Abstract: In this paper, we consider stochastic dynamics of a two-dimensional stochastic differential equation with additive noise. When the strength of the noise is zero, this equation undergoes a Bautin bifurcation. We obtain the main conclusions including the existence and uniqueness of the solution, synchronization of system and property of the random equilibrium, where going through some processes like deducing the stationary probability density of the equation and calculating the Lyapunov exponent. For better understanding of the effect under noise, we make a clear comparison between the stochastic system and the deterministic one and make precise numerical simulations to show the slight changes at Bautin bifurcation point. Furthermore, we take a real model as an example to present the application of our theoretical results.

Keywords: stochastic dynamics, random equilibrium, the stationary probability density, Lyapunov exponent, Bautin bifurcation

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1 Introduction

Stochastic bifurcation is a sublimation and generalization based on deterministic bifurcation theory in the presence of noise. Its behavior can be widely found in physics and economics, and studying stochastic dynamics can be used to correct error caused by noise of the machine or make a risk evaluation or prediction in their related fields. Compared with deterministic bifurcation, apart from concerning about topological properties of the trajectories before and after the critical values of parameters, stochastic bifurcations focus more on the qualitative changes of the stationary probability density, which is the essential difference generated by noises. For this reason, there are only a few pioneer studies about stochastic bifurcation.

In 2002, Berglund and Gentz [2] considered pitchfork bifurcation with additive noise and got accurate results on both the behavior of the individual paths and the probability for exceptional paths. Afterwards, Callaway et al. [3] continued to further study pitchfork bifurcation for random dynamical system (RDS) through the method of dichotomy spectrum. In 2013, Huang and Chen [7] investigated a classical Rayleigh-van der Pol equations by means of asymptotic expansions and discussed its stochastic bifurcation under small noise. In 2018, Doan et al. [4] described the properties of the normal form of a Hopf bifurcation with additive white noise, showing three dynamical phases in the full parameter space. Besides the theoretical analysis, some numerical simulations about stability and synchronization under different noises have been done [6,16] over the past decade.
These works are all about codimension-one stochastic bifurcations and even some results observed by numerically or experimentally. In sharp contrast to achievements on bifurcation phenomena of deterministic dynamical systems \[11,12,17\], stochastic bifurcations are still on the incipient stage whether theory or applications. Even if in the deterministic system, it is harder to prove the higher codimension bifurcations \[15,18\], including more complex deduction, huge amount of computation, more transformation techniques, and so on.

Motivated by the aforementioned researches, our work is devoted to exploring the dynamics of a codimension-two Bautin bifurcation with additive noise. In the absence of noise, Bautin bifurcation has the general normal form as follows:

\[
\begin{align*}
\dot{v}_1 &= \beta_1 v_1 - v_2 + \beta_2 v_1 (v_1^2 + v_2^2) - (av_1 - bv_2)(v_1^2 + v_2^2)^2, \\
\dot{v}_2 &= \beta_1 v_2 + v_1 + \beta_2 v_2 (v_1^2 + v_2^2) - (av_2 + bv_1)(v_1^2 + v_2^2)^2,
\end{align*}
\]

where \(\beta_1 \in \mathbb{R}, \beta_2 \in \mathbb{R}, a > 0, b \in \mathbb{R}\) are parameters. To better understand its dynamics, we show its bifurcation diagram in Figure 1 with \(a = 1, b = 1\), which is the same as Figure 8.7 in \[9\], but the different condition is \(b = 0\). We confirm this does not affect the bifurcation diagram, but has an impact on the angular rotation under the format of the polar coordinate.

With additive noise, the equation for Bautin bifurcation becomes a stochastic differential equation,

\[
\begin{align*}
\dot{v}_1 &= (\beta_1 v_1 - v_2 + \beta_2 v_1 (v_1^2 + v_2^2) - (av_1 - bv_2)(v_1^2 + v_2^2)^2)dt + \sigma dW_1^t, \\
\dot{v}_2 &= (\beta_1 v_2 + v_1 + \beta_2 v_2 (v_1^2 + v_2^2) - (av_2 + bv_1)(v_1^2 + v_2^2)^2)dt + \sigma dW_2^t
\end{align*}
\]

where \(\sigma \geq 0\) is the strength of the noise and \(W_1^t, W_2^t\) are independent one dimensional Brownian motions. It is clear that Bautin bifurcation in the deterministic system occurs when the first Lyapunov coefficient of Hopf bifurcation equals zero. Comparing with the codimension-one Hopf bifurcation, the power of the high-order terms in the normal form equation (1) increases by five and there are two extra parameters \(a\) and \(b\) except more bifurcation parameters \(\beta_1\) and \(\beta_2\). Besides these differences in high-order terms and parameters, the addition of random terms all increase the difficulties, not only in the process of the proof of existence and uniqueness of the solution, but also in getting the corresponding stationary probability

![Figure 1: Around Bautin point (0, 0) counterclockwise, there are three regions \(\{\beta_1 < 2\sqrt{-\beta_2}\} \cup \{\beta_1 < 0, \beta_2 < 0\}\), \(\{\beta_1 > 0\}\), and \(\{\beta_1 < 0, \beta_2 > 0, \beta_3 > 2\sqrt{-\beta_2}\}\), where \(T = \beta_2 + 4\beta_1 = 0, \beta_2 > 0\). There is only a stable equilibrium point (pink orbit) in region \(\circ\). From region \(\circ\) to region \(\bigcirc\), a unique reseda unstable limit cycle appears when cross the Hopf boundary \(H_1 = \{\beta_1 = 0, \beta_2 < 0\}\) (subcritical case). In the whole region \(\bigcirc\), the state of equilibrium turns unstable to stable. A unique bottle green stable limit cycle emerges at the Hopf boundary \(H = \{\beta_1 = 0, \beta_2 > 0\}\) (supercritical case). Especially, there exist two red limit circles with opposite stability in region \(\bigcirc\). Only one blue limit cycle remains when touching the blue curve \(T\).](image-url)
density when exploring the condition of synchronization. It is also crucial to make sure the condition of globally uniformly attracting and locally uniformly attracting for random equilibrium.

In this paper, we first prove the existence and uniqueness of the solution for equation (2) and then study the linearization part of equation (2) to calculate its stationary density function and the largest Lyapunov exponent. Besides, we acquire the conditions of the stability of random equilibrium and compare it with deterministic system. Then we make some numerical simulations to show randomness at the Bautin bifurcation point and give an example as the application of our results.

For simplification, equation (2) can be written as follows:
\[
dV_t = f(V_t)dt + \sigma dW_t,
\]
where \( V_t = (v_t, v_2)^T \), \( W_t = (W_1^t, W_2^t)^T \), \( W_1^t \) and \( W_2^t \) are the Wiener processes, \( f(V_t) = \left( \begin{array}{c} \beta_1 \beta_2 \end{array} \right)(v_t^2 + v_2^2) \), \( \beta_1 = -1 \beta_2 = 1 \), \( v_t \in \mathbb{R}^2 \). Let \( \Omega = C_0(\mathbb{R}, \mathbb{R}^2) \) be the space of all continuous functions \( \omega : \mathbb{R} \rightarrow \mathbb{R}^2 \), \( \omega(0) = 0 \), \( \mathcal{F} = \mathcal{B}(\Omega) \) be the Borel \( \sigma \)-algebra on \( \Omega \). There exists Wiener probability measure \( \mathbb{P} \) on \( (\Omega, \mathcal{F}) \) to ensure \( W_1^t \), \( W_2^t \) are independent one-dimensional Brownian motions referred to Appendix A.1. Define sub \( \sigma \)-algebra \( \mathcal{F}_{s,t} \) generated by \( \omega(u) - \omega(v) \), \( s \leq v \leq u \leq t \). For each \( t \in \mathbb{R} \), define \( \theta_t : \Omega \rightarrow \Omega \) by \( \theta_t(\omega)(s) = \omega(s + t) - \omega(t) \). \( (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}) \) is an ergodic dynamical system referred to Appendix A.2.

## 2 The existence and uniqueness of solution for random dynamical system

For better understanding, we give classical definitions about RDS, tempered random set and random attractor in Appendix A. Before obtaining the results of the existence and uniqueness of the solution for equation (2), we give two lemmas as follows.

