Research Article

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Homoclinic solutions for a differential inclusion system involving the $p(t)$-Laplacian

Abstract: The aim of this article is to study nonlinear problem driven by the $p(t)$-Laplacian with nonsmooth potential. We establish the existence of homoclinic solutions by using variational principle for locally Lipschitz functions and the properties of the generalized Lebesgue-Sobolev space under two cases of the nonsmooth potential: periodic and nonperiodic, respectively. The resulting problem engages two major difficulties: first, due to the appearance of the variable exponent, commonly known methods and techniques for studying constant exponent equations fail in the setting of problems involving variable exponents. Another difficulty we must overcome is verifying the link geometry and certifying boundedness of the Palais-Smale sequence. To our best knowledge, our theorems appear to be the first such result about homoclinic solution for differential inclusion system involving the $p(t)$-Laplacian.

Keywords: $p(t)$-Laplacian, homoclinic solution, locally Lipschitz, nonsmooth critical point theory

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1 Introduction

In this article, we study the following nonlinear second-order $p(t)$-Laplacian system with nonsmooth potential

\begin{equation}
\begin{cases}
\frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) - a(t)|u(t)|^{p(t)-2}u(t) \in \partial f(t, u(t)), \\
u(t) \to 0, \quad \text{as } |t| \to \infty,
\end{cases}
\end{equation}

where $p, a : \mathbb{R} \to \mathbb{R}^+$, $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $u \mapsto f(t, u)$ is locally Lipschitz. Here $\partial f(t, x)$ denotes the subdifferential of the locally Lipschitz function $u \mapsto f(t, u)$.

In recent years, the study on $p(t)$-Laplacian problems has attracted more and more attention. The $p(t)$-Laplacian possesses more complicated phenomena than the $p$-Laplacian. For example, it is inhomogeneous, which causes many difficulties, and some classical theories and methods, such as the theory of Sobolev spaces, are not applicable. The study of various mathematical problems with variable exponent growth condition has received considerable attention in recent years; see [26,30,50,52]. One of the most studied models leading to problems of this type is the model of motion of electro-rheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field [59]. Problems with variable exponent growth conditions also appear in the...
mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the filtration processes of an ideal barotropic gas through a porous medium [2,3]. Another field of application of equations with variable exponent growth conditions is image processing [10]. We refer the reader to [23,55–58,60,61] for an overview and references on this subject, and to [12–15,28,29,41,45,53,54] for the study of the $p(t)$-Laplacian equations and the corresponding variational problems.

Since many free boundary problems and obstacle problems may be reduced to partial differential equations (PDEs) with discontinuous nonlinearities, the existence of solutions for the problems with discontinuous nonlinearities has been widely investigated in recent years. Chang [4] extended the variational methods to a class of nondifferentiable functionals. In 2000, Kourogenis and Papageorgiou [35] obtained some nonsmooth critical point theorems. Subsequently, the nonsmooth version of the three critical points theorem and the nonsmooth Ricceri-type variational principle was established by Marano and Motreanu [36], who gave an application to elliptic problems involving the $p$-Laplacian with discontinuous nonlinearities. Kandilakis et al. [34] obtained the local linking theorem for locally Lipschitz functions. Dai [16] elaborated a nonsmooth version of the fountain theorem and gave an application to a Dirichlet-type differential inclusion. In 2019, Ge and Rădulescu [27] obtained infinitely many solutions for a nonhomogeneous differential inclusion with lack of compactness with the $p(x)$-Laplacian.

It is well known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, and its perturbed system probably produces chaotic phenomena. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic orbits of problem (1.1). When $p(t) \equiv p$, (1.1) reduces to $p$-Laplacian system:

$$
\frac{d}{dt}|(u(t)|^{p-2}u(t)) - a(t)|u(t)|^{p-2}u(t) \in \partial f(t, u(t)),
$$

(1.2)

Hu and Papageorgiou studied the existence of homoclinic solution using the theory of nonsmooth critical points and the idea of approximation [31,32] in the case of periodic nonsmooth potential with scalar equation. However, they did not prove the existence of homoclinic solutions and approached the problem differently from ours. Particularly, none of the works addressed the issues in the case of non-periodic nonsmooth potential. With regard to the results of (1.1) in PDE, please refer to the literature [6–9,17,19,22,33,42,47,48].

To the best of our knowledge, there is few paper discussing the homoclinic solutions of problem (1.1) with nonsmooth potential via nonsmooth critical point theory can be found in the existing literature. In order to fill in this gap, inspired by [28,37,41,57], we study problem (1.1) from a more extensive viewpoint. More precisely, we would study the existence of nontrivial homoclinic solutions of problem (1.1) with the generalized subquadratic and superquadratic in two cases of the nonsmooth potential: periodic and non-periodic, respectively. Moreover, our results generalize and improve the ones in (1.2). The resulting problem engages two major difficulties: first, due to the appearance of the variable exponent, which is not homogeneous, some special techniques and sharp estimation of inequality will be needed to study this type of problem (1.1). Another difficulty we must overcome is verifying the link geometry and certifying boundedness of the sequence of solutions $\{u_n\}$ associated with problem (1.1). It is worth to point out that commonly known methods and techniques for studying constant exponent equations fail in the setting of problems involving variable exponents. In these cases, we have to use techniques which are simpler and more direct in this article.

Throughout this article, we formulate the hypotheses on $p(t)$, $a(t)$ and basic assumptions on $f(t, u)$: $H(p)$ $p \in C(\mathbb{R}, \mathbb{R}^+)$ and

$$
1 < p^- = \inf_{t \in \mathbb{R}} p(t) \leq \sup_{t \in \mathbb{R}} p(t) = p^+ < \infty;
$$

$H(a)$ $a \in C(\mathbb{R}, \mathbb{R}^+)$ and there exists $a_0 > 0$ such that $a(t) \geq a_0 > 0$ for $t \in \mathbb{R}$;
H(f)
(i) the function \( f(t, \cdot) : \mathbb{R} \to \mathbb{R} \) is measurable for all \( u \in \mathbb{R}^N \) and \( f(t, 0) = 0 \);
(ii) the function \( f(\cdot, u) : \mathbb{R}^N \to \mathbb{R} \) is locally Lipschitz for a.e. \( t \in \mathbb{R} \).

Our approach is variationally based on the nonsmooth critical point theory (see Rădulescu and Repovš [51], Diening et al. [18] and the papers by Chang, Fan, Rădulescu, Papageorgiou, Papageorgiou and Zhao et al. [4, 21, 38, 40, 44]). For the convenience of the reader, in the next section we recall some basic definitions and facts from the theory, which we shall use in the sequel.

This article is organized as follows. In Section 2, we present some necessary preliminary knowledge on the generalized gradient of the locally Lipschitz function and variable exponent Sobolev spaces. In Section 3, we establish and prove the existence of nontrivial homoclinic solution related to periodic problem (1.1). In Section 4, we establish and prove the existence of nontrivial homoclinic solution corresponding to nonperiodic problems (1.1) and (4.2), respectively.

Throughout the article, we make use of the following notations:
• \( L^s(\mathbb{R}) (1 \leq s < \infty) \) denotes the Lebesgue space with the norm \( \|u\|_s = \left( \int_{\mathbb{R}} |u|^s \, dt \right)^{1/s} \);
• For any \( x \in \mathbb{R} \) and \( r > 0 \), \( B_r(x) = \{ y \in \mathbb{R} : |y - x| < r \} \) and \( B_r = B_r(0) \);
• \( C_i, C_2, \ldots \) denote positive constants possibly different in different places.

2 Preliminaries

We start with some preliminary basic results on variable exponent Sobolev spaces. For more details we refer the readers to the book of Rădulescu and Repovš [51], Diening et al. [18] and the papers by Chang, Fan, Rădulescu, Papageorgiou, Papageorgiou and Zhao et al. [4, 21, 38, 40, 44].

2.1 Weighted variable exponential \( W^{1,p(t)}_a \) space

In order to discuss problem (1.1), we recall some known results from critical point theory and the properties of space \( W^{1,p(t)}_a \) for the convenience of the readers.

Let \( \Omega \) be a subset of \( \mathbb{R} \),
\[
S(\Omega, \mathbb{R}^N) = \{ u : \text{the function } u : \Omega \to \mathbb{R}^N \text{ is measurable} \}
\]
and any two elements in \( S(\Omega, \mathbb{R}^N) \) which are almost equal are considered the same element. Let \( p, a \) satisfy assumptions \( H(p) \), \( H(a) \), respectively.

Define \( L^{p(t)}_a(\Omega, \mathbb{R}^N) \) (denoted by \( L^{p(t)}_a \)) as follows:
\[
L^{p(t)}_a(\Omega, \mathbb{R}^N) = \left\{ u \in S(\Omega, \mathbb{R}^N) : \int_{\Omega} a(t) |u(t)|^{p(t)} \, dt < \infty \right\}
\]
endowed with the norm
\[
|u|_{p(t), a} = \inf \left\{ A > 0 : \int_{\Omega} a(t) \left| \frac{u}{A} \right|^{p(t)} \, dt \leq 1 \right\}.
\]
If \( a(t) \equiv 1 \), \( L^{p(t)}_a \) and the corresponding norm \( |u|_{p(t), a} \) are written simply by \( L^{p(t)}, |u|_{p(t)} \).
Define \( \mathcal{W}_a^{\rho_p(t)}(\Omega, \mathbb{R}^N) \) (denoted by \( \mathcal{W}_a^{\rho_p(t)} \)) as follows:
\[
\mathcal{W}_a^{\rho_p(t)}(\Omega, \mathbb{R}^N) = \{ u \in L_a^{\rho_p(t)}(\Omega, \mathbb{R}^N) : \dot{u} \in L_a^{\rho_p(t)}(\Omega, \mathbb{R}^N) \}
\]
with the norm
\[
\| u \| = \inf \left\{ A > 0 : \int_{\Omega} \left( \frac{\| u \|}{A} + \alpha(t) \frac{\| u \|}{A} \right) dt \leq 1 \right\}.
\]

In particular, if \( \alpha(t) \equiv 1 \), \( \mathcal{W}_a^{\rho_p(t)} \) is reduced to
\[
W^{\rho_p(t)}(\Omega, \mathbb{R}^N) = \{ u \in L^{p(t)}(\Omega, \mathbb{R}^N) : \dot{u} \in L^{p(t)}(\Omega, \mathbb{R}^N) \}
\]
and the norm
\[
\| u \| = | u |_{p(t)} + | \dot{u} |_{p(t)}.
\]

We use \( \mathcal{W}_a^{\rho_p(t)} \) to represent the space of \( C_0^{\infty}(\Omega, \mathbb{R}^N) \) consisting of infinitely continuous differentiable functions with compact supports on \( \Omega \) completion in \( W^{\rho_p(t)} \). We call the space \( L^{\rho_p(t)} \) a generalized Lebesgue space, and it is a special kind of generalized Orlicz spaces. The space \( W^{\rho_p(t)} \) is called a generalized Sobolev space, it is a special kind of generalized Orlicz-Sobolev spaces. For more details on the general theory of generalized Orlicz spaces and generalized Orlicz-Sobolev spaces, see [18,20,51] and references therein.

The following propositions summarize the main properties of this norm (see Alves and Liu [1], Rădulescu and Repovš [51] and Fan and Zhao [21]).

**Proposition 2.1.** \( L_a^{\rho_p(t)}, \mathcal{W}_a^{\rho_p(t)}, \mathcal{W}_0^{\rho_p(t)} \) are reflexive Banach spaces with norms defined above when \( p^- > 1 \).

