nabla u_m\|_{\frac{d + 2\alpha}{d - 2\alpha}} \leq C\|u_m\|_{V_\alpha}$ and $\|\nabla A^{\alpha-1} u_m\|_{\frac{d + 2\alpha}{d - 2\alpha}} \leq C\|u_m\|_{V_\alpha}$. Note $V_\alpha \hookrightarrow L^{p+1}(\Omega)$ (see the assumptions on $p$ in (fass1)), $\|u_m\|_{V_\alpha} \leq C_{\alpha,T}$ (see (2.29)), $m = 1, 2, \ldots$, by Hölder’s inequality (note $\frac{p - 1}{p + 1} + \frac{d - 2\alpha}{(p + 1)(d - 2\alpha)} + \frac{d - 2\alpha}{(p + 1)(d - 2\alpha)} = 1$), we obtain from (1.15) and (2.33) that

$$
|\langle f(u_m), A^\alpha u_m \rangle| \leq C \int_{\Omega} (1 + |u_m|^p)|\nabla u_m| |\nabla A^{\alpha-1} u_m| \, dx
$$

$$
\leq C(1 + \|u_m\|_{V_\alpha}^p)\|\nabla u_m\|_{\frac{d + 2\alpha}{d - 2\alpha}} \|\nabla A^{\alpha-1} u_m\|_{\frac{d + 2\alpha}{d - 2\alpha}} \leq C_{\alpha,T} \|u_m\|_{V_\alpha}^2.
$$

In view of the above analysis, $g \in H$, and (2.32), we obtain

$$
\frac{d}{dt} \left[ (u_{int}, A^\alpha u_m) + \frac{1}{2} \|u_m\|_{V_\alpha}^2 \right] \leq C_{\alpha,T} \|u_m\|_{V_\alpha}^2 + \|u_m\|_{V_\alpha}^2 + C_{\alpha,T}, \quad \forall t \in [0, T], \ m = 1, 2, \ldots.
$$

(2.34)
Let
\[ Y(t) = E(u_m(t), u_m(t)) + \frac{1}{4} \left( (u_m, A^\alpha u_m) + \frac{1}{2} \| u_m \|^2_{V_m} \right), \quad \forall t \in [0, T], \ m = 1, 2, \ldots, \]
where \( E(\cdot, \cdot) \) is defined in (2.9). By Cauchy-Schwartz’s inequality, Young’s inequality, \( \| u_m \|^2_{V_m} \leq C_{R,T} \) (see (2.29)), and (A.8), we obtain
\[ Y(t) \geq \frac{1}{2} \| u_m \|^2 - \frac{1}{2} \| u_m \|^2_{V_m} - \| u_m \|_{V_m}, \quad \forall t \in [0, T], \ m = 1, 2, \ldots. \]

Then it follows,
\[ Y(t) \geq \frac{1}{16} \| u_m \|^2 - \| u_m \|^2_{V_m} - C_{R,T}, \quad \forall t \in [0, T], \ m = 1, 2, \ldots. \] (2.35)

Therefore, we obtain from (2.24), (2.34), and (2.35) that
\[ \frac{d}{dt} Y(t) = \frac{d}{dt} \left[ E(u_m(t), u_m(t)) + \frac{1}{4} \left( (u_m, A^\alpha u_m) + \frac{1}{2} \| u_m \|^2_{V_m} \right) \right] \\
\leq - \| u_m \|_{V_m}^2 + \frac{1}{4} \| u_m \|^2_{V_m} + C_{R,T} \| u_m \|_{V_m} + C_{R,T} \\
\leq C_{R,T} Y(t) + C_{R,T}, \quad \forall t \in [0, T], \ m = 1, 2, \ldots. \] (2.36)

For \( t \in (0, \min(1, T)] \), by the above inequality, we have
\[ \frac{d}{dt} \left( t^\frac{1}{2} Y(t) \right) = t^\frac{1}{2} \frac{d}{dt} Y(t) + \frac{1}{2} t^\frac{1}{2} Y(t) \leq t^\frac{1}{2} (C_{R,T} Y(t) + C_{R,T}) + \frac{1}{2} t^\frac{1}{2} Y(t) \\
= \left( C_{R,T} + \frac{1}{2} \right) \left( t^\frac{1}{2} Y(t) \right) + C_{R,T} t^\frac{1}{2}. \]

Then, it follows from Gronwall’s inequality that, for \( t \in (0, \min(1, T)] \),
\[ t^\frac{1}{2} Y(t) \leq C_{R,T} \int_0^t e^{C_{R,T} (t-s)^{\frac{1}{2}}} ds \\
\leq C_{R,T} \left( e^{C_{R,T} t + \frac{1}{2} \ln(t)} \right) s^\frac{1}{2} ds \\
\leq C_{R,T} \sup_{0 < s < t} \left( e^{C_{R,T} (t-s)^{\frac{1}{2}}} \right) \int_0^t s^\frac{1}{2} ds \\
\leq C_{R,T} \int_0^t s^\frac{1}{2} ds \leq C_{R,T} t^{\frac{1}{2} + 1} \leq C_{R,T} t^{\frac{1}{2}}. \] (2.37)

If \( T > 1 \) and \( t \in (1, T] \), it follows from Gronwall’s inequality, (2.36), and (2.37) that
\[ Y(t) \leq Y(1) e^{C_{R,T} (t-1)} + C_{R,T} \int_1^t e^{C_{R,T} (t-s)} ds \leq C_{R,T} e^{C_{R,T} (t-1)} + C_{R,T} e^{C_{R,T} (t-1)} \leq (C_{R,T} + 1) e^{C_{R,T} (T-1)} \leq C_{R,T}. \] (2.38)
In view of (2.35), (2.37), and (2.38), it follows

$$\|u_m\|_{L_2}^2 \leq 16Y(t) + 16C_{R,T} \leq C_{R,T}, \quad \forall t \in (0, T], \ m = 1, 2, \ldots. \quad (2.39)$$

Then,

$$\int_0^T \|u_m(t)\|_{L_2}^2 \, dt \leq C_{R,T}. \quad (2.40)$$

Let

$$v_m(t) = u_m(t), \quad m = 1, 2, \ldots. \quad (2.41)$$

Differentiating (2.21) with respect to \( t \), it follows

$$v_{mt} + A^a v_m + 2\phi'(\|u_m\|_{L_2}^2)(A^a u_m, v_m)A^a u_m + \phi(\|u_m\|_{L_2}^2)A^a v_m + f'(u_m)v_m = 0. \quad (2.42)$$

By multiplying both sides of (2.41) by \( v_m \) and integrating over \( \Omega \), we obtain from (2.29), \( m = 1, 2, \ldots \), that

$$\left( \int \phi'(|u_m|^{p-1})v_m^2 \, dx \right) \leq \left\{ \begin{array}{ll}
C_{R,T}|u_m|^2, & \text{if } d < 2a, \\
C \int (1 + |u_m|^{p-1})v_m^2 \, dx, & \text{if } d \geq 2a,
\end{array} \right.$$

$$\left( \int \phi'(|u_m|^{p-1})v_m^2 \, dx \right)^{1/2} \leq \left( \int \phi'(|u_m|^{p-1})v_m^2 \, dx \right)^{1/2}, \quad \forall t \in [0, T], \ m = 1, 2, \ldots. \quad (2.43)$$

Then it follows from (2.42) that

$$\left( \int \phi'(|u_m|^{p-1})v_m^2 \, dx \right) \leq \|u_m\|_{L_2}^2 + C_{R,T} \|v_m\|_{L_2}^2, \quad \forall t \in [0, T], \ m = 1, 2, \ldots. \quad (2.44)$$

By multiplying both sides of (2.41) by \( A^a v_{mt} \), we obtain

$$(v_{mt}, A^a v_m) + \phi(\|u_m\|_{L_2}^2)(A^a v_m, A^a v_m) + 2\phi'(\|u_m\|_{L_2}^2)(A^a u_m, v_m, A^a u_m, A^a v_m) + (A^a v_m, A^a v_m) \quad (2.45)$$

Then, by Cauchy-Schwartz’s inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|v_m\|_{L_2}^2 + \|v_m\|^2 \leq |\phi(\|u_m\|_{L_2}^2)|\|v_m\|_{L_2}^2 |v_m|_{L_2} + 2|\phi'(\|u_m\|_{L_2}^2)|\|u_m\|_{L_2}^2 \|v_m\|_{L_2}^2 \|u_m\|_{L_2}^2 |v_m|_{L_2} + |f'(u_m)| |A^a v_m|.$$
By similar proof to (2.43), we have
\[ \|F(u_m, \lambda^a v_m)\| \leq C_{R,T}\|v_m\|_V \|A^a v_m\|_V = C_{R,T}\|v_m\|_V \|v_m\|_{\alpha}. \]
Note \(\|u_m\|_V \leq C_{R,T}\) (see (2.29)), \(m = 1, 2, \ldots\), it follows from the above analysis and Cauchy’s inequality that
\[ \frac{d}{dt}\|v_m\|_{V_a}^2 + 2\|v_m\|^2 \leq C_{R,T}(\|v_m\|_{V_a}^2 + \|v_m\|_V^2), \quad \forall t \in [0, T], \ m = 1, 2, \ldots \] (2.45)

Let
\[ \Psi(t) = \|v_m\|_{V_a}^2 + \frac{1}{2}(v_m, v_m) + \frac{1}{2}\|v_m\|_V^2. \]
We have, by Cauchy-Schwartz’s inequality and Young’s inequality,
\[ \Psi(t) \leq \|v_m\|_{V_a}^2 + \frac{1}{4}\|v_m\|_V^2 + \frac{1}{2}\|v_m\|_V\|v_m\|_V = \frac{5}{4}(\|v_m\|_{V_a}^2 + \|v_m\|_V^2), \]
\[ \Psi(t) \geq \|v_m\|_{V_a}^2 + \frac{1}{4}\|v_m\|_V^2 - \frac{1}{2}\|v_m\|_V\|v_m\|_V = \frac{1}{8}(\|v_m\|_{V_a}^2 + \|v_m\|_V^2), \]
\[ \frac{1}{8}(\|v_m\|_{V_a}^2 + \|v_m\|_V^2) \leq \Psi(t) \leq \frac{5}{4}(\|v_m\|_{V_a}^2 + \|v_m\|_V^2). \] (2.46)

Using (2.44) and (2.45) we have that
\[ \frac{d}{dt}\Psi(t) + \frac{3}{2}\|v_m\|^2 \leq C_{R,T}\Psi(t), \quad \forall t \in (0, T), \ m = 1, 2, \ldots \] (2.47)

Then, by similar proof of (2.39), it follows from (2.46) that
\[ \Psi(t) \leq C_{R,T}, \quad \forall t \in (0, T] \] (2.48)
and
\[ \|u_{m\alpha}\|_{V_a}^2 + \|u_{m\alpha}\|_V^2 = \|v_m\|_{V_a}^2 + \|v_m\|_V^2 \leq C_{R,T}, \quad \forall t \in [0, T], \ m = 1, 2, \ldots \] (2.49)
For any \(a \in (0, T)\), it follows from (2.47), (2.46), (2.49)
\[ \frac{3}{2}\int_a^T \|u_{m\alpha}\|^2 \, dt \leq C_{R,T}\int_a^T \Psi(t) \, dt + \Psi(a) \leq C_{R,T}\int_a^T (\|v_m\|_{V_a}^2 + \|v_m\|_V^2) \, dt + C_{R,T} \leq C_{R,T}. \]