**Lemma 2.1.** The RDS \( \varphi \): \( \hat{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}^2 \), defined by
\[
\varphi(t, \omega, V_0) = K(\theta_t \omega, \Psi(t, \omega, K(\omega)^{-1}V_0))
\]
is generated by equation (2), where \( \hat{\Omega} \subset \Omega \) is \( \theta \)-invariant \( \mathcal{F} \)-measurable set of full probability, \( K \) is a translation operator from \( \hat{\Omega} \times \mathbb{R}^2 \) to \( \mathbb{R}^2 \), defined by \( K(t, V_0) = V_0 + \sigma V^*(\omega) \), \( V^*(\omega) \) is a random variable \( V^* = \int_{-\infty}^{0} \exp(cs)dW_s \), and \( \varphi(t, \omega, V_0) \) is the solution of the random differential equation \( \dot{V}_t = f(K(\theta_t \omega, V_0)) + \sigma V^*(\theta_t \omega) \).

**Proof.** It is well known that Ornstein-Uhlenbeck process \( V \) satisfies the stochastic differential equation:
\[
dV = -cVdt + dW_t,
\]
Its solution is \( V^*(\theta_t \omega) \), satisfying \( V^*(\theta_t \omega) = V^*(\omega) - c \int_0^t V^*(\theta_h \omega)dh + \omega(t) \) and \( |V^*(\theta_t \omega)|^2 \leq M(\omega) + N(\omega) \ln(1 + |t|) \), for all \( t \in \mathbb{R} \) and \( \omega \in \hat{\Omega} \), where \( M(\omega) \) and \( N(\omega) \) are two random variables. Through the translation operator \( K \) and combining with the Ornstein-Uhlenbeck process, equation (2) becomes a random differential equation \( \dot{V}_t = f(K(\theta_t \omega, V_0)) + \sigma V^*(\theta_t \omega) \), whose solution is \( \Psi(t, \omega, V_0) = V_0 + \int_0^t f(K(\theta_s \omega, V_s)) + \sigma V^*(\theta_s \omega)ds \), for all \( t \geq 0 \).

Next step is to prove that \( \varphi(t, \omega, V_0) = K(\theta_t \omega, \Psi(t, \omega, K(\omega)^{-1}V_0)) \) is the solution of equation (2).
\[
\varphi(t, \omega, V_0) = K(\theta_t \omega, \Psi(t, \omega, K(\omega)^{-1}V_0))
= \Psi(t, \omega, V_0 - \sigma V^*(\omega)) + \sigma V^*(\theta_t \omega)
= V_0 - \sigma V^*(\omega) + \int_0^t f(K(\theta_s \omega, V_s - \sigma V^*(\omega))) + \sigma V^*(\theta_s \omega)ds + \sigma V^*(\theta_t \omega)
\]
\[ V_t = V_0 + \int_0^t \left[ f(K(\theta, \omega, K^{-1}(\theta, \omega)(\varphi(s, \omega, V_t)))) + \sigma V^*(\varphi(t, \omega)) \right] ds + \sigma V^*(\varphi(t, \omega)) - \sigma V^*(\omega) \] 

\[ = V_0 + \int_0^t f(\varphi(s, \omega, V_t)) ds + \omega(t). \]

This proves that \( \varphi(t, \omega, V_t) \) satisfies the integral equation of equation (2). \( \square \)

**Lemma 2.2.** There exists a tempered stochastic processes \( (\Psi_t)_{t \in \mathbb{R}} \) such that for all \( \omega \in \hat{\Omega} \) and \( V_t \in \mathbb{R}^2 \), we have 
\[ \|\Psi(t, \omega, V_t)\|^2 \leq \zeta(t, \omega, \|V_t\|^2), \]
which implies that the solution \( \Psi(t, \omega, V_t) \) exists for all \( t \geq 0 \).

**Proof.** By replacing \( V_t \) with \((v_1, v_2)^T\) and \( V^* \) with \((v_1^*, v_2^*)^T\), we rewrite equation \( \dot{V}_t = f(K(\theta, \omega, V_t)) + \sigma V^*(\varphi(t, \omega)) \) as follows:

\[ \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} \beta_1 & -1 \\ 1 & \beta_1 \end{pmatrix} \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} + \sigma \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} + \beta_2 \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} + \beta_2 \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} \]

Let \( \eta_t = \frac{1}{2}(v_1^2 + v_2^2) \). Then a direct computation yields that

\[ \dot{\eta}_t = v_1 \nu v_1 + v_2 \nu v_2 \]

\[ = 2 \left( \beta_1 + \beta_2 \right) \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} \eta_t + \sigma \beta_2 \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} + \sigma \nu v_2^*(\theta, \omega) [(\beta_1 + c)v_1 + v_2] - \sigma \nu v_1^*(\theta, \omega) [v_1 - (\beta_1 + c)v_2]. \]

Note that \( \max(|(\beta_1 + c)v_1 + v_2, v_1 - (\beta_1 + c)v_2|) \leq \sqrt{2}(\beta_1 + c)^2 + 1 \). Thus,

\[ |v_1^*(\theta, \omega)| |(\beta_1 + c)v_1 + v_2| - v_2^*(\theta, \omega) |v_1 - (\beta_1 + c)v_2| \leq \sqrt{2}(\beta_1 + c)^2 + 1 \left( |v_1^*(\theta, \omega)| + |v_2^*(\theta, \omega)| \right) \leq 2 \sqrt{2} |(\beta_1 + c)^2 + 1| \eta \|V^*(\theta, \omega)\|. \]

On the other hand,

\[ \left\| \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} \right\|^2 = 2 \eta_t + \sigma^2 \|V^*(\theta, \omega)\|^2 + 2 \sigma v_1^*(\theta, \omega)v_1 + 2 \sigma v_2^*(\theta, \omega)v_2, \]

which together with the fact that \( |v_1^*(\theta, \omega)v_1 + v_2^*(\theta, \omega)v_2| \leq \sqrt{2} \|V^*(\theta, \omega)\| \) implies that

\[ \left\| \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} \right\|^2 - 2 \eta_t - \sigma^2 \|V^*(\theta, \omega)\|^2 \leq 2 \sqrt{2} \|V^*(\theta, \omega)\|. \]

Similarly, we have

\[ \left\| \begin{pmatrix} v_1 + \nu v_1^*(\theta, \omega) \\ v_2 + \nu v_2^*(\theta, \omega) \end{pmatrix} \right\|^2 = 4 \eta_t^2 + 4(\sigma^2 \|V^*(\theta, \omega)\|^2 + 2 \sigma v_1^*(\theta, \omega)v_1 + 2 \sigma v_2^*(\theta, \omega)v_2) \eta_t + (\sigma^2 \|V^*(\theta, \omega)\|^2 + 2 \sigma v_1^*(\theta, \omega)v_1 + 2 \sigma v_2^*(\theta, \omega)v_2)^2, \]
\[
\left\| \begin{bmatrix} v_1 + \sigma_1'(\theta \omega) \\ v_2 + \sigma_2'(\theta \omega) \end{bmatrix} \right\|_F^2 - 4r_f^2 - (\sigma^2 \| V'(\theta \omega) \|^2 + 2\sigma\sigma'(\theta \omega)v_1 + 2\sigma\sigma'(\theta \omega)v_2)^2 - 4\sigma \| V'(\theta \omega) \|^2 r_f \\
\leq 8\sigma\sqrt{2r_f \| V'(\theta \omega) \|}.
\]

Consequently,
\[
ar_f \left\| \begin{bmatrix} v_1 + \sigma_1'(\theta \omega) \\ v_2 + \sigma_2'(\theta \omega) \end{bmatrix} \right\| \geq -8\sigma\sigma^2\sqrt{2r_f \| V'(\theta \omega) \|} + 4ar_f^2 + 4a\sigma^2 \| V'(\theta \omega) \|^2 r_f^2.
\]

Hence, we obtain
\[
\frac{1}{2} \| \Psi(t, \omega, V_\eta) \|^2 \leq \zeta(t, \omega, \| V \|),
\]
where \( t \mapsto \zeta(t, \omega, \| V \|^2) = \zeta(t) \) is the solution of the following scalar differential equation:
\[
\dot{\zeta} = a_1(\omega)\zeta^\frac{1}{2} + b_1(\omega)\zeta + c_1(\omega)\zeta^2 + d_1(\omega)\zeta^2 + e_1(\omega)\zeta^2 - 8\alpha\zeta^3,
\]
with initial condition \( \zeta_0 = \| V \|^2 \). Here, the functions \( a_1, b_1, c_1, d_1, \) and \( e_1 \) are defined by
\[
a_1(\omega) = 2\sqrt{\beta_1 + c^2} + 1 \| V'(\theta \omega) \| + \sqrt{2}\beta_1 \sigma^3 \| V'(\theta \omega) \|^3 + 2\sqrt{a^2 + b^2} \sigma^3 \| V'(\theta \omega) \|^5,
\]
\[
b_1(\omega) = 2\beta_1 + 6\beta_1 \sigma^2 \| V'(\theta \omega) \|^2 + 8\sqrt{2}(a^2 + b^2) \sigma^3 \| V'(\theta \omega) \|^6,
\]
\[
c_1(\omega) = 6\sqrt{2}\beta_1 \sigma \| V'(\theta \omega) \| + (16\sigma^2 + 8\sigma^2 \sqrt{a^2 + b^2}) \| V'(\theta \omega) \|^3,
\]
\[
d_1(\omega) = 4\beta_1 + (16\sqrt{2}(a^2 + b^2) \sigma - 8\sigma^3) \| V'(\theta \omega) \|^2,
\]
\[
e_1(\omega) = (16\sqrt{2}a \sigma + 8 \sqrt{a^2 + b^2}) \| V'(\theta \omega) \|.
\]

