The evolution of immersed locally convex plane curves driven by anisotropic curvature flow

Abstract: In this article, we study the evolution of immersed locally convex plane curves driven by anisotropic flow with inner normal velocity $V = \frac{1}{a} \psi(x) \kappa^a$ for $a < 0$ or $a > 1$, where $x \in [0, 2\pi] \cap \mathbb{R}$ is the tangential angle at the point on evolving curves. For $-1 \leq a < 0$, we show the flow exists globally and the rescaled flow has a full-time convergence. For $a < -1$ or $a > 1$, we show only type I singularity arises in the flow, and the rescaled flow has subsequential convergence, i.e. for any time sequence, there is a time subsequence along which the rescaled curvature of evolving curves converges to a limit function; furthermore, if the anisotropic function $\psi$ and the initial curve both have some symmetric structure, the subsequential convergence could be refined to be full-time convergence.

Keywords: curvature flow, anisotropy, long-time behaviour, singularity

MSC 2020: 35B40, 35K15, 35K55, 53E10

1 Introduction

Let $\gamma_0$ be a locally convex immersed closed plane curve with total curvature $2m\pi (m \in \mathbb{N})$, parameterized by a smooth immersion $X_0(u) : S^1 \to \mathbb{R}^2$. Here “locally convex” means that $\gamma_0$ has no inflection point (i.e. curvature is positive everywhere). In general, $\gamma_0$ can have self-intersection (if $m \geq 2$). For such an initial curve, we consider the long-time behaviour of evolving curves $X(u, t) : S^1 \times [0, T) \to \mathbb{R}^2$ under the following anisotropic curvature flow:

$$\frac{\partial X}{\partial t} = \frac{1}{a} \psi(x) \kappa^a \cdot n, \quad t \in (0, T),$$

$$X(u, 0) = X_0(u), \quad u \in S^1,$$

(1.1)

where $\psi$ is a $2m\pi$-periodic smooth positive function of tangential angle $x$ at $X(u, t)$, and $\kappa(u, t)$ is the curvature at $X(u, t)$ with respect to unit inner normal vector $n(u, t)$. Throughout this article we assume the anisotropic function $\psi$ satisfies

$$0 < A_1 \leq \psi \leq |\psi'| + |\psi''| \leq A_2$$

(1.2)

for some positive constants $A_1$ and $A_2$. 

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When $\psi \equiv 1$, (1.1) is usually called isotropic flow. If $\psi \equiv 1$ and $\alpha \equiv 1$, (1.1) is the well-known curve shortening flow. It was proposed in Mullins’s work [32] to describe the motion of grain boundaries. Gage and Hamilton [18] proved that under the curve shortening flow any convex embedded curve shrinks into a point asymptotically like a circle in finite time. Then Grayson [22] showed this result holds for arbitrary embedded closed curves. Here, embeddedness implies the curve does not have self-intersection and the total curvature is $2\pi$.

Later on, Gage [20] studied the anisotropic flow (1.1) with $\alpha = 1$ and $\psi$ being some particular nonconstant positive function of tangential angle $x \in [0, 2\pi]$ in the setting of Minkowski geometry, and Gage and Li [21] studied the anisotropic flow (1.1) with $\alpha = 1$ and $\psi$ being any positive function defined on circle. They showed that the fluid (1.1) shrinks any embedded convex curve to a point in finite time. After rescaling about the final point to keep the enclosed area fixed, the flow converges to some self-similar limiting shape along some subsequence for any given time sequence going to infinity. And the subsequential convergence could be refined to be full-time convergence if $\psi(x) = \psi(x + \pi)$ for $x \in [0, 2\pi]$. Then Andrews carried a systematical study on isotropic or anisotropic embedded convex flows in [4] for general $\alpha \neq 0$ and smooth positive anisotropic function $\psi$. For negative $\alpha$, under the flow (1.1) any embedded convex curve expands to infinity in finite time when $\alpha < -1$ or in infinite time when $-1 \leq \alpha < 0$. After rescaling about the origin to keep their enclosed area constant, the evolving curves converge to a unique limiting shape which is independent of initial curve. For positive $\alpha$, the flow (1.1) shrinks any embedded convex curve to a point in finite time. The asymptotic behaviour of flow becomes complicated and depends crucially on the value of exponent $\alpha$. For $\alpha > \frac{1}{2}$, the evolving curves converge subsequentially to a limiting shape after rescaling. Particularly, if $\alpha > \frac{1}{3}$ and $\psi \equiv 1$, or $\alpha \geq 1$ and $\psi$ is symmetric, i.e. $\psi(x) = \psi(x + \pi)$ for $x \in [0, 2\pi]$, there is only one possible limiting shape and hence the above subsequential convergence could be improved to be full-time convergence. For $0 < \alpha \leq \frac{1}{3}$, the evolving curves have a sub-sequential convergence if their isoperimetric ratios are time-independently bounded. Furthermore, Andrew’s another work [5] says that for $0 < \alpha \leq \frac{1}{7}$, the isoperimetric ratios of the evolving curves generically approach infinity, even if anisotropic function $\psi$ is symmetric. A complete classification of possible limiting shape is given by Andrews in [6] for the case of $\alpha \neq 0$ and $\psi \equiv 1$. Particularly, if $\alpha = \frac{1}{3}$ and $\psi \equiv 1$, the flow is natural in affine geometry and has been applied to image processing and related problems. It can be shown that the rescaled flow converges to an ellipse as time goes to infinity, see [2,11,37].

As a natural generalization, people are interested in the evolution of the flow (1.1) for immersed locally convex curves. The curve may have intersection and the long-time behaviour of flow becomes more complicated. We first introduce the study on isotropic case $\psi \equiv 1$. In this case, if $\alpha = 1$, Angenent [8] carried out a classification on the singularity according to the blow-up rate of curvature in finite time and showed that the profile of type I singularity could be characterized via self-similar solutions given by Abresch-Langer [1] while the profile of type II singularity is a translating solution – grim reaper. In [29] Lin et al. generalized Angenent’s result to the case $\alpha \in (0, 1)$. When $\alpha > 1$, Poon and Tsai [35] showed there is only type I singularity and the curvature function converges to nonnegative Lipschitz limit function after rescaling. When $\alpha < 0$, (1.1) becomes an expanding flow. If $\alpha < -1$, the flow expands to infinity in finite time and the radius of curvature blows up, converging to nonnegative Lipschitz limit function after rescaling, see [28]. If $-1 \leq \alpha < 0$, the flow exists for all time and expands to infinity asymptotically like an $m$-fold circle as time goes to infinity, see [39]. The related study can be found in [40] by Urbas.

