Existence of nontrivial solutions for the Klein-Gordon-Maxwell system with Berestycki-Lions conditions

Abstract: In this article, we study the following Klein-Gordon-Maxwell system:

\[
\begin{align*}
-\Delta u - (2\omega + \phi)\phi u &= g(u), & \text{in } \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi)u^2, & \text{in } \mathbb{R}^3,
\end{align*}
\]

where \(\omega\) is a constant that stands for the phase; \(u\) and \(\phi\) are unknowns and \(g\) satisfies the Berestycki-Lions condition \([\text{Nonlinear scalar field equations. I. Existence of a ground state}, \text{Arch. Rational Mech. Anal. 82 (1983), 313–345;} \text{Nonlinear scalar field equations. II. Existence of infinitely many solutions}, \text{Arch. Rational Mech. Anal. 82 (1983), 347–375}].\) The Klein-Gordon-Maxwell system is a model describing solitary waves for the nonlinear Klein-Gordon equation interacting with an electromagnetic field. By using variational methods and some analysis techniques, the existence of positive solution and multiple solutions can be obtained. Moreover, we study the properties of decay estimates and asymptotic behavior for the positive solution.

Keywords: Klein-Gordon-Maxwell system, positive solution, multiple solutions, decay estimate, asymptotic behavior

MSC 2020: 35J47, 35J60, 35J50

1 Introduction and main results

In this article, we investigate the following Klein-Gordon-Maxwell system:

\[
\begin{align*}
-\Delta u - (2\omega + \phi)\phi u &= g(u), & \text{in } \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi)u^2, & \text{in } \mathbb{R}^3,
\end{align*}
\]

where \(g \in C(\mathbb{R}, \mathbb{R})\) satisfies the following conditions:

\((g_1)\) \(\liminf_{s \to 0} \frac{g(s)}{s} \leq \limsup_{s \to 0} \frac{g(s)}{s} = -m < 0.\)

\((g_2)\) \(\limsup_{s \to \infty} \frac{g(s)}{s^2} \leq 0.\)

\((g_3)\) there exists \(\zeta > 0\) such that \(G(\zeta) = \int_0^\zeta g(s)ds > 0,\) where \(G(s) = \int_0^s g(t)dt.\)
Such conditions were first raised by Berestycki and Lions [8,9] in the study of the nonlinear Schrödinger equations. In view of [8,9], \( (g_0) \) is almost necessary, while \((g_0)\) and \((g_0)\) are necessary for the existence of a nontrivial solution. There are many problems discussing the existence of solutions under Berestycki-Lions conditions, such as Schrödinger-Poisson system, Choquard equation, and Kirchhoff equation. For more details, we refer readers to [2,3,33,39].

The system (1.1) comes from the following system of nonlinear Klein-Gordon equations coupled with Maxwell’s equation:

\[
\begin{align*}
\Box u + \left( |\nabla S - eA|^2 - \left( \frac{\partial S}{\partial t} + e\phi \right)^2 + m_0^2 \right) u - |u|^{p-2}u &= 0, \\
\frac{\partial}{\partial t} \left( \frac{\partial S}{\partial t} + e\phi \right) u^2 &= -\nabla \cdot \left( \nabla S - eA \right) u^2, \\
\nabla \cdot \left( \frac{\partial A}{\partial t} + \nabla \phi \right) &= \frac{\partial S}{\partial t} + e\phi \left| u \right|^2, \\
\nabla \times \left( \nabla \times A \right) + \frac{\partial A}{\partial t} + \nabla \phi &= e\left( \nabla S - eA \right) u^2,
\end{align*}
\]

where \( \Box \) is wave operator, \( e, m_0, \omega > 0, 2 < p < 6, u(x, t) \in \mathbb{R}, (\phi(x, t), A(x, t)) \in \mathbb{R} \times \mathbb{R}^3 \). Such type of system appears in a very interesting physical context: as a model describing solitary waves for the nonlinear Klein-Gordon equation interacting with an electromagnetic field. Here, \( m_0, e, \) and \( \omega \) represent the mass, the charge of the particle, and the phase, respectively. The unknowns of the system (1.2) are the field \( u \) associated to the particle and the electric potential \( \phi \). In order to look for the standing wave solutions for (1.2), we consider the system (1.2) in its electrostatic form, namely, set \( u = u(x), S = \omega t, A = 0, \phi = \phi(x), e = 1, \) and then the following form can be obtained:

\[
\begin{align*}
-\Delta u + (m_0^2 - \omega^2)u - (2\omega + \phi)\phi u &= |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi)u^2, & \text{in } \mathbb{R}^1.
\end{align*}
\]

For more mathematical and physical background about the Klein-Gordon-Maxwell system, we refer the readers to [6,7] and the references therein.

Benci and Fortunato in [7] first studied the Klein-Gordon-Maxwell system by using the variational method. They obtained infinitely many radially symmetric solutions for the system (1.3) when \( m_0 > \omega > 0 \) and \( 4 < p < 6 \). Later, it is improved by the result in [19] if one of the following conditions holds:

- \( m_0 > \omega > 0 \) and \( 4 \leq p < 6 \),
- \( m_0\sqrt{p-2} \geq \sqrt{2}\omega > 0 \) and \( 2 < p < 4 \).

With the aid of the Pohozaev identity and monotonicity trick (see [22,31,32]), the range of \( p \) is further extended in [5]. Azzollini and Pomponio in [5] proved the existence of a nontrivial solution of system (1.3), and one of the assumptions is satisfied

- \( m_0 > \omega > 0 \) and \( 3 \leq p < 6 \),
- \( m_0\sqrt{(p-2)(4-p)} \geq \omega > 0 \) and \( 2 < p < 3 \).