From tempered of \( V'(\theta \omega) \), all stochastic processes \((a_1)_{t \in \mathbb{R}}, (b_1)_{t \in \mathbb{R}}, (c_1)_{t \in \mathbb{R}}, (d_1)_{t \in \mathbb{R}}, \) and \((e_1)_{t \in \mathbb{R}}\) are also tempered. Noted that
\[
\alpha_3 \zeta^3 + 5\frac{a_3^3(\omega)}{6^3 a^6} \geq |a_3(\omega)|\zeta^2, \quad \alpha_3 \zeta^3 + 2\frac{b_3^3(\omega)}{3^3 a^3} \geq |b_3(\omega)|\zeta^2, \quad \alpha_3 \zeta^3 + \frac{c_3^3(\omega)}{4a} \geq |c_3(\omega)|\zeta^2,
\]
\[
\alpha_3 \zeta^3 + \frac{2^2d_3^3(\omega)}{3^3 a^3} \geq |d_3(\omega)|\zeta^2, \quad \alpha_3 \zeta^3 + \frac{5^2e_3^3(\omega)}{6^3 a^5} \geq |e_3(\omega)|\zeta^2.
\]

Therefore,
\[
a_1(\omega)\zeta^2 + b_1(\omega)\zeta + c_1(\omega)\zeta^2 + d_1(\omega)\zeta^2 + e_1(\omega)\zeta^2 - 8\alpha_3 \zeta^3 \\
\leq 5\frac{a_3^3(\omega)}{6^3 a^6} + 2\frac{b_3^3(\omega)}{3^3 a^3} + \frac{c_3^3(\omega)}{4a} + \frac{2d_3^3(\omega)}{3^3 a^3} + \frac{5^2e_3^3(\omega)}{6^3 a^5} - 3a_3 \zeta^3 \\
\leq \gamma(\omega) - \sqrt{a} \zeta,
\]
where
\[
\gamma(\omega) = \frac{2}{9} + 5\frac{a_3^3(\omega)}{6^3 a^6} + 2\frac{b_3^3(\omega)}{3^3 a^3} + \frac{c_3^3(\omega)}{4a} + \frac{2d_3^3(\omega)}{3^3 a^3} + \frac{5^2e_3^3(\omega)}{6^3 a^5}
\]
is tempered.

Next we have the following theorem to address the existence and uniqueness of the solution for equation (2) and the existence of a random attractor for RDS \(( \theta, \varphi )\) generated by equation (2).

**Theorem 2.3.** For the stochastic differential equation (2), there exists a \( \theta \)-invariant \( \mathcal{F} \)-measurable set \( \hat{\Omega} \subset \Omega \) of full probability such that the following statements hold.
(i) For all $\omega \in \tilde{\Omega}$ and $V_i \in \mathbb{R}^2$, equation (2) admits a unique solution $\varphi(t, \omega, V_i)$ for all $t \geq 0$.

(ii) There exists a random attractor $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2)$ of RDS $(\theta, \varphi)$ such that $\omega \mapsto A(\omega)$ is measurable with respect to $\mathcal{F}^0_{\omega}$.

**Proof.** (i) Due to Lemma 2.2, the solution $\Psi(t, \omega, V_i)$ exists for all $t \geq 0$. Therefore, this proves the fact that we assumed to prove Lemma 2.1. Lemma 2.1 completes the proof of (i).

(ii) Let $D \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2)$ be tempered. Then there exists a tempered random variable $R : \tilde{\Omega} \to \mathbb{R}^+$ such that $D(\omega) \subset B_{R(\omega)}(0)$. By Lemma 2.2, for all $V_i \in D(\theta, \omega)$, we have $\|\Psi(t, \theta, \omega, V_i)\|^2 \leq 2\zeta(t, \theta, \omega, R(\theta, \omega)) \leq 2\exp(-\sqrt{\lambda}t)R(\theta, \omega) + 2\int_0^t \exp(\sqrt{\lambda}s)\gamma(\omega)ds$. Define $r(\omega) = \sqrt{1 + 2\int_0^\infty \exp(\sqrt{\lambda}s)\gamma(\omega)ds}$. For each $\omega \in \tilde{\Omega}$, there exists $T > 0$ such that $\Psi(t, \theta, \omega, D(\theta, \omega)) \subset B_{r(\omega)}(0)$ for all $t \geq T$. This means that $B_{r(\omega)}(0)$ is an absorbing set. Applying theorem 14.4 in [8] and following the definitions of the random attractor and the translation operator $K(t, V_i)$, there exists a random attractor $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2)$ of RDS $(\theta, \varphi)$. \hfill $\square$

### 3 Main results

After obtaining the solution of equation (2), we further study the linearization of the system. Its linearization part is denoted as follows:

$$\Phi(t, \omega, V) = Dv \varphi(t, \omega, V).$$

It is easy to obtain that $\Phi(0, \omega, V) = id$ and $\Phi(t, \omega, V) = Df(\varphi(t, \omega, V))Dv \varphi(t, \omega, V)$. Define a skew product flow $(\Theta_\lambda)_{\lambda \in \mathbb{R}^1}$ on $\Omega \times \mathbb{R}^2$, then $\Phi(t, \omega, V)$ is a linear cocycle over $\Theta_\lambda(\omega, V) = (\Theta_\lambda \omega, \varphi(t, \omega, V))$. Therefore, $(\Theta, \Phi)$ is a linear RDS.

First, we obtain statistical information about the dynamics of equation (2) in the presence of noise ($\sigma \neq 0$). In statistical mechanics, the Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle. In the simplification of equation (2), the Fokker-Planck equation for the probability density $p(V_i, t)$ of the random variable $V_i$ expressed as follows:

$$\frac{\partial p(V_i, t)}{\partial t} = -\sum_{i=1}^2 \frac{\partial}{\partial V_i} \left[ f(V_i) p(V_i, t) \right] + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial^2}{\partial V_i \partial V_j} \left[ \sigma^2 p(V_i, t) \right],$$

where $V_i = v_i, V_j = v_j, f(V_i) = \beta_1 v_i - v_2 + \beta_2 v_i (v_i^2 + v_j^2) - (a v_1 - b v_2) (v_i^2 + v_j^2)^2, f(V_j) = \beta_2 v_j + v_1 + \beta_3 v_i (v_i^2 + v_j^2) - (a v_2 + b v_1) (v_i^2 + v_j^2)^2$. Let $\frac{\partial \Psi(V_i, t)}{\partial t} = 0$, we deduce the stationary probability density $p(v_i, v_j)$ of the Fokker-Planck equation associated with equation (2) as follows:

$$p(v_i, v_j) = \kappa \exp \left( \frac{2}{\sigma^2} \left[ \frac{\beta_1}{2} (v_i^2 + v_j^2) + \frac{\beta_2}{4} (v_i^2 + v_j^2)^2 - \frac{a}{6} (v_i^2 + v_j^2)^3 \right] \right),$$

where $\kappa$ is a constant satisfying $\int_{\mathbb{R}^2} \kappa p(v_i, v_j) dv_i dv_j = 1$.

In fact, substituting $p(v_i, v_j), f(V_i), f(V_j)$, we have

$$\frac{\partial p(V_i, V_j)}{\partial t} = \left[ 2 \beta_1 + 4 \beta_2 (v_i^2 + v_j^2) - 6 a (v_i^2 + v_j^2)^2 \right] p(v_i, v_j) - f(V_i) p(v_i, v_j) \left[ \frac{2}{\sigma^2} \left( \beta_1 v_1 + \beta_2 (v_i^2 + v_j^2) v_1 - a (v_i^2 + v_j^2)^2 v_1 \right) \right]$$

$$- f(V_j) p(v_i, v_j) \left[ \frac{2}{\sigma^2} \beta_2 v_2 + \beta_3 (v_i^2 + v_j^2) v_2 - a (v_i^2 + v_j^2)^2 v_2 \right]$$

$$+ \frac{\sigma^2}{2} \left[ 2 \beta_1 + 4 \beta_2 (v_i^2 + v_j^2) - 6 a (v_i^2 + v_j^2)^2 \right] p(v_i, v_j).$$

(3)
Besides, \( \kappa \) is bounded on the base of mathematical analysis.