Furthermore, for immersed locally convex curves, the anisotropic case of flow is studied. For any smooth positive function $\psi$, Poon and Tsai [36] studied the flow (1.1) with $0 < \alpha \leq 1$ by following the lines of [8,29]. In this article, we focus on studying the remaining case of $\alpha < 0$ and $\alpha > 1$.

We note that the study on embedded convex curve flow in the literature depends crucially on stronger isoperimetric inequality, e.g. Bonnesen inequality. However, these stronger isoperimetric inequalities do not always hold for immersed locally convex curves. On the other hand, the appearance of anisotropy also brings challenge since there is no translating invariance property any more.

To proceed, we reformulate our problem as the following. When locally convex solution $X(\cdot, t)$ is considered, each point on it has a unique tangent and one can use the tangent angle
to parameterize it. Generally speaking, \( x \) is a function depending on \( t \). In order to make \( x \) independent of time \( t \), one can attain that by adding a tangential component to the velocity vector \( \partial X / \partial t \), which does not affect the geometric shape of the evolving curve (see, for instance, [17, 18]). Then the evolution equations can be expressed in the coordinates of \( x \) and \( t \). If we denote by \( \kappa(x, t) \) the curvature function of \( X(x, t) \), Problem (1.1) can be reformulated by

\[
\frac{\partial x}{\partial t} = \kappa \left( \frac{1}{\alpha} \psi(x) x^a \right)_{xx} + \frac{1}{\alpha} \psi(x) x^a, \quad x \in S^1_m, \ t > 0,
\]

where \( \kappa(x, 0) = \kappa_0(x) > 0 \) is the curvature function of initial curve \( X_0 \).

Set

\[
v(x, t) = \psi(x) x^a.
\]

Then we obtain

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \varphi(x) v^p (v_{xx} + v), \\
v(x, 0) &= v_0(x) = \psi(x) \kappa_0^p(x),
\end{align*}
\]

where \( T_{\text{max}} \) is the maximal existence time of the flow and the following notations are used:

\[
\varphi(x) = \psi^{-\frac{1}{p}}(x) \quad \text{and} \quad p = 1 + \frac{1}{\alpha}.
\]

We divide our study into two cases. One case is \( \alpha > 1 \) or \( \alpha < -1 \). The other is \( -1 \leq \alpha < 0 \).

(i) **The case of \( \alpha > 1 \) or \( \alpha < -1 \)** (resp. \( 1 < p < 2 \) or \( 0 < p < 1 \)).

Under the flow (1.1), the curvature function of evolving curves touches zero in finite time for \( \alpha < -1 \) and blows up in finite time for \( \alpha > 1 \). In both the cases, the solution \( v \) to (1.4) blows up in finite time, that is there exists a sequence \( \{x_j, t_j\}_{j=1}^{\infty} \) such that

\[
v(x_j, t_j) \to \infty, \quad j \to \infty
\]

and

\[
x_j \to x_0 \in S^1_m, \quad t_j \to T_{\text{max}} < \infty, \quad j \to \infty.
\]

To investigate asymptotic behaviour of the solution \( v(x, t) \) near time \( T_{\text{max}} \), we rescale the solution in the following way. Define \( R(t) \) as the unique solution to the ordinary differential equation (ODE)

\[
\frac{dR}{dt} = R^{p+1}(t), \quad R(T_{\text{max}}) = \infty.
\]

Then \( R(t) = [p(T_{\text{max}} - t)]^{-1/p} \). Carry the rescaling

\[
u(x, t) = v(x, t) / R(t)
\]

and take \( \tau \in [0, \infty) \) be the new time parameter defined by the relation

\[
t = T_{\text{max}} (1 - e^{-\tau}), \quad \tau \in [0, T_{\text{max}}).
\]

Then we conclude that the function

\[
u(x, \tau) = p^{1/p} T_{\text{max}}^{1/p} e^{-\tau} \cdot v(x, T_{\text{max}} (1 - e^{-\tau}))
\]

is a positive, bounded solution of the equation

\[
\begin{align*}
\frac{du}{d\tau} &= \varphi(x) u^p (u_{xx} + u(x, \tau) - \frac{1}{\varphi(x) u^{p-1}(x, \tau)}), \quad (x, \tau) \in S^1_m \times (0, \infty), \\
u(x, 0) &= u_0(x) = p^{1/p} T_{\text{max}}^{1/p} v_0(x), \quad x \in S^1_m
\end{align*}
\]
where $0 < p < 1$ or $1 < p < 2$.

If there exists a constant $C$ such that
\[
\max_{x \in S^m_n} v(x, t) \leq C(T_{\max} - t)^{-1/p}, \quad \forall t \in [0, T_{\max}),
\]
then we say $v$ has **type I blow-up**.

**Theorem 1.1.** Suppose that $\alpha < -1$ or $\alpha > 1$. There exists a unique family of smooth, locally convex immersed curves $X(t)$ satisfying (1.1) for $t \in (0, T_{\max})$ and $T_{\max} < \infty$. For $\alpha < -1$, the flow (1.1) expands to infinity and $\min_{x \in S^m_n} \kappa(x, t)$ grows to zero in finite time. For $\alpha > 1$, the singularity appears during the evolution of the flow (1.1) and $\max_{x \in S^m_n} \kappa(x, t)$ blows up in finite time. In both cases, $v = \frac{1}{|\kappa|^\alpha}$ has type I blow-up. The asymptotic behaviour of $v(\theta, t)$ could be described by the solution $u(x, \tau)$ to (1.7) on $S^1_m \times [0, \infty)$. For any sequence $(t_j)_{j=1}^{\infty}$, there exists a subsequence $(\tau_j)_{j=1}^{\infty}$ such that $u(x, \tau_j)$ converges uniformly to a nonnegative Lipschitz limit function $w(x)$ as $k \to \infty$, and $w(x)$ is a solution to the ODE
\[
w^{\prime\prime\prime}(x)w(x) + w''(x) - \frac{1}{\varphi(x)} = 0,
\]
which holds for $x \in S^1_m$ when $\alpha < -1$ and for $x \in \{x \in S^1_m : w(x) > 0\}$ when $\alpha > 1$.

Moreover, if we assume that $\varphi(x)$ satisfies:
\[\varphi(x) = \varphi(-x) \quad \text{and} \quad \varphi'(x) \geq 0 \quad \text{for} \quad x \in (-m\pi, 0)\]
and $u_0(x)$ (or equivalently $\nu_0(x)$) satisfies (1.9) with $u_0(x) > 0$ (or equivalently $\nu_0(x) > 0$) somewhere, the above convergence can be refined to be full-time convergence, that is $u(x, \tau)$ converges uniformly to $w(x)$ as $\tau \to \infty$.

(ii) **The case of $-1 \leq \alpha < 0$ (resp. $p \leq 0$).**

In this case, we can show that there is a unique smooth positive solution $v$ to (1.4) which exists for all time. To study the long-time behaviour, we need to rescale the solution in the following way.