**Proposition 2.2.** Let \( \rho(u) = \int_{\Omega} \alpha(t)|u(t)|^{p(t)} dt \) for any \( u, v \in L_a^{\rho_p(t)} \), then the following properties hold:
(i) \( \rho(u) = 0 \iff u = 0 \);
(ii) \( \rho(u) = \rho(-u) \);
(iii) \( \rho(\alpha u + \beta v) \leq \alpha \rho(u) + \beta \rho(v) \) for any \( \alpha, \beta \geq 0, \alpha + \beta = 1 \);
(iv) \( \rho(u + v) \leq 2^p(\rho(u) + \rho(v)) \);
(v) If \( \lambda > 1 \), then
\[
\lambda^p \rho(u) \leq \rho(\lambda u) \leq \lambda^p \rho(u) \leq \lambda \rho(u) \leq \rho(u);
\]
(vi) if \( \rho(u)_{p(t),a} = 1 \) if and only if \( \rho \left( \frac{u}{\lambda} \right) = 1 \), for any \( u \in L_a^{\rho_p(t)} \). \( \{0\} \).

**Proposition 2.3.** For any \( u \in L_a^{\rho_p(t)} \), the following properties hold:
(i) \( \| u \|_{p(t),a} < 1 (=1;>1) \iff \rho(u) < 1 (=1;>1) \);
(ii) \( \| u \|_{p(t),a} > 1, \text{then} |u|^{p(t)}_{p(t),a} \leq \rho(u) \leq \| u \|^{p(t)}_{p(t),a} \);
(iii) \( \| u \|_{p(t),a} < 1, \text{then} |u|^{p(t)}_{p(t),a} \leq \rho(u) \leq \| u \|^{p(t)}_{p(t),a} \);
(iv) \( |u|_{p(t),a} \to 0 \iff \rho(u) \to 0 \);
(v) \( |u|_{p(t),a} \to \infty \iff \rho(u) \to \infty \).

**Proposition 2.4.** Let \( \phi(u) = \int_{\Omega} (|u|^{p(t)} + \alpha(t)|u|^{p(t)}) dt \) for any \( u \in W_a^{\rho_p(t)} \), then the following properties hold:
(i) \( \| u \| < 1 (=1;>1) \implies \phi(u) < 1 (=1;>1) \);
(ii) \( \| u \| > 1, \text{then} \| u \|^{p'} \leq \phi(u) \leq \| u \|^{p'} \);
(iii) \( \| u \| < 1, \text{then} \| u \|^{p'} \leq \phi(u) \leq \| u \|^{p'} \);
(iv) \( \| u \| \to 0 \iff \phi(u) \to 0 \);
(v) \( \| u \| \to \infty \iff \phi(u) \to \infty \).
Proposition 2.5. Let \( \rho(u) = \int_\Omega a(t)|u|^{p(t)}\,dt \) for any \( u \in L^p_a(\Omega) \), \( \{u_n\} \subset L^p_a(\Omega) \), then the following properties are equivalent:

(i) \( \lim_{n \to \infty} |u_n - u|_{p(t),a} = 0 \);
(ii) \( \lim_{n \to \infty} \rho(u_n - u) = 0 \);
(iii) \( u_n \to u \text{ a.e. } t \in \Omega \) and \( \lim_{n \to \infty} \rho(u_n) = \rho(u) \).

Proposition 2.6. \((L^p(t))^* = L^q(t)\) with \( 1/p(t) + 1/q(t) = 1 \) and

\[
\left| \int_\Omega u(t)v(t)\,dt \right| \leq 2|u|_{p(t)}|v|_{q(t)} , \quad \forall u \in L^p(t), \quad v \in L^q(t),
\]

where \((L^p(t))^*\) is the dual space of \(L^p(t)\).

Proposition 2.7. \(C_0^\infty(\mathbb{R}, \mathbb{R}^N)\) is dense in space \(W^{1,1}_a(\Omega)\).

Proposition 2.8. Let \( u \in W^{1,p(t)}_a(\Omega) \), then

(i) \( u \in C(\mathbb{R}, \mathbb{R}^N) \) and \( u(t) \to 0 \) as \( |t| \to \infty \). Moreover, the embedding \( W^{1,p(t)}_a(\Omega) \hookrightarrow L^\infty(\mathbb{R}, \mathbb{R}^N) \) is continuous, and there exists a constant \( k > 0 \) such that

\[
||u||_{L^\infty} \leq k||u|| , \quad \forall u \in W^{1,p(t)}_a(\Omega);
\]

(ii) If \( H(p), H(a) \) hold and \( a(t) \to +\infty \) as \( |t| \to \infty \), then the embedding \( W^{1,p(t)}_a(\Omega) \hookrightarrow L^\infty(\mathbb{R}, \mathbb{R}^N) \) is compact.

Consider the following functional:

\[
I(u) = \int_\Omega \frac{1}{p(t)}(|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})\,dt , \quad \forall u \in W^{1,p(t)}_a(\Omega).
\]

We know that \( I \in C(W^{1,p(t)}_a(\Omega), \mathbb{R}) \) under condition \( H(a) \). Moreover,

\[
\langle I'(u), v \rangle = \int_\Omega (|\dot{u}|^{p(t)-2}\dot{u}\dot{v} + a(t)|u|^{p(t)-2}uv)\,dt , \quad \forall u, v \in W^{1,p(t)}_a(\Omega).
\]

Proposition 2.9. \( I' \) is a mapping of type \((S)_+\), i.e., if

\[
\lim_{n \to \infty} (I'(u_n) - I'(u), u_n - u) \leq 0,
\]

then \( u_n \) has a convergent subsequence in \( W^{1,p(t)}_a(\Omega) \).

Denote

\[
A = I' : W^{1,p(t)}_a(\Omega) \to (W^{1,p(t)}_a)^*,
\]

then we have

\[
\langle A(u), v \rangle = \int_\Omega (|\dot{u}'(t)|^{p(t)-2}\dot{u}'(t)v'(t) + a(x)|u(t)|^{p(t)-2}uv)\,dt
\]

for all \( u, v \in W^{1,p(t)}_a(\Omega) \).

Proposition 2.10. The mapping \( A \) is a strictly monotone, bounded homeomorphism and is of type \((S)_+\) in \( W^{1,p(t)}_a(\Omega) \).
2.1.1 Periodic variable exponential $W^{1,p(t)}_{2nb}$ space

For any $b > 0$, $n \geq 1$, let $T_n = [-nb, nb]$. Define $L^{p(t)}_{2nb}(T_n, \mathbb{R}^N)$ (denoted by $L^{p(t)}_{2nb}$) as follows:

$$L^{p(t)}_{2nb}(T_n, \mathbb{R}^N) = \left\{ u \in S(T_n, \mathbb{R}^N) : \int_{-nb}^{nb} a(t)|u(t)|^{p(t)}dt < \infty \right\}$$

endowed with the norm

$$|u|_{p(t)} = \inf \left\{ \lambda > 0 : \int_{-nb}^{nb} a(t) \left| \frac{u}{\lambda} \right|^{p(t)}dt \leq 1 \right\}.$$

Moreover, $L^{co}_{2nb}(T_n, \mathbb{R}^N)$ (denoted by $L^{co}_{2nb}$) be a Banach space with the norm

$$\|u\|_{L^{co}_{2nb}} = \text{ess sup}|u(t)| : t \in [-nb, nb]|.$$

Define $W^{1,p(t)}_{2nb}(T_n, \mathbb{R}^m)$ (denoted by $W^{1,p(t)}_{2nb}$) as follows:

$$W^{1,p(t)}_{2nb}(T_n, \mathbb{R}^m) = \{ u \in L^{p(t)}_{2nb}(T_n, \mathbb{R}^N) : \dot{u} \in L^{p(t)}_{2nb}(T_n, \mathbb{R}^N) \}$$

endowed with the norm

$$|u|_1 = \inf \left\{ \lambda > 0 : \int_{-nb}^{nb} \left( \frac{|u|}{\lambda} + a(t) \left| \frac{u}{\lambda} \right|^{p(t)} \right)dt \leq 1 \right\}.$$

In particular, if $b \to +\infty$, $L^{p(t)}_{2nb}(T_n, \mathbb{R}^N)$, $L^{co}_{2nb}(T_n, \mathbb{R}^N)$, $W^{1,p(t)}_{2nb}(T_n, \mathbb{R}^m)$ are written simply by $L^{p(t)}_{a}(\mathbb{R}, \mathbb{R}^N)$, $L^{co}_{a}(\mathbb{R}, \mathbb{R}^N)$, $W^{1,p(t)}_{a}(\mathbb{R}, \mathbb{R}^m)$, respectively.

2.2 Nonsmooth analysis theory

The nonsmooth critical point theory for locally Lipschitz functionals is based on the subdifferential theory of Clark [11], Rădulescu [49], Gasiński and Papageorgiou [25].

**Definition 2.11.** Let $X$ be a Banach space and let $X^*$ be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(X, X^*)$. A function $\phi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exist $U \in \mathcal{N}(x)$ and a constant $k_U > 0$, such that

$$|\phi(y) - \phi(z)| \leq k_U \|y - z\|_X, \quad \forall y, z \in U.$$

**Definition 2.12.** For a given locally Lipschitz function $\phi : X \to \mathbb{R}$, the generalized directional derivative of $\phi$ at $x \in X$ in the direction $h \in X$ is defined by

$$\phi^0(x; h) \doteq \lim_{y \to x; \lambda \downarrow 0} \sup \frac{\phi(y + \lambda h) - \phi(y)}{\lambda}$$

$$= \inf_{\varepsilon, \delta > 0} \sup_{|x - y| < \varepsilon : \|x - y\| < \delta} \frac{\phi(y + \lambda h) - \phi(y)}{\lambda}.$$

Based on Definition 2.12, one can easily verify that the function $h \mapsto \phi^0(x; h)$ is sublinear, Lipschitz continuous (see [11, Proposition 2.1.1]).

**Definition 2.13.** Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function. Then generalized subdifferential of $\phi$ at $x \in X$ is the nonempty set $\partial \phi(x) \subseteq X^*$ defined by
\[ \partial \phi(x) = \{ x^* \in X^* : \langle x^*, h \rangle \leq \phi(x; h), \quad \forall h \in X \}. \]

The multifunction \( x \to \partial \phi(x) \) is known as the generalized (or Clarke) subdifferential of \( \phi \). If \( \phi, \psi : X \to \mathbb{R} \) are locally Lipschitz functions, then \( \partial(\phi + \psi)(x) \subseteq \partial \phi(x) + \partial \psi(x) \) and for every \( \lambda \in \mathbb{R} \), \( \partial(\lambda \phi)(x) = \lambda \partial \phi(x) \).

**Definition 2.14.** Let \( \phi : X \to \mathbb{R} \) be a locally Lipschitz function. A point \( x \in X \) is said to be a critical point of \( \phi \) if \( 0 \in \partial \phi(x) \).

If \( x \in X \) is a critical point of \( \phi \), then \( c = \phi(x) \) is a critical value of \( \phi \). It is easy to see that, if \( x \in X \) is a local extremum of \( \phi \), then \( 0 \in \partial \phi(x) \). Moreover, the multifunction \( x \to \partial \phi(x) \) is upper semicontinuous from \( X \) into \( X^* \) equipped with the \( w^* \) topology, i.e., for any \( U \subseteq X^* \) open, the set \( \{ x \in X : \partial \phi(x) \subseteq U \} \) is open in \( X \). For more details we refer to Clarke [11, Proposition 2.1.2].

**Definition 2.15.** The locally Lipschitz function \( \phi : X \to \mathbb{R} \) satisfies the nonsmooth Palais-Smale (PS) condition, if any sequence \( \{ x_n \}_{n \geq 1} \subseteq X \) such that
\[
\{ \phi(x_n) \}_{n \geq 1}
\]
is bounded and \( m(x_n) \to 0 \) as \( n \to \infty \), has a strongly convergent subsequence, where \( m(x_n) = \min \| x^* \| : x^* \in \partial \phi(x_n) \).