The stationary probability density \( p(v_1, v_2) \) with an ergodic probability measure \( \mu \) gives rise to an invariant measure \( \nu \) for the skew product flow \( \Theta_\sigma \), where \( \nu(C) = \int_C \nu_\omega(d\nu) \), for all \( C \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2) \), \( C_\omega = V \in \mathbb{R}^2 : (\omega, V) \in C \), and \( \nu_\omega = \lim_{t \to \infty} \phi(t, \theta_\omega) \). Reversely, the stationary measure \( \nu(B) = \int_B \nu_\omega(d\nu) \), for all \( B \in \mathcal{B}(\mathbb{R}^2) \). Therefore, the largest Lyapunov exponent \( \lambda_{\text{top}} \) can be given by

\[
\lambda_{\text{top}} = \lim_{t \to \infty} \frac{1}{t} \ln \| \Phi(t, \omega, V) \|, \quad \text{for } \nu - \text{a. a. } (\omega, V) \in \Omega \times \mathbb{R}^2.
\]

For the calculation of the largest Lyapunov exponent, define \( \lambda(V) = \max_{(r, r) \in \mathbb{R}^2} (Df(V) r, r) \), and then we have \( \frac{d}{dt} \| \Phi(t, \omega, V) \|^2 \leq 2 \lambda'(\phi(s, \omega, V)) \| \Phi(t, \omega, V) \|^2 \), where \( z \in \mathbb{R}^3 \setminus \{0\} \). Due to the arbitrariness of \( z \), we obtain that \( \| \Phi(t, \omega, V) \|^2 \leq \| z \|^2 \exp \left( 2 \int_0^t \lambda'(\phi(s, \omega, V))ds \right) \). Thus, for almost all \( \omega \in \Omega, V \in \mathbb{R}^2 \), and \( t \geq 0 \), \( \| \Phi(t, \omega, V) \| \leq \exp(\lambda'(\phi(\omega, \omega, V))) \).

**Proposition 3.1.** For any \( V = (v_1, v_2)^T \in \mathbb{R}^2 \), \( \lambda'(V) \leq \beta_1 + 3 \beta_2 (v_1^2 + v_2^2) + (2 \sqrt{a^2 + b^2} - 3a)(v_1^2 + v_2^2) \) and the linear RDS \( \Phi \) satisfies the integrability condition of the multiplicative ergodic theorem.

**Proof.** \( Df(V) \) has explicit form as follows:

\[
Df(V) = \begin{pmatrix}
\beta_1 & -1 \\
1 & \beta_1
\end{pmatrix} + \frac{1}{2} \begin{pmatrix}
3v_1^2 + v_2^2 & 2v_1v_2 \\
v_1^2 + 3v_2^2
\end{pmatrix} - \begin{pmatrix}
-4b_1v_2 - 4bv_1^2 & 4av_1v_2 + 4av_1^3 \\
4av_1v_2 + 4av_1^3 & 4bv_2 + 4bv_1^2
\end{pmatrix} - \begin{pmatrix}
5av_1^4 + av_2^4 + 6av_1v_2^2 & -5bv_2 - bv_1^4 - 6bv_1v_2^2 \\
5bv_2 + bv_1^4 + 6bv_2v_2^2 & 5av_1^4 + av_2^4 + 6av_1v_2^2
\end{pmatrix}.
\]

For any \( r \in \mathbb{R}^2 \) with \( ||r|| = 1 \), where \( r = (r_1, r_2)^T \) and \( r_1^2 + r_2^2 = 1 \), let \( \eta_1 = \cos \phi \) and \( \eta_2 = \sin \phi \), for some \( \phi \in [0, 2\pi) \). Derive that

\[
(Df(v_1, v_2)r, r) = \beta_1 + 2\beta_2 (v_1^2 + v_2^2) + 2\beta_1v_1v_2 \sin 2\phi + \beta_2(v_1^2 - v_2^2) \cos 2\phi
- 2(2v_1^2 - v_2^2) + 2av_1v_2^2 + 2av_1^3v_2) \sin 2\phi - 3a(v_1^2 + v_2^2)
+ 2(2v_1^2 - v_2^2) + 2av_1v_2^2 + 2av_1^3v_2) \cos 2\phi
\leq (v_1^2 + v_2^2)\sqrt{\beta_2^2 + 4(a^2 + b^2)(v_1^2 + v_2^2)^2 - 2\beta_2a(v_1^2 + v_2^2)}
+ \beta_1 + 2\beta_2(v_1^2 + v_2^2) - 3a(v_1^2 + v_2^2)^2
\leq (v_1^2 + v_2^2)\sqrt{\beta_2^2 + 4(a^2 + b^2)(v_1^2 + v_2^2)^2 + 4\beta_2\sqrt{a^2 + b^2}(v_1^2 + v_2^2)}
+ \beta_1 + 2\beta_2(v_1^2 + v_2^2) - 3a(v_1^2 + v_2^2)^2
\leq \beta_1 + 2\beta_2(v_1^2 + v_2^2) - 3a(v_1^2 + v_2^2)^2 + 2\sqrt{a^2 + b^2}(v_1^2 + v_2^2)\beta_2(v_1^2 + v_2^2)
= \beta_1 + 3\beta_2(v_1^2 + v_2^2) + (2\sqrt{a^2 + b^2} - 3a)(v_1^2 + v_2^2)^2.
\]
Due to $\sup_{0 \leq t \leq 1} \ln \| \Phi(t, \omega, V) \| \leq \int_0^1 |A^r(\varphi(s, \omega, V))| \, ds$, which implies that

$$
\int \sup_{\Omega \times R^2} \ln \| \Phi(t, \omega, V) \| \, dv(\omega, V)
\leq \int \sup_{\Omega \times R^2} \int_0^1 |A^r(\varphi(s, \omega, V))| \, ds \, dv(\omega, V)
\leq \int |A^r(\varphi(\omega, V))| \, dv(\omega, V)
\leq \int |A^r(V)| \, d\mu(V)
\leq \int |\beta_1 + 3\beta_2(v_1^2 + v_2^2) + (2\sqrt{a^2 + b^2} - 3a)(v_1^2 + v_2^2)|p(v_1, v_2) \, dv_1 \, dv_2
\leq \beta_1 + 3\beta_2 \int_R (v_1^2 + v_2^2)p(v_1, v_2) \, dv_1 \, dv_2 + |2\sqrt{a^2 + b^2} - 3a| \int_R (v_1^2 + v_2^2)^2p(v_1, v_2) \, dv_1 \, dv_2.
$$

Therefore, the linear RDS $\Phi$ satisfies the integrability condition of the multiplicative ergodic theorem (Theorem 3.4.11 in [1]).

**Proposition 3.2.** If $2\beta_1 + 4\beta_2 \kappa \kappa_3 - 6\alpha \kappa \kappa_3 < 0$, then $\lambda_2 < 0$ and the disintegrations of the Markov measure $\nu$ are singular with respect to the Lebesgue measure on $R^2$, where $\lambda_2$ be the sum of the two Lyapunov exponents of the linear RDS $\Phi$, $k_1 = \int_0^\infty r \exp\left(\frac{6\beta_1 r - 22a \kappa^2}{6a^2}\right) \, dr$ and $k_2 = \int_0^\infty r^2 \exp\left(\frac{6\beta_1 r - 22a \kappa^2}{6a^2}\right) \, dr$.

**Proof.** According to the formula $\lambda_2 = \lim_{t \to \infty} \frac{1}{t} \ln \det\Phi(t, \omega, V)$, the sum of Lyapunov exponents is calculated as follows:

$$
\lambda_2 = 2\beta_1 + 4\beta_2 \int_{R^2} (v_1^2 + v_2^2)p(v_1, v_2) \, dv_1 \, dv_2 - 6a \int_{R^2} (v_1^2 + v_2^2)^2p(v_1, v_2) \, dv_1 \, dv_2
\leq 2\beta_1 - 6a \int_{R^2} (v_1^2 + v_2^2) \exp\left(\frac{2}{\sigma^2} \left[ \frac{\beta_1}{2}(v_1^2 + v_2^2) + \frac{\beta_2}{4}(v_1^2 + v_2^2)^2 - \frac{a}{6}(v_1^2 + v_2^2)^3 \right] \right) \, dv_1 \, dv_2
+ 4\beta_2 \int_{R^2} (v_1^2 + v_2^2) \exp\left(\frac{2}{\sigma^2} \left[ \frac{\beta_1}{2}(v_1^2 + v_2^2) + \frac{\beta_2}{4}(v_1^2 + v_2^2)^2 - \frac{a}{6}(v_1^2 + v_2^2)^3 \right] \right) \, dv_1 \, dv_2.
$$

Let $v_1 = r \sin \phi$, $y_1 = r \cos \phi$, and make variable substitution again $r^2 \mapsto r$, we have $\lambda_2 = 2\beta_1 + 4\beta_2 \kappa \kappa_3 - 6a \kappa \kappa_3$. If $2\beta_1 + 4\beta_2 \kappa \kappa_3 - 6a \kappa \kappa_3 < 0$, then $\lambda_2 < 0$. If $\lambda_2 < 0$, the disintegration of the Markov measure $\nu$ is singular with respect to $\mu$ [10].