If $\alpha = -1$, i.e. $p = 0$, we let $u(x, \tau) = e^{-\tau}v(x, t)$ and $t = \tau + 1$, $\tau \in [0, \infty)$. Then $u(x, \tau)$ satisfies
\[
u_\tau(x, \tau) = \varphi(x)\left(u_{\tau\tau} + u - \frac{u}{\varphi(x)}\right), \quad (x, \tau) \in S^1_m \times (0, \infty),
\]
with initial condition $u(x, 0) = v(x, 1), x \in S^1_m$.

If $-1 < \alpha < 0$, i.e. $p < 0$, we let $u(x, \tau) = t^{1/p}v(x, t)$ and $t = e^\tau$, $\tau \in [0, \infty)$. Then $u(x, \tau)$ satisfies
\[
u_\tau(x, \tau) = \varphi(x)\nu(\tau)\left(u_{\tau\tau} + u + \frac{1}{\nu\varphi(x)\nu^{p-1}}\right), \quad (x, \tau) \in S^1_m \times (0, \infty),
\]
with initial condition $u(x, 0) = v(x, 1), x \in S^1_m$.

**Theorem 1.2.** Suppose that $-1 \leq \alpha < 0$. There exists a unique family of smooth, locally convex immersed curves $X(t)$ satisfying (1.1) for all $t \in (0, \infty)$. As $t \to \infty$, the curvature function $\kappa(x, t)$ converges uniformly to zero, and the evolving curves expand to infinity in an asymptotically self-similar way, which can be described via the solution $u(x, \tau)$ to (1.10) or (1.11) for $\alpha = -1$ or $-1 < \alpha < 0$, respectively. For $\alpha = -1$, $u(x, \tau)$ converges smoothly to the function $w(x) \in C^\infty(S^1_m)$ as $\tau \to \infty$, and $w(x)$ is the unique positive solution to the ODE
\[
w_{xx}(x) + w(x) - \frac{w(x)}{\varphi(x)} = 0, \quad x \in S^1_m.
\]
For $-1 < \alpha < 0$, $u(x, \tau)$ converges smoothly to the function $w(x) \in C^\infty(S^1_m)$ as $\tau \to \infty$, and $w(x)$ is the unique positive solution to the ODE
\[
w_{xx}(x) + w(x) + \frac{1}{p\varphi(x)w^{p-1}(x)} = 0, \quad x \in S^1_m.
\]
Before ending this section, we would like to say more about the existing literature on the curvature flows. Since anisotropy is indispensable in dealing with crystalline materials, Mullin's theory was generalized by Gurtin [23,24] and by Angenent and Gurtin [9,10] (see also the monograph of Gurtin [25]) to include the anisotropy and the possibility of a difference in bulk energies between phases. For perfect conductors, the temperatures in both phases are constant. Set $V$ to be the velocity along inner normal. The equation of the interface's motion becomes

$$\beta(\theta)V = g(\theta)x + F,$$

where $\theta$ is the tangential angle of the interface, $\beta > 0$ is a kinetic coefficient, $g$ is of the form $g = d^2f/d\theta^2 + f$, $f > 0$ is the interfacial energy, and the constant $F$ is the energy of the solid relative to its surrounding. Chou and Zhu studied this problem in [16] and gave a complete classification result on long-time behaviour of interface according to the value range of $F$. We note that equation (1.14) could be regarded as a curvature flow with forcing term. In this respect, we refer readers to [27] for crystalline eikonal-curvature flow, [34] for segmentation flow of colour images, and [13,19,33] for constrained flows.

2 Preparations

In this section, we shall assume that $p \in \mathbb{R}$ and $0 < T_{\text{max}} \leq \infty$, and state some lemmas. The proofs for Lemmas 2.1 to 2.3 are similar to that of the previous papers [8,18], and we would not go into details.

**Lemma 2.1.** Let $p \in \mathbb{R}$ and $v(x, t) > 0$ be the $2\pi$-periodic solution to (1.4). Then at each point $(x, t) \in S^1_m \times [0, T_{\text{max}})$, we have either

$$v_x + v(x, t) > 0,$$

or

$$v^2_t(x, t) + v^2(x, t) \leq A^2, \quad A = \max_{S^1_m} \sqrt{(v_0)^2 + v_0^2}. \quad (2.2)$$

**Lemma 2.2.** Let $p \in \mathbb{R}$ and $v(x, t) > 0$ be the $2\pi$-periodic solution to (1.4). For any $(x, t) \in S^1_m \times [0, T_{\text{max}})$, we have

$$|v(x, t)| \leq A + 2\pi \max_{x \in S^1_m} v(x, t). \quad (2.3)$$

**Lemma 2.3.** Let $p \in \mathbb{R}$ and $v(x, t) > 0$ be the $2\pi$-periodic solution to (1.4). Then there exists a constant $C$ depending only on $v_0(x)$, such that

$$\int_{-\pi}^{\pi} v^2_t(x, t)dx \leq \int_{-\pi}^{\pi} v^2(x, t)dx + C \quad (2.4)$$

for all $t \in [0, T_{\text{max}})$. Moreover, if $\max_{x \in S^1_m} v(x, t) = v(x(t), t)$ for some $x(t) \in S^1_m$, then for any small $\varepsilon > 0$, there exists a number $\delta > 0$, depending only on $\varepsilon$, such that

$$(1 - \varepsilon)v_{\text{max}}(t) \leq v(x, t) + \sqrt{2\pi C}, \quad (2.5)$$

for all $x \in (x(t) - \delta^2, x(t) + \delta^2)$ and $t \in [0, T_{\text{max}})$. 


3 The case of $0 < p < 1$ and $1 < p < 2$

In this section, we study the evolution of the flow (1.1) in the case of $0 < p < 1$ or $1 < p < 2$ and prove Theorem 1.1. In the first part, we discuss the convergence of curvature function along a time sequence, that is sub-convergence. In the second part, we show that the sub-convergence can be refined to the convergence along whole time, that is full-time convergence, under some symmetric conditions on anisotropic function and initial data.

3.1 Sub-convergence

**Lemma 3.1. (Preserved convexity)** Let $0 < p < 1$ or $1 < p < 2$. If initial curve $X_0$ is locally convex, then $X(\cdot, t)$ continues to be locally convex as long as the flow (1.1) exists.