Then, in view of (2.49), we obtain that
\[ \|u_{m\alpha}\|_{V_a}^2 + \|u_{m\alpha}\|_V^2 + \int_a^T \|u_{m\alpha}\|^2 \, dt \leq C_{R,T}, \quad \forall a \in (0, T), \ m = 1, 2, \ldots \] (2.50)
In view of (2.29), (2.30), (2.31), (2.39), (2.40), (2.50), and Remark 3, there exists a subsequence of \( \{u_{m_{j+1}}\} \) again and

- \( u \in L^\infty(0, T; V_a) \cap L^2(0, T; V_{2a}) \cap L^\infty(a, T; V_{2a}) \) with \( u_t \in L^\infty(0, T; H) \cap L^\infty(a, T; V_a) \cap L^2(0, T; V_a) \) and \( u_{tt} \in L^\infty(a, T; V_a) \cap L^2(0, T; V_a) \), and \( \zeta \in L^\infty(0, T; L^p(\Omega)) \), where \( a \in (0, T) \) is any positive constant and \( y \) is any constant satisfying (2.4),

such that, as \( m \to \infty \),

1. \( u_m \to u \) weakly in \( L^2(0, T; V_{2a}) \),
2. \( u_m \to u \) weakly star in \( L^\infty(0, T; V_a) \) and weakly in \( L^q(0, T; V_a) \) for any \( q > 1 \),
3. \( u_m \to u \) weakly star in \( L^\infty(a, T; V_{2a}) \) and weakly in \( L^2(a, T; V_{2a}) \) for any \( q > 1 \),
4. \( u_{mt} \to u_t \) weakly star in \( L^\infty(0, T; H) \) and weakly in \( L^q(0, T; H) \) for any \( q > 1 \),
5. \( u_{mt} \to u_t \) weakly in \( L^2(0, T; V_a) \),
6. \( u_{mt} \to u_t \) weakly star in \( L^\infty(a, T; V_a) \) and weakly in \( L^q(0, T; V_a) \) for any \( q > 1 \),
7. \( u_{mt} \to u_t \) weakly in \( L^2(0, T; V_a) \),
8. \( u_{mt} \to u_t \) weakly star in \( L^\infty(a, T; V_a) \) and weakly in \( L^q(a, T; V_a) \) for any \( q > 1 \),
9. \( f(u_m) \to \zeta \) weakly star in \( L^\infty(0, T; L^p(\Omega)) \) and weakly in \( L^q(0, T; L^p(\Omega)) \) for any \( q > 1 \).

In view of Lemma 1, we know that there exists a subsequence of \( \{u_{m_{j+1}}\} \), denoted by \( \{u_m\}_{j+1} \) again such that

10. \( u_m \to u \) strongly in \( L^2(0, T; V_{2a}) \) (by (1) and (5)),
11. \( u_m \to u \) strongly in \( C(\{0, T\}; V_a) \) for any \( a' \in (0, a) \) (by (2) and (5)),
12. \( u_m \to u \) strongly in \( C([a, T]; V_a) \) for any \( a \in (0, T) \) (by (3) and (5)),
13. \( u_{mt} \to u_t \) strongly in \( C([0, T]; V_{a''}) \) for any \( a'' \in (0, a) \) (by (4) and (7)),
14. \( u_{mt} \to u_t \) strongly in \( L^2(0, T; V_a) \) for any \( a' \in (0, a) \) (by (5) and (7)),
15. \( u_{mt} \to u_t \) strongly in \( C([a, T]; V_{a''}) \) for any \( a' \in (0, a) \) (by (6) and (7)).

Then we obtain from (10) that

16. \( u_m(x, t) \to u(x, t) \) for a.e. \( (x, t) \in \Omega \times [0, T] \),

which implies \( f(u_m(x, t)) \to f(u(x, t)) \) a.e. \( (x, t) \in \Omega \times [0, T] \). Then by (9), we obtain

\[
\zeta = f(u). \tag{2.51}
\]

Moreover, we obtain from \( u \in L^\infty(0, T; V_a) \cap C([0, T]; V_a) \) (see (2) and (11)), \( u_t \in L^\infty(0, T; H) \cap C([0, T]; V_{a''}) \) (see (4) and (13)) and Lemma 2 that

17. \( u \in C_w(\{0, T\}; V_a) \) and \( u_t \in C_w(\{0, T\}; H) \).

By weak lower semicontinuity of norms, it follows from (2) and (2.29) that

\[
|u|_{L^q(0, T; V_a)} \leq \liminf_{m \to \infty} |u_m|_{L^q(0, T; V_a)} \leq C_{R, T} T^4, \quad \forall q \geq 1.
\]

Then we obtain

\[
\|u\|_{L^\infty(0, T; V_a)} = \lim_{q \to \infty} \|u\|_{L^q(0, T; V_a)} \leq C_{R, T}.
\]

Similarly, it follows from (4) and (2.29) that

\[
|u_t|_{L^\infty(0, T; H)} \leq C_{R, T}.
\]

Therefore, we obtain from the above two inequalities, weak lower semicontinuity of norms, (1), (5), (7), (2.30), and (2.31) that (2.5) holds. Similarly, we obtain from (2.39) and (2.50) that (2.6) holds.

Next, we show

\[
\lim_{m \to \infty} \int_0^T \phi(|u_m|_{V_a}^2) dt = \int_0^T \phi(|u|_{V_a}^2) dt. \tag{2.52}
\]
In fact, by (2.29), (2.5), and (10), it follows from Hölder’s inequality that

\[
\left| \int_0^T \phi(\|u_m\|_{\Omega}^2) \, dt - \int_0^T \phi(\|u\|_{\Omega}^2) \, dt \right|^2 \\
\leq T \int_0^T \left| \phi(\|u_m\|_{\Omega}^2) - \phi(\|u\|_{\Omega}^2) \right|^2 \, dt \\
\leq T \int_0^T \left( \int_0^1 \left( \phi(\|A^\alpha u_m\|_{\Omega}^2) + (1 - \lambda)\|u\|_{\Omega}^2 \right) \, d\lambda \right) \left( \|u_m\|_{\Omega}^2 - \|u\|_{\Omega}^2 \right)^2 \, dt \\
\leq C_{R,T} \int_0^T \left( \|u_m\|_{\Omega}^2 + \|u\|_{\Omega}^2 \right) \left( \|u_m\|_{\Omega}^2 - \|u\|_{\Omega}^2 \right)^2 \, dt \\
\leq C_{R,T} \|u_m - u\|_{(0,T; V_\alpha)}^2 \rightarrow 0 \text{ as } m \rightarrow \infty.
\]

With the above preparations, now we show the function \(u\) got above is a weak solution. For any \(t \in C[0, T]\), it follows from (2.20) that

\[
\int_0^T \left[ (u_{nt}, \psi(t)e_i) + (A^\alpha u_{nt}, \psi(t)e_i) + \phi(\|u_m\|_{\Omega}^2)(A^\alpha u_m, \psi(t)e_i) + (f(u_m), \psi(t)e_i) \right] \, dt \\
= \int_0^T \left( g, \psi(t)e_i \right) \, dt, \quad i = 1, 2, \ldots, m.
\]

Since \(\psi(t)e_i \in L^1(0, T; V_\alpha)\) for all \(i = 1, \ldots, m\), letting \(m \rightarrow \infty\), we obtain from the above analysis that

\[
\int_0^T \left[ (u_{nt}, \psi(t)e_i) + (A^\alpha u_{nt}, \psi(t)e_i) + \phi(\|u_m\|_{\Omega}^2)(A^\alpha u_m, \psi(t)e_i) + (f(u), \psi(t)e_i) \right] \, dt = \int_0^T \left( g, \psi(t)e_i \right) \, dt, \quad i = 1, 2, \ldots.
\]

So by the arbitrariness of \(\psi(t)\) and \(\{e_i\}_{i=1}^\infty\) is a basis of \(V_\alpha\), we obtain (2.3). By (11) and (13), as \(m \rightarrow \infty\), we obtain \(u_m(0) \rightarrow u(0)\) strongly in \(V_\alpha\) for any \(\alpha' \in (0, \alpha)\) and \(u_{nt}(0) \rightarrow u(t)\) strongly in \(V_{\alpha'}\) for any \(\alpha'' \in (0, \alpha]\), which, together with (2.22), implies \(u(0) = u_0\) and \(u_t(0) = u_t\). So, by Definition 1, \(u\) is a weak solution to problem (1.20) on the time interval \([0, T]\).

**Step 2: Proof of** \((2.7), (2.8), (2.11), \text{ and the continuity of } t \mapsto (u(t), u_t(t)) \text{ in } H\). By (12), we obtain \(u(t) \in C([a, T], V_\Omega)\) for every \(a \in (0, T)\). Then we have \(\int_\Omega F(u(t)) \, dx \in C[a, T]\). In fact, for any \(t \in [a, T]\), and any \(\{t_n\}_{n=1}^{\infty} \in [a, T]\) such that \(t_n \rightarrow t\) as \(n \rightarrow \infty\), we have \(u(t_n) \rightarrow u(t)\) in \(V_\alpha\). Note \(\|u(t_n)\|_{V_\alpha} \leq C_{R,T} \|u(t)\|_{V_\alpha} \leq C_{R,T}\) (see (2.5)), by similar proof to Remark 3, it follows,

\[
\int_\Omega F(u(t_n)) \, dx - \int_\Omega F(u(t)) \, dx \leq \int_\Omega |F(u(t_n)) - F(u(t))| \, dx \\
= \int_0^1 \int_\Omega f(\lambda u(t_n) + (1 - \lambda)u(t))(u(t_n) - u(t)) \, dx \, d\lambda \leq C_{R,T} \|u(t_n) - u(t)\|_{V_\alpha} \\
\rightarrow 0 \quad \text{as } n \rightarrow \infty,
\]

which means

\[
\int_\Omega F(u(t)) \, dx \in C[a, T] \quad \text{for any } a \in (0, T),
\]

(2.53)
which, together with \( u \in C([a, T], V_a) \) and \( u_\ell \in C([a, T], H) \) (see (12) and (15)), implies

\[
E(u_\ell(t), u(t)) = \frac{1}{2} \| u_\ell(t) \|^2 + \Phi(\| u(t) \|_{V_a}^2) + \int_\Omega F(u(t))dx - \int_\Omega gu(t)dx \in C[a, T]
\]  

(2.54)

for any \( a \in (0,T) \). Since \( u_\ell \in L^2(0,T; V_a) \) (see (5)), \( u_\ell(t) \in V_a \) for a.e. \( t \in [0, T] \). By taking \( \psi = u_\ell \) in (2.3), we find

\[
\frac{d}{dt} E(u(t), u_\ell(t)) + \| u_\ell(t) \|_{V_a}^2 = 0 \quad \text{for a.e.} \ t \in [0, T].
\]

For any \( 0 < s < t \leq T \), we have \( E(u(t), u_\ell(t)) \in C[s, t \) (by (2.54)) and \( \| u_\ell(t) \|_{V_a}^2 \in L^2(s, t) \) (by \( u_\ell \in L^2(0,T; V_a) \)). Then integrating the above equality from \( s \) to \( t \), it follows

\[
E(u(t), u_\ell(t)) + \int_s^t \| u_\ell(t) \|_{V_a}^2 dt = E(u(s), u_\ell(s)), \quad \forall 0 < s < t \leq T,
\]

which is exactly (2.7).