Next we give the condition when system (2) admits synchronization, i.e., its largest Lyapunov exponent $\lambda_{\text{top}} = \lim_{t \to \infty} \frac{1}{t} \ln \| \Phi(t, \omega, V) \|$ is negative. Noted that when the largest Lyapunov exponent is negative and according to Lemma 3.1 and Proposition 3.10 in [5], our system is asymptotically stable and swift transitive. Next we need to show contraction on large sets for $\varphi$. By the definition of $f$, we have that

$$
(f(x) - f(y), x - y) = \beta_2 (\| x \|^2 x_1 - \| y \|^2 y_1)(x_1 - y_1) + \beta_2 (\| x \|^2 x_2 - \| y \|^2 y_2)(x_2 - y_2)
- (b \| x \|^4 x_1 - b \| y \|^4 y_1 + a \| x \|^4 x_2 - a \| y \|^4 y_2)(x_2 - y_2)
- (a \| x \|^4 x_1 - a \| y \|^4 y_1 + b \| x \|^4 x_2 - b \| y \|^4 y_2)(x_1 - y_1) + \beta_1 \| x - y \|^2.
$$
Fix \( r > 0 \), and consider \( B_r(z) \), where \( z = (R, 0) \) for some \( R > 0 \) to be chosen large enough. For any \( x, y, e B_r(z) \), we obtain that \( (f(x) - f(y), x - y) < 0 \) since the coefficient of the higher-order term is negative. Due to the property of monotonicity in large sets, this induces contraction on large sets. Thus, the fibers of the random attractor \( A \) for RDS \( \varphi \) are singletons. According to Proposition B.1 in Appendix B, we obtain the existence of the attracting random equilibrium \( A : \Omega \rightarrow R^2 \). Using Proposition B.1 and Theorem B.2 in Appendix B for the following theorem, we obtain a parameterized condition for having a negative largest Lyapunov exponent. This theorem guarantees the effective-achievable simulations for the synchronization of system.

**Theorem 3.3.** The largest Lyapunov exponent \( \lambda_{top} < 0 \), if \( \beta_1 + 3\beta_2\pi\kappa_1 + (2\sqrt{a^2 + b^2} - 3a)\pi\kappa_2 < 0 \).

**Proof.** \( \lambda_{top} = \limsup_{t \to -\infty} \frac{1}{t} \ln \| \Phi(t, \omega, s) \| \leq \limsup_{t \to -\infty} \frac{1}{t} \int_0^t \lambda^*(\varphi(s, \omega, z)) ds \). Note that the skew product flow \( \Theta_\omega(V) = (\theta_\omega, \varphi(s, \omega, \hat{V})) \) preserves the probability measure \( \nu \), and \( \lambda^* \) is integrable. By using Birkhoff's ergodic theorem [14], we obtain that \( \lambda_{top} \leq \int_{R^2} \lambda^*(v_1, v_2)p(v_1, v_2)dv_1dv_2 \). By Propositions 3.1 and 3.2, we obtain that

\[
\lambda_{top} \leq \beta_1 + 3\beta_2 \int_{R^2} (v_1^2 + v_2^2)p(v_1, v_2)dv_1dv_2 + (2\sqrt{a^2 + b^2} - 3a) \int_{R^2} (v_1^2 + v_2^2)p(v_1, v_2)dv_1dv_2
\]

Since \( \beta_1 + 3\beta_2\pi\kappa_1 + (2\sqrt{a^2 + b^2} - 3a)\pi\kappa_2 < 0 \), then \( \lambda_{top} < 0 \). \( \Box \)

The last part of the main results is about the random equilibrium of equation (2). We mainly give two theorems of globally uniformly attracting and locally uniformly attracting. The definitions of globally uniformly attracting and locally uniformly attracting are presented in Appendix A.6.

**Theorem 3.4.** For \( \beta_1 < 0, \beta_2 \leq 0, \) and \( |b| \leq a, \) the random equilibrium of equation (2) is globally uniformly attracting.

**Proof.** Denoted \( \begin{pmatrix} x \\ y \end{pmatrix} = \varphi(t, \omega, Z_1) \), and \( \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \varphi(t, \omega, Z_2) \), where \( Z_1, Z_2 \in R^2, \omega \in \Omega \). We have

\[
\frac{d}{dt} \begin{pmatrix} x - \dot{x} \\ y - \dot{y} \end{pmatrix} = \begin{pmatrix} \beta_1 & -1 \\ 1 & \beta_1 \end{pmatrix} \begin{pmatrix} x - \dot{x} \\ y - \dot{y} \end{pmatrix} + \beta_2 \left[ (x^2 + y^2)(x, y) - (\dot{x}^2 + \dot{y}^2)(\dot{x}, \dot{y}) \right]
\]

\[
- (x^2 + y^2) \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (\dot{x}^2 + \dot{y}^2) \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.
\]

Therefore,

\[
\frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} x - \dot{x} \\ y - \dot{y} \end{pmatrix} \right\|^2 = \beta_1 \left\| \begin{pmatrix} x - \dot{x} \\ y - \dot{y} \end{pmatrix} \right\|^2 + \beta_2 [r^2 + \hat{r}^2 - (r + \hat{r})(x\ddot{x} + y\ddot{y})] - a[r^3 + \hat{r}^3 - (x\ddot{x} + y\ddot{y})(r^2 + \hat{r}^2)]
\]

\[
+ b(x\ddot{y} - \ddot{y}x)(r^2 - \hat{r}^2),
\]

where \( r = x^2 + y^2, \hat{r} = \dot{x}^2 + \dot{y}^2. \)

From the inequality \( (ab|^2 + |cd|^2 \leq (a^2 + c^2)(b^2 + d^2) \), we derive that

\[
|(x\ddot{x} + y\ddot{y})(r^2 + \hat{r}^2)| + |(x\ddot{y} - \ddot{y}x)(r^2 - \hat{r}^2)| \leq \sqrt{2r^2 + \hat{r}^2} \leq r^3 + \hat{r}^3.
\]

If \( \beta_1 < 0, \beta_2 \leq 0, \) we acquire \( \frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} x - \dot{x} \\ y - \dot{y} \end{pmatrix} \right\|^2 \leq \beta_1 \left\| \begin{pmatrix} x - \dot{x} \\ y - \dot{y} \end{pmatrix} \right\|^2 \). The proof of the globally uniformly attracting is completed. \( \Box \)
Theorem 3.5. If $\beta_2 > 0$ or $\beta_1 > 0$, $\beta_2 < 0$, the random attractor is not locally uniformly attracting.

Proof. Suppose there exists a $\delta > 0$ such that

$$\sup_{x \in B_\delta(0)} \omega \in \Omega \sup_{x \in B_\delta(0)} \| \phi(t, \omega, A(\omega) + x) - A(\theta_t\omega) \| = 0.$$ 

From Proposition B.3 in Appendix B, there exists a positive measure set $H_0$ such that $A(\omega) \in B_\delta(0)$ for all $\omega \in H_0$. Let $\phi(t, \omega)$ as the solution of the deterministic equation (2) for $A(\omega)$ with initial condition $x(0) = x_0$, when $Y = \sqrt{\beta_2^2 + 4a\beta_1}$, there exists a $T > N$ such that

$$\| \phi(T, (\pm \delta, 0)) \| > \frac{\beta_2 + Y}{8a}, \quad \| \phi(T, (\delta, 0)) - \phi(T, (-\delta, 0)) \| > \frac{\beta_2 + Y}{2a}.$$ 

- If $\beta_2 > 0$, $\beta_1 > 0$ or $\beta_1 > 0$, $\beta_2 < 0$, choosing $\omega \in H_0 = \{ \omega : \sup_{t \in [0, T]} \| \phi(t, \omega) \| < \varepsilon \}$, we obtain that

$$\| \phi(T, (\delta, 0)) - \phi(T, (\delta, 0)) \| > \frac{1}{4} \sqrt{\frac{\beta_2 + Y}{2a}}, \quad \| \phi(T, (-\delta, 0)) - \phi(T, (-\delta, 0)) \| > \frac{1}{4} \sqrt{\frac{\beta_2 + Y}{2a}}.$$ 

$$\| \phi(T, (\delta, 0)) - \phi(T, (\delta, 0)) \| > \frac{1}{4} \sqrt{\frac{\beta_2 + Y}{2a}}.$$ 

For all $\omega \in H = H_0 \cap H_\epsilon$, we have

$$\sup_{x \in B_\delta(0)} \| \phi(t, \omega, A(\omega) + x) - A(\theta_t\omega) \| \geq \max\{ \| \phi(t, \omega, (\delta, 0)) - A(\theta_t\omega) \|, \| \phi(t, \omega, (-\delta, 0)) - A(\theta_t\omega) \| \} > \frac{1}{4} \sqrt{\frac{\beta_2 + Y}{2a}}.$$ 

This contradicts the definition of the locally uniformly attracting.