**Proof.** For the solution $v$ to equation (1.4), we set $v_{\max}(t) = \min_{x \in S_0^1} v(x, t)$ for $t \in [0, T_{\max})$. We show $v(x, t)$ preserves to be positive by illustrating that $v_{\max}(t)$ is a nondecreasing function. Suppose by contradiction that this is not true. Assume $v_{\min}(0) > \varepsilon > 0$. Then we use the continuity to obtain that there is a first time, say $t_0 < T_{\max}$, such that $v_{\min}(t) = v_{\min}(0) - \varepsilon$. Let $\min_{x \in S_0^1} v(x_0, t_0) = v_{\min}(0) - \varepsilon$. Then at the point $(x_0, t_0)$,

$$\frac{\partial v}{\partial t}(x_0, t_0) \leq 0, \quad \frac{\partial v}{\partial x^2}(x_0, t_0) \geq 0, \quad v(x_0, t_0) > 0.$$ 

This contradicts with equation (1.4). Since $v = \psi k^a$, we know that the curvature function $\kappa > 0$ for all $t \in [0, T_{\max})$. □

**Lemma 3.2. (Unique existence)** Let $0 < p < 1$ or $1 < p < 2$, and the initial curve be immersed, locally convex, closed and smooth. The flow (1.1) has a unique smooth, locally convex solution on a time interval, which can be continued as long as $v = \psi k^a$ is bounded from above.

**Proof.** By applying the classical Leray-Schauder fixed point theory to Problem (1.4), there exists a smooth solution $v$ on $[0, T)$ for some time $T$. See details in [30], where a generalized area-preserving flow is studied. Recalling the proof of Lemma 3.1, we know that $v$ is bounded by a positive constant from below. As long as $v$ is bounded from above, the equation for $v$ is uniformly parabolic. According to the Schauder theory of parabolic type partial differential equation, the solution $v$ can be continued beyond the time $T$. □

**Lemma 3.3. (Type I blow-up)** Let $0 < p < 1$ or $1 < p < 2$, and $v(x, t)$ be a $2\pi n$-periodic solution of (1.4) on $S_0^1 \times [0, \infty)$. Then there exists a positive constant $C$ independent of time, such that $v(x, t)$ satisfies

$$\max_{x \in S_0^1} v(x, t) \leq C(T_{\max} - t)^{-1/p}, \quad \forall t \in [0, T_{\max}),$$

(3.1)

that is $v(x, t)$ has only type I blow-up.

**Proof.** This result has been proved in [36]. The idea of proof is to show that the integral

$$I(t) = \int_{-\pi}^{\pi} \frac{\psi^2 - p}{\psi} \, dx$$

satisfies

$$I(t) \leq C(T_{\max} - t)^{-(2-p)/p}, \quad t \in [0, T_{\max}),$$

for some constant $C$. Then from Lemma 2.3, we can estimate $v_{\max}$ from above by $I(t)$ and obtain the desired result. □
From Lemmas 2.2 and 3.3, we can establish the following gradient estimate and the convergence result.

**Lemma 3.4.** Let \(0 < p < 1\) or \(1 < p < 2\), and \(u(x, \tau) > 0\) be a bounded 2-periodic solution of (1.7) on \(S^1_m \times [0, \infty)\). Then there exists a constant \(C > 0\), independent of time, such that
\[
|u_\tau(x, \tau)| \leq C,
\]
for all \((x, \tau) \in S^1_m \times [0, \infty)\).

**Lemma 3.5.** Let \(0 < p < 1\) or \(1 < p < 2\), and \(u(x, \tau) > 0\) be a bounded 2-periodic solution of (1.7) on \(S^1_m \times [0, \infty)\). For each sequence \(\{\tau_n\}_{n=1}^{\infty}\), \(\tau_n \to \infty\) as \(n \to \infty\), there is a subsequence, also denoted by \(\{\tau_n\}_{n=1}^{\infty}\), so that \(u(x, \tau_n)\) converges uniformly to a Lipschitz continuous function \(w(x) \geq 0\) in \(S^1_m\).

To show the limit function \(w(x)\) is a steady state, we prove the following lemma.

**Lemma 3.6.** Let \(0 < p < 1\) or \(1 < p < 2\) and suppose that \(u(x, \tau_n)\) converges uniformly to a Lipschitz continuous function \(w(x) \geq 0\) as \(\tau_n \to \infty\). For any bounded sequence \(s_n \to \infty\), \(u(x, \tau_n + s_n)\) also converges uniformly to \(w(x)\) in \(S^1_m\) as \(n \to \infty\).

**Proof.** Similar to the proof of [36], we may assume that \(0 \leq s_n \leq 1\) for all \(n\) without loss of generality. From (1.7), for \(\tau \in (0, \infty)\), we find that
\[
0 \leq \int_{-m\pi}^{m\pi} \frac{u^2}{\phi} \, dx = -\frac{1}{2} \frac{d}{d\tau} \int_{-m\pi}^{m\pi} \left[ u^2 - u^2 + \frac{2}{2 - p} \frac{u^2 - p}{\phi} \right] \, dx \quad \tau \in (0, \infty),
\]
which indicates that the quantity
\[
\varepsilon(u(\cdot, \tau)) = \int_{-m\pi}^{m\pi} \left( u^2 - u^2 + \frac{2}{2 - p} \frac{u^2 - p}{\phi} \right) \, dx
\]
is nonincreasing. Recall that both \(u\) and \(u_\tau\) are uniformly bounded and \(p \in (0, 2)\). Therefore, we obtain that
\[
0 \leq \int_{0}^{\infty} \int_{-m\pi}^{m\pi} \frac{u^2}{\phi} \, dx \, d\tau = \frac{1}{2} \varepsilon(u(\cdot, 0)) - \frac{1}{2} \lim_{s \to \infty} \varepsilon(u(\cdot, s)) < \infty.
\]
For any \(x \in S^1_m\), we have
\[
\frac{1}{\sqrt{\phi(x)}} \left[ u^{(2-p)/2}(x, \tau_n + s_n) - u^{(2-p)/2}(x, \tau_n) \right] = \frac{2 - p}{2} \int_{\tau_n}^{\tau_n + s_n} \frac{u_\tau(x, \tau)}{\sqrt{\phi(x)}} \, d\tau.
\]
In addition, according to the Cauchy-Schwarz inequality, and \(0 \leq s_n \leq 1\), we have
\[
\frac{1}{\sqrt{\phi(x)}} \left[ u^{(2-p)/2}(x, \tau_n + s_n) - u^{(2-p)/2}(x, \tau_n) \right] \, dx \leq \frac{2 - p}{2} \int_{-m\pi}^{m\pi} \frac{u_\tau(x, \tau)}{\sqrt{\phi(x)}} \, dx \, d\tau \leq \frac{2 - p}{2} (2\pi)^{1/2} \left( \int_{-m\pi}^{m\pi} \frac{u^2(x, \tau)}{\phi(x)} \, dx \, d\tau \right)^{1/2}.
\]
Therefore,
\[
\int_{-m\pi}^{m\pi} \frac{1}{\sqrt{\phi(x)}} \left[ u^{(2-p)/2}(x, \tau_n + s_n) - u^{(2-p)/2}(x, \tau_n) \right] \, dx \to 0, \quad n \to \infty.
\]
On the contrary, if \( u(x, \tau_n + s_n) \) does not converge uniformly to \( w(x) \), as \( n \to \infty \), then there exist \( \varepsilon > 0 \), and a subsequence of \( \{\tau_n + s_{\eta_n} \} \), still denoted by \( \{\tau_n + s_{\eta_n} \} \), such that \( \max_{x \in S_m^n} |u(x, \tau_n + s_n) - w(x)| \geq \varepsilon \) for all \( n \). Suppose that \( u(x, \tau_n + s_n) \) converges uniformly on \( S_m^n \) to another function \( \tilde{w} \geq 0 \). In view of (3.6), we have \( \tilde{w} = \tilde{w} \), which is a contradiction. Hence, we get the conclusion.