**Lemma 2.16.** (Lebourg’s mean value Theorem [39]). Given the points \( x \) and \( y \) in \( X \) and a real-valued function \( \phi \) which is Lipschitz continuous on an open set containing the segment \( \{ x, y \} = \{(1 - t)x + ty : t \in [0, 1] \} \), there exist \( z = x + t_0(y - x) \), with \( 0 < t_0 < 1 \), and \( x^* \in \partial \phi(z) \) such that
\[
\phi(y) - \phi(x) = \langle x^*, y - x \rangle.
\]

If \( \phi \in C^1(X, \mathbb{R}) \), then as we already mentioned \( \partial \phi(x) = \{ \phi'(x) \} \) and so the above definition of the PS condition coincides with the classical (smooth) one. In the context of the smooth theory, Cerami introduced a weaker compactness condition which in our nonsmooth setting has the following form:

**Definition 2.17.** The locally Lipschitz function \( \phi : X \to \mathbb{R} \) satisfies the nonsmooth Cerami condition (C-condition), if any sequence \( \{ x_n \}_{n \geq 1} \subseteq X \) such that
\[
\{ \phi(x_n) \}_{n \geq 1}
\]
is bounded and \( (1 + \| x_n \|) m(x_n) \to 0 \) as \( n \to \infty \), has a strongly convergent subsequence.

**Lemma 2.18.** (Weierstrass theorem [43]). Assume that \( \varphi \) is a locally Lipschitz functional on a Banach space \( X \) and \( \varphi : X \to \mathbb{R} \) satisfies:

(i) \( \varphi \) is weakly lower semicontinuous;
(ii) \( \varphi \) is coercive.

Then there exists \( x^* \in X \) such that \( \varphi(x^*) = \min_{x \in X \varphi(x)} \).

**Lemma 2.19.** (Nonsmooth mountain pass theorem [35]). Let \( X \) be a reflexive Banach space, \( \phi : X \to \mathbb{R} \) a locally Lipschitz functional satisfying the PS-condition. Assume that there exist \( x_0, x_1 \in X, c_0 \in \mathbb{R} \) and \( \vartheta > 0 \) such that \( \| x_1 - x_0 \| > \vartheta \) and
\[
\max \{ \phi(x_0), \phi(x_1) \} < c_0 = \inf \{ \phi(y) : \| y - x_0 \| = \vartheta \}.
\]

Then \( \phi \) has a critical point \( x \in X \) with \( c = \phi(x) \geq c_0 \), where \( c \) is given by
\[
c = \inf_{y \in \Gamma} \max_{t \in T} \phi(y(t)),
\]
\[
\Gamma = \{ y \in C([0, 1], X) : y(0) = x_0, y(1) = x_1 \}.
\]
3 Periodic $p(t)$-Laplacian inclusion system

In this section, we establish the existence of homoclinic solutions with periodic assumption for problem (1.1). In this situation, our hypotheses on $p$, $a$, and $f$ are the following:

$H(p)_1$: $p(t)$ is $2b$-periodic;

$H(a)_1$: $a(t)$ is $2b$-periodic;

$H(f)_1$:

(i) the function $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is $2b$-periodic;

(ii) for almost all $t \in T = [-b, b]$, there exists a function $a(t) \in C(\mathbb{R}) \cap L^p(\mathbb{R})$ such that

$$|\omega| \leq a(t)(1 + |\omega|^{p(t)-1}), \quad \forall \omega \in \mathbb{R}$, $a(0) = 0$;

where $a \in L^\infty(\mathbb{R})$, $a^* < y^- < y(t) < y^* < p^-$;

(iii) there exist constants $M, a, \beta > 0$ such that

$$0 \leq \left(p^* + \frac{1}{a + \beta|\omega|^p}\right)f(t, u) \leq -f^0(t, u; -u) \quad \forall t \in T, \quad |u| \geq M,$$

where $\nu < p^-$;

(iii') there exist constants $\mu > p^*$, $M > 0$ such that

$$\mu f(t, u) \leq -f^0(t, u; -u) \quad \forall t \in T, \quad |u| \geq M;$$

(iv) there exists a function $q(t) > 0$ such that

$$\lim_{|u| \to 0} \frac{(w, u)}{|u|^{p(t)}} \leq 0, \quad \lim_{|u| \to +\infty} \frac{f(t, u)}{|u|^{p(t)}} > 0, \quad \forall t \in T, \quad w \in \partial f(t, u),$$

where $p^* < q^-$.

Our main results can be stated as follows.

**Theorem 3.1.** If hypotheses $H(p), H(p)_1$, $H(a), H(a)_1$, $H(f)$, and $H(f)_1$: (i), (ii), (iii), (iv) hold, then problem (1.1) has a nontrivial homoclinic solution.

**Theorem 3.2.** If hypotheses $H(p), H(p)_1$, $H(a), H(a)_1$, $H(f)$, and $H(f)_1$: (i), (ii), (iii'), (iv) hold, then problem (1.1) has a nontrivial homoclinic solution.

**Proof of Theorem 3.1.** We consider the following auxiliary periodic problem:

\[
\begin{cases}
-\frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) + a(t)|u(t)|^{p(t)-2}u(t) \in \partial f(t, u(t)), & \text{a.e., } t \in T_n, \\
u(-nb) = u(nb), \quad \nu(-nb) = u(nb) \quad (3.1)
\end{cases}
\]

From [5], we know that problem (3.1) has a nontrivial solution $u_n \in C^1_{2nb}(T_n, \mathbb{R})$. Let $\varphi_\alpha : W^{1,p(t)}_{2nb}(T_n, \mathbb{R}) \to \mathbb{R}$ be defined by

$$\varphi_\alpha(u) = \int_{-nb}^{nb} \frac{1}{p(t)}|u(t)|^{p(t)} + \alpha(t)|u(t)|^{p(t)} - \int_{-nb}^{nb} f(t, u(t))dt = \varphi_\alpha^0(u) - \int_{-nb}^{nb} f(t, u(t))dt. \quad (3.2)$$

We claim that $\varphi_\alpha$ be the locally Lipschitz functional. In fact, for all $u_1, u_2 \in W^{1,p(t)}_{2nb}(T_n, \mathbb{R})$, one has

$$|\varphi_\alpha(u_1) - \varphi_\alpha(u_2)| = |\varphi_\alpha^0(u_1) - \varphi_\alpha^0(u_2)|,$$

where $\bar{u} = su_1 + (1-s)u_2$, $s \in (0, 1)$. Let $\Omega \subset T_n$, fix $u_0 \in W^{1,p(t)}_{2nb}(\Omega, \mathbb{R})$ and $B_r = \{u \in W^{1,p(t)}_{2nb}(T_n, \mathbb{R}) : \|u - u_0\|_1 \leq r\}$. 

$\varphi_\alpha$
Note that $B_r$ is compact, which yields that there exists $C_1 > 0$ such that
\[
\| \check{\varphi}_n(\mathbf{u}) \|_{W^{1,p(t)}(\mathbb{R}^N)} \leq C_1 \tag{3.4}
\]
as $r \to 0$. Then, it follows from (3.3) and (3.4), we obtain
\[
\check{\varphi}_n(u_1) - \check{\varphi}_n(u_2) = |\check{\varphi}_n(\mathbf{u})|_1 = \| \check{\varphi}_n(\mathbf{u}) \|_{W^{1,p(t)}(\mathbb{R}^N)} \leq C_1 \| u_1 - u_2 \|_1,
\]
for all $u_1, u_2 \in W^{1,p(t)}(\Omega, \mathbb{R}^N)$.

On the other hand, it follows from $H(f)_i$: (ii) and Lemma 2.16, for all $u_1, u_2 \in W^{1,p(t)}_{2nb}(\Omega, \mathbb{R}^N)$, we have
\[
|f(t, u_1) - f(t, u_2)| \leq \alpha(t)(1 + |\check{u}|^{p(t)-1})\| u_1 - u_2 \|
\]
and
\[
\alpha(t)|\check{u}|^{p(t)-1} \leq \frac{(\gamma(t) - \alpha(t))(a(t)|\check{u}|^{p(t)-1})}{\gamma(t) - 1} + \alpha(t) - 1 |\check{u}|^{p(t)-1},
\]
which imply that there exist some constants $C_2, C_3 > 0$ such that
\[
(a(t)|\check{u}|^{p(t)-1})^{\frac{1}{p(t)}} \leq C_2|a(t)|^{\frac{1}{p(t)-1}} + C_3|\check{u}|^{p(t)}.
\]

Then, in virtue of (3.6), (3.7) and Hölder inequality, one has
\[
\left\| \int_{-nb}^{nb} f(t, u)dt - \int_{-nb}^{nb} f(t, u_2)dt \right\| \leq \int_{-nb}^{nb} |a(x)(1 + |\check{u}|^{p(t)-1})|u_1 - u_2|dt
\]
\[
\leq \left[ |a(t)|^{\frac{1}{p(t)}} + |\check{u}|^{\frac{p(t)-1}{p(t)}} \right] |u_1 - u_2|^{\gamma(t)}
\]
\[
\leq C_4|u_1 - u_2|_1.
\]
Hence, from (3.2), (3.5) and (3.8), we obtain
\[
\varphi_n(u) = \check{\varphi}_n(u) - \int_{-nb}^{nb} f(t, u)dt \leq C_5|u_1 - u_2|_1 + C_4|u_1 - u_2|_1 \leq C_5|u_1 - u_2|_1
\]
which yields that $\varphi_n$ be the nonsmooth locally Lipschitz energy functional corresponding to problem (3.1).

Therefore, it follows from (3.2), $H(f)_i$: (ii) and $H(f)_i$: (iii), (iv), for $\sigma \geq 1$, there exist $C_6, C_7 > 0$ such that
\[
\varphi_n(\sigma u) \leq \int_{-b}^{b} \frac{1}{p(t)}(\sigma|\check{u}|^{p(t)} + a(t)|\sigma u|^{p(t)})dt - \int_{-b}^{b} f(t, au)dt
\]
\[
\leq \sigma^p \int_{-b}^{b} \frac{1}{p(t)}(\check{u}|^{p(t)} + a(t)|u|^{p(t)})dt - \sigma^q \int_{-b}^{b} |u|^{q(t)}dt - C_0\sigma \int_{-b}^{b} |u|^{p(t)} - 2b\sigma C_7.
\]

Since $p^* < q^*$, there exists a constant $\sigma_0 > 0$ such that $\sigma > \sigma_0$ and $\check{u} \in W^{1,p(t)}_{2nb}(T_i, \mathbb{R}^N)$, we have $\varphi_n(\sigma \check{u}) < 0$.

Let $\check{u} \in W^{1,p(t)}_{2nb}(T_i, \mathbb{R}^N)$ be defined as follows:
\[
\check{u}(t) = \begin{cases} 
\check{u}(t), & \text{if } t \in T_i; \\
0, & \text{if } t \in T_{\ast} \setminus T_i
\end{cases}
\]
Note that $f(t, 0) = 0$, then for all $\sigma \geq \sigma_0$, we deduce that $\varphi_n(\sigma \check{u}) = \varphi_n(\sigma u)$.