- If $\beta_2 > 0$, $\beta_1 < 0$, similarly, we have

$$\sup_{x \in B_\delta(0)} \| \phi(t, \omega, A(\omega) + x) - A(\theta_t\omega) \| > \frac{1}{4} \sqrt{\frac{\beta_2 - Y}{2a}}.$$ 

\[\square\]

Remark 3.6. Comparing with the deterministic case, we conclude the differences between random equilibrium and non-random equilibrium. The addition of stochastic terms has an impact on different regions. If $\beta_1 > 0$ (the first and forth quadrant), the diversity is from the uncertainty of stability for non-random equilibrium to becoming a not locally uniformly attracting random equilibrium. Instead, when parameters satisfy $\beta_1 < 0$, $\beta_2 > 0$, stable non-random equilibrium becomes a not locally uniformly attracting random equilibrium; when parameters satisfy $\beta_1 < 0$, $\beta_2 < 0$, stable non-random equilibrium becomes a globally uniformly attracting random equilibrium. By this token, the most influential region caused by noise is the region $\beta_1 < 0$, $\beta_2 > 0$. That is exactly where we are looking at two special cases, region $\ominus$ and $T$ curve in Figure 1.

4 Numerical simulation

In this section, we mainly discuss two special cases, the region $\ominus$ and $T$ curve in Figure 1. Figure 2(b) and Figure 3(a) and (b) focus on the Bautin bifurcation point of the region $\ominus$ in Figure 1 with $\beta_1 = -1.16725 \times 10^{-22}$ and $\beta_2 = 1.16725 \times 10^{-7}$. Figure 4(a) and (b) focus on the point of $T$ curve in Figure 1.
with \( \beta_1 = -1.16725 \times 10^{-21} \) and \( \beta_2 = 6.8330 \times 10^{-11} \). Here, we fix the values of \( a \) and \( b \), where \( a = b = 1 \).

Denoted \( Q_1 = 2\beta_1 + 4\beta_2\sigma\kappa_6 - 6\alpha\pi\kappa_6 \), \( Q_2 = \beta_1 + 3\beta_2\pi\kappa_5 + (2\sqrt{a^2 + b^2} - 3a)\pi\kappa_5 \), we plot \( Q_1 \) and \( Q_2 \) to make sure the conditions in Proposition 3.2 and Theorem 3.3 can be satisfied. We mainly judge the tend of \( Q_1 \) and \( Q_2 \) among the other parameters and \( a \) as a center parameter. It is worth noting that we do not select \( b \) and \( a \) as a pair of variables, because \( b \) is unrelated to \( \kappa_1 \), \( \kappa_2 \), and \( \kappa_3 \) and only related to \( Q_2 \). The situations are similar when centering on other parameters and hence, we omitted here.

In Figure 5(a), the upper pink surface represents \( Q_2 \), and the lower blue one is \( Q_1 \). The parameters’ regions of four subgraphs are in the order from left to right, top to bottom as follows:

\[
\{\beta_1 \in [-1.2 \times 10^{-21}, -1.1 \times 10^{-21}], \beta_2 \in [1.1 \times 10^{-7}, 1.2 \times 10^{-7}], a = b = 1, \sigma = 1.2 \times 10^{-5}\};
\{a \in [1, 2], \beta_2 \in [1.1 \times 10^{-7}, 1.2 \times 10^{-7}], \beta_1 = -1.16725 \times 10^{-21}, b = 1, \sigma = 1.2 \times 10^{-5}\};
\{\beta_1 \in [-1.2 \times 10^{-23}, -1.1 \times 10^{-21}], a \in [1, 2], \beta_2 = 1.16725 \times 10^{-7}, b = 1, \sigma = 1.2 \times 10^{-5}\};
\{\sigma \in [1.1 \times 10^{-3}, 1.2 \times 10^{-5}], a \in [1, 2], \beta_1 = -1.16725 \times 10^{-21}, \beta_2 = 1.16725 \times 10^{-7}, b = 1\}.
\]
Figure 4: (a) The phase portraits and (b) time series and hists for the single limit cycle.
In Figure 5(b), the upper brown surface represents $Q_2$, and the lower one is $Q_1$. The parameters’ regions of four subgraphs are as follows:

- $\{ \beta_1 \in [-1.2 \times 10^{-21}, -1.1 \times 10^{-21}], \beta_2 \in [6.8 \times 10^{-11}, 6.9 \times 10^{-11}], a = b = 1, \sigma = 10^{-7} \};$
- $\{ a \in [1, 2], \beta_1 \in [6.8 \times 10^{-11}, 6.9 \times 10^{-11}], \beta_2 = -1.6675 \times 10^{-21}, b = 1, \sigma = 10^{-7} \};$
- $\{ \beta_1 \in [-1.2 \times 10^{-21}, -1.1 \times 10^{-21}], a \in [1, 2], \beta_2 = 6.833 \times 10^{-7}, b = 1, \sigma = 10^{-7} \};$
- $\{ \sigma \in [1 \times 10^{-7}, 2 \times 10^{-7}], a \in [1, 2], \beta_1 = -1.6675 \times 10^{-21}, \beta_2 = 6.833 \times 10^{-7}, b = 1 \}.$

From the eight subgraphs, we can see that both $Q_1$ and $Q_2$ are negative. This lays the foundation for the later numerical simulation.

Figure 2(a) shows the codimension-two bifurcation diagrams with $\beta_1$ and $\beta_2$ as the bifurcation parameters, where $H$ represents Hopf bifurcation and $GH$ represents Bautin bifurcation. In Figure 2(a), the red line represents the Hopf bifurcation curve, while the blue and green lines are the forward and backward Bautin bifurcation curves. We found the Bautin bifurcation point with accurate parameters values $\beta_1 = -1.16725 \times 10^{-21}$ and $\beta_2 = 1.16725 \times 10^{-7}$.

In Figure 2(b), the first two subgraphs are time series of $v_1$ and $v_2$ with small limit circle $r = 3.1623 \times 10^{-4}$ and $\sigma = 1.2 \times 10^{-5}$; the last two subgraphs are time series of $v_1$ and $v_2$ with large limit circle $r = 3.4165 \times 10^{-4}$ and $\sigma = 1.2 \times 10^{-5}$, where blue mark $\circ$ is the original orbit without noise and another five curves are five sample trajectories with different colors.

Figure 3(a) shows the phase diagrams of $v_1$ and $v_2$ with small and large limit circles. From this graph, whether the large limit circle (red color) or small one (green) without noise, five sample trajectories are near to original orbits. It keeps certain synchronous but not stable.

In Figure 3(b), above two subgraph are hists of sample points of five trajectories and the others are ten sample trajectories, where the four hists are statistics of these sample points whose ranges are divided into 30 intervals. These are the distribution of sample points comparing the original limit circles from another perspective.

For the point on $T$ curve, we also draw its phase diagrams, time series diagrams and hists with limit circle $r = 5.8451 \times 10^{-6}$ and $\sigma = 10^{-7}$ in Figure 4(a) and (b). It has a similar phenomenon with Bautin bifurcation point. What the biggest difference between them is the strength of noise. The only limit circle
on T curve is more sensitive to the noise, at least lowering 100 orders of magnitude. If we choose larger noise $\sigma$, the synchronism of system will be destroyed. Due to larger noise, the negativity of the largest Lyapunov exponent can easily be broken.

Remark 4.1. For either Bautin bifurcation point or the point on $T$ curve, the parameter $\beta_2 > 0$ satisfies the condition in the Theorem 3.5. Therefore, not locally attracting random equilibrium will lead to the deviation of trajectories eventually even if it appears the synchronism in the early stage.