Now we can show the limit function \( w(x) \) is a steady state.

**Lemma 3.7.** Let \( 0 < p < 1 \) and suppose that \( u(x, \tau_n) \) converges uniformly to a Lipschitz continuous function \( w(x) \geq 0 \) as \( \tau_n \to \infty \). Then \( w(x) \in C^1(S_m^n) \) is a solution to the ODE

\[
wx(x) + w(x) - \frac{1}{\varphi(x)w^{p-1}(x)} = 0.
\]

**Proof.** Let \( \phi(x) \in C_0^\infty(S_m^n) \) be a test function and let \( h(t) \in C_0^\infty((0,1)) \). We can rewrite (1.7) as

\[
\frac{1}{1-p}\int_0^1 \left( \frac{1}{\varphi(x)w^{p-1}(x, \tau_n + \tau)} \right)h'(t)\phi(x) dx dr
\]

\[
= \int_0^1 \phi(x)w(x, \tau_n + \tau) + u(x, \tau_n + \tau) - \frac{1}{\varphi(x)w^{p-1}(x, \tau_n + \tau)} h(t)\phi(x) dx dr
\]

\[
= \int_0^1 \phi(x)w(x, \tau_n + \tau) + u(x, \tau_n + \tau)\phi(x) - \frac{\phi(x)}{\varphi(x)w^{p-1}(x, \tau_n + \tau)} h(t)\phi(x) dx dr.
\]

By bounded convergence theorem,

\[
\frac{1}{1-p}\int_0^1 \left( \frac{1}{\varphi(x)w^{p-1}(x)} \right)h'(t)\phi(x) dx dr = \int_0^1 \left( \frac{w(x)\phi_x(x) + w(x)\phi(x) - \frac{\phi(x)}{\varphi(x)w^{p-1}(x)}}{\varphi(x)w^{p-1}(x)} \right)h(t)\phi(x) dx dr,
\]

and as \( n \to \infty \). In particular, since \( h \in C_0^\infty((0,1)) \) is arbitrary, we conclude

\[
0 = \int_0^b \left( w(x)\phi_x(x) + w(x)\phi(x) - \frac{\phi(x)}{\varphi(x)w^{p-1}(x)} \right) dx = \int_0^b \left( w(x) + w(x) - \frac{\phi(x)}{\varphi(x)w^{p-1}(x)} \right)\phi(x) dx,
\]

which, owing to the arbitrariness of \( \phi(x) \in C_0^\infty(S_m^n) \), illustrates that \( w(x) \) solves the ODE (3.7) in weak sense over \( S_m^n \). On the basis of regularity theory and the fact that \( w(x) \geq 0 \) is Lipschitz continuous over \( S_m^n \), we have \( w(x) \in C^1(S_m^n) \) anywhere. The proof is accomplished.

**Lemma 3.8.** Let \( 1 < p < 2 \). Suppose that \( u(x, \tau) > 0 \) is the \( 2m \eta \)-periodic solution of (1.7) on \( S_m^n \times [0, \infty) \). Then there is a time sequence \( \{\tau_n\}_{n=1}^\infty \), \( \tau_n \to \infty \) as \( n \to \infty \), such that \( u(x, \tau_n) \) converges uniformly to a nonnegative Lipschitz limit function \( w(x) \) as \( n \to \infty \). Let \( K = \{x \in S_m^n : w(x) > 0 \} \) and let \( I \) be any compact subset of \( K \). Then \( u(x, \tau_n) \) converges in \( C^0(I) \) to \( w(x) \) as \( n \to \infty \), and \( w(x) \) is the solution to the ODE (3.7) on \( K \).

**Proof.** Following the step of proof for Lemma 3.5, we know that there exists a time sequence \( \{\tau_n\}_{n=1}^\infty \), \( \tau_n \to \infty \) as \( n \to \infty \), such that \( u(x, \tau_n) \) converges uniformly to a nonnegative Lipschitz limit function \( w(x) \) as \( n \to \infty \). Then we adopt the idea of proof for Lemma 3.7 to show the conclusion holds. Indeed, since \( u(x, \tau_n) \) may
approach zero as $n \to \infty$ and $1 < p < 2$, our proof has to be confined to compact subset of the region $K = \{ x \in S^1_m : w(x) > 0 \}$ to guarantee that the bounded convergence theorem could be applied.

### 3.2 Full-time convergence

Let $0 < p < 1$ or $1 < p < 2$. Note that the anisotropic plane curve flow may evolve in a complicated way due to the fact that $\psi(x)$ in (1.1) is not a constant, and it is not easy to study its full-time convergence behaviour for the rescaled evolving curves. However, if Problem (1.1) is studied under some symmetric conditions on $\psi(x)$ and initial data, we can verify that the solution $u(x, \tau)$ to (1.7) eventually converges uniformly to a limit function as $\tau \to \infty$.

**Lemma 3.9.** Let $p > 0$, $p < 1$ or $1 < p < 2$, and $v(x, t) > 0$ be the $2\pi$-periodic solution of (1.4) on $S^1_m \times [0, T_{\max})$. Suppose $\phi(x) = a^{1,p} \psi^{-1}(x)$ and initial data $v_0(x) = \frac{1}{a} \psi(x) x_0(x)$ satisfy:

$$
\begin{align*}
\frac{\partial v}{\partial t} &= \psi v G_p \phi v G_p \phi v G_p \phi v G_p \phi v, \\
\phi(x) &= \phi(-x), \quad \phi'(x) \geq 0, \quad \text{for } x \in [-m\pi, 0]; \\
v_0(x) &= v_0(-x), \quad v_0'(x) \geq 0, \quad \text{for } x \in [-m\pi, 0].
\end{align*}
$$

We further assume $v_0'(x) > 0$ somewhere. Then we can conclude that

$$
v(x, t) = v(-x, t) \quad \text{and} \quad v_t(x, t) > 0, \quad \text{for } x \in (-m\pi, 0), \quad t \in (0, T_{\max}).
$$