As in [5], we see that the solution $u_\rho \in C^1_{2nb}(T, \mathbb{R}^N)$ of problem (3.1) is obtained via the nonsmooth mountain pass theorem. One will immediately obtain the fact that there exists $\rho > 0$ such that
\[
c_\rho = \inf_{\varphi_n(u) \geq \varphi_n(\check{u}(t))} \sup_{t \in [0,1]} \varphi_n(y(t)) = \varphi_n(u_\rho) \geq \inf \{ \varphi_n(\check{u}(u) : \|u\| = \rho \} > 0.
\]
where
\[ \Gamma_n = \{ y \in C([0, 1], W_{2n}^{1,p(t)}) : y(0) = 0, y(1) = \sigma\bar{u} \} \]
for \( \sigma \geq \sigma_0 \) and \( 0 \in \partial\varphi_n(u) \) for all \( n \geq 1 \). Extending by constant, as \( n_1 \leq n_2 \) we see that
\[ W_{n_1}^{1,p(t)} \subseteq W_{n_2}^{1,p(t)} \quad \text{and} \quad \Gamma_{n_1} \subseteq \Gamma_{n_2} \]
and consequently
\[ c_{n_2} \leq c_{n_1}, \quad \forall n_1 < n_2. \]
This way we have produced a decreasing sequence \( \{c_{n_1}\}_{n_1 \geq 1} \) of critical values. For every \( n \geq 1 \), from (3.2), we have
\[ c_n = \varphi_n(u_n) = \frac{1}{p(t)} \int_{-n}^{n} (|\dot{u}|^{p(t)} + a(t)|u_n|^{p(t)}) \, dt - \int_{-n}^{n} f(t, u_n(t)) \, dt \leq c_1, \quad (3.9) \]
which implies that
\[ \int_{-n}^{n} \frac{p(t)}{p(t)} (|\dot{u}|^{p(t)} + a(t)|u_n|^{p(t)}) \, dt - \int_{-n}^{n} p(t) f(t, u_n(t)) \, dt \leq p^* c_1. \quad (3.10) \]
Then, it follows from (3.10), we obtain
\[ \int_{-n}^{n} (|\dot{u}|^{p(t)} + a(t)|u_n|^{p(t)}) \, dt - \int_{-n}^{n} p(t) f(t, u_n(t)) \, dt \leq p^* c_1. \quad (3.11) \]
Using (2.1), one has
\[ A(u_n) = w_n, \quad w_n \in L_{2n}^{\infty}, \]
\[ w_n(t) \in \partial f(t, u_n(t)), \quad \text{a.e. } t \in T_n, \]
which yields that
\[ -\int_{-n}^{n} (|\dot{u}|^{p(t)} + a(t)|u_n|^{p(t)}) \, dt = \int_{-n}^{n} (w_n(t), u_n(t)) \, dt \leq \int_{-n}^{n} f^0(t, u_n(t) ; u_n(t)) \, dt. \quad (3.12) \]
Combining (3.11) with (3.12), we obtain
\[ \int_{-n}^{n} (-p^* f(t, u_n) - f^0(t, u_n ; u_n)) \, dt \leq p^* c_1. \quad (3.13) \]
By virtue of H(f); (iii), one has
\[ f(t, u_n) \leq (\alpha + \beta|u_n|^\gamma)(-p^* f(t, u_n) - f^0(t, u_n ; u_n)), \quad \forall |u_n| > M. \]
Hence, from H(f); (ii), (3.13) and Proposition 2.8 (i), there exists a constant \( \xi_0 \) which is independent of \( n \) such that
\[ \int_{-n}^{n} f(t, u_n) \, dt = \int_{T_n} f(t, u_n) \, dt + \int_{T_n} f(t, u_n) \, dt \leq \int_{T_n} (\alpha + \beta|u_n|^\gamma)(-p^* f(t, u_n) - f^0(t, u_n ; u_n)) \, dt + \xi_0 \]
\[ \leq (\alpha + \beta|u_n|_{L^\infty}) \int_{-n}^{n} (-p^* f(t, u_n) - f^0(t, u_n ; u_n)) \, dt + \xi_0 \]
\[ \leq (\alpha + \beta|u_n|_{L^\infty}) \int_{-n}^{n} (-p^* f(t, u_n) - f^0(t, u_n ; u_n)) \, dt + \xi_0 \]
\[ \leq p^* c_1(\alpha + \beta|u_n|_{L}) + \xi_0, \]
which, together with (3.2), (3.9) and Proposition 2.4 (ii), imply that

\[
\frac{1}{p^*} \|u_n\|^{p^*} \leq \frac{1}{p^*} \int_{-nb}^{nb} (|u_n|^{p(t)} + a(t)|u_n|^{p(t)})dt \\
\leq \int_{-nb}^{nb} \frac{1}{p(t)} (|u_n|^{p(t)} + a(t)|u_n|^{p(t)})dt \\
= q_n(u_n) + \int_{-nb}^{nb} f(t, u_n)dt \\
\leq c_1 + p^* c_2 (\alpha + \beta \xi_1^{p^*}) + \xi_0,
\]

for \(\|u_n\| > 1,\) since \(\nu < p^*\), there exists a constant \(\xi_2\) which is independent of \(n\) such that

\[
\int_{-nb}^{nb} \frac{1}{p(t)} (|u_n|^{p(t)} + a(t)|u_n|^{p(t)})dt \leq c_1 + p^* c_2 (\alpha + \beta \xi_1^{p^*}) + \xi_0 = \xi_2,
\]

where \(\xi_2\) is a constant which is independent of \(n\). Thus, by (3.14), we obtain

\[
\|u_n\|_{W^{1,p}_{2nb}} \leq \xi_3,
\]

where \(\xi_3 > 0\) is independent of \(n\). Moreover, by an argument as in the proof of [46, (2.19)], there exists a constant \(\xi_4\) which is independent of \(n\) such that

\[
\|u_n\|_{L^{\infty}_{2nb}} \leq \xi_4.
\]

In what follows, we extend by periodicity \(u_n\) and \(w_n\) to all of \(\mathbb{R}\). From (3.15) and the fact that the embedding \(W^{1,p(t)}_{2nb} \hookrightarrow C(T_n, \mathbb{R}^N)\) is compact, we may assume that

\[
u_n \rightarrow u \quad \text{in} \quad C_0(\mathbb{R}, \mathbb{R}^N),
\]

hence \(u \in C(\mathbb{R}, \mathbb{R}^N).\) In view of hypothesis H(f): (ii), one has

\[
\|w_{\delta}(t)| \| \leq \|a\|_{\infty} (1 + \|u_{\delta}(t)\|^{\alpha(t)-1}) \leq \xi_5,
\]

where \(\xi_5 > 0\) is a constant which is independent of \(n\). Passing to a subsequence if needed, we may assume that \(w_n \rightharpoonup w\) in \(L^{\infty}_{\alpha(t)}(\mathbb{R}, \mathbb{R}^N)\) and \(w_n \rightarrow w\) in \(L^{q(t)}_{\alpha(t)}(T_n, \mathbb{R}^N)\), where \(1/p(t) + 1/q(t) = 1\). It is obvious that \(w \in L^{\infty}_{\alpha(t)}(\mathbb{R}, \mathbb{R}^N) \cap L^{q(t)}_{\alpha(t)}(\mathbb{R}, \mathbb{R}^N)\), thus \(w(t) \in \partial f(t, u(t))\) in \(T_n\) for all \(n \geq 1\) and \(w(t) \in \partial f(t, u(t))\) on \(\mathbb{R}\).

For any \(\tau > 0\), we have

\[
\int_{-\tau}^{\tau} |u_n(t) - u(t)|^{p(t)}dt \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty,
\]

which shows that

\[
\lim_{n \rightarrow \infty} \int_{-\tau}^{\tau} |u_n(t)|^{p(t)}dt = \int_{-\tau}^{\tau} |u(t)|^{p(t)}dt.
\]

Choose \(n_0 \geq 1\) such that \([-\tau, \tau] \subseteq T_{n_0} = [-n_0b, n_0b]\) and for \(n \geq n_0,\) from (3.15), we derive

\[
\int_{-\tau}^{\tau} |u_n(t)|^{p(t)}dt \leq \int_{-\tau}^{\tau} |u_n(t)|^{p(t)}dt \leq \max\{\xi_3^{q^*}, \xi_3^{p^*}\} = \xi_3^0.
\]

Thus, from (3.18) and (3.19), one has

\[
\int_{-\tau}^{\tau} |u(t)|^{p(t)}dt \leq \xi_3^0,
\]

where \(\xi_3^0\) is a constant which is independent of \(n\).
By the arbitrariness of \( \tau > 0 \), from (3.20), we deduce that \( u \in L^p(t) \).

Let \( \theta \in C^0_c(\mathbb{R}, \mathbb{R}^N) \), then \( \supp \theta \subseteq T_n \) for large \( n \geq 1 \), which together with (3.15), performs integration by parts and Proposition 2.6, we have

\[
\left| \int_{\mathbb{R}} (u_n(t), \dot{\theta}(t)) \, dt \right| = \left| \int_{\mathbb{R}} (\theta(t), u_n(t)) \, dt \right| = \left| \int_{-nb}^{nb} (u_n(t), \theta(t)) \, dt \right| \leq \|u_n\|_{L^2_{loc}} \|\theta\|_{L^\infty_{2nb}} \leq \xi \|\theta\|_{L^\infty_{2nb}}.
\]

Note that \((u_n(t), \dot{\theta}(t)) \to (u(t), \dot{\theta}(t))\) in \( C_{loc}(\mathbb{R}, \mathbb{R}^N) \). By (3.16), we have

\[
|(u_n(t), \dot{\theta}(t))| \leq \|u_n\|_{L^\infty_{2nb}} |\dot{\theta}(t)| \leq \xi |\dot{\theta}(t)| \quad \text{a.e. on} \ T_n = [-nb, nb].
\]

Let

\[
\eta(t) = \begin{cases} 
\xi |\dot{\theta}(t)|, & \text{if } t \in \supp \theta; \\
0, & \text{if } t \notin \supp \theta.
\end{cases}
\]

Then, \( \eta \in L^1(\mathbb{R}) \) and \(|(u_n(t), \dot{\theta}(t))| \leq \eta(t) \) a.e. on \( \mathbb{R} \) for all \( n \). Therefore, by generalized Lebesgue-dominated convergence theorem, we see that

\[
\int_{\mathbb{R}} (u_n(t), \dot{\theta}(t)) \, dt \to \int_{\mathbb{R}} (u(t), \dot{\theta}(t)) \, dt
\]

and

\[
\left| \int_{\mathbb{R}} (u_n(t), \dot{\theta}(t)) \, dt \right| \leq \xi \|\theta\|_{L^\infty_{2nb}}.
\]

It follows from [5] that \( u \in W^{1,p(t)}_{2nb}(\mathbb{R}, \mathbb{R}^N) \).

Recall that \( w_n \to w \in L^p(t)_{loc}(\mathbb{R}, \mathbb{R}^N) \), which leads to

\[
\int_{\mathbb{R}} (w_n(t), \theta(t)) \, dt \to \int_{\mathbb{R}} (w(t), \theta(t)) \, dt. \tag{3.21}
\]

Furthermore, by (3.17), one has

\[
\int_{\mathbb{R}} (\dot{u}(t)|u_n(t)|^{(p(t)-2)} u_n(t), \theta(t)) \, dt \to \int_{\mathbb{R}} (\dot{u}(t)|u(t)|^{(p(t)-2)} u(t), \theta(t)) \, dt. \tag{3.22}
\]

Since \( u_n \) is a solution of problem (3.1), then we obtain \( \frac{d}{dt}(|\dot{u}_n(t)|^{p(t)-2} \dot{u}_n(t)) \in L^p(t)_{2nb} \). So it follows that \( |\dot{u}_n(t)|^{p(t)-2} \dot{u}_n(t) \in W^{1,p(t)}_{2nb} \) for all \( n \geq 1 \). By (3.22) and Proposition 2.1, let

\[
|\dot{u}_n| |^{p(t)-2} \dot{u}_n \to \nu \quad \text{in} \ W^{1,p(t)}_{loc}(\mathbb{R}, \mathbb{R}^N),
\]

then we have

\[
|\dot{u}_n| |^{p(t)-2} \dot{u}_n \to \nu \quad \text{in} \ C_{loc}(\mathbb{R}, \mathbb{R}^N). \tag{3.23}
\]

Define the continuous functional \( \lambda : \mathbb{R} \to \mathbb{R} \) by

\[
\lambda(u) = |u|^{p(t)-2} u , \quad \forall u \in C_{loc}(\mathbb{R}, \mathbb{R}^N).
\]

Then, from (3.23) that

\[
\dot{u}_n = \lambda^{-1}(|\dot{u}_n|^{p(t)-2} \dot{u}_n) \to \lambda^{-1}(\nu) \quad \text{in} \ L^1_{loc}(\mathbb{R}, \mathbb{R}^N),
\]

which shows that \( \lambda^{-1}(\nu) = \dot{u} \), so we have \( \nu = |\dot{u}|^{p(t)-2} \dot{u} \). Also, integration by parts yields that

\[
\int_{\mathbb{R}} (|\dot{u}(t)|^{p(t)-2} \dot{u}(t), \theta(t)) \, dt = -\int_{\mathbb{R}} \frac{d}{dt}(|\dot{u}(t)|^{p(t)-2} \dot{u}(t), \theta(t)) \, dt. \tag{3.24}
\]
Hence, it follows from (3.24), we obtain
\[
\int_{\mathbb{R}} (|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{\theta}(t))dt \rightarrow -\int_{\mathbb{R}} \left(\frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)), \theta(t)\right)dt.
\] (3.25)

For any \( n \geq 1 \), we have
\[
\int_{\mathbb{R}} (|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{\theta}(t))dt + \int_{\mathbb{R}} (a(t)|u(t)|^{p(t)-2}u(t), \theta(t))dt = \int_{\mathbb{R}} (w(t), \theta(t))dt.
\] (3.26)

Letting \( n \rightarrow \infty \), then from (3.21), (3.22), (3.25), and (3.26), we obtain
\[
-\int_{\mathbb{R}} \frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t))\dot{\theta}(t)dt + \int_{\mathbb{R}} a(t)|u(t)|^{p(t)-2}u(t)\theta(t)dt = \int_{\mathbb{R}} (w(t), \theta(t))dt.
\]

From the arbitrary of \( \theta \in C_{0}^{\infty}(\mathbb{R}, \mathbb{R}^{N}) \), we can deduce that
\[
-\frac{d}{dt}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) + a(t)|u(t)|^{p(t)-2}u(t) = w(t), \quad \text{a.e. } t \in \mathbb{R}
\]
and \( w \in L_{loc}^{p(t)}(\mathbb{R}, \mathbb{R}^{N}) \), \( w(t) \in \partial f(t, u(t)) \).