5 Application

In this section, we take a classical model of a laser with controllable resonator [13] as an example of application. In laser systems, the resonator that controls the lasing processes is a nonlinear function of the radiation field. Laser system can be extensively used in industry, agriculture, precision measurement and detection, communication and information processing, medical treatment, military, and other fields. In practical production, the generation of noise is unavoidable. It is vital to keep sophisticated equipment stable by judging how noise affects the system. Therefore, we investigate the model with additive noise as follows:

$$\begin{align*}
\dot{m} &= Gm\left(n - \frac{a_1}{\rho m + 1} - 1\right), \\
\dot{n} &= a_2 - (m + 1)n,
\end{align*}$$

(5)

where $m \geq 0, \rho > 0, G > 1, a_1, a_2 > 0$. The system has three equilibria,

$$(m_1, n_1) = (0, a_2),$$

$$(m_2, n_2) = \left(1 - \frac{a_2 - 1}{\rho} - 1, \frac{a_2 - 1}{\rho} - 1, \frac{a_2 - 1}{\rho} - 1\right) + \frac{1}{2} \left(\frac{a_2 - 1}{\rho} - 1, \frac{a_2 - 1}{\rho} - 1\right)^2 + \frac{4}{\rho} (a_2 - a_1 - 1, a_2)
\frac{a_2}{m_2 + 1},$$

$$(m_3, n_3) = \left(1 - \frac{a_2 - 1}{\rho} - 1, \frac{a_2 - 1}{\rho} - 1\right) - \frac{1}{2} \left(\frac{a_2 - 1}{\rho} - 1, \frac{a_2 - 1}{\rho} - 1\right)^2 + \frac{4}{\rho} (a_2 - a_1 - 1, a_2)
\frac{a_2}{m_3 + 1}.$$}

Especially, the Jacobian matrix at $(m_2, n_2)$ has a pair of complex roots. We transfer the origin to the equilibrium $(m_2, n_2)$ and expand fractional part in the form of series in powers with the new variables $\bar{m}$ and $\bar{n}$. The system becomes

$$\begin{align*}
\dot{\bar{m}} &= a_1\bar{m} + a_2\bar{n} + a_3\bar{m}^2 + G\bar{m}\bar{n} + a_6\bar{m}, \\
\dot{\bar{n}} &= a_2\bar{m} + a_3\bar{n} - \bar{m}\bar{n},
\end{align*}$$

where $a_1 = \frac{G_{m+m_2}}{m_2 + 1}, a_2 = Gm_2, a_3 = -\frac{a_2}{m_2 + 1}, a_4 = -m_2 - 1, a_5 = \frac{G_{m+n_2}}{(m_2 + 1)^2},$ and $a_6 = \frac{G_{m+n_2}}{(m_2 + 1)^2}$. Furthermore, let $z = \bar{m} + i\bar{n}$, and the system can be transformed into a complex system:

$$\dot{z} = \hat{A}z + \sum_{k,l=2}^{\infty} \frac{1}{k!l!} g_{kl} p^{k} q^{l},$$

where $g_{kl} = \frac{q^{kl}}{p^{2kl}} (p, F(zq + \tilde{q})), p, q$ satisfying $Aq = \hat{A}, A^2 p = \hat{p},$ where $A$ is the Jacobian matrix of the linear part and $F$ contains the higher order terms above quadratic. Eliminating quadratic terms by complex coordinate change and the time parametrization seeing Lemma 8.3 in [9], the system becomes the following form:

$$\dot{z} = \hat{A}z + l_1|z|^2 + l_2|z|^4,$$
where \( \text{Re} \hat{\lambda} = \frac{1}{2}(a_1 + a_2) \) and \( \text{Im} \hat{\lambda} = 1 \), \( l_1 \), \( l_2 \), are, respectively, the first and the second Lyapunov coefficient.

\[
l_1 = \frac{1}{2w} \left( \text{Re} g_{11} - \frac{1}{w} \text{Im} (g_{50} g_{11}) \right),
\]

\[
12l_2 = -\frac{1}{w^2} g_{50} g_{12} + \frac{1}{w} \left( \text{Re} \left[ g_{10} \left( 3g_{12} - g_{50} \right) + g_{12} \left( g_{10} - \frac{1}{2} g_{50} \right) + \frac{1}{3} g_{50} g_{11} \right] \right) + 3 \text{Im} (g_{50} g_{11} \text{Im} g_{11})
\]

\[
+ \frac{1}{w^4} \left[ \text{Im} \left( g_{10} \left( g_{50}^2 - 3g_{50} g_{11} - 4g_{11}^2 \right) \right) + 3 \text{Re} (g_{50} g_{11}) - 2 |g_{50}|^2 \right],
\]

where \( w = \sqrt{|a_1 a_2 - a_1 a_3 - \frac{1}{4} (a_1 + a_3)^2|} \). It is equivalent to the system as follows

\[
\begin{align*}
\dot{m} &= \text{Re} \hat{\lambda} m - \text{Im} \hat{\lambda} n + l_1 m (\bar{m}^2 + \bar{n}^2) + l_2 m (\bar{m}^2 + \bar{n}^2)^2, \\
\dot{n} &= \text{Re} \hat{\lambda} n + \text{Im} \hat{\lambda} m + l_1 n (\bar{m}^2 + \bar{n}^2) + l_2 n (\bar{m}^2 + \bar{n}^2)^2 + \sigma dW^1, \\
\end{align*}
\]

We find the system undergoes Bautin bifurcation at \((m_2, n_2)\) on the Hopf curve if \( \rho < 1 \) and \( \rho - 1 + G \rho > 0 \).

To study the laser system with noise, it has the following form:

\[
\begin{align*}
\dot{m} &= \text{Re} \hat{\lambda} m - \text{Im} \hat{\lambda} n + l_1 m (\bar{m}^2 + \bar{n}^2) + l_2 m (\bar{m}^2 + \bar{n}^2)^2 + \sigma dW^1, \\
\dot{n} &= \text{Re} \hat{\lambda} n + \text{Im} \hat{\lambda} m + l_1 n (\bar{m}^2 + \bar{n}^2) + l_2 n (\bar{m}^2 + \bar{n}^2)^2 + \sigma dW^2. \\
\end{align*}
\]

From Theorem 3.4 and 3.5, we have conclusions

(i) If \( l_2 < 0, \text{Re} \hat{\lambda} < 0, l_1 \leq 0, \text{Re} \hat{\lambda} + 3l_1 \bar{n} \bar{k} < 0 \), then the random equilibrium of equation (6) is globally uniformly attracting;

(ii) If \( l_2 > 0, l_1 < 0, \text{Re} \hat{\lambda} > 0, l_1 < 0, l_2 < 0, \text{Re} \hat{\lambda} + 3l_1 \bar{n} \bar{k} < 0 \), then the random equilibrium of equation (6) is not locally uniformly attracting,

where \( \bar{k} \) is a constant satisfying

\[
\int_0^\infty \int_0^\infty \bar{k} \exp \left( \frac{2}{\sigma^2} \left( \frac{\text{Re} \hat{\lambda} r + \frac{l_1}{2} r^2 + \frac{l_2}{3} r^3}{\sigma^2} \right) \right) dr dt = 1,
\]

\[
\bar{k} = \int_0^\infty r \exp \left( \frac{\text{Re} \hat{\lambda} r + \frac{l_1}{2} r^2 + \frac{l_2}{3} r^3}{\sigma^2} \right) dr.
\]

Now we give some simulations to show the stability at the random Bautin bifurcation point.

First, we obtain a Hopf bifurcation point \((0.73205, 1.73205)\) with parameters valuing \( G = 8.83013, \rho = 0.5, a_1 = 1, a_2 = 3 \). Then we find a Bautin bifurcation point \((1.21832, 1.35238)\) along the Hopf curve with \( G = 9.72065, \rho = 1.50854, a_1 = 1, a_2 = 3 \). By a series of calculations, we obtain \( \text{Re} \hat{\lambda} = 1.40515, l_1 = 0.11254, l_2 = 3.45014, \) corresponding to the coefficients \( \beta_1 = 1.45073, \beta_2 = 0.11254, a = -3.45014, \) and \( b = 0 \) in equation (2), respectively. It should be noted that the bifurcation diagram in Figure 1 is aimed at \( a > 0 \). For the case \( a < 0 \), it is the opposite of the case \( a > 0 \). According to our previous analysis, when the strength of noise \( \sigma \) is in the range of \((0, 10^{-6})\), it is a very small interference to the system. When \( \sigma > 10^{-6} \), the stability starts to be destroyed because it does not satisfy the condition in Theorem 3.3.