Moreover, we have

$$
u(x, \tau) = u(-x, \tau) \quad \text{and} \quad u_t(x, \tau) > 0, \quad \text{for } x \in (-m\pi, 0), \quad \tau \in (0, \infty).
$$

**Proof.** From (3.10) and the uniqueness of solution to (1.4), we can obtain that $v(x, t) = v(-x, t)$ for all $(x, t) \in (-m\pi, 0) \times (0, T_{\max})$. A direct computation shows that $G(x, t) = v_t(x, t)$ satisfies the following problem:

$$
\begin{cases}
G_t &= \psi v G_p \phi v G_p \phi v G_p \phi v G_p \phi v, \\
G(-m\pi, t) &= G(0, t) = 0, \\
G(x, 0) &= v_0'(x) \geq 0,
\end{cases}
$$

where $\phi'(x) \geq 0$ on $(-m\pi, 0)$ and the last term in the equation is nonnegative. This guarantees that we can apply the Strong Maximum Principle to conclude that $G(x, t) > 0$ on $(-m\pi, 0) \times (0, T_{\max})$. The proof is complete.

To go further, we need the following theorem on zero number theory for parabolic-type equation, see [7,14,31].

**Lemma 3.10.** Let $z(x, t)$ be a solution of the equation

$$z_t = a(x, t) z_{xx} + b(x, t) z_x + c(x, t) z, \quad \text{in } (x_1, x_2) \times (0, T),$$

where $a(x, t) > 0$, $a_0$, $a_{xx}$, $a_b$, $b_0$, $b_1$, $c \in L^\infty((x_1, x_2) \times (0, T))$. If $z(x_1, t) \neq 0$ and $z(x_2, t) \neq 0$ and define $l(t)$ as the number of zeros of $z(x, t)$ in $(x_1, x_2)$, then $l(t)$ is finite and is nonincreasing for all $t \in (0, T)$ and if $(x_0, t_0)$ is a multiple zero of $z$, then for $t_0 < t < t_2$, we have $l(t_2) < l(t_0)$.

Based on the zero number theory, Brunovský, Polácik and Sandstede [12] proved the following useful lemma.
Lemma 3.11. Let \( z(x, t) \) be a solution of the equation
\[
\dot{z}_t = a(x, t)z_{xx} + b(x, t)z_x + c(x, t)z, \quad 0 < x < 1, \quad t > 0,
\] (3.14)
with the boundary conditions
\[
z(i, t) = 0, \quad i = 0, 1, \quad t > 0,
\] (3.15)
or
\[
z_x(i, t) = a(t)z(i, t), \quad i = 0, 1, \quad t > 0,
\] (3.16)
where \( a(x, t) > 0, \ a, \ a_x, \ a_t, \ b, \ b_x, \ b_t, \ c \) are continuous on \([0, 1] \times (0, \infty)\), and \( a(t), i = 0, 1, \) are bounded \( C^1 \) functions on \([0, \infty)\). Let \( z(x, t) \neq 0 \) be a classical solution of (3.14), (3.15) or (3.14), (3.16). Then there exists a \( t^* \), such that for any \( t > t^* \), we have
\[
z(0, t) \neq 0,
\] (3.17)
in the case of (3.15) and
\[
z(0, t) \neq 0,
\] (3.18)
in the case of (3.16).

Now we can show the uniqueness of the limit function by using the above lemma.

Lemma 3.12. Let \( 0 < p < 1 \) or \( 1 < p < 2 \). Suppose that \( \varphi(x) = a^{\frac{1}{\alpha}}[\psi \varphi^{-\frac{1}{p}}(x)] \) and initial data \( v_0(x) = \frac{1}{a} \varphi(x)\kappa_0^m(x) \) satisfy (3.10). Suppose \( u(x, \tau) > 0 \) is a bounded \( 2m\pi \)-periodic solution of (1.7) on \( S_m^1 \times [0, \infty) \), and there exist two sequences \( u(x, \tau_n) \) and \( u(x, s_n) \) converging uniformly to Lipschitz continuous limit functions \( w(x) \) and \( \bar{w}(x) \), respectively. Then \( w(x) \) is identical to \( \bar{w}(x) \) on \( S_m^1 \).

Proof. By Lemma 3.9, we know that \( w(x) \) and \( \bar{w}(x) \) are nondecreasing function on \([-m\pi, 0] \), and symmetric with respect to 0. Therefore, if \( w(x) \) and \( \bar{w}(x) \) are not constant in \([-m\pi, m\pi] \), they have the same minimal period \( 2m\pi \), with
\[
w(0) = \max_{x \in S_m^1} w(x), \quad w(-m\pi) = \min_{x \in S_m^1} w(x), \quad w(0) = \max_{x \in S_m^1} w(x), \quad w(-m\pi) = \min_{x \in S_m^1} w(x).
\] (3.19)
First, we need to verify that
\[
w(0) = \bar{w}(0).
\] (3.20)
Let \( z(x, \tau) = u_t(x, \tau) \). From (1.7), we can see
\[
z_t = \varphi u^p z_{xx} + [p\varphi u^{p-1}(u_{xx} + u) + \varphi u^p - 1]z.
\] (3.21)
Since \( u_t(0, \tau) = 0 \) and \( u_t(m\pi, \tau) = 0 \), we have \( (u_t)_t(0, \tau) = 0 \) and \( (u_t)_t(m\pi, \tau) = 0 \), i.e.
\[
z_t(0, \tau) = 0 \quad \text{and} \quad z_t(m\pi, \tau) = 0.
\]
By Lemma 3.12, there exists a time \( \tau^* \), such that for any \( \tau > \tau^* \),
\[
z(0, \tau) \neq 0.
\]
This means that, for any \( \tau > \tau^* \),
\[
u_t(0, \tau) > 0 \quad \text{or} \quad u_t(0, \tau) < 0,
\]
i.e. \( u(0, \tau) \) is increasing or decreasing on \((\tau^*, \infty)\). Since \( u(x, \tau) \) is bounded, we obtain that
\[
\lim_{n \to \infty} u(0, \tau_n) = \lim_{n \to \infty} u(0, s_n).
\]
Thus, (3.20) is proved.

Now, we prove the conclusion of lemma for the case of \( 0 < p < 1 \).
(i) When \( w(0) = \tilde{w}(0) = 0 \), it is obvious that \( w(x) = \tilde{w}(x) \) holds on \( S_m^1 \).

(ii) When \( w(0) = \tilde{w}(0) > 0 \), the uniqueness of solution to the ODE (3.7) implies that there exists an \( x_0 \in (0, m\pi] \) such that

\[
w(x) = \tilde{w}(x) > 0, \quad x \in (-x_0, x_0),
\]

and

\[
w(x_0) = \tilde{w}(x_0) = 0.
\]

Since \( w(x) \) and \( \tilde{w}(x) \) are nondecreasing on \( (-m\pi, 0] \), we have \( w(x) = \tilde{w}(x) \) on \( S_m^1 \).