Next, we show that \( u(t) \neq 0 \). For all \( n \geq 1 \), from (3), we have
\[
a_{0} \int_{-n}^{n} |u_{n}(t)|^{p(t)}dt \leq \int_{-n}^{n} (w_{n}(t), u_{n}(t))dt.
\] (3.28)

Let
\[
h_{n}(t) = \begin{cases} (w_{n}(t), u_{n}(t)), & \text{if } u_{n}(t) \neq 0; \\ |u_{n}(t)|^{p(t)}, & \text{if } u_{n}(t) = 0. \end{cases}
\] (3.29)

Then, it follows from (3.28) and (3.29) that
\[
a_{0} \int_{-n}^{n} |u_{n}(t)|^{p(t)}dt \leq \int_{-n}^{n} h_{n}(t)|u_{n}(t)|^{p(t)}dt \leq \text{esssup}_{n}(h_{n}) \int_{-n}^{n} |u_{n}(t)|^{p(t)}dt.
\]

By virtue of hypothesis H(f): (iv), for a given \( \varepsilon > 0 \), we can find \( \delta > 0 \) such that
\[
\frac{(w_{n}(t), u_{n}(t))}{|u_{n}(t)|^{p(t)}} \leq \varepsilon, \quad |u_{n}(t)|^{p(t)} \leq \delta, \quad \forall t \in T_{n}, \omega \in \partial f(t, u_{n}).
\] (3.30)

If \( u = 0 \), then it follows from (3.17) that
\[
u_{n}(t) \rightarrow 0 \text{ in } C_{0}^{\infty}(\mathbb{R}, \mathbb{R}^{N})
\] (3.31)

and so we can find \( n_{0} \geq 1 \) such that
\[
|u_{n}(t)|^{p(t)} \leq \delta, \quad \forall t \in T_{n}, \ n \geq n_{0}.
\]
Thus for all \( n \geq n_0 \) and almost all \( t \in T_n \), we have \( h_n(t) \leq \varepsilon \) and so

\[
a_0 \leq \esssup_{T_n} h_n = \esssup_{\mathbb{R}} h_n \leq \varepsilon, \quad \forall n \geq n_0.
\]

In the above derivation process, we use the fact that \( u_n \) and \( \omega_n \) are extended by periodicity to all of \( \mathbb{R} \). Let \( \varepsilon > 0 \) reach a contradiction since \( 0 < a_0 \). This proves that \( u \neq 0 \). \( \square \)

**Proof of Theorem 3.2.** Similar to the proof of Theorem 3.1, we can prove Theorem 3.2, so we omit its course. \( \square \)

**Example 3.1.** Let \( p(t) = \frac{5}{2} + \frac{1}{2} \sin t \) for \( t \in \mathbb{R} \), and

\[
f(t, u) = a(t)|u|^p \ln(1 + |u|), \quad \forall (t, u) \in [-\pi, \pi] \times \mathbb{R}^N,
\]

where \( a(t) \) satisfies \( H(a), H(a)_1 \) with \( b = \pi \). It is evident that \( f \) is locally Lipschitz and

\[
\partial f(t, u) = a(t) \left[ 3|u| u \ln(1 + |u|) + \frac{|u|^2 u}{1 + |u|} \right].
\]

Then

\[
-f^0(t, u; -u) = \left[ 3 + \frac{|u|}{(1 + |u|) \ln(1 + |u|)} \right] f(t, u)
\]

\[
\geq \left( 3 + \frac{1}{1 + |u|} \right) f(t, u)
\]

\[
\geq \left( p^* + \frac{1}{1 + |u|} \right) f(t, u),
\]

which implies that \( f \) satisfies \( H(f)_1; (iii) \) with \( a = \beta = \nu = 1 \). Hence, from Theorem 3.1, problem (1.1) has a nontrivial homoclinic solution.

**Example 3.2.** Let \( p(t) = \frac{5}{2} + \frac{1}{2} \sin t \) for \( t \in \mathbb{R} \), and

\[
f(t, u) = a(t)|u|^{7/2} \ln(1 + |u|), \quad \forall (t, u) \in [-\pi, \pi] \times \mathbb{R}^N.
\]

Similar to Example 3.1, problem (1.1) has a nontrivial homoclinic solution by Theorem 3.2.

### 4 Nonperiodic \( p(t) \)-Laplacian inclusion system

In this section, we investigate the question of existence of homoclinic (to zero) solutions without periodic assumptions. Namely, we examine the following two types of problems:

\[
\frac{d}{dt}([u(t)]^{p(t)-2} \dot{u}(t)) - a(t)[u(t)]^{p(t)-2} u(t) \in \partial f(t, u(t)), \quad (4.1)
\]

and another problem

\[
\frac{d}{dt}([u(t)]^{p(t)-2} \dot{u}(t)) - a(t)[u(t)]^{p(t)-2} u(t) \in \partial f_1(t, u(t)) - \partial f_2(t, u(t)), \quad (4.2)
\]

where \( a \) satisfy the following hypothesis:

\[
H(a)_2 \quad \lim_{t \to +\infty} a(t) = +\infty.
\]

Note that \( a \in C(\mathbb{R}, \mathbb{R}^+) \) is coercive, then \( H'(a)_2 \) is satisfied, namely, \( H'(a)_2 \) there exists \( r > 0 \) such that

\[
\lim_{|p| \to +\infty} \text{meas}\{t \in B_r(y) : a(t) \leq b\} = 0 \quad \text{for any } b > 0.
\]
For the nonlinearity \( f \), we suppose the following hypotheses:

**H(f)2**

(ii) for almost all \( t \in \mathbb{R} \), there exist a function \( a(t) \in C(\mathbb{R}) \cap L^{p(t)}(\mathbb{R}) \) such that

\[
|\omega| \leq a(t)(1 + |u|^{a(t)-1}), \quad \forall u \in \mathbb{R}^N, \quad \omega \in \partial f(t, u(t)),
\]

where \( a \in L^{\infty}(\mathbb{R}), a^+ < y^+ < y(t) < y^+ < p^- \);

(ii') there exist two functions \( a_i(t) < a(t) (i = 1, 2) \) such that

\[
|\omega| \leq a_i(t)u_i(t) |u|^{a_i(t)-1}, \quad \forall t \in \mathbb{R}, \quad |u| \leq 1,
\]

and

\[
|\omega| \leq a_i(t)u_i(t) |u|^{a_i(t)-1}, \quad \forall t \in \mathbb{R}, \quad |u| \geq 1,
\]

where \( \omega \in \partial f(t, u(t)), a_i(t) \in C(\mathbb{R}, \mathbb{R}^+) \) and \( a_i(t) \in C(\mathbb{R}, \mathbb{R}^+) \cap L^{p_i(t)}(\mathbb{R}), a_i^+ < y^- < y(t) < y^+ < p^- \);

(iii) there exist constants \( M, a, \beta > 0 \) such that

\[
0 \leq \left( p^+ + \frac{1}{a + \beta |u|^p} \right) f(t, u) \leq -f^0(t, u; -u) \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N,
\]

where \( \nu \leq p^- \);

(iii') there exist constants \( \mu > p^+, M > 0 \) such that

\[
\mu f(t, u) \leq -f^0(t, u; -u) \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N;
\]

(iv) there exists a function \( q(t) > 0 \) such that

\[
\lim_{|u| \to 0} \frac{w}{|u|^{p^-}} = 0, \quad \lim_{|u| \to \infty} \inf_{|u|^{p(t)}} f(t, u) > 0, \quad \forall t \in \mathbb{R}, w \in \partial f(t, u),
\]

where \( p^+ < q^+ \).

**Theorem 4.1.** If hypotheses \( H(p), H(a), H(f) \) and \( H(f)_2: (ii), (iii), (iv) \) hold, then problem (4.1) has at least a nontrivial homoclinic solution.

**Theorem 4.2.** If hypotheses \( H(p), H(a), H(f) \) and \( H(f)_2: (ii), (iii'), (iv) \) hold, then problem (4.1) has at least a nontrivial homoclinic solution.

**Remark 4.3.** Note that \( H(f)_2: (ii) \) is weaker than [27, \( f_1 \)], by virtue of hypothesis \( H'(a)_2 \), we cannot immediately obtain [27, Theorem 1.2] for problem (4.1) with symmetrical condition \( f(t, u) = f(t, u) \) for all \( (t, u) \in \mathbb{R} \times \mathbb{R}^N \).

**Theorem 4.4.** If hypotheses \( H(p), H(a), H(a)_2, H(f), H(f)_2: (ii') \) and the following condition hold:

(v) there exist an open subset \( \Omega \subset \mathbb{R} \) and function \( \gamma(t) \) such that

\[
f(t, u) \geq \eta |u|^{p(t)}, \quad \forall (t, u) \in \Omega \times \mathbb{R}^N, \quad |u| \leq 1,
\]

where \( \gamma(t) \) satisfies \( H(p) \) and \( \gamma^+ < p^+, \eta > 0 \) is a constant.

Then problem (4.1) has at least a nontrivial homoclinic solution.

With regard to problem (4.2), in this situation, assume that \( p, a \) and \( f_1 \) satisfy all the conditions in Theorem 4.2 and \( f_2 \) satisfies the following conditions:
\( H(f) \)

(i) the function \( f_2(t, \cdot) : \mathbb{R} \to \mathbb{R} \) is measurable for all \( u \in \mathbb{R}^N \) and \( f_2(t, 0) = 0 \);

(ii) the function \( f_3(\cdot, u) : \mathbb{R}^N \to \mathbb{R} \) is locally Lipschitz for a.e. \( t \in \mathbb{R} \);

(iii) for almost all \( t \in \mathbb{R} \), there exists a function \( a(t) \in C(\mathbb{R}) \cap L^{p(t)}(\mathbb{R}) \) such that

\[
|\omega| \leq a(t)(1 + |u|^{p(t)-1}), \quad \forall u \in \mathbb{R}^N, \quad \omega \in \partial f_2(t, u(t)),
\]

where \( a \in L^{\infty}(\mathbb{R}), a^* < \gamma < \gamma(t) < \gamma^* < p^* \);

(iv) there exists a constant \( q \in [p^*, \mu) \) such that

\[
f^2(t)(t, u; u) \leq q f_2(t, u), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N.
\]

**Theorem 4.5.** If hypotheses \( H(p), H(a), H(f) \) hold, then problem (4.2) has at least a nontrivial homoclinic solution.