Next we give figures with two kinds strength of noises. Figure 6(a) with \( \sigma = 9 \times 10^{-5} \) (the aforementioned subgraph) and \( \sigma = 10^{-6} \) (the other) describe the phase diagrams near the Bautin bifurcation point. The blue line is the original orbits with \( \sigma = 0 \); the green one is with noise. The green orbits in the aforementioned subgraph of Figure 6(a) has been out of the blue orbits. We can only see the green curves in the peripheral areas. For the smaller noise, both curves are almost coincident in the subgraph shown in Figure 6(a).
Figure 6: (a) The phase portraits with noise of different intensity and (b) the partial enlarged drawing of Figure 6(a).

Figure 6(b) is the partial enlarged drawing of Figure 6(a). It is clear to see the departure from original orbits because of the larger noise.

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References


Appendix

Here, we list some classical definitions for the convenience of the reader.

**Definition A.1.** (Brownian motion) The stochastic process \( \{ X(t) : t \geq 0 \} \) is called Brownian motion if the following three conditions hold:

(i) \( X(0) = 0 \);

(ii) For \( 0 \leq t_0 < t_1 < t_2 < \infty \), the increments \( X(t_2) - X(t_1) \) and \( X(t_1) - X(t_0) \) are independent;

(iii) For \( 0 \leq t_1 < t_2 < \infty \), \( X(t_2) - X(t_1) \) is normally distributed with mean zero and variance \( t_2 - t_1 \); that is, \( X(t_2) - X(t_1) \sim N(0, t_2 - t_1) \).

Standard Brownian motion is Brownian motion with \( X(0) = 0 \) and the variance equals one. Standard Brownian motion is also referred to as the Wiener process. A two-sided Brownian motion is the process \( \{ X'_t : t \geq 0 \} \) such that \( \{ X_t : t \geq 0 \} \) and \( \{ X_t : t \geq 0 \} \) are independent standard Brownian motions.

**Definition A.2.** (Ergodic dynamical system) A metric dynamical system \((\Omega, \mathcal{F}, \mu, (\theta(t))_{t \in \mathbb{T}})\) with time \( \mathbb{T} \) is called ergodic if all sets in \( \mathcal{F} \) have probability 0 or 1, where \( \mathcal{I} \subset \mathcal{F} \) is a sub-\( \sigma \)-algebra formed by the family of measurable invariant sets of \((\theta(t))_{t \in \mathbb{T}}\).

**Definition A.3.** (Random dynamical system [1]) A measurable random dynamical system on the measurable space \((X, \mathcal{B})\) over a metric dynamical system \((\Omega, \mathcal{F}, \mu, (\theta(t))_{t \in \mathbb{T}})\) with time \( \mathbb{T} \) is a mapping \( \varphi : \mathbb{T} \times \Omega \times X \rightarrow X, (t, \omega, x) \mapsto \varphi(t, \omega, x) \), with the following properties:

(i) Measurability: \( \varphi \) is \( \mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B} \)-measurable.

(ii) Cocycle property: The mappings \( \varphi(t, \omega) = \varphi(t, \omega, \cdot) : X \rightarrow X \) form a cocycle over \( \theta(\cdot) \), i.e., they satisfy \( \varphi(0, \omega) = \text{id} \), for all \( \omega \in \Omega \) and \( \varphi(t + s, \omega) = \varphi(t, \theta(s)\omega) \circ \varphi(s, \omega) \), for all \( s, t \in \mathbb{T} \), \( \omega \in \Omega \).

**Definition A.4.** (Random set [8]) Let \( (X, d) \) be a complete and separable metric space.

(i) A family \( \mathcal{D} = \{ D_\omega \}_{\omega \in \Omega} \) of nonempty subsets of \( X \) is called a random set if the mapping \( \omega \mapsto \text{dist}(x, D_\omega) \) is \( \mathcal{F} \)-measurable for all \( x \in X \), where \( \text{dist}(x, D_\omega) = \inf_{y \in D_\omega} d(x, y) \).

(ii) A random set \( \mathcal{D} \) is called a random closed set if \( D_\omega \) is closed for each \( \omega \in \Omega \) and a random compact set if \( D_\omega \) is compact for each \( \omega \in \Omega \).

(iii) A random set \( \mathcal{D} \) is said to be tempered if there exists a \( x_0 \in X \) such that \( D_\omega \subset \{ x \in X : d(x, x_0) \leq r(\omega) \} \) for all \( \omega \in \Omega \), where the random variable \( r(\omega) > 0 \) is tempered, i.e., \( \sup_{t \in \mathbb{R}} r(\theta(t)\omega) e^{-r(t)} < \infty \) for all \( y > 0 \) and \( \omega \in \Omega \).

The collection of all tempered random sets in \( X \) will be denoted by \( \mathcal{D} \).

**Definition A.5.** (Random attractor [8]) A random compact set \( \mathcal{A} = \{ A_{\omega} \}_{\omega \in \Omega} \) form \( \mathcal{D} \) is called a random attractor of RDS \((\theta, \varphi)\) on \( \Omega \times X \) in \( \mathcal{D} \) if \( \mathcal{A} \) is a \( \varphi \)-invariant set, i.e., \( \varphi(t, \omega, A_\omega) = A_{\theta(t)\omega} \) for all \( t \geq 0 \) and \( \mathcal{D} \in \mathcal{D} \), and pathwise pullback attracting in \( \omega \in \Omega \), i.e.,

\[
\lim_{t \to \infty} \text{dist}(\varphi(t, \theta(t)\omega, D_{\theta(t)\omega}), A_{\omega}) = 0, \quad \text{for all } \omega \in \Omega \quad \text{and} \quad \mathcal{D} \in \mathcal{D}.
\]

**Definition A.6.** (Globally uniformly attracting) We call the random attractor \( \mathcal{A} \) globally uniformly attracting if for any \( \delta > 0 \), there exists \( \delta > 0 \) such that

\[
\lim_{t \to \infty} \sup_{\omega \in \Omega} \sup_{x \in \mathcal{B}(0)} \| \varphi(t, \omega, A_\omega(x)) - A(\theta(t)\omega) \| = 0.
\]

We call it locally uniformly attracting if there exists a \( \delta > 0 \) such that the above holds.
Appendix B includes some known results in given [4], which we have used.

**Proposition B.1.**

\[
\begin{align*}
\frac{dx}{dt} &= (ax - \beta y) + (ax - by)(x^2 + y^2) dt + \sigma_1 dW(t), \\
\frac{dy}{dt} &= (ay - \beta x) + (ay + bx)(x^2 + y^2) dt + \sigma_2 dW(t).
\end{align*}
\] (A1)

Suppose that the largest Lyapunov exponent \( \lambda_{\text{top}} \) of RDS generated by the stochastic differential equation (A1) is negative. Then the fibers of the random attractor are singletons, given by \( \mathcal{F}^{-\infty}_\omega \)-measurable map \( A : \Omega \to \mathbb{R}^2 \). Furthermore, the following statements hold:

(i) \( A(\theta_\omega) \) is a random equilibrium of RDS, for all \( t \geq 0 \) and almost all \( \omega \in \Omega \).

(ii) The random equilibrium is distributed according to the stationary density.

(iii) The largest Lyapunov exponent of the linearization along the random equilibrium \( a, \xi = Df(A(\theta_\omega))\xi \) is equal to \( \lambda_{\text{top}} = \lim_{t \to \infty} \frac{1}{t} \ln \| \Phi(t, \omega, V) \| \).

**Theorem B.2.** Suppose that \( \lambda_{\text{top}} < 0 \). Then the random attractor \( A \) for the stochastic differential equation (A1) is given by a random equilibrium, i.e., \( A(\omega) \) is a singleton for almost all \( \omega \in \Omega \). In addition, the stochastic differential equation admits exponentially fast synchronization, i.e., for all \( V_1, V_2 \in \mathbb{R}^2 \), for almost all \( \omega \in \Omega \),

\[
\lim_{T \to \infty} \sup_{T} \frac{1}{2} \ln \| \varphi(t, \omega, V_1) - \varphi(t, \omega, V_2) \| < 0.
\]

**Proposition B.3.** Consider system such that \( b \leq \kappa \) (which means the largest Lyapunov exponent \( \lambda_{\text{top}} < 0 \)). Then for any \( y \in \mathbb{R}^2, \varepsilon > 0, \) and \( T \geq 0 \), there exists a set \( E \in \mathcal{F}^{-\infty}_\omega \) with \( P(E) > 0 \) such that \( A(\theta_\omega) \in B_\varepsilon(y) \), for all \( s \in [0, T] \) and \( \omega \in E \), where \( A(\omega) \) is the unique random equilibrium for system from Proposition B.1 here.