Next we prove (2.8). By (2.55) and \( u_\ell \in L^2(0,T; V_a) \), one can see \( \lim_{n \to 0} E(u(s), u_\ell(s)) \) exists, which is denoted by \( \bar{E} \), i.e.,

\[
\lim_{s \to 0^+} E(u(s), u_\ell(s)) = \bar{E} = E(u(t), u_\ell(t)) + \int_0^t \| u_\ell(t) \|_{V_a}^2 dt.
\]

(2.56)

So, in order to show (2.8), we only need to show

\[
\bar{E} \leq E(u_0, u_\ell).
\]

(2.57)

For any \( 0 < s < T \), by (2.24), we have

\[
E(u_m(s), u_m(s)) \leq E(u_{m_0}, u_{m_1}), \quad m = 1, 2, \ldots.
\]

(2.58)

By (12) and (15), we have

\[
u_m(s) \to u(s) \quad \text{strongly in} \ V_a \quad \text{and} \quad u_{m_\ell}(s) \to u_\ell(s) \quad \text{strongly in} \ H \quad \text{as} \ m \to \infty.
\]

(2.59)

Note \( \| u_m(s) \|_{V_a} \leq C_{R,T} \) and \( |u(s)| \|_{V_a} \leq C_{R,T} \) (see (2.5) and (2.29)), by similar proof to Remark 3, it follows

\[
\left| \int_\Omega F(u_m(s))dx - \int_\Omega F(u(s))dx \right| \\
\leq \int_\Omega \left| F(u_m(s)) - F(u(s)) \right| dx \\
= \int_0^1 \int_\Omega f(\lambda u_m(s) + (1-\lambda)u(s))(u_m(s) - u(s))dx \lambda \leq C_{R,T} \| u_m(s) - u(s) \|_{V_a} \to 0 \quad \text{as} \ n \to \infty.
\]

Consequently, we obtain from (2.9) and (2.59) that

\[
\lim_{m \to \infty} E(u_m(s), u_m(s)) = \lim_{m \to \infty} \left[ \frac{1}{2} \| u_m(s) \|^2 + \Phi(\| u(s) \|_{V_a}^2) \right] + \int_\Omega F(u_m(s))dx - \int_\Omega gu_m(s)dx \\
= \frac{1}{2} \| u(s) \|^2 + \Phi(\| u(s) \|_{V_a}^2) + \int_\Omega F(u(s))dx - \int_\Omega gu(s)dx = E(u(s), u_\ell(s)).
\]

Note \( u_{m_0} \to u_0 \) strongly in \( V_a \) and \( u_{m_1} \to u_1 \) strongly in \( H \) as \( m \to \infty \), by the same proof as above, we obtain

\[
\lim_{m \to \infty} E(u_{m_0}, u_{m_1}) = E(u_0, u_1).
\]
Then, let $m \to \infty$ in (2.58), it follows

$$E(u(s), u_t(s)) \leq E(u_0, u_t),$$

which implies

$$\bar{E} = \limsup_{s \to 0^+} E(u(s), u_t(s)) \leq \limsup_{s \to 0^+} E(u(s), u_t(s)) \leq E(u_0, u_t).$$

(2.60)

So, (2.57) is true, i.e., (2.8) holds.

Next, we prove (2.11) with the assumption that $\phi$ is nonnegative. Then, it follows from the definition of $\Phi$ (see (2.10)) that

- $\Phi(s)$ is a nonnegative increasing function for $s \geq 0$.

In view of (2.7) and (2.8), to show (2.11), we only need to prove

$$\bar{E} \geq E(u_0, u_t).$$

(2.61)

Since $(t, u) \in C([0, \infty), V_0)$ (see (12)) and $u(t) \in C([0, T], V_0)$ (see (15)), $\Phi(\|u(t)\|_{V_0}^2) \geq 0$, and $\int_\Omega F(u(t))dx \geq -\frac{\epsilon^2}{2}$ (see (A.3)), it follows that $\liminf_{t \to 0^+} \|u(t)\|^2_{V_0}$, $\liminf_{t \to 0^+} \|u(t)\|^2$, and $\liminf_{t \to 0^+} \int_\Omega F(u(t))dx$ exist, which are denoted by $\sigma_1, \sigma_2, \text{ and } \sigma_3$, respectively, i.e.,

$$\sigma_1 = \liminf_{t \to 0^+} \Phi(\|u(t)\|_{V_0}^2),$$

(2.62)

$$\sigma_2 = \liminf_{t \to 0^+} \|u(t)\|^2,$$

(2.63)

$$\sigma_3 = \liminf_{t \to 0^+} \int_\Omega F(u(t))dx.$$  

(2.64)

By the property of the limit inferior, there exists a sequence $\{t_n\}_{n=1}^{\infty} \subset (0, T)$, $t_n \to 0$ as $n \to \infty$ such that

$$\sigma_1 = \lim_{n \to \infty} \Phi(\|u(t_n)\|_{V_0}^2),$$

(2.65)

$$\sigma_2 = \lim_{n \to \infty} \|u(t_n)\|^2,$$

(2.66)

$$\sigma_3 = \lim_{n \to \infty} \int_\Omega F(u(t_n))dx.$$  

(2.67)

Since $u \in C_u([0, T], V_0)$ and $u_t \in C_u([0, T]; H)$ (see (17)), then

$$\lim_{n \to \infty} \langle \phi, u(t_n) \rangle = \langle \phi, u_0 \rangle, \quad \lim_{n \to \infty} \langle \psi, u(t_n) \rangle \to \langle \psi, u_t \rangle, \quad \forall \phi \in V_a \text{ and } \psi \in H,$$

i.e., as $n \to \infty$,

(18) $u(t_n) \to u_0$ weakly in $V_a$ and $u_t(t_n) \to u_t$ weakly in $H$.

Note $V_a \hookrightarrow H$, we obtain

(19) $u(t_n) \to u_0$ strongly in $H$, and then $u(x, t_n) \to u_0(x)$ for a.e. $x \in \Omega$.

Since $\Phi(\cdot)$ is an increasing continuous function, by Lemma 3, (18), and weak lower semicontinuous of norms, we obtain

$$\sigma_1 = \liminf_{n \to \infty} \Phi(\|u(t_n)\|_{V_0}^2) \geq \Phi(\liminf_{n \to \infty} \|u(t_n)\|_{V_0}^2) \geq \Phi(\|u_0\|_{V_0}^2),$$

(2.68)

$$\sigma_2 = \liminf_{n \to \infty} \|u(t_n)\|^2 \geq \|u_t\|^2.$$  

(2.69)
It follows from (19), (A.3), and Fatou’s lemma that

\[
\sigma_3 = \liminf_{n \to \infty} \int_{\Omega} F(u(t_n))dx
\]

\[
= \liminf_{n \to \infty} \int_{\Omega} \left[ F(u(t_n)) - \frac{1}{2}(\mu_f - 2)|u(t_n)|^2 + \frac{\sigma^2}{2} \right]dx + \frac{1}{2}(\mu_f - 2) \lim_{n \to \infty} \|u(t_n)\|^2 - \frac{\sigma^2}{2} |\Omega|
\]

\[
\geq \int \liminf_{n \to \infty} \left[ F(u(t_n)) - \frac{1}{2}(\mu_f - 2)|u(t_n)|^2 + \frac{\sigma^2}{2} \right]dx + \frac{1}{2}(\mu_f - 2)\|u_0\|^2 - \frac{\sigma^2}{2} |\Omega|
\]

\[
= \int F(u(t)) - \frac{1}{2}(\mu_f - 2)|u(t)|^2 + \frac{\sigma^2}{2} \right]dx + \frac{1}{2}(\mu_f - 2)\|u_0\|^2 - \frac{\sigma^2}{2} |\Omega| = F(u_0).
\]

Then, by the above analysis, (2.9), (2.62), (2.63), (2.64), (2.56), and \( u_t \in C_{w}(0, T; H) \) (see (17)) it follows

\[
E(u_0, u_0) = \frac{1}{2}\|\mathbf{u}\|^2 + \Phi(\|\mathbf{u}\|\|\|\|_{\mathbf{v}}^2) + \int_{\Omega} F(u_0)dx - \int_{\Omega} g_0dx \leq \frac{1}{2}(\sigma_1 + \sigma_2) + \sigma_3 - \lim_{t \to 0} \int_{\Omega} g(t)dx
\]

\[
\leq \frac{1}{2}\left[ \liminf_{t \to 0} \|u(t)\|^2 + \liminf_{t \to 0} \Phi(\|u(t)\|\|\|_{\mathbf{v}}^2) + \liminf_{t \to 0} \int F(u(t))dx - \lim_{t \to 0} \int g(t)dx
\]

\[
\leq \liminf_{t \to 0} \left[ \|u(t)\|^2 + \Phi(\|u(t)\|\|\|_{\mathbf{v}}^2) + \int F(u(t))dx - \int g(t)dx \right] \leq \lim_{t \to 0} E(u(t), u(t)) = E,
\]

which is exactly (2.61).

Finally, we show the continuity of \( t \mapsto (u(t), u(t)) \) in \( H \) with the assumption that \( (\phi_{ass}) \), i.e., \( \phi \) is nonnegative and \( \int_{a_1}^{a_2} \phi(s)ds > 0 \) for any \( 0 \leq a_1 < a_2 < \infty \). Since we have to prove \( u_t \in C((0, T]; H) \) (see (15)), we only need to show \( u_t \) is right continuous at 0 in \( H \). By (2.11), (2.68), (2.69), (2.70), (2.62), (2.63), (2.64), we have

\[
\limsup_{t \to 0} \frac{1}{2}\|u(t)\|^2 + \liminf_{t \to 0} \left( \int_{\Omega} F(u(t))dx + \frac{1}{2}\Phi(\|u(t)\|\|\|_{\mathbf{v}}^2) - \int g(t)dx \right)
\]

\[
\leq \lim_{t \to 0} E(u(t), u(t))
\]

\[
= E(u_0, u_0) = \frac{1}{2}\|u_0\|^2 + \int_{\Omega} F(u_0)dx + \frac{1}{2}\Phi(\|u_0\|\|\|_{\mathbf{v}}^2) - \int g_0dx
\]

\[
\leq \frac{1}{2}\sigma_1 + \frac{1}{2}\sigma_2 - \lim_{t \to 0} \int g(t)dx
\]

\[
\leq \frac{1}{2}\liminf_{t \to 0} \|u(t)\|^2 + \liminf_{t \to 0} \left( \int_{\Omega} F(u(t))dx + \frac{1}{2}\liminf_{t \to 0} \Phi(\|u(t)\|\|\|_{\mathbf{v}}^2) - \lim_{t \to 0} \int g(t)dx \right)
\]

\[
\leq \liminf_{t \to 0} \frac{1}{2}\|u(t)\|^2 + \liminf_{t \to 0} \left( \int_{\Omega} F(u(t))dx + \frac{1}{2}\Phi(\|u(t)\|\|\|_{\mathbf{v}}^2) - \int g(t)dx \right),
\]

thus, we obtain that

\[
\lim_{t \to 0} \|u(t)\|^2 = \|u_0\|^2.
\]

Similarly,

\[
\lim_{t \to 0} \Phi(\|u(t)\|\|\|_{\mathbf{v}}^2) = \Phi(\|u_0\|\|\|_{\mathbf{v}}^2),
\]

which is exactly (2.61).
which implies

\[
\lim_{t \to 0^+} \left| \int_{a}^{t} \phi(s) \, ds \right| = \lim_{t \to 0^+} |\Phi(\|u(t)\|_{V_a^2}) - \Phi(\|u_0\|_{V_a^2})| = 0.
\]

Note that \( \phi \) is nonnegative and \( \int_{a}^{t} \phi(s) \, ds > 0 \) for any \( 0 \leq a_1 < a_2 < \infty \), it follows

\[
\lim_{t \to 0^+} \|u(t)\|_{V_a}^2 = \|u_0\|_{V_a}^2. \tag{2.72}
\]

Since \( u \in C_0([0, T]; V_a) \) and \( u_\epsilon \in C_0([0, T]; H) \) (see (17)), \( V_a \) and \( H \) are both Hilbert spaces, it follows (2.71), (2.72), \( u \in C((0, T]; V_a) \) (see (12)), and \( u_\epsilon \in C((0, T]; H) \) (see (15)) that \( u \in C_0([0, T]; V_a) \) and \( u_\epsilon \in C_0([0, T]; H) \), i.e., \( t \mapsto (u(t), u_\epsilon(t)) \) is continuous in \( \mathcal{H} \).