**Proof of Theorem 4.1.** In order to prove Theorems 4.1, 4.2 and 4.4, we first define a functional \( \varphi : W^{1,p(t)}_a \to \mathbb{R} \) as follows:

\[
\varphi(u) = \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}(t)|^{p(t)} + a(t)|u(t)|^{p(t)})dt - \int_{\mathbb{R}} f(t, u(t))dt.
\]

(4.3)

Using the same type of reasoning as the proof of Theorem 3.1, it is easy to verify that \( \varphi \) is the nonsmooth Lipschitz energy functional corresponding to problem (1.1).

Let \( \{u_n\}_{n \geq 1} \subseteq W^{1,p(t)}_a \) be such that

\[
|\varphi(u_n)| \leq M \quad \text{and} \quad m(u_n) \to 0 \quad \text{as} \quad n \to +\infty,
\]

(4.4)

where \( M > 0 \) is a constant. Our proofs are divided into two steps.

**Step 1:** \( u_n \to u \) in \( W^{1,p(t)}_a \).

Since \( \partial \varphi(u_n) \subseteq W^{-1,p(t)} \) is weakly compact and norm is weakly lower semicontinuous, according to Weierstrass theorem (Lemma 2.18). We can find \( u_n^* \in \partial \varphi(u_n) \) such that \( m(u_n) = \|u_n^*\| \) for \( n \geq 1 \).

Define nonlinear operator \( \mathcal{L} : W^{1,p(t)}_a \to (W^{1,p(t)}_a)^* \) as follows:

\[
\langle \mathcal{L}(u), v \rangle = \int_{\mathbb{R}} |\dot{u}(t)|^{p(t)-2}(\dot{u}(t), \dot{v}(t))dt, \quad \forall u, v \in W^{1,p(t)}_a.
\]

According to the literature [21], \( \mathcal{L} \) is monotonic and semicontinuous, so it is maximal monotone (see also [24]), therefore, \( u_n^* = \mathcal{L}(u_n) - w_n \) for \( n \geq 1 \), and \( w_n \in \partial \varphi(t, u_n) \), \( w_n \in L^{p'(t)} \), where \( 1/p'(t) + 1/p(t) = 1 \).

In another way, by the selection of sequence \( \{u_n\}_{n \geq 1} \subseteq W^{1,p(t)}_a \), we obtain

\[
|\langle u_n^*, u_n \rangle| \leq \varepsilon_n, \quad \varepsilon_n \downarrow 0,
\]

(4.5)

which yields that

\[
-\int_{\mathbb{R}} (|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)})dt + \int_{\mathbb{R}} \omega_n u_n dt \leq \varepsilon_n.
\]

(4.6)

Note that \( \langle w_n, -u_n \rangle \leq f^0(t, u_n; -u_n) \), using this fact and by (4.3), (4.4) and (4.6), one has

\[
p^* M \geq p^* \varphi(u_n) - \langle u_n^*, u_n \rangle
\]

\[
\geq \int_{\mathbb{R}} \frac{p^*}{p(t)}(|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)})dt - p^* \int_{\mathbb{R}} f(t, u_n)dt
\]

\[
- \int_{\mathbb{R}} (|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)})dt + \int_{\mathbb{R}} w_n u_n dt,
\]

(4.7)

\[
\geq \int_{\mathbb{R}} [-p^* f(t, u_n) - f^0(t, u_n; -u_n)]dt.
\]
For any \((t, u_n) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\})\), by H(f) 2: (iii), we have

\[
f(t, u_n(t)) \leq (\alpha + \beta|u_n(t)|^p)[ -p^* f(t, u_n) - f_0(t, u_n; -u_n)].
\]  

(4.8)

Hence, without loss of generality, we assume that \(|u_n| \geq 1\). Otherwise, \(|u_n|\) is bounded. It follows from (4.3), (4.4), (4.7), (4.8), Propositions 2.4 (ii) and 2.8 (i) that

\[
\frac{1}{p^*}\|u_n\|^p \leq \int R \frac{1}{p(t)}(|u_n|^{p(t)} + a(t)|u_n|^{p(t)})dt
\]

\[
= \varphi(u_n) + \int \varphi(t, u_n(t))dt
\]

\[
\leq \varphi(u_n) + \int \varphi(t, u_n(t))dt
\]

\[
\leq M_1 + (\alpha + \beta\|u_n\|_{\infty}^p) \int \varphi(t, u_n(t))dt
\]

\[
\leq M_1 + p^* M_1(\alpha + \beta\|u_n\|_{\infty}^p)
\]

\[
\leq M_1 + p^* M_1(\alpha + \beta\|u_n\|_{\infty}^p)
\]

\[
(4.9)
\]

Note that \(p < p^*\), in light of (4.9), it is easy to show that \(|u_n|\) is bounded. So passing to a subsequence if necessary, it can be assumed that \(u_n \rightarrow u\) in \(W_0^{1, p(t)}\), \(u_n \rightarrow u\) in \(L^{p(t)}\). Because (4.5), then

\[
\langle L(u_n), u_n - u \rangle - \int R w_n(u_n - u)dt \leq \varepsilon_n, \quad \forall n \geq 1.
\]

By virtue of the fact that \(|w_n|_{\infty, 1}\) is bounded in \(L^{p(t)}\), then \(\limsup_{n \rightarrow \infty} \langle L(u_n), u_n - u \rangle \leq 0\). By Proposition 2.9, we deduce that \(u_n \rightarrow u\) in \(W_0^{1, p(t)}\).

**Step 2:** \(\varphi\) satisfies nonsmooth mountain pass theorem.

Let \(\varepsilon > 0\) be small enough, in view of hypotheses H(f) 2: (ii), (iv), one has

\[
f(t, u) \leq \varepsilon |u|^p + c(\varepsilon)|u|^{p(t)}, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N.
\]

Let \(|u| = \rho\) be small enough, then by Proposition 2.4, we have

\[
\varphi(u) \geq \frac{1}{p^*} \int R (|u|^{p(t)} + a(t)|u|^{p(t)})dt - \varepsilon \int R |u|^{p'} dt - c(\varepsilon) \int R |u|^{p(t)}dt
\]

\[
\geq \frac{1}{p^*}\|u\|^{p^*} - \varepsilon c_{\rho}^{p^*}\|u\|^{p^*} - c(\varepsilon)c_{\rho}^{p^*}\|u\|^{p^*}
\]

(4.10)

for the above \(\varepsilon\), let \(\varepsilon c_{\rho}^{p^*} < \frac{1}{2p^*}\), where \(c_{\rho}(c_{\rho}^{p'})\) is the embedding constant from \(W_0^{1, p(t)}\) to \(L^{p^*}(L^p)\). Then, from (4.10), we obtain

\[
\varphi(u) \geq \frac{1}{2p^*}\|u\|^{p^*} - c(\varepsilon)c_{\rho}^{p^*}\|u\|^{p^*}
\]

Note that \(p^* < \alpha\), then there exists a constant \(r > 0\) such that \(\varphi(u) \geq r\) when \(|u| = \rho\) for \(\rho\) small enough.

By H(f) 2: (ii), (iv), we see that

\[
f(t, u) \geq |u|^{p(t)}, \quad \forall t \in \mathbb{R}, \quad |u| \geq M
\]

(4.11)

and

\[
|f(t, u)| \leq c_0 a(t), \quad \forall t \in \mathbb{R}, \quad |u| < M,
\]

(4.12)

respectively. Thus, for any \(u \in W_0^{1, p(t)} \setminus \{0\}\) and \(\sigma > 1\), from (4.3), (4.11) and (4.12), we obtain

\[
\varphi(\sigma u) = \int R \frac{1}{p(t)}(|\sigma u|^{p(t)} + a(t)|\sigma u|^{p(t)})dt - \int R f(t, \sigma u)dt
\]

\[
\leq \sigma^{p^*} \int R \frac{1}{p(t)}(|\sigma u|^{p(t)} + a(t)|u|^{p(t)})dt - \sigma^{q^*} \int R |u|^{p(t)}dt + c_0 \int R a(t)dt
\]
Since $p^* < q^-$, it is easy to show that $\varphi(\sigma u_n) \to -\infty$ as $\sigma \to +\infty$. Because $\varphi(0) = 0$ and $\varphi$ satisfies Lemma 2.19, hence there exists at least one nontrivial critical point, that is, system (4.1) has at least one homoclinic orbit. □

**Proof of Theorem 4.2.** Applying the proof of Theorem 4.1, we deduce that $\varphi$ is the nonsmooth Lipschitz energy functional corresponding to problem (1.1).

Let $\{u_{n}\}_{n \geq 1} \subseteq W^{1,p(t)}_0$ be such that

$$|\varphi(u_n)| \leq M_1 \quad \text{and} \quad m(u_n) \to 0 \quad \text{as} \quad n \to +\infty,$$

(4.13)

where $M_1 > 0$ as given in (4.4). As $\partial \varphi(u_n) \subseteq W^{-1,p(t)}_0$ is weakly compact and the norm is weakly lower semicontinuous, by Lemma 2.18, we can choose $u^*_n \in \partial \varphi(u_n)$ such that $m(u_n) = \|u^*_n\|$ for $n \geq 1$.

Define nonlinear operator $\mathcal{L} : W^{1,p(t)}_0 \to (W^{1,p(t)}_0)^*$ as follows:

$$\langle \mathcal{L}(u), v \rangle = \int_{\mathbb{R}} |\dot{u}(t)|^{p(t)-2}(\dot{u}(t), \dot{v}(t))dt, \quad \forall u, v \in W^{1,p(t)}_0.$$  

According to the literature [21], $\mathcal{L}$ is monotonic and semicontinuous, so it is maximal monotone (see also [24]), therefore $u^*_n = \mathcal{L}(u_n) - w_n$ for $n \geq 1$, and $w_n \in \partial f(t, u_n)$, $w_n \in L^p(t)$, where $1/p'(t) + 1/p(t) = 1$.

In another way, by the selection of sequence $\{u_{n}\}_{n \geq 1} \subseteq W^{1,p(t)}_0$, we obtain

$$|\langle u^*_n, u_n \rangle| \leq \varepsilon_n, \quad \varepsilon_n \downarrow 0,$$

(4.14)

which yields

$$-p^\ast \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)})dt + \int_{\mathbb{R}} \omega_n u_n dt \leq \varepsilon_n.$$  

(4.15)

Note that $\langle w_n, -u_n \rangle \leq f^0(t, u_n; -u_n)$, using this fact and by (4.3), (4.13) and (4.15), one has

$$\mu M_1 + \varepsilon_n \geq \mu \varphi(u_n) + \langle u^*_n, -u_n \rangle$$

$$= \mu \left[ \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)})dt - \int_{\mathbb{R}} f(t, u_n) dt \right] + \langle u^*_n, -u_n \rangle$$

$$= \mu \left[ \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)})dt - \int_{\mathbb{R}} f(t, u_n) dt \right] - p^\ast \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)})dt$$

$$+ \int_{\mathbb{R}} \omega_n u_n dt$$

$$\geq (\mu - p^\ast) \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)})dt - \int_{\mathbb{R}} (\mu f(t, u_n) + f^0(t, u_n; -u_n)) dt,$$

which leads to

$$(\mu - p^\ast) \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)})dt \leq \mu M_1 + \varepsilon_n + \int_{\mathbb{R}} (\mu f(t, u_n) + f^0(t, u_n; -u_n)) dt.$$  

(4.16)

By H(f)$_2$: (ii), there exist two functions $a_0(t), b_0(t) \in L^\infty(\mathbb{R})$, such that

$$|f(t, u_0(t))| \leq a_0(t) + b_0(t)|u_0(t)|^{p(t)}.$$  

Recall that $u \mapsto f(t, u)$ is local Lipschitz, there exists $c(t) \in L^\infty(\mathbb{R})$, such that

$$f^0(t, u_n(t); -u_n(t)) \leq c(t)|u_n(t)|, \quad \forall u \in \mathbb{R}^N.$$  

Thus, there exist $a_2 > 0, b_2 \in L^\infty(\mathbb{R})$, such that

$$\mu f(t, u_n(t)) + f^0(t, u_n(t); -u_n(t)) \leq a_2 b_2(t), \quad \forall t \in \mathbb{R}, \quad |u_n| < M.$$
which implies that there exists a constant $C_3 > 0$ such that

$$\int_{\{|u_n| < M\}} (p\phi(f(t, u_n(t))) + f^0(t, u_n(t); -u_n(t)))dt \leq C_3. \quad (4.17)$$