For the case of \( 1 < p < 2 \), if \( w(0) = \tilde{w}(0) > 0 \), then we can apply the similar arguments as above on the interval \( K = \{x \in S_m^1 : w(x) = \tilde{w}(x) > 0\} \) to derive the desired result. ☐

Proof of Theorem 1.1. According to Lemma 3.6, for any time sequence \( \{\tau_n\}_{n=1}^{\infty} \) with \( \tau_n \to \infty \), there exists a sub-sequence \( \{\tau_{n_k}\}_{k=1}^{\infty} \) such that \( u(x, \tau_{n_k}) \) converges uniformly to a nonnegative Lipschitz continuous limit function \( w(x) \) as \( k \to \infty \). Then Lemmas 3.7 and 3.8 tell us that \( w(x) \) is a solution to the ODE (3.7), which holds for \( x \in S_m^1 \) when \( 0 < p < 1 \), and for \( x \in \{x \in S_m^1 : w(x) > 0\} \) when \( 1 < p < 2 \). Furthermore, if anisotropic function and initial data satisfy symmetric structure conditions, Lemma 3.12 illustrates that the limit function \( w(x) \) does not depend on the choice of time sequence. Hence, the solution \( u(x, \tau) \) converge uniformly to \( w(x) \) as \( \tau \to \infty \). The proof of Theorem 1.1 is finished. ☐

Remark. The ingradient in the proof of full-time convergence is to show the uniqueness of element in the \( \omega \)-limit set. Here, the zero number theory plays a crucial role. We note that in some cases the sub-sequential convergence can also be improved to be full-time convergence by using the method of Simon, i.e. to establish a Lojasiewicz-Simon inequality to give an estimate on the distance that the solution can move away from the limit, see [3]. For related study on this topic, we refer the reader to [38].

4 The case of \( p \leq 0 \)

In this section, we study the evolution of the flow (1.1) in the case of \( p \leq 0 \). We show the smooth sub-convergence of curvature function at first and then refine it to be full-time convergence.

4.1 Sub-convergence

We begin by stating a useful comparison principle, which is from [26].

**Lemma 4.1.** Assume \( f(t) \) is Lipschitz and \( g(t) \) is smooth on \( [a, b] \) with

\[
f(a) \leq g(a) \quad \text{and} \quad \frac{df}{dt} \leq \frac{dg}{dt} \quad \text{on} \ [a, b].
\]

Then we have \( f(t) \leq g(t) \) on \( [a, b] \).

The \( L^\infty \) a priori estimates for the solution can be established.

**Lemma 4.2.** Let \( p \leq 0 \) and \( \nu(x, t) > 0 \) be the \( 2m\pi \)-periodic solution of (1.4). Then there exist two positive constants \( C_1, C_2 \) independent of time, such that

\[
C_1 e^t \leq \nu(x, t) \leq C_2 e^t, \quad \text{when} \ p = 0;
\]
or
\[
C_d t^{-1/p} \leq v(x, t) \leq C_d t^{-1/p}, \quad \text{when } p < 0
\]
for \((x, t) \in S^1_m \times [1, \infty)\). And the solution to (1.10) \((p = 0)\) or (1.11) \((p < 0)\) satisfies
\[
C_1 \leq u(x, \tau) \leq C_2, \quad (4.2)
\]
for all \((x, \tau) \in S^1_m \times [0, \infty)\).

**Proof.** From (1.4), we have
\[
v_t = \varphi(x)v^p v_{xx} + \varphi(x)v^{p+1}.
\]
Suppose \(\varphi(x)\) satisfies \(0 < C_4 \leq \varphi(x) \leq C_3\) for two constants \(C_3\) and \(C_4\). Then we have
\[
\frac{d}{dt}v_{\max}(t) \leq C_3v_{\max}^{p+1}(t) \quad \text{and} \quad \frac{d}{dt}v_{\min}(t) \geq C_4v_{\min}^{p+1}(t). \quad (4.3)
\]
So the conclusion can be proved by Lemma 4.1. \(\square\)

The *a priori* estimate for the gradient of solution \(u\) can also be established.

**Lemma 4.3.** Let \(p \leq 0\) and \(u(x, \tau) > 0\) be a bounded 2πn-periodic solution of (1.10) \((p = 0)\) or (1.11) \((p < 0)\) on \(S^1_m \times [0, \infty)\). Then there exists a constant \(C > 0\), independent of time, such that
\[
|u_x(x, \tau)| \leq C, \quad (4.4)
\]
for all \((x, \tau) \in S^1_m \times [0, \infty)\).

**Proof.** The conclusion is a direct result of Lemmas 2.2 and 4.2. \(\square\)

Now we can show the global existence of the flow (1.1) for \(p \leq 0\).

**Lemma 4.4.** (Global existence) Let \(p \leq 0\). When the initial curve \(X_0\) is smooth, locally convex, closed, and immersed, the flow (1.1) has a unique smooth, locally convex solution on the time interval \([0, \infty)\).

**Proof.** The estimate in Lemma 4.2 implies that the evolving curves in the flow (1.1) preserve to be locally convex as long as the flow exists. Also, the two-side bound on \(v(x, t)\) means that the equation for \(v\) is uniformly parabolic on time interval \([0, T]\) for any given \(T > 0\). The *a priori* estimates on \(L^\infty\) norm of \(v\) and its gradient guarantee that we can obtain higher order estimates on \(v\) if \(\psi(x)\) and initial curve are smooth enough. Then we can use fixed point theorem to show the existence of solution \(v\) on \([0, T]\) for any given \(T > 0\). The uniqueness of the solution can be shown via maximum principle. \(\square\)

In the following, we consider the long-time behaviour of the flow. Let \(U(x, t)\) be the support function of evolving curves \(X(., t)\) in the flow (1.1), that is
\[
U(x, t) = \langle X(x, t), -n(x, t) \rangle, \quad (x, t) \in S^1_m \times [0, \infty).
\]

The evolution of support function \(U\) satisfies the following equation:
\[
\begin{cases}
\frac{\partial U}{\partial t} = -\frac{1}{\alpha}\psi(x)\xi^\alpha, & (x, t) \in S^1_m \times (0, \infty), \\
U(x, 0) = U_0(x), & x \in S^1_m,
\end{cases} \quad (4.5)
\]
where \(U_0(x)\) is the support function of initial curve. Noting that \(\psi(x)\) has a positive lower bound and \(v = \psi x^\alpha \geq C e^{\epsilon(p = 0)}\) or \(C t^{p/2}(p < 0)\), it is easy to show that the support function \(U\) approaches infinity as \(t \to \infty\) and the following lemma holds.
Lemma 4.5. (Expanding to infinity) Let $p \leq 0$. When the initial curve $X_0$ is smooth, locally convex, closed, and immersed, the flow $(1.1)$ expands to infinity as $t \to \infty$.