**Step 3: proof of** (2.12). Let \( u(t) \) and \( w(t) \) be two weak solutions with initial data \( (u_0, u_\epsilon) \) and \( (w_0, w_\epsilon) \), respectively, such that \( \|u_0, u_\epsilon\|_{H^1} \leq R \) and \( \|w_0, w_\epsilon\|_{H^1} \leq R \), where \( R > 0 \) is a constant. Then by step 1, \( u(t) \) and \( w(t) \) are both defined on \([0, T]\) for any \( T > 0 \). Let \( z(t) = u(t) - w(t) \). Then

\[
(z(t), \psi) + \lambda (A^2 z(t), \psi) + \phi(\|u(t)\|_{V_a}^2) - \phi(\|w(t)\|_{V_a}^2) + ((f(u) - f(w), \psi) = 0 \tag{2.73}
\]

holds every \( \psi \in V_a \), almost every \( t \in [0, T] \), and

\[
z(0) = u_0 - w_0, \quad z(0) = u_\epsilon - w_\epsilon. \tag{2.74}
\]

Let

\[
\phi(t) = \frac{1}{2} (\phi(\|u(t)\|_{V_a}^2) + \phi(\|w(t)\|_{V_a}^2)), \quad t \in [0, T] \tag{2.75}
\]

and

\[
\dot{\phi}(t) = \frac{1}{2} \int_{0}^{t} \phi'(\|u(s)\|_{V_a}^2 + (1 - \lambda)\|w(s)\|_{V_a}^2) \, ds
\]

\[
= \begin{cases} \phi'(\|u(t)\|_{V_a}^2) - \phi'(\|w(t)\|_{V_a}^2), & \text{if } \|u(t)\|_{V_a} \neq \|w(t)\|_{V_a}, \\ \frac{1}{2} \phi'(\|u(t)\|_{V_a}^2), & \text{if } \|u(t)\|_{V_a} = \|w(t)\|_{V_a}, \\ t \in [0, T]. \tag{2.76} \end{cases}
\]

Then by \( u, w \in C([0, T], V_a) \) (see step 2) and \( \phi \in C^1 \) (see assumption \( [\phi_{\text{ass}}] \)), we obtain \( \phi(t) \in C([0, T]) \) and \( t \phi(t) \in C([0, T]) \). Moreover, since \( \|u(t)\|_{V_a} \leq C_{R,T} \) and \( \|w(t)\|_{V_a} \leq C_{R,T} \) (see steps 1 and 2), it follows

\[
|\phi(t)| \leq C_{R,T} \quad \text{and} \quad |t \phi(t)| \leq C_{R,T}, \quad t \in [0, T]. \tag{2.77}
\]

By using the above notations, and noting that

\[
\|u\|_{V_a}^2 - \|w\|_{V_a}^2 = (A^2 u, A^2 u) - (A^2 w, A^2 w) = (A^2(u + w), A^2)z,
\]

(2.73) can be re-written as

\[
(z(t), \psi) + \phi(t) (A^2 z(t), \psi) + (A^2 z(t), \psi) + \phi(t) (A^2(u + w), A^2 z((u + w), \psi)) + ((f(u) - f(w), \psi) = 0. \tag{2.78}
\]

Since \( z \in C([0, T], V_a) \) and \( \dot{z} \in C([0, T], H) \) (see step 2), we obtain \( A^2 z + \varepsilon z \in V_a \) for every \( t \in [0, T] \), where \( \varepsilon \in (0, 1) \) is a constant to be determined later. By taking \( \psi = A^2 z + \varepsilon \zeta \in V_a \) in (2.78), we obtain

\[
\frac{d}{dt} \|z\|_{V_a}^2 + \frac{d}{dt} (z, \zeta) - \varepsilon \|z\|_{V_a}^2 + \phi(t) (z, \zeta) + \varepsilon \phi(t) \|z\|_{V_a}^2 + \|z\|_{V_a}^2 + \frac{\varepsilon}{2} \frac{d}{dt} \|z\|_{V_a}^2
\]

\[
= \phi(t) (A^2(u + w), A^2 z((u + w), \zeta) + \varepsilon \phi(t) (A^2(u + w), A^2 z)^2 + (f(u) - f(w), A^2 z) + \varepsilon = 0,
\]
i.e.,
\[
\frac{d}{dt}H_t(z, z_t) + (1 - \varepsilon)\|z_t\|^2 + (f(u) - f(w), A^a_z t + \varepsilon z) = K(t), \quad t \in [0, T],
\]
(2.79)
where
\[
K(t) = -\phi_\varepsilon(t)(z, z_t) - \varepsilon\bar{\phi}_\varepsilon(t)(A^\varepsilon_t(u + w), A^\varepsilon_t \varepsilon)(u + w, z_t) - \varepsilon\bar{\phi}_\varepsilon(t)(A^\varepsilon_t(u + w), A^\varepsilon_t \varepsilon)(u + w, z_t) \leq 0, \quad t \in [0, T],
\]
(2.80)
and
\[
H_t(z, z_t) = \frac{1}{2}\|z_t\|^2 + \varepsilon(z, z_t) + \frac{\varepsilon}{2}\|z_t\|^2.
\]
(2.81)
Since \(\varepsilon \in (0, 1)\), then \(\sigma = (\varepsilon + 1)/2 \in (\varepsilon, 1)\). By Young’s inequality it follows
\[
H_t(z, z_t) \geq \frac{1}{2}\|z_t\|^2 + \varepsilon\|z_t\|\|z_t\| + \frac{\varepsilon}{2}\|z_t\|^2
\]
(2.82)
and
\[
H_t(z, z_t) \leq \frac{1}{2}\|z_t\|^2 + \|z_t\|\|z_t\| + \frac{\varepsilon}{2}\|z_t\|^2.
\]
(2.83)
By \(\|u\|_{V_a} \leq C_{R,T}\) and \(\|w\|_{V_a} \leq C_{R,T}\) (see steps 1 and 2), (2.77), Young’s inequality, and \(V_a \rightarrow H\), we can infer that
\[
K(t) \leq C_{R,T}\|z_t\| + \varepsilon C_{R,T}\|z_t\|^2 + C_{R,T}\|z_t\|\|z_t\| + \varepsilon C_{R,T}\|z_t\|^2 \leq C_{C_{R,T}}\|z_t\|^2 + \frac{1}{2}\|z_t\|^2.
\]
(2.84)
Note (1.14), (1.15), (A.5), (A.7), we obtain from \(\|u\|_{V_a} \leq C_{R,T}\) and \(\|w\|_{V_a} \leq C_{R,T}\), Hölder’s inequality, and Young’s inequality that
\[
|f(u) - f(w), A^a_z t + \varepsilon z|
\]
\[
\left| \int_0^1 f'(t(1 - \lambda)w)dz, A^a_z t + \varepsilon z \right|
\]
\[
\leq C_{R,T}\|z\|\|z_t\| + \varepsilon C_{R,T}\|x\|^2, \quad \text{if } d < 2\alpha,
\]
(2.85)
\[
\leq C_{R,T}\|u\|_{V_a} \|z\|\|z_t\| + \varepsilon\|z_t\|^2, \quad \text{if } d < 2\alpha,
\]
\[
\leq C_{R,T}\left(\int_0^1 (1 + |u|^{p-1} + |w|^{p-1})dx\right)^{1/2} \|z\|_{L^\alpha} \|A^t z_t + \varepsilon z\|_{L^\alpha}, \quad \text{if } d = 2\alpha,
\]
\[
\leq C_{R,T}\left(\int_0^1 (1 + |u|^{p-1} + |w|^{p-1})^2dx\right)^{1/2} \|z\|_{L^\alpha} \|A^t z_t + \varepsilon z\|_{L^\alpha}, \quad \text{if } d > 2\alpha.
\]
\[
C_{R,T}(\|z_i\|_a^2 + (1 + \varepsilon)\|z_i\|^2), \quad \text{if } d < 2\alpha,
\]
\[
C_{R,T}\|z_i\|_a A^{-\alpha}z_i + \varepsilon \|z_i\|_a, \quad \text{if } d = 2\alpha,
\]
\[
C_{R,T}\|z_i\|_a A^{-\alpha}z_i + \varepsilon \|z_i\|_a \alpha A^{-\alpha}z_i + \varepsilon \|z_i\|_a, \quad \text{if } d > 2\alpha
\]
\[
\leq C_{c,R,T}\|z_i\|_a^2 + C_{c,R,T}\|z_i\|_a^2.
\]

Applying (2.84) and (2.85) in (2.79), we obtain from (2.82) that
\[
\frac{d}{dt}H_t(z, z_i) + \left(\frac{1}{2} - \varepsilon\right)\|z_i\|^2 \leq C_{c,R,T}H_t(z, z_i), \quad t \in [0, T].
\]

So, if \(\varepsilon \in (0, \frac{1}{2})\), by applying Gronwall’s inequality, we obtain
\[
H_t(z(t), z_i(t)) + \left(\frac{1}{2} - \varepsilon\right)\int_0^t \|z_i\|^2 \, dt \leq C_{c,R,T}H_t(z(0), z_i(0)), \quad t \in [0, T],
\]
which, together with (2.82) and (2.83), implies (2.12). \(\square\)

### 3 Existence of the global attractor

In this section, we will study the existence of global attractor for problem (1.20). First, we recall some definitions and results related to the global attractor. More details can be found in [69].

**Definition 2.** The pair \((X, S_t)\) with \(X\) being a complete metric space is said to be a continuous dynamical system if
(i) \(\{S_t\}_{t \geq 0}\) is a family of continuous mappings of \(X\) into itself and it satisfies the semigroup property: \(S_0 = I\) and \(S_{t+s} = S_t \circ S_s\) for all \(t, s \geq 0\); and
(ii) the mapping \(x \mapsto S_t x\) is continuous from \(R_+ = [0, \infty)\) into \(X\) for every \(x \in X\).

**Definition 3.** Let \((X, S_t)\) be a continuous dynamical system, where \(X\) is complete metric space with metric \(d\).
- \((X, S_t)\) is said to be (bounded) dissipative if it possesses a bounded absorbing set \(B\), and a closed set \(B \subset X\) is said to be absorbing for \(S_t\) if for any bounded set \(D \subset X\) there exists \(t_0(D)\) such that \(S_tD \subset B\) for \(t \geq t_0(D)\);
- \((X, S_t)\) is said to be asymptotically smooth if for every bounded set \(D\) such that \(S_tD \subset D\) for \(t > 0\) there exists a compact set \(K\) in the closure \(\overline{D}\) of \(D\), such that \(S_tD\) converges uniformly to \(K\) in the sense that
  \[
  \lim_{t \to \infty} \sup_{x \in D} \inf_{y \in K} d(S_t x, y) = 0;
  \]
- A compact set \(\mathcal{A}\) is said to be a global attractor if
  (i) \(\mathcal{A}\) is an invariant set; that is, \(S_t \mathcal{A} = \mathcal{A}\) for \(t \geq 0\); and
  (ii) \(\mathcal{A}\) is uniformly attracting; that is, for all bounded set \(D \subset X\),
  \[
  \lim_{t \to \infty} \sup_{x \in D} \inf_{y \in \mathcal{A}} d(S_t x, y) = 0.
  \]

The following theorem is used to study the existence of global attractor, which can be found in [58].