Combining with (4.17) and $H(f)_2$: (iii'), we can deduce

$$\int_{\{|u_n| < M\}} (p\phi(f(t, u_n(t))) + f^0(t, u_n(t); -u_n(t)))dt + \int_{\{|u_n| < M\}} (p\phi(f(t, u_n(t))) + f^0(t, u_n(t); -u_n(t)))dt \leq C_3. \quad (4.18)$$

Hence, from (4.16) and (4.18), we obtain

$$\langle \mu - p^* \rangle \int_{\mathbb{R}} \frac{1}{p(t)}(|\tilde{u}_n|^p(t)) + a(t)|u_n|^p(t)dt \leq C_4. \quad (4.19)$$

Note that $\mu > p^*$, it follows from (4.19) that $\{|u_n| \leq 1\} \subseteq W^{l,p(t)}$ is bounded. So passing to a subsequence if necessary, it can be assumed that $u_n \to u$ in $W^{l,p(t)}$, $u_n \to u$ in $L^p(t)$. Because (4.14), then

$$\langle \mathcal{L}(u_n), u_n - u \rangle - \int_{\mathbb{R}} w_n(u_n - u)dt \leq \varepsilon_n, \quad \forall n \geq 1.$$ 

By virtue of the fact that $\{w_n\}_{n \geq 1}$ is bounded in $L^p(t)$, then

$$\limsup_{n \to \infty} \langle \mathcal{L}(u_n), u_n - u \rangle \leq 0.$$ 

By Proposition 2.9, we obtain $u_n \to u$ in $W^{l,p(t)}$.

Next, we need only to verify that $\varphi$ satisfies nonsmooth mountain pass theorem, i.e., Lemma 2.19, the proof is similar to Theorem 4.1, so we omitted its course. \hfill \Box

**Proof of Theorem 4.4.** Consider the functional $\varphi : W^{l,p(t)}_a \to \mathbb{R}$ be defined as (4.3), i.e.,

$$\varphi(u) = \int_{\mathbb{R}} \frac{1}{p(t)}(|\tilde{u}(t)|^p(t) + a(t)|u(t)|^p(t))dt - \int_{\mathbb{R}} f(t, u(t))dt = \tilde{\varphi}(u) - \int_{\mathbb{R}} f(t, u(t))dt. \quad (4.20)$$

First, we prove that $\varphi$ is the nonsmooth Lipschitz energy functional corresponding to problem (1.1). However, similar arguments as Theorem 4.1, it is easy to see that $\tilde{\varphi}$ is the locally Lipschitz functional. So we only need to show $\int_{\mathbb{R}} f(t, u(t))dt$ is the locally Lipschitz functional.

Let $\Omega \subset \mathbb{R}$, from $H(f)_2$: (ii') and Lemma 2.16, for all $u_1, u_2 \in W^{l,p(t)}_a(\Omega, \mathbb{R}^N)$, one has

$$|f(t, u_1) - f(t, u_2)| \leq a(t)|u_1|^{p(t)-1}|u_1 - u_2|, \quad i = 1, 2 \quad (4.21)$$

and

$$a(t)|u_1|^{p(t)-1} \leq \frac{y(t) - a(t))a(t)|u_1|^{p(t)-1}}{y(t) - 1} + a(t) - \frac{1}{y(t) - 1}|\tilde{u}|^{p(t)-1}, \quad i = 1, 2,$$

which yields that

$$\langle a(t)|\tilde{u}|^{p(t)-1} \rangle \leq C_0\tilde{a}(t) + C_0|\tilde{u}|^{p(t)}, \quad i = 1, 2, \quad (4.22)$$

where $\tilde{u} = su_1 + (1 - s)u_2$, $s \in (0, 1)$, $C_8, C_9 > 0$. Then, from (4.21), (4.22) and Hölder inequality, we obtain
Hence, \( \phi \) is the nonsmooth Lipschitz energy functional corresponding to problem (1.1).

Next, our proofs are divided into three steps.

**Step 1: \( \phi \) is coercive.**

It follows from (ii') that

\[
\phi(u) = \int_{\mathbb{R}} \frac{1}{p(t)} (|\dot{u}|^{p(t)} + a(t)|u|^{p(t)}) \, dt - \int_{\mathbb{R}} f(t, u) \, dt \\
\geq \frac{1}{p^*} \|u\|^{p^*} - \int_{\{t|u| \leq 1\}} f(t, u) \, dt - \int_{\{t|u| > 1\}} f(t, u) \, dt \\
\geq \frac{1}{p^*} \|u\|^{p^*} - \int_{\{t|u| \leq 1\}} a_1(t)|u|^{p_1(t)} \, dt - \int_{\{t|u| > 1\}} a_2(t)|u|^{p_2(t)} \, dt \\
\geq \frac{1}{p^*} \|u\|^{p^*} - C_{1,1} \int_{\{t|u| \leq 1\}} b^{\frac{\theta(t)}{p^*}} a^{\frac{\theta(t)}{p^*}} |u|^{q_1(t)} \, dt - C_{1,2} \int_{\{t|u| > 1\}} b^{\frac{\theta(t)}{p^*}} a^{\frac{\theta(t)}{p^*}} |u|^{q_2(t)} \, dt \\
\geq \frac{1}{p^*} \|u\|^{p^*} - 2C_{1,1} b^{\frac{\theta(t)}{p^*}} |L^{q_1(t)}| u^{\tilde{a}_1(t),a} - 2C_{1,2} b^{\frac{\theta(t)}{p^*}} |L^{q_2(t)}| u^{\tilde{a}_2(t),a} \\
\geq \frac{1}{p^*} \|u\|^{p^*} - 2C_{1,1} b^{\frac{\theta(t)}{p^*}} \|u\|^\tilde{a}_1 - 2C_{1,2} b^{\frac{\theta(t)}{p^*}} \|u\|^\tilde{a}_2,
\]

where \( C_{1,10} = \sup_{t \in \mathbb{R}} a_1(t) (i = 1, 2), \ b(t) = a(t)^{-1}, \ 1/ \gamma(t) + a(t)/p(t) = 1, \) and \( \tilde{a}_i = [ \tilde{a}_i' , \tilde{a}_i''] \) is a constant.

From H(f)2 (ii'), we know \( \tilde{a}_1' < \tilde{a}'_2 < p^* \), so \( \tilde{a}_1 < p^* \). Hence, by H(a)2, we have \( \phi(u) \rightarrow +\infty \) as \( |u| \rightarrow +\infty \).

Thus, \( \phi \) is bounded below.

**Step 2: \( \phi \) is sequence weakly lower semicontinuous.**

Without loss of generality, we assume that \( u_n \rightharpoonup u \) in \( W^{L,p(t)}_a \), so from Proposition 2.8 (ii), we have \( u_n \rightharpoonup u \) in \( L^{\infty}(\mathbb{R}) \), which implies that

\[
u_n \rightarrow u \quad \text{and} \quad f(t, u_n(t)) \rightarrow f(t, u(t)), \quad \forall t \in \mathbb{R}.
\]

By Fatou lemma, we have

\[
\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \int_{\mathbb{R}} f(t, u_n(t)) \, dt \leq \int_{\mathbb{R}} f(t, u(t)) \, dt.
\]

Hence, from (4.3) and (4.25), we obtain

\[
\liminf_{n \to \infty} \phi(u_n) \geq \liminf_{n \to \infty} \int_{\mathbb{R}} \frac{1}{p(t)} (|\dot{u}_n|^{p(t)} + a(t)|u_n|^{p(t)}) \, dt - \limsup_{n \to \infty} \int_{\mathbb{R}} f(t, u_n(t)) \, dt \\
\geq \int_{\mathbb{R}} \frac{1}{p(t)} (|\dot{u}|^{p(t)} + a(t)|u|^{p(t)}) \, dt - \int_{\mathbb{R}} f(t, u(t)) \, dt \\
= \phi(u),
\]

which shows that \( \phi \) is sequence weakly lower semicontinuous.
Using Lemma 2.18, there is a global minimum point \( u_0 \in W_a^{1,p(t)} \) such that
\[
\varphi(u_0) = \min_{u \in W_a^{1,p(t)}} \varphi(u).
\]

**Step 3: \( \varphi(u_0) < 0 \).**
Let \( u_0 \in (W_a^{1,p(t)} \cap W_a^{1,p(t)} \setminus \{0\} with \|u_0\| = 1, by (4.3) and condition (v), for \( 0 < s < 1 \), we obtain
\[
\varphi(su_0) = \frac{1}{p(t)} \int_{\mathbb{R}} (|su_0|^{p(t)} + a(t)|su_0|^{p(t)})dt - \int_{\mathbb{R}} f(t, su_0(t))dt
\leq \frac{s^p}{p} - \int_{\Omega} f(t, su_0(t))dt
\leq \frac{s^p}{p} - \eta s^r \int_{\Omega} |u_0(t)|^{p(t)}dt.
\]

Note that \( 1 < y^* < p^- \), it is easy to show that \( \varphi(su_0) < 0 \) as \( s > 0 \) small enough. 

**Proof of Theorem 4.5.** Define a functional \( \psi : W_a^{1,p(t)} \to \mathbb{R} \) as follows:
\[
\psi(u) = \int_{\mathbb{R}} \frac{1}{p(t)} (|u(t)|^{p(t)} + a(t)|u(t)|^{p(t)})dt - \int_{\mathbb{R}} f(t, u(t))dt + \int_{\mathbb{R}} f(t, u(t))dt.
\] (4.27)

Arguments as in proof of Theorems 4.1 and 4.5, we can easily obtain \( \psi \) as the nonsmooth Lipschitz energy functional corresponding to problem (4.2).

Let \( \{u_n\}_{n \geq 1} \subseteq W_a^{1,p(t)} \) be such that
\[
|\psi(u_n)| \leq M_2 \quad \text{and} \quad m(u_n) \to 0 \quad \text{as} \quad n \to +\infty,
\] (4.28)
where \( M_2 > 0 \) is a constant. Because \( \partial \psi(u_n) \subseteq W_a^{1,p(t)} \) is weakly compact and the norm is weakly lower semicontinuous. By Lemma 2.18, we can choose \( u_n^* \in \partial \psi(u_n) \) such that \( m(u_n) = \|u_n^*\| \) for \( n \geq 1 \).