**Remark.** For $\alpha < -1$ (i.e. $0 < p < 1$), it can be shown via similar idea that

$$\lim_{t \to \infty} \max_{x \in S^1} U(x, t) = \infty.$$  

To look at the behaviour of curvature function for evolving curves, we can use the similar proof for Lemma 3.7 to show the following convergence result holds. Here, we note that $u(x, \tau)$ has a positive lower bound and one does not need to worry about the degeneracy of limit function.

Lemma 4.6. Let $p \leq 0$ and $u(x, \tau) > 0$ be a bounded $2\pi$-periodic solution of $(1.10)\ (p = 0)$ or $(1.11)\ (p < 0)$. For each sequence $\tau_n, \tau_n \to \infty$ as $n \to \infty$, there is a subsequence $\tau_{n_k}$ such that $u(x, \tau_{n_k})$ converges in $C^0(S^1)$ to a limit function $w(x) \in C^0(S^1)$, which is a positive $2\pi$-periodic solution to the ODE

$$w_{xx}(x) + w(x) - \frac{w(x)}{\varphi(x)} = 0 \quad (p = 0), \quad (4.6)$$

or

$$w_{xx}(x) + w(x) + \frac{1}{pp(x)\varphi(x)p(x)} = 0 \quad (p < 0).$$

(4.7)

4.2 Full-time convergence

Lemma 4.7. Let $p \leq 0$. Suppose there are two sequences $\{\tau_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$, both of which go to $\infty$ as $n \to \infty$, such that $u(x, \tau_n)$ converges uniformly to $w(x)$ and $u(x, s_n)$ converges uniformly to $\bar{w}(x)$. Then we have $w(x) = \bar{w}(x)$.

**Proof.** We adopt the idea of proof from [15]. Set $G(x) = \frac{\varphi(x)}{w(x)}$. Suppose $G(x_0) = \max_{x \in S^1} G(x)$. Then at $x_0$,

$$0 = G(x_0) = \frac{w_{xx}(x_0)\bar{w}(x_0) - w(x_0)\bar{w}_{xx}(x_0)}{\bar{w}^2(x_0)} \quad (4.8)$$

and

$$0 \geq G_{xx}(x_0) = \frac{w_{xx}(x_0)\bar{w}(x_0) - w(x_0)\bar{w}_{xx}(x_0)}{\bar{w}^2(x_0)}, \quad (4.9)$$

i.e.

$$\frac{w_{xx}(x_0)}{\bar{w}(x_0)} \leq \frac{w_{xx}(x_0)}{\bar{w}(x_0)}. \quad (4.10)$$

First, we consider the case of $p < 0$. By (4.7),

$$g(x_0, w(x_0)) = -\frac{1}{pp(x_0)}w^{p-p}(x_0)$$

$$= w^p(x_0) \left[ \frac{w_{xx}(x_0)}{\bar{w}(x_0)} + 1 \right]$$

$$\leq w^p(x_0) \left[ \frac{\bar{w}_{xx}(x_0)}{\bar{w}(x_0)} + 1 \right]$$

$$= \frac{w^p(x_0)}{\bar{w}^p(x_0)}g(x_0, \bar{w}(x_0)).$$
i.e.
\[
\frac{g(x_0, w(x_0))}{w''(x_0)} \leq \frac{g(x_0, \bar{w}(x_0))}{\bar{w}''(x_0)}.
\]

Therefore,
\[
w(x_0) = \left[ -p\varphi(x_0) \frac{g(x, w(x_0))}{w''(x_0)} \right]^{\frac{1}{p}} \leq \left[ -p\varphi(x_0) \frac{g(x, \bar{w}(x_0))}{\bar{w}''(x_0)} \right]^{\frac{1}{p}} = \bar{w}(x_0),
\]

and \( G(x) = \frac{w(x)}{\bar{w}(x)} \leq \frac{w(x_0)}{\bar{w}(x_0)} \leq 1 \), i.e. \( w(x) \leq \bar{w}(x) \). By carrying the similar analysis on the function \( \bar{w}/w \), one can show that \( \bar{w}(x) \leq w(x) \).

Then, we consider the case of \( p = 0 \). For any \( x \in S_m^1 \), we have
\[
\frac{w_{x_0}(x)}{w(x)} = \frac{1}{\varphi(x)} - 1 = \frac{\bar{w}_{x_0}(x)}{\bar{w}(x)},
\]
i.e. \( G_{x_0}(x) = 0 \). So there exists a constant \( C \) such that \( G_x = C \). Since \( G \) is a periodic function, we have \( G_x = 0 \).

Therefore, \( G(x) = \frac{w(x)}{\bar{w}(x)} = C_0 > 0 \) and \( C_0 \) is a constant.

We claim that
\[
w \equiv \bar{w},
\]
i.e. \( C_0 = 1 \). If not, we suppose that \( C_0 > 1 \). By the continuity, there also exists a time sequence \( \{\eta_n\}_{n=1}^{\infty} \), \( \eta_n \to \infty(n \to \infty) \), such that \( u(x, \eta_n) \) converges uniformly to another limit function \( w_1(x) \), which satisfies
\[
\bar{w}(0) < w_1(0) < w(0),
\]
and
\[
\frac{w(x)}{w_1(x)} = C_1 < C_0 = \frac{w(x)}{\bar{w}(x)}, \quad x \in S_m^1.
\]

Then we consider the evolution of \( \xi(x, \tau) = u(x, \tau) - w_1(x) \), which satisfies
\[
\xi_{\tau} = \varphi(x) \left( \xi_{xx} + \xi - \frac{\xi}{\varphi(x)} \right), \quad (x, \tau) \in S_m^1 \times (0, \infty).
\]

When \( n \) is sufficiently large, we have \( \xi(x, \tau_n) > 0 \) and \( \xi(x, \tau_n) < 0 \). A contradiction with zero number theory! Therefore, \( w(x) = \bar{w}(x) \) on \( S_m^1 \).

**Proof of Theorem 1.2.** According to Lemma 3.7, we know that for any time sequence \( \{\tau_n\}_{n=1}^{\infty} \), there exists a subsequence \( \{\tau_{n_k}\}_{k=1}^{\infty} \) converging smoothly to a positive function \( w(x) \), which is a solution to the ODE (4.6) for \( p = 0 \) and the ODE (4.7) for \( p < 0 \). Lemma 4.7 tells us that \( w(x) \) does not depend on the choice of time sequence. Hence, we obtain the full-time convergence and finish the proof of Theorem 1.2.

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