**Theorem 2.** Let \((X, S_t)\) be a dissipative asymptotically smooth continuous dynamical system on a complete metric space \(X\). Then \(S_t\) possesses a unique global attractor \(\mathcal{A}\) and \(\mathcal{A} = \omega(B_0)\), where \(\omega(B_0)\) is the omega-limit set of \(B_0\), and \(B_0\) is the absorbing set of \((X, S_t)\), i.e.,
\[
\mathcal{A} = \omega(B_0) = \bigcap_{t \geq 0} \bigcup_{s \geq t} S_s u B_0,
\]
(3.1)
and bar over a set means the closure in $X$. Furthermore, $\omega \in \mathcal{A}$ if and only if there exist sequences $\{t_n\}_{n=1}^\infty \subset \mathbb{R}$, and $\{x_n\}_{n=1}^\infty \subset \mathcal{B}$, such that $t_n \to \infty$ as $n \to \infty$ and $S_{t_n} x_n \to \omega$ in $X$ as $n \to \infty$. Moreover, if $\mathcal{B}$ is connected, then $\mathcal{A}$ is connected.

The following theorem is used to study the dissipativity.

**Theorem 3.** [58] Let $(X, S_t)$ be a continuous dynamical system in some Banach space $X$ with norm $\|x\|_X$. Assume that

(i) there exists a continuous function $W(x)$ on $X$ possessing the properties

$$\phi_1(\|x\|_X) \leq W(x) \leq \phi_2(\|x\|_X), \quad \forall x \in X,$$

where $\phi_1$ are continuous functions on $\mathbb{R}_+$, $[0, \infty)$ such that $\phi_1(r) \to \infty$ as $r \to \infty$;

(ii) there exists a derivative $\frac{d}{dt} W(S_t y)$ for $t > 0$ and $y \in X$, a positive function $\kappa(r)$ on $\mathbb{R}_+$, and a positive constant $\chi$ such that

$$\frac{d}{dt} W(S_t y) \leq -\kappa(\|y\|_X) \text{ provided } \|S_t y\|_X \geq \chi.$$

Then the dynamical system $(X, S_t)$ is dissipative with an absorbing set of the form

$$B_0 = \{x : \|x\|_X \leq R\},$$

where the constant $R$ depends on the functions $\phi_1$ and $\phi_2$ and the constant $\chi$ only.

By Theorem 1, we obtain the following corollary.

**Corollary 1.** Let the assumptions $(\phi_1), (\phi_2), (\phi_3)$, and $(f)$ hold, $\frac{1}{2} \leq \alpha \leq 1$, $g \in H$, and $(u_0, u_t) \in \mathcal{H}$ satisfying $\|(u_0, u_t)\|_H \leq R$, where $R > 0$ is a constant. Then for every $T > 0$, problem (1.20) has a unique weak solution $u$ on $[0, T]$ such that

$$(u, u_t) \in C([0, T]; \mathcal{H}), u \in L^2(0, T; V_{a0}), u_t \in L^2(0, T; V_a), u_{tt} \in L^2(0, T; V_{a}),$$

and (2.3) holds. Moreover, there exists a constant $C_{R,T}$ such that

$$\|u\|_{L^2(0, T; V_{a0})} + \max_{0 \leq t \leq T} \|u(t)\|_{V_{a0}} + \max_{0 \leq t \leq T} \|u_t(t)\|_{V_{a0}} + \|u_{tt}(t)\|_{V_{a0}} \leq C_{R,T}. \quad (3.2)$$

For every $a \in (0, T)$, $u \in L^\infty(a, T; V_{a0})$, $u_t \in L^\infty(a, T; V_a)$, $u_{tt} \in L^\infty(a, T; V_{a})$, and there exists a constant $C_{a,T}$ such that

$$\|u\|_{L^\infty(a, T; V_{a0})} + \|u_t\|_{L^\infty(a, T; V_a)} + \|u_{tt}\|_{L^\infty(a, T; V_{a})} \leq C_{a,T}, \quad (3.3)$$

and it holds the following energy equality

$$E(u(t), u_t(t)) + \int_s^t \|u(t)\|_{V_{a0}}^2 \, dt = E(u(s), u_t(s)), \quad \forall 0 \leq s < t \leq T; \quad (3.4)$$

where $E$ is defined in (2.9) and (2.10). Moreover, if $u(t)$ and $w(t)$ are two weak solutions with initial data $(u_0, u_t)$ and $(w_0, w_t)$, respectively, such that $\|(u_0, u_t)\|_H \leq R$ and $\|(w_0, w_t)\|_H \leq R$, then for every $T > 0$ and $\varepsilon \in (0, 1/2]$, there exists a constant $C_{\varepsilon,R,T} > 0$ such that $z(t) = u(t) - w(t)$ satisfies the relation

$$\frac{1 - \varepsilon}{2(1 + \varepsilon)} \|z(t)\|_{V_{a0}}^2 + \frac{\varepsilon(1 - \varepsilon)}{4} \|z(t)\|_{V_a}^2 + \left(1 - \varepsilon\right) \int_0^t \|z\|^2 \, dt \leq C_{\varepsilon,R,T}(\|z(0)\|_{V_{a0}}^2 + \|z(0)\|_{V_a}^2), \quad t \in [0, T]. \quad (3.5)$$

---

1 This function $\kappa(r)$ may tend to zero as $r \to \infty$. 

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Let the assumptions of Corollary 1 be in force, and define \( \mathcal{S}_t \) by
\[
\mathcal{S}_t(u_0, u_t) = (u(t), u_t(t)), \quad \forall (u_0, u_t) \in \mathcal{H},
\]
where \( u \) is the solution got in the above corollary. Then by Corollary 1, \( (\mathcal{H}, \mathcal{S}_t) \) is a continuous dynamical system. Then we obtain the following result:

**Theorem 4.** Let the assumptions of Corollary 1 be in force. Then the continuous dynamical system \( (\mathcal{H}, \mathcal{S}_t) \) defined by (3.6) has a connected global attractor, which is bounded in \( V_{\beta_1} \times V_{\beta_2} \) for any \( \beta_1 < 2\alpha \) and \( \beta_2 < \alpha \).

**Proof.** We will prove this theorem by using Theorem 2, and the proof is divided into two steps.

**Step 1:** \( (\mathcal{H}, \mathcal{S}_t) \) *is dissipative.* By assumption \( (\phi_{ass})_1 \), there exists constants \( r_0 > 0 \) and
\[
0 < \epsilon_1 \leq \frac{2|\mu_f - 2| + \lambda_1^{2\alpha}}{|\mu_f - 2| + \lambda_1^{2\alpha}}
\]
such that
\[
\phi(r) \geq \epsilon = \max \left\{ |\mu_f - 2|\lambda_1^{-\alpha} + \epsilon_1, \frac{1}{2}(|\mu_f - 2|\lambda_1^{-\alpha} + \eta\lambda_1^{-\alpha} + \eta) + \epsilon_1 \right\}, \quad r \geq r_0.
\]
We define
\[
W(u_0, u_1) = E(u_0, u_1) + \eta \left( (u_0, u_1) + \frac{1}{2}\|u_0\|_{V_\alpha}^2 \right), \quad (u_0, u_1) \in \mathcal{H},
\]
where \( E \) and \( \eta \) are defined in (2.9) and (1.16), respectively.

For \( \|u_0\|_{V_\alpha}^2 \geq r_0 \), by (3.8) and \( \phi \) is nonnegative we obtain
\[
\Phi(\|u_0\|_{V_\alpha}^2) = \int_0^{\|u_0\|_{V_\alpha}^2} \phi(\xi) d\xi \geq \epsilon \int_{r_0}^{\|u_0\|_{V_\alpha}^2} d\xi = \epsilon(\|u_0\|_{V_\alpha}^2 - r_0),
\]
which implies
\[
\Phi(\|u_0\|_{V_\alpha}^2) \geq \epsilon(\|u_0\|_{V_\alpha}^2 - r_0) + \zeta_1, \quad \forall u_0 \in V_\alpha,
\]
where \( \zeta_1 = \min(0, \min_{s \leq 0} \Phi(s)) \). By (A.3) and (1.18),
\[
\int_\Omega F(u_0) dx \geq -\frac{1}{2}|\mu_f - 2|\|u_0\|_2^2 - \frac{\tilde{C}^2}{2}|\Omega| \geq -\frac{1}{2}\lambda_1^{-\alpha}|\mu_f - 2|\|u_0\|_{V_\alpha}^2 - \frac{\tilde{C}^2}{2}|\Omega|.
\]
So, by (2.9), (3.9), (3.10), (1.18), Young’s inequality, (3.8), and (3.7), we obtain
\[
W(u_0, u_1) = \frac{1}{2}\|u_1\|^2 + \Phi(\|u_0\|_{V_\alpha}^2) + \int_\Omega F(u_0) dx - \int_\Omega g u_0 dx + \eta \left( (u_0, u_1) + \frac{1}{2}\|u_0\|_{V_\alpha}^2 \right)
\]
\[
\geq \frac{1}{2}\|u_1\|^2 + \epsilon(\|u_0\|_{V_\alpha}^2 - r_0) + \zeta_1 - \frac{1}{2}\lambda_1^{-\alpha}|\mu_f - 2|\|u_0\|_{V_\alpha}^2 - \frac{\tilde{C}^2}{2}|\Omega|
\]
\[
- \frac{\epsilon_1}{2}\|u_0\|_{V_\alpha}^2 - \frac{1}{2}\|g\|_2^2 - \eta \left( \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\lambda_1^{-\alpha}\|u_0\|_{V_\alpha}^2 + \frac{1}{2}\|u_0\|_{V_\alpha}^2 \right)
\]
\[
\begin{align*}
&= \frac{1}{2}(1 - \eta)\|u_t\|^2 + \left( e - \frac{1}{2}(\|\mu - 2| + 2| + \eta \lambda_1 - \eta - \eta) - \frac{\eta}{2}\right)\|u_0\|_{V_0}^2 + \gamma_1 - \epsilon_0 - \frac{\epsilon_1}{2}|\Omega| - \frac{1}{2}\varepsilon_1\|g\|^2
\end{align*}
\]

\[
\geq \frac{2|\mu_t - 2| + \lambda_1}{4(|\mu_t - 2| + \lambda_1)}\|u_t\|^2 + \frac{\epsilon_1}{2}\|u_0\|_{V_0}^2 + \gamma_2 \geq \frac{\epsilon_1}{2}(\|u_0\|_{V_0}^2 + \|u_t\|^2) + \gamma_2.
\]

Let

\[
\phi_i(r) = \frac{\epsilon_1}{2}r^2 + \gamma_2,
\]

we obtain

\[
\lim_{r \to \infty} \phi_i(r) = \infty \quad \text{and} \quad W(u_0, u_i) \geq \phi_i(\|u_0, u_i\|_{H^1}).
\]