Define nonlinear operator \( \mathcal{L} : W_a^{1,p(t)} \to (W_a^{1,p(t)})^* \) as follows:
\[
\langle \mathcal{L}(u), v \rangle = \int_{\mathbb{R}} (|u(t)|^{p(t)} - 2 \langle \dot{u}(t), \dot{v}(t) \rangle)dt, \quad \forall u, v \in W_a^{1,p(t)}.
\]

According to the literature [21], \( \mathcal{L} \) is monotonic and semicontinuous, so it is maximal monotone (see also [24]), therefore, \( u_n^* = \mathcal{L}(u_n) - w_n + w_n^- \) for \( n \geq 1 \), and \( w_n \in \partial f_1(t, u_n), w_n^- \in \partial f_2(t, u_n), w_n^0, w_n^- \in L^p(t) \), where
\[
1/p(t) + 1/p(t) = 1.
\]

In another way, by the selection of sequence \( \{u_n\}_{n \geq 1} \subseteq W_a^{1,p(t)} \), we obtain
\[
|\langle u_n^*, u_n \rangle| \leq \varepsilon_n, \quad \varepsilon_n \downarrow 0.
\] (4.29)

Then, it follows from (4.27), (4.28), (4.29), H(f)\_2; (iii) and H(f)\_3; (iv), we can show that
\[
M_2 + \frac{\varepsilon_n}{\mu} \geq \psi(u_n) - \frac{1}{\mu} \langle u_n^*, u_n \rangle
+ \int_{\mathbb{R}} \left( \frac{1}{p(t)} - \frac{1}{\mu} \right) (|\dot{u}_n(t)|^{p(t)} + a(t)|u_n(t)|^{p(t)})dt
- \int_{\mathbb{R}} f_1(t, u_n(t))dt + \int_{\mathbb{R}} f_2(t, u_n(t))dt + \frac{1}{\mu} \int_{\mathbb{R}} (|w_n^1 - w_n^-|_{p(t)} - |w_n^1, u_n|_{p(t)})dt
\geq \int_{\mathbb{R}} \frac{1}{p(t)} (|\dot{u}_n(t)|^{p(t)} + a(t)|u_n(t)|^{p(t)})dt
+ \frac{1}{\mu} \int_{\mathbb{R}} [-f_1^0(t, u_n; u_n) - \mu f_1(t, u_n)|dt + \frac{1}{\mu} \int_{\mathbb{R}} [\mu f_2(t, u_n) - f_2^0(t, u_n; u_n)]dt
\geq \left( \frac{1}{p(t)} - \frac{1}{\mu} \right) \int_{\mathbb{R}} (|\dot{u}_n(t)|^{p(t)} + a(t)|u_n(t)|^{p(t)})dt.
\] (4.30)
Note that \( \mu > p^* \), by virtue of (4.30) and Proposition 2.4, we have \( \{ u_n \}_{n=1} \subseteq W_{a}^{1,p(t)} \) is bounded, so we assume that \( u_n \rightharpoonup u \) in \( W_{a}^{1,p(t)} \) and \( u_n \rightarrow u \) in \( L^{p(t)} \).

Thanks to (4.29), thus

\[
\langle L(u_n), u_n - u \rangle - \int_{\mathbb{R}} w_1^*(u_n - u) dt + \int_{\mathbb{R}} w_2^*(u_n - u) dt \leq \varepsilon_n, \quad \forall n \geq 1. \tag{4.31}
\]

Recall that \( w_1^1, w_2^2 \in L^{p(t)}(\mathbb{R}, \mathbb{R}^N) \), so we have

\[
\int_{\mathbb{R}} w_1^1(u_n - u) dt \rightarrow 0, \quad \int_{\mathbb{R}} w_2^2(u_n - u) dt \rightarrow 0, \tag{4.32}
\]
as \( n \rightarrow \infty \). Then, by (4.31) and (4.32), we obtain

\[
\limsup_{n \rightarrow \infty} \langle L(u_n), u_n - u \rangle \leq 0. \tag{4.33}
\]

Combining with (4.33) and Proposition 2.9, we deduce that \( u_n \rightarrow u \) in \( W_{a}^{1,p(t)} \). So \( \varphi \) satisfies PS condition.

**Step 2**: \( \psi \) satisfies nonsmooth mountain pass theorem.

For any \( \varepsilon > 0 \), by H(f)2: (ii), (iv), one has

\[
f_1(t, u) \leq c|u|^p + c(\varepsilon)|u|^{p(t)}, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^{N}. \tag{4.34}
\]

Choose \( \|u\| = \rho \) is small enough, from (4.27), (4.34) and Proposition 2.4, we obtain

\[
\psi(u) \geq \frac{1}{p^*} \int_{\mathbb{R}} (|u|^{p(t)} + \alpha(t)|u|^{p(t)}) dt - \varepsilon \int_{\mathbb{R}} |u|^p dt - c(\varepsilon) \int_{\mathbb{R}} |u|^{p(t)} dt \\
\geq \frac{1}{p^*} |u|^p - \varepsilon c_p^p \|u\|^p - c(\varepsilon)c_a^p \|u\|^p
\]

for \( \varepsilon > 0 \), let \( \varepsilon c_p^p < \frac{1}{2p^*} \), where \( c_p(c_a) \) is the embedding constant from \( W_{a}^{1,p(t)} \) to \( L^{p(L^p)} \). Then,

\[
\psi(u) \geq \frac{1}{2p^*} |u|^p - c(\varepsilon)c_a^p \|u\|^p.
\]

Note that \( p^* < a^* \), there exists a constant \( r > 0 \) such that \( \varphi(u) \geq r \) as \( \|u\| = \rho \), when \( \rho \) is small enough.

As in [41], it follows from H(f)2: (iiif) and H(f)3: (iv) that

\[
f_1(t, au) \geq \sigma^p f_1(t, u), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^{N}. \tag{4.35}
\]

and

\[
f_2(t, au) \leq \sigma^p f_2(t, u), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^{N}. \tag{4.36}
\]

Let \( \omega \in W_{a}^{1,p(t)} \) be such that

\[
|\omega(t)| = \begin{cases} 
1, & \text{as } |t| \leq 1; \\
0, & \text{as } |t| \geq 2; \\
\leq 1, & \text{as } |t| \in (1, 2].
\end{cases} \tag{4.37}
\]

In view of (4.36) and Proposition 2.8 (i), one has

\[
\int_{-2}^{2} f_2(t, \omega) dt = \int_{\{t \in [-2, 2] : |\omega| > 1\}} f_2(t, \omega) dt + \int_{\{t \in [-2, 2] : |\omega| \leq 1\}} f_2(t, \omega) dt \\
\leq \int_{\{t \in [-2, 2] : |\omega| > 1\}} f_2(t, \omega) |\omega|^p dt + \int_{-2}^{2} \max_{|\omega| \leq 1} f_2(t, \omega) |\omega|^p dt
\]
\[
\frac{1}{p(t)} \left| \frac{d}{dt} (|u|^{p(t)-2} u) \right| \leq \max_{|\omega|=1} f_2(t, \omega) dt + \int_{-2}^{2} \max_{|\omega|=1} f_2(t, \omega) dt,
\]
}\leq \kappa \|\omega\|^{p(t)} \int_{-2}^{2} \max_{|\omega|=1} f_2(t, \omega) dt + \int_{-2}^{2} \max_{|\omega|=1} f_2(t, \omega) dt = M_3 \|\omega\|^{p(t)} + M_4,
\]

where
\[
M_3 = \kappa \int_{-2}^{2} \max_{|\omega|=1} f_2(t, \omega) dt, \quad M_4 = \int_{-2}^{2} \max_{|\omega|=1} f_2(t, \omega) dt.
\]

When \( \sigma > 1 \), by (4.35), we have
\[
\int_{-2}^{2} f_1(t, \sigma \omega(t)) dt \geq \sigma^\mu \int_{-2}^{2} f_1(t, \omega(t)) dt = m \sigma^\mu,
\]
where \( m = \int_{-1}^{1} f(t, \omega) dt > 0 \). Thus, it follows from (4.27), (4.36), (4.37), (4.38), (4.39) and Proposition 2.4 (ii), we derive
\[
\psi(\sigma \omega) = \int_{\mathbb{R}} \left( (|\sigma \omega|^{p(t)}) + \alpha(t) |\sigma \omega|^{p(t)} \right) dt + \int_{\mathbb{R}} (f_2(t, \sigma \omega(t)) - f_1(t, \sigma \omega(t))) dt
\]
\[
\leq \frac{\sigma^\mu}{p} \int_{\mathbb{R}} (|\omega|^{p(t)}) dt + \int_{-2}^{2} f_2(t, \sigma \omega) dt - \int_{-2}^{2} f_1(t, \sigma \omega) dt
\]\[
\leq \frac{\sigma^\mu}{p} \|\omega\|^{p(t)} + M_3 \sigma \|\omega\|^{p(t)} + M_4 \sigma - m \sigma^\mu.
\]

Since \( \mu > \varphi > p^\ast, m > 0 \), from (4.40), we can choose \( \sigma_0 > 1 \) such that \( e = \sigma_0 \mu \in W_0^{1, p(t)} \) and \( \varphi(e) < 0 \). Hence, from Lemma 2.19, there exists at least one nontrivial critical point, that is, system (4.2) has at least one homoclinic orbit.

**Example 4.1.** Let \( p(t) = \frac{3}{2} + \frac{2}{n} \arctan |t| \) for \( t \in \mathbb{R} \), and
\[
f(t, u) = a(t)(1 + a(t)^{-1}) |u|^{\alpha(t)} \ln(1 + |u|), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N,
\]
where \( a(t) \) satisfies \( H_a, H(\alpha) \). It is evident that \( f \) is locally Lipschitz and
\[
\partial f(t, u) = a(t)(1 + a(t)^{-1}) \left( \frac{5}{2} |u|^{1/2} \ln(1 + |u|) + \frac{|u|^{3/2}}{1 + |u|} \right).
\]

As
\[
-f^0(t, u; -u) = \left[ \frac{5}{2} + \frac{|u|}{(1 + |u|) \ln(1 + |u|)} \right] f(t, u)
\]
\[
\geq \left( \frac{5}{2} + \frac{1}{1 + |u|} \right) f(t, u)
\]
\[
> \left( p^\ast + \frac{1}{1 + |u|} \right) f(t, u),
\]

\( f \) satisfies \( H(f) \) (iii) with \( \alpha = \beta = \nu = 1 \). Thus, we can show that \( f \) satisfies the hypothesis of Theorem 4.1.

Moreover, it is similar to obtain that \( f \) satisfies the hypothesis of Theorem 4.2 with \( \mu = \frac{5}{2} \).
Example 4.2. Let $p(t) = 5 + \frac{1}{1+t^2}$ for $t \in \mathbb{R}$, and

$$f(t, u) = a(t)\{2|\sin t| + 2|2u|^{4|\sin t|+4} + |u|^{2|\sin t|+2}\}, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N,$$

where $a(t) = 1 + t^2$ satisfies $H(a), H(a)_{\infty}$. Obviously, $f$ is locally Lipschitz and

$$\partial f(t, u) = a(t)\{2|\sin t| + 2|2u|^{4|\sin t|+4} + |u|^{2|\sin t|+2}\}.$$

Since

$$|\partial f(t, u)| \leq \begin{cases} \frac{3(2|\sin t| + 2)}{1 + t^2}|u|^{2|\sin t|+1}, & |u| \leq 1; \\ \frac{3(4|\sin t| + 4)}{2(1 + t^2)}|u|^{4|\sin t|+3}, & |u| \geq 1. \end{cases}$$

Then $f$ satisfies $H(f)_{\infty}$ with

$$a_1(t) = 2|\sin t| + 2, \quad a_2(t) = 4|\sin t| + 4, \quad a_3(t) = \frac{3}{1 + t^2}, \quad a_4(t) = \frac{3}{2(1 + t^2)}.$$

Let $\Omega = (-2, -2), \gamma(t) = 2|\sin t| + 2$, one has

$$f(t, u) \geq \frac{1}{5}|u|^{2|\sin t|+2}, \quad \forall |u| \leq 1.$$

Hence, from Theorem 4.4, problem (4.1) has at least a nontrivial homoclinic solution.

Example 4.3. Let $p(t) = \frac{3}{2} + \frac{2}{\pi} \arctan t$ for $t \in \mathbb{R}$, $f = f_1 - f_2$ and

$$f_1(t, u) = a_1(t)(1 + a_2(t))^{-1}|u|^{1/2} \log(1 + |u|), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N,$$

$$f_2(t, u) = a_2(t)|\sin t|^{2|u| + |u|^3}, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N,$$

where $a_1(t)(i = 1, 2)$ satisfies $H(a), H(a)_{\infty}$ and $H(f)_{\infty}$ $(ii'), (v)$, respectively. It is visible that $f$ is locally Lipschitz, $\partial f \subset \partial f_1 - \partial f_2$ and

$$f_2'(t, u; u) = a_2(t)[2|\sin t|u|^{2} + 3|u|^3] \geq 3f_2(t, u),$$

which implies, $f_2$ satisfies $H(f)_{\infty}$ $(iv)$ with $q = 3$ and $\mu = \frac{3}{2}$. It is easy to verify that $f$ satisfies the hypothesis of Theorem 4.5.

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References