On the other hand, by \((3.9), (A.8), (1.18)\), Young’s inequality, \(\eta \in (0, 1)\), and \(\Phi(\cdot)\) is nondecreasing (since \(\phi\) is nonnegative), we obtain

\[
W(u_0, u_i) = \frac{1}{2}\|u_t\|^2 + \Phi(\|u_0\|_{V_0}^2) + \int_\Omega F(u_0)dx - \int_\Omega g u_0 dx + \eta \left( \|u_0, u_i\| + \frac{1}{2}\|u_0\|_{V_0}^2 \right)
\]

\[
\leq \frac{1}{2}\|u_t\|^2 + \Phi(\|u_0\|_{V_0}^2) + \tilde{G}(\|u_t\|_{V_0}^2) + \frac{1}{2}\|g\|^2 + \frac{1}{2}\|u_0\|_{V_0}^2 + \frac{1}{2}\|u_t\|^2 + \Phi(\|u_0\|_{V_0}^2 + \|u_t\|^2) + \tilde{G}(\|u_0\|_{V_0}^2 + \|u_t\|^2) + \frac{1}{2}\|g\|^2,
\]

where \(\tilde{G}\) is given in \((A.8)\), which is positive. Let

\[
\phi_2(r) = \max\left\{1, \frac{1}{2} + \lambda_1 r^2 + \Phi(r^2) + \tilde{G}(r^2) + \frac{1}{2}\|g\|^2, \right\}
\]

we obtain

\[
\lim_{r \to \infty} \phi_2(r) = \infty \quad \text{and} \quad W(u_0, u_i) \leq \phi_2(\|u_0, u_i\|_{H^1}).
\]

Let \((u, u_i) = S_t(u_0, u_i)\), which is defined in \((3.6)\). Since \(u \in C([0, \infty); V_0)\) (see Corollary 1), by taking \(\psi = u(t)\) in \((2.3)\), we obtain that

\[
\frac{d}{dt}\left( (u, u_i) + \frac{1}{2}\|u\|_{V_0}^2 \right) - \|u_t\|^2 + \Phi(\|u\|_{V_0}^2)\|u\|_{V_0}^2 + (f(u), u) - (g, u) = 0.
\]

If

\[
\|S_t(u_0, u_i)\|_{H^1}^2 = \|u, u_i\|_{H^1}^2 = \|u_t\|^2 + \|u\|_{V_0}^2 > 2\chi
\]

with

\[
\chi = \max\left\{r_0, \frac{2\lambda_1(|g| + \hat{C}^2|\Omega|)}{|\mu_t - 2| + \lambda_1}, \frac{|g| + \hat{C}^2|\Omega|}{\varepsilon_1} \right\},
\]

where \(r_0, \hat{C}, \mu_t, \) and \(\varepsilon_1\) are the positive constants given \((3.8), (A.2), (1.6),\) and \((3.7)\), respectively, we have

\[
(i) \|u\|_{V_0}^2 > \chi; \quad \text{or} \quad (ii) \|u\|_{V_0}^2 \leq \chi \quad \text{and} \quad \|u_t\|^2 > \chi.
\]
By (3.4), (3.13), (A.1), (1.18), (3.15), (1.16), (3.8), and Young’s inequality, we obtain
\[
\frac{dW}{dt} = -\|u(t)\|_{V_\alpha}^2 + \eta \frac{d}{dt} \left( \|u(t)\|_{V_\alpha}^2 + \frac{1}{2}\|u(t)\|_{V_\alpha}^2 \right)
\]
\[
= -(\eta - 1)\|u(t)\|_{V_\alpha}^2 - \eta \phi(\|u(0)\|_{V_\alpha})\|u(t)\|_{V_\alpha}^2 - \eta (f(u), u) + \eta (g, u)
\]
\[
\leq (\eta - 1)\|u(t)\|_{V_\alpha}^2 - \eta \phi(\|u(0)\|_{V_\alpha})\|u(t)\|_{V_\alpha}^2 - \frac{\eta}{2}[(2\mu - 3)\|u\|^2 - \tilde{C}_1\|u\|^2] + \frac{\eta}{2}(\|g\|^2 + \|u\|^2)
\]
\[
\leq -(1 - \eta)\lambda_0\|u(t)\|^2 - \frac{\eta}{2}[2\phi(\|u(0)\|_{V_\alpha})\|u(t)\|^2 - 2\mu \|u\|^2 - 2\lambda_0\|u\|^2 - \|g\|^2 - \tilde{C}_1\|u\|^2]
\]
\[
\leq \left\{ \begin{array}{ll}
(1 - \eta)\lambda_0\|u(t)\|^2 + \eta \mu \|u\|^2 - 2\lambda_0\|u\|^2 + \eta \|g\|^2 + \tilde{C}_1\|u\|^2), & \text{if (ii) of (3.15) holds},

-\frac{\eta}{2}[2\phi(\|u(0)\|_{V_\alpha})\|u(t)\|^2 - 2\mu \|u\|^2 - 2\lambda_0\|u\|^2 - \|g\|^2 - \tilde{C}_1\|u\|^2], & \text{if (i) of (3.15) holds}
\end{array} \right.
\]
\[
\leq \left\{ \begin{array}{ll}
\frac{\lambda_0}{2(\mu - 2) + \lambda_0} (\|g\|^2 + \tilde{C}_1\|u\|^2) & \text{(since } \chi \geq \chi_0),

\frac{\lambda_0}{2(\mu - 2) + \lambda_0} (\|g\|^2 + \tilde{C}_1\|u\|^2), & \text{(since } \|u\|_{V_\alpha}^2 > \chi \geq \chi_0),
\end{array} \right.
\]
\[
\leq \frac{\lambda_0}{4(\mu - 2) + \lambda_0} (\|g\|^2 + \tilde{C}_1\|u\|^2).
\]

In view of Theorem 3, it follows from (3.11), (3.12), and the above ordinary differential inequality, the dynamical system \((\mathcal{H}, S_t)\) is dissipative and has a bounded absorbing set \(B\), defined as
\[
B_s = \{(u_0, u_1) \in \mathcal{H} : \|(u_0, u_1)\|_{\mathcal{H}} \leq R_s\}. \tag{3.16}
\]

**Step 2: Existence of global attractor.** By Definition 3, there exists \(t_s = t(B') \geq 0\) such that \(S(t)B_s \subset B\) for \(t \geq t_s\). Thus, we can define a set
\[
B = \bigcup_{t \geq t_s} S(t)B_s \subset B_s.
\]
then \(B\) is a forward invariant (i.e., \(S_tB \subset B\) for every \(t \geq 0\)) closed bounded connect absorbing set. Thus,
\[
\|(u(t), u(t))\|_{\mathcal{H}} = \|S_t(u_0, u_1)\|_{\mathcal{H}} \leq R_s, \quad t \geq t_s, (u_0, u_1) \in B_s, \tag{3.17}
\]
\[
\|(u(t), u(t))\|_{\mathcal{H}} = \|S_t(u_0, u_1)\|_{\mathcal{H}} \leq R_s, \quad t \geq 0, (u_0, u_1) \in B. \tag{3.18}
\]

Now let \((u_0, u_1) \in \bigcup_{t \geq t_s} S(t)B_s\), then there exists \(t_0 \geq t_s\) and \((u_0', u_1') \in B_s\), such that \((u_0, u_1) = S_{t_0}(u_0', u_1')\). Then
\[
(u_0, u_1(t)) = S_{t_0}(u_0, u_1) = S_{t_0}(u_0', u_1') = S_{t_0-t}(u_0', u_1').
\]

Then for \(t \in [0, 1]\),
\[
(u(t), u(t)) = S_{t}(u_0, u_1) = S_{t_0-t}(u_0', u_1').
\]

Note \(|S_{t_0-t}(u_0', u_1')|_{\mathcal{H}} \leq R_s\) (see (3.17)) and \(t + 1 \in [1, 2]\), by choosing \(T = 2\) and \(\alpha = 1\) in (3.3), it follows
\[
\|u\|_{L^\infty(0, 1; v_\alpha)} + \|u_t\|_{L^\infty(0, 1; v_\alpha)} + \|u_t\|_{L^\infty(0, 1; v_\alpha)} + \|u_t\|_{L^\infty(0, 1; u_\alpha)} \leq C_{R, 2} = C_{R_s} \tag{3.19}
\]

Then for any \(\beta_1 \in (0, 2\alpha)\) and \(\beta_2 \in (0, \alpha)\), the above inequality and Lemma 1 imply
\[
\|u\|_{L^\infty(0, 1; v_\beta_1)} + \|u_t\|_{L^\infty(0, 1; v_\beta_2)} \leq C_{R_s}.
\]
Especially,
\[ \|u_0\|_V^\beta + \|u\|_V^\beta \leq C_R. \]

Since \( V_{\beta_i} \times V_{\beta_j} \hookrightarrow \mathcal{H} \), the above analysis shows that \( B \) is a forward invariant compact connect absorbing set for \((\mathcal{H}, S_t)\). Then it is easy to see \((\mathcal{H}, S_t)\) is asymptotically smooth, and the conclusion follows from Theorem 2.

\[ \square \]

4 Infinite fractal dimension of the global attractor

In this section, we will estimate the fractal dimension of the global attractor got in Theorem 4 with assumption that \( g(x) \equiv 0 \), i.e., we consider the problem

\[
\begin{cases}
u_t + \phi(\|u\|^2)A^p u + A^q u_t + f(u) = 0, \\
u(0) = u_0, & u_t(0) = u_t
\end{cases}
\]  (4.1)

with the following assumption on \( \phi \) and \( f \):

**Assumptions 5.1:** The functions \( \phi \) and \( f \) of model (1.20) satisfy

(i) \( \phi(s) = L s^q \) for any \( s \geq 0 \), there \( L > 0 \) and \( q \geq 1 \) are constants;

(ii) \( f(u) = a|u|^{-\alpha} - b|u|^{-\beta} u + h(u) \), where \( h \) is a \( C^1 \) function such that \( h(-u) = -h(u) \),

\[
\mu_h = \liminf_{|s| \to \infty} \frac{h(s)}{s} > -\infty, \quad \lim_{|s| \to 0} \frac{|h(s)|}{|s|^q} = K \geq 0,
\]  (4.2)

where the constants \( a \geq 0, b > 0, \zeta \geq 2, \nu \geq 2, q \geq 2 \), and

(ii-1) if \( d \leq 2\alpha \), then there exist constants \( C > 0 \) and \( p \geq 1 \) such that \( |h'(u)| \leq C(1 + |u|^{p-1}) \);

(ii-2) if \( d > 2\alpha \), then there exist constants \( C > 0 \) and \( p \in \left[ 1, \frac{d+2\alpha}{d-2\alpha} \right] \) such that \( |h'(u)| \leq C(1 + |u|^{p-1}) \).

Moreover, it holds \( \zeta \leq \frac{2d}{d-2\alpha} \) and \( \nu < \min\{p+1, \zeta, q, 2\beta + 2\} \).

Let

\[ H(u) = \int_0^u h(s) ds. \]  (4.3)

By the above assumptions, we can see there exists positive constants \( M_1 \) and \( M_2 \) such that

\[ |H(u)| \leq M_1|u|^{q+1} + M_2|u|^{p+1}. \]  (4.4)

With Assumption 5.1, one can easily see the assumptions in Theorem 4 are in force. Moreover, we have the following theorem:

**Theorem 5.** Let Assumption 5.1 be in force, then the semigroup \( S_t \) defined in (3.6) is odd, i.e., \(-S_t(u_0, u_t) = S_t(-u_0, -u_t)\) for all \((u_0, u_t) \in \mathcal{H} \) and \( t \geq 0 \), and the global attractor \( \mathcal{A} \) got in Theorem 4 is symmetric, i.e., \((-u_0, -u_t) \in \mathcal{A} \) for any \((u_0, u_t) \in \mathcal{A} \), where the space \( \mathcal{H} \) is defined in (1.19).

**Proof.** Let \((u(t), u_t(t)) = S_t(u_0, u_t)\) be defined in (3.6). Then \( u(t) \) is the weak solution of (4.1) with initial value \((u_0, u_t)\). Since \( f(u) \) is odd (see assumption (ii) in Assumption 5.1), one can easily see \(-u(t)\) is the solution of (4.1) with initial value \((-u_0, -u_t)\). By the uniqueness of solutions (see Corollary 1), \((-u(t), -u_t(t)) = S_t(-u_0, -u_t)\), so \( S_t \) is odd.
By (3.1), we have
\[ \mathcal{A} = \omega(B_n) = \bigcap_{t \geq 0} \bigcup_{B_i} S_t u B_i, \]
where \( B_i \) is the symmetric set defined in (3.16). By Theorem 2, for any \((u, v) \in \mathcal{A}\), there exist sequences \( \{t_n\}_{n=1}^{\infty} \subseteq \mathbb{R} \) and \( \{(u_n, v_n)\}_{n=1}^{\infty} \subseteq \mathcal{B} \), such that \( t_n \to \infty \) as \( n \to \infty \) and \( S_{t_n}(u_n, v_n) \to (u, v) \) in \( \mathcal{H} \) as \( n \to \infty \). Since \( B_i \) is symmetric and \( S_t \) is odd, we obtain \((-u_n, -v_n) \in B_i \) and \( S_{t_n}(-u_n, -v_n) = -S_{t_n}(u_n, v_n) \to (-u, -v) \) in \( \mathcal{H} \) as \( n \to \infty \). Then by Theorem 2 again, \((-u, -v) \in \mathcal{A}\), so \( \mathcal{A} \) is symmetric.

Now we consider the fractal dimension \( \dim_{\mathcal{H}} \mathcal{A} \) of the global attractor \( \mathcal{A} \), which is defined as
\[ \dim_{\mathcal{H}} \mathcal{A} = \limsup_{\varepsilon \to 0} \frac{\ln n(\mathcal{A}, \varepsilon)}{\ln(1/\varepsilon)}, \]
where \( n(\mathcal{M}, \varepsilon) \) is the minimal number of closed balls of radius \( \varepsilon \) which cover the set \( \mathcal{A} \) (note by Theorem 4 \( \mathcal{A} \) is compact).

Next we study \( \dim_{\mathcal{H}} \mathcal{A} \) by \( Z_2 \) index theory. To this end, in the following, we give an outline of \( Z_2 \) index and its properties. The concept is most explained for even functional \( F \) on some Banach space \( Y \), with symmetry group \( Z_2 = \{id, -id\} \). Define
\[ \mathcal{B} = \{ B \subset Y : B \text{ is closed, } \mathcal{B} = -B \} \]
(4.5)
to be the class of closed symmetric subsets of \( Y \).

**Definition 4.** For \( B \in \mathcal{B}, B \neq \emptyset \), following Coffman [73], let
\[ \gamma(B) = \begin{cases} \inf \{ m \in \mathbb{N} : \exists h \in C(B, \mathbb{R}^{m}\{0\}), \ h(-u) = -h(u) \}, \\ \infty, \text{ if } \{\cdots\} = \emptyset, \end{cases} \]
in particular, if \( 0 \in B \)
and define \( \gamma(\emptyset) = 0 \). \( \gamma(B) \) is called the \( Z_2 \) index or Krasnoselskii genus.

The following properties of \( Z_2 \) index can be found in [68]:

**Proposition 1.** For any bounded symmetric neighborhood \( \Omega \) of the origin in \( \mathbb{R}^d \) there holds \( \gamma(\partial \Omega) = d \), where \( \partial \Omega \) is the boundary of \( \Omega \).

**Proposition 2.** Let \( B, B_1, B_2 \in \mathcal{B}, \) and \( h \in C(Y, Y) \) be an odd map. Then the following hold:
(i) \( \gamma(B) \geq 0 \), and \( \gamma(B) = 0 \) if and only if \( B = \emptyset \);
(ii) if \( B_1 \subset B_2 \), then \( \gamma(B_1) \leq \gamma(B_2) \);
(iii) \( \gamma(B_1 \cup B_2) \leq \gamma(B_1) + \gamma(B_2) \);
(iv) \( \gamma(B) \leq \gamma(h(B)) \);
(v) If \( B \) is compact and \( 0 \notin B \), then \( \gamma(B) < \infty \) and there exists a neighborhood \( N(B) \) of \( B \) in \( Y \) such that \( \gamma(N(B)) \in \mathbb{N} \).

Let \( Y = \mathcal{H} \). By Theorem 5, \( \mathcal{A} \in \mathcal{B} \), where \( \mathcal{B} \) is defined in (4.5). Then \( \mathcal{A}\setminus N(0) \in \mathcal{B} \) and \( \gamma(\mathcal{A}\setminus N(0)) \) is well defined, where \( N(0) \) is a neighborhood of \( 0 = (0, 0) \) in \( \mathcal{H} \). Moreover, we have

**Theorem 6.** Let Assumption 5.1 be in force. Then for \( m = 1, 2, \ldots \), there exists a neighborhood \( N(0) \) of \( 0 \) in \( \mathcal{H} \) such that \( \gamma(\mathcal{A}\setminus N(0)) \geq m \).

By using the above theorem, we obtain
Theorem 7. Let Assumption 5.1 be in force. Then \( \dim_\mathcal{A} = \infty \).

To prove the above two theorems, we need the following lemma, which can be found in [74]:

Lemma 4. Let \((Y, S_t)\) be a continuous dynamical system in some Banach space \(Y\). Assume \(S_t\) is odd and \(\mathcal{A}\) is a symmetric global attractor. If there exists a bounded set \(B \in \mathcal{B}\) such that \(\gamma(B) \geq m, m < \infty\), and \(\omega(B) \in \mathcal{A} \setminus \{0\}\), then there exists a neighborhood \(N(0)\) of \(0\) in \(Y\) such that \(\gamma(\mathcal{A} \setminus N(0)) \geq m\), where \(\mathcal{B}\) is defined in (4.5), and the omega-limit set \(\omega(\cdot)\) is defined in (3.1).

Proof of Theorem 6. By Lemma 4, we only need to prove for \(m = 1, 2, \ldots\), there exists \(B_m \in \mathcal{B}\) such that \(\gamma(B_m) \geq m\) and \(\omega(B_m) \in \mathcal{A} \setminus \{0\}\).

Let \((u(t), u(t)) = S_t(u_0, u_0)\) be defined in (3.6), then \(u(t)\) is a weak solution of (4.1) with initial value \((u_0, u_0)\) and \((u(t), u(t))\) satisfies the conclusions got in Corollary 1. By the energy equality (3.4), it follows

\[
E(u(t), u(t)) + \int_0^t \|u_r(t)\|_{\mathcal{V}}^2 \, dr = E(u(s), u(s))
\]

holds for all \(t > s \geq 0\), where (see (2.9) and (2.10) and note \(g \equiv 0\) and Assumption 5.1)

\[
E(u_0, u_0) = \frac{1}{2} \|u_0\|^2 + \int_0^1 \Phi(\xi) \, d\xi
\]

and \(\mathcal{H}\) is defined in (4.3).

Let \(\{e_j\}_{j=1}^m\) be given in (1.17). Then, by (1.18), \((\lambda_j^{-1/2} e_j, \lambda_j^{-1/2} e_j)\) = \(\delta_{i,j}\). So, \(\{e_j\}_{j=1}^m\) is a complete orthonormal family of \(V_a\), where \(\tilde{e}_j = \lambda_j^{-1/2} e_j\). For \(m = 1, 2, \ldots\), and \(e > 0\), let \(V_a^m = \text{span}\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_m\}\). Because \(V_a^m\) is \(m\)-dimensional, all norms on \(V_a^m\) are equivalent. In particular, there exists a constant \(\sigma > 0\) such that

\[
\|u_0\|_{V_a^m} \geq \sigma \|u_0\|_V.
\]

Let

\[
B_m^e = \bigcup_{(a_0, \ldots, a_m) \in \partial \mathcal{B}_m} \left\{ (e u_0, 0) : u_0 = \sum_{j=1}^m a_j \tilde{e}_j \right\} \subset \mathcal{B},
\]

where \(\partial \mathcal{B}_m\) is the boundary of the unit ball \(\mathcal{B}_m\) of \(\mathbb{R}^m\), i.e.,

\[
\mathcal{B}_m = \left\{ (a_1, \ldots, a_m) \in \mathbb{R}^m : \sum_{j=1}^m a_j^2 \leq 1 \right\}.
\]

Let

\[
\mathcal{B}_m^e = \left\{ (e a_1, \ldots, e a_m) \in \mathbb{R}^m : \sum_{j=1}^m a_j^2 \leq 1 \right\}.
\]

Then \(\mathcal{B}_m^e\) is a bounded symmetric neighborhood of the origin in \(\mathbb{R}^m\). By Proposition 1, we obtain

\[
\gamma(B_m^e) = \gamma(\partial \mathcal{B}_m^e) = m.
\]
For any \((v_0, 0) = (\varepsilon u_0, 0) \in B_m^\varepsilon\), there exists \((a_1, a_2, \ldots, a_m) \in \partial B_m\) such that \(u_0 = \sum_{j=1}^m a_j \tilde{e}_j \in V_m\). Then it follows
\[
|u_0|_{V_m}^2 = \sum_{j=1}^m a_j^2 = 1,
\]
which, together with (4.7), implies
\[
\|u_0\| \geq \omega.
\]
Finally, it follows from the above analysis, \(V_a \hookrightarrow L^r, V_a \rightharpoonup L^{p+1}\) (see Assumption 5.1), and (4.4) that there exists a positive constant \(C\) such that
\[
E(v_0, 0) \leq \frac{L \varepsilon^{2\theta+2}}{2(\theta + 1)} \|u_0\|_{V_a}^{2(\theta + 1)} + \frac{1}{\zeta} \|u_0\| \left( - \frac{b \varepsilon^\nu}{v} \|u_0\|_V^\nu + M_2 \varepsilon^{p+1} \|u_0\|_{V_0}^{p+1} \right)
\leq C \varepsilon^{(\theta+2)\varepsilon^\nu} \|u_0\|_{V_a}^{2\theta+1} + \varepsilon^\nu \|u_0\|_V^\nu + \varepsilon^{p+1} \|u_0\|_{V_0}^{p+1} - \|u_0\|_V^\nu (4.9)
\leq C \varepsilon^{(\theta+2)\varepsilon^\nu} + \varepsilon^\nu + \varepsilon^{p+1} - \omega^\nu.
\]
Since \(v < \min\{2\theta + 2, \zeta, q, p+1\}\), for \(\varepsilon > 0\) small, we have \(E(v_0, 0) < 0\) for all \((v_0, 0) \in B_m^\varepsilon\). Let \((v(t), v_0(t)) = S_t(v_0, 0)\), by (4.6), it follows \(E(v(t), v_0(t)) = E(v_0, 0) < 0\) for all \(t \geq 0\), which, together with \(E(0, 0) = 0\), implies \(u(B_m^\varepsilon) \in \mathcal{A} \setminus \{0\}\). In view of (4.8), the conclusion follows from Lemma 4. \(\square\)

**Proof.** For \(m = 1, 2, \ldots\), by Theorem 6, there exists a neighborhood \(N(0)\) of \(0\) in \(\mathcal{H}\) such that \(\gamma(\mathcal{A} \setminus N(0)) \geq m\). So thanks to the Mane projection theorem (see [75]), we obtain \(\dim_\gamma \mathcal{A} \geq \left\lfloor \frac{m-1}{2} \right\rfloor\), where \(\left\lfloor \frac{m-1}{2} \right\rfloor\) is the integer part of \(\frac{m-1}{2}\). Then the conclusion follows from the arbitrary of \(m\). \(\square\)

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**References**


Appendix: Proof of (2.18)

By (1.6), there exists a constant $M > 0$ such that

$$\frac{f(s)}{s} \geq (\mu_f - 1), \quad \forall |s| > M,$$

which implies

$$f(s) \begin{cases} 
\geq (\mu_f - 1)s + C_1 - X_{(\mu_f - 1)}M(\mu_f - 1), & \text{if } s \geq 0 \\
\leq (\mu_f - 1)s + C_2 + X_{(\mu_f - 1)}M(\mu_f - 1), & \text{if } s \leq 0,
\end{cases} \quad (A.1)$$

where

$$X_{(\mu_f - 1)} = \begin{cases} 
1, & \text{if } \mu_f > 1; \\
0, & \text{if } \mu_f \leq 1,
\end{cases} \quad C_1 = \min \left\{ 0, \min_{0 \leq s \leq M} f(s) \right\}, \quad C_2 = \max \left\{ 0, \max_{-M \leq s < 0} f(s) \right\}.$$

Let

$$\tilde{C} = \max\{|C_1 - X_{(\mu_f - 1)}M(\mu_f - 1)|, |C_2 + X_{(\mu_f - 1)}M(\mu_f - 1)|\}. \quad (A.2)$$

Then it follows

$$F(s) = \int_0^s f(\tau)d\tau \geq \frac{1}{2}(\mu_f - 1)s^2 - \tilde{C}|s| \geq \frac{1}{2}(\mu_f - 2)s^2 - \frac{\tilde{C}^2}{2},$$

which implies

$$\int_\Omega F(u_0)dx \geq \frac{1}{2}(\mu_f - 2)||u_0||^2 - \frac{\tilde{C}^2}{2}||\Omega||. \quad (A.3)$$

On the other hand, by assumption $(f_{ass})$, we obtain when $d \geq 2$, there exist positive constants $\hat{C}_1$ and $\hat{C}_1$ such that

$$|f(s)| \leq \hat{C}_1 + \hat{C}_1|s|^p, \quad (A.4)$$

where

$$1 \leq p \begin{cases} 
< \infty & \text{if } d = 2a; \\
\leq \frac{d + 2a}{d - 2a} & \text{if } d > 2a.
\end{cases} \quad (A.5)$$

Then there exist positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$|F(s)| = \left| \int_0^s f(\tau)d\tau \right| \leq \tilde{C}_1 + \tilde{C}_2|s|^{p+1}. \quad (A.6)$$

Since

$$V_0 \hookrightarrow \begin{cases} 
C(\Omega), & \text{if } d < 2a \\
L^q(\Omega), \forall q \in [1, \infty), & \text{if } d = 2a \\
L^q(\Omega), \forall q \in \left[ 1, \frac{2d}{d - 2a} \right], & \text{if } d > 2a,
\end{cases} \quad (A.7)$$
it follows from (A.5) and (A.6) that
\[
\int_{\Omega} F(u_0) \, dx \leq \left| \Omega \right| \left[ \max_{|s| \leq \max \{ |u_0(x)| \}} |F(s)| \right] \leq \left| \Omega \right| \left[ 1 + \max_{|u| \leq C(u_0) \|u\|_{\Omega}^{1/2}} |F(s)| \right], \quad \text{if } d < 2\alpha;
\]
\[
\left( C_{1}\left| \Omega \right| + C_{2}\|u_0\|_{\Omega}^{p + 1} \right) \leq \left( C_{1}\left| \Omega \right| + C_{2}\|u_0\|_{\Omega}^{p + 1} \right)^{\max} \leq \|u_0\|_{\Omega}^{p + 1}, \quad \text{if } d \geq 2\alpha
\]
(\text{A.8})

where \( C_{i} \) is the optimal embedding constant of the embedding given in (A.7). It is obvious that \( \tilde{G}(\cdot) \) is a monotone positive function.

Now we prove (2.18). First we prove for every \( \eta \geq \eta_{0}, \) there exists constants \( a_{i} > 0 \) and \( v_{i}^{\eta_{0}, \eta} \geq 0, \) such that for any \( \nu \geq v_{i}^{\eta_{0}, \eta} \) it holds
\[
\frac{1}{4} E^{\eta}_{\nu}(u_0, u_{i}) - a_{i} \leq W^{\eta, \nu}(u_0, u_{i}).
\] (A.9)

By the definitions of \( E, E^{\eta} \) and \( W^{\eta, \nu} \) in (2.9), (2.13), and (2.17), respectively, we have
\[
\frac{1}{4} E^{\eta}_{\nu}(u_0, u_{i}) - W^{\eta, \nu}(u_0, u_{i}) = \frac{1}{4} [\|u\|^{2} + \Phi(\|u\|_{\Omega}^{2}) - \eta \|u\|_{\nu}^{2} - a(\eta) + \|u_{0}\|^{2} - \frac{1}{2} \|u\|^{2} + \Phi(\|u\|_{\nu}^{2})]
\]
\[
- \int_{\Omega} F(u_0) \, dx + \int_{\Omega} g u_0 \, dx - \eta \left[ (u_0, u_{i}) + \frac{1}{2} \|u_{0}\|^{2} \right] - \nu \|u_{0}\|^{2}
\]
\[
\leq -\frac{1}{4} [\|u\|^{2} + \Phi(\|u\|_{\nu}^{2}) - \eta \|u\|_{\nu}^{2}] - \frac{1}{4} a(\eta) + \left( \frac{1}{4} - \nu \right) \|u_{0}\|^{2} - \int_{\Omega} F(u_0) \, dx
\]
\[
+ \frac{1}{2} (\|g\|^{2} + \|u\|^{2}) + \frac{1}{4} \|u_{i}\|^{2} + \eta^{2} \|u_{0}\|^{2}.
\]

By the definition of \( a(\eta) \) (see (2.14)), it follows
\[
-\frac{1}{4} [\Phi(\|u\|_{\nu}^{2}) + \eta \|u\|_{\nu}^{2}] \leq -\frac{1}{4} a(\eta).
\]

Then, by (A.3), \( \eta \geq \eta_{0}, \) and (2.16), we have, for
\[
v \geq \max \left\{ 0, \frac{7}{4} + \eta^{2} - \frac{1}{2} \mu_{1} \right\}
\]
\[
\frac{1}{4} E^{\eta}_{\nu}(u_0, u_{i}) - W^{\eta, \nu}(u_0, u_{i}) \leq -\frac{1}{2} a(\eta) + \left( \frac{3}{4} - \nu + \eta^{2} \right) \|u_{0}\|^{2} - \int_{\Omega} F(u_0) \, dx + \frac{1}{2} \|g\|^{2}
\]
\[
\leq -\frac{1}{2} a(\eta) + \left( \frac{7}{4} - \nu + \eta^{2} - \frac{1}{2} \mu_{1} \right) \|u_{0}\|^{2} + \frac{1}{2} \|g\|^{2} + \frac{\tilde{C}_{2}^{2}}{2} \left| \Omega \right|
\]
\[
\leq -\frac{1}{2} a(\eta) + \frac{1}{2} \|g\|^{2} + \frac{\tilde{C}_{2}^{2}}{2} \left| \Omega \right|.
\]

So, (A.9) follows with \( v_{i}^{\eta_{0}, \eta} \) and
\[
a_{i} = \max \left\{ 1, -\frac{1}{2} a(\eta) + \frac{1}{2} \|g\|^{2} + \frac{\tilde{C}_{2}^{2}}{2} \left| \Omega \right| \right\}.
\]

Next we prove for every \( \eta \geq \eta_{0}, \) there exists a constant \( v_{i}^{\eta_{0}, \eta} \) such that for any \( \nu \geq v_{i}^{\eta_{0}, \eta} \), there exists a monotone positive function \( G(s) \) satisfying
\[
W^{\eta, \nu}(u_0, u_{i}) \leq \frac{3}{4} E^{\eta}_{\nu}(u_0, u_{i}) + G(\|u_{0}\|_{\nu}^{2}).
\] (A.10)
By the definitions of $E$, $E^\eta$, and $W^{\nu, \eta}$ in (2.9), (2.13), and (2.17), respectively, we have

\[
W^{\nu, \eta}(u_0, u_i) - \frac{3}{4}E^\eta(u_0, u_i) = \frac{1}{2}||u_0||^2 + \Phi(||u_0||_{L^2}) + \int_\Omega F(u_0)dx - \int_\Omega gu_0dx + \eta \left( u_0 + \frac{1}{2}||u_0||_{L^2}^2 \right) + \frac{3}{4}||u_0||^2 + \Phi(||u_0||_{L^2}) + \eta ||u_0||_{L^2}^2 - \alpha(\eta) + ||u_0||^2 \\
\leq \frac{1}{4}a(\eta) - (||u_0||^2 + \Phi(||u_0||_{L^2}^2) + \eta ||u_0||_{L^2}^2) + \frac{1}{2}a(\eta) + \int_\Omega F(u_0)dx \\
+ \frac{1}{2}||g||^2 + \frac{1}{2}||u_0||^2 + \frac{1}{4}||u_i||^2 + \eta^2 ||u_0||^2 + (\nu - \frac{3}{4})||u_0||^2 \\
\leq \frac{1}{4}a(\eta) - (\Phi(||u_0||_{L^2}^2) + \eta ||u_0||_{L^2}^2) + \frac{1}{2}a(\eta) + \int_\Omega F(u_0)dx \\
+ \frac{1}{2}||g||^2 + \left( \nu + \eta^2 - \frac{1}{4} \right) ||u_0||^2.
\]

Then by (A.5), $\eta \geq \eta_0$, we have, for

\[
\nu \geq \max\left\{ 0, \frac{1}{4} - \eta^2 \right\},
\]

it holds

\[
W^{\nu, \eta}(u_0, u_i) - \frac{3}{4}E^\eta(u_0, u_i) \leq \frac{1}{2}a(\eta) + \tilde{G}(||u_0||_{L^2}^2) + \frac{1}{2}||g||^2 + \left( \nu - \frac{1}{4} + \eta^2 \right) ||u_0||^2,
\]

by (1.18), we obtain

\[
||u_0||^2 = \sum_{j=1}^{\infty} (u_0, e_j)^2 \leq \lambda_1^{-\alpha} \sum_{j=1}^{\infty} \beta_j^*(u_0, e_j)^2 = \lambda_1^{-\alpha} ||u_0||_{L^2}^2.
\]

Then it follows

\[
W^{\nu, \eta}(u_0, u_i) - \frac{3}{4}E^\eta(u_0, u_i) \leq \frac{1}{2}a(\eta) + \tilde{G}(||u_0||_{L^2}^2) + \frac{1}{2}||g||^2 + \left( \nu - \frac{1}{4} + \eta^2 \right) \lambda_1^{-\alpha} ||u_0||_{L^2}^2.
\]

Let

\[
G(s) = \frac{1}{2}a(\eta) + \tilde{G}(s) + \frac{1}{2}||g||^2 + \left( \nu - \frac{1}{4} + \eta^2 \right) \lambda_1^{-\alpha} s.
\]

It is obvious that $M(s)$ is a monotone positive function and (A.10) holds with $v_{\nu, \eta}^l = \max\left\{ 0, \frac{1}{4} - \eta^2 \right\}$.

Let

\[
\nu_{\eta, \eta} = \max\{v_{\eta, \eta}^l, v_{\eta, \eta}^r\} = \max\left\{ 0, \eta^2 + \frac{1}{2}H_f - \frac{1}{4}, \frac{1}{4} - \eta^2 \right\}.
\]

Then (2.18) follows from (A.9) and (A.10).