In [4], the existence of ground state solutions for the system (1.3) was first studied by using the Nehari manifold if one of the assumptions holds

- \( m_0 > \omega > 0 \) and \( 4 \leq p < 6 \),
- \( m_0\sqrt{p-2} \geq \omega\sqrt{6-p} > 0 \) and \( 2 < p < 4 \).

Afterward, Wang in [36] extended the result in [4]. He obtained the same conclusion based on the properties of \( \phi \), when one of the following hypotheses is satisfied

- \( m_0 > \omega > 0 \) and \( 4 \leq p < 6 \),
• \[ m_0 \left(1 + \frac{(q - p^2)}{4p - 2} \right) > \omega > 0 \] and \( 2 < p < 4 \).

Also, the paper in [16] also got the existence of ground state solutions if one of the following assumptions holds:
• \( m_0 > \omega > 0 \) and \( 4 \leq p < 6 \),
• \( m_0 \sqrt{\frac{(8p - 2)}{p^2}} > \omega > 0 \) and \( 2 < p < 4 \).

Chen and Tang in [17] generalized \(|u|^{p-2}u\) to the nonlinear term \( g(u) \), which satisfies the following conditions:

\begin{align*}
\text{(AR) } & \text{There exists } \mu > 2 \text{ such that } g(t)t \geq \mu G(t), \text{ for all } t \in \mathbb{R}. \\
\end{align*}

They proved the existence of ground state solutions when one of the assumptions is satisfied
• \( m_0 > \omega > 0 \) and \( 4 \leq \mu < 6 \),
• \( m_0 \sqrt{\frac{m - 2}{\omega} - \mu} > 0 \) and \( 2 < \mu < 4 \).

Replacing \(|u|^{p-2}u\) with \( \lambda |u|^{p-2}u + |u|^{2-2}u \), [11,13,35] researched the existence of solutions for the Klein-Gordon-Maxwell system with the critical case. Meanwhile, there are many literature studying the Klein-Gordon-Maxwell system with the potential \( V \), namely, \( V(x) \neq m_0^2 - \omega^2 \). The potentials \( V \) to be coercive, periodic, steep well, and vanishing at infinity are introduced by different mathematicians, which can be found in [12,14–16,24–27,29,34,38].

Motivated by the works mentioned earlier, the first purpose of this article is to study the existence of a positive solution for the Klein-Gordon-Maxwell system by using the variational method when the nonlinear term satisfies the Berestycki-Lions conditions. Here, we consider the situation where \( m_0 = \omega \), and then system (1.3) turn into the system (1.1). Moreover, we study the properties of decay estimates at infinity and asymptotic behavior for the positive solution in the next two theorems. Furthermore, multiple solutions can be gained through some analytical techniques at last.

Now we can present the main results of this article.

**Theorem 1.1.** Assume that \((g_1)-(g_3)\) hold. Then there exists \( \omega_0 > 0 \) such that system \((1.1)\) has a nontrivial positive solution for \( 0 < \omega < \omega_0 \).

**Remark 1.1.** To our best knowledge, the existence of nontrivial solutions for the autonomous Klein-Gordon-Maxwell system (namely, \( V(x) = m_0^2 - \omega^2 \)) was studied in the case where \( g \) satisfies (AR) condition in [17]. Evidently, Berestycki-Lions conditions is weaker than (AR) condition. The (AR) condition is used to obtain the boundedness of Palais-Smale sequence and to recover the compactness. It seems difficult to show the Palais-Smale condition for the energy functional corresponding to system (1.1) when \( g \) satisfies Berestycki-Lions conditions. Here, we make use of the monotonicity trick method, which was developed by Struwe [31,32] and Jeanjean [22] to overcome the difficulty.

**Remark 1.2.** Notice that the nonlocal term \( \phi \) of the Klein-Gordon-Maxwell system has no specific expression. Influenced by the nonlocal term \( \phi \), we need to modify the energy functional corresponding to system (1.1) by using the truncation technique in this article. In the previous study of the Klein-Gordon-Maxwell system, we know that the integral with nonlocal term \( \phi \) is controlled by the \( L^2(\mathbb{R}^3) \) norm of \( u \). But \( H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3) \) is not compact, which brings difficulties to obtain the (PS) condition (the short for Palais-Smale condition). In order to obtain our conclusion smoothly, we also have to rely on a new property, which is introduced in Lemma 2.4 of Section 2.

Next we research the properties of the positive solution gained in Theorem 1.1. First, we introduce the conclusion of decay estimates at infinity for the positive solution and its derivatives.
Theorem 1.2. Suppose that \((g_1) - (g_3)\) hold and \(u_\omega\) is a positive solution of system (1.1) for any \(\omega \in (0, \omega_0)\) in \(H^1(\mathbb{R}^3)\). Then there exist constants \(M, \zeta, R_0 > 0\), such that for all \(|x| > R_0\),
\[
|D^i u_\omega| \leq Me^{-\zeta|x|}, \quad i = 0, 1, 2.
\]

Second, we show that the positive solution of system (1.1) tends to the positive solution of the Schrödinger equation as \(\omega \to 0\).

Theorem 1.3. Suppose that \((g_1) - (g_3)\) hold and \(\{u_{\omega_n}\} \subset H^1(\mathbb{R}^3)\) are the positive solutions obtained in Theorem 1.1 for each \(n \in \mathbb{N}\). Then \(u_{\omega_n} \to u_\ast\) in \(H^1(\mathbb{R}^3)\) as \(\omega_n \to 0\), where \(u_\ast\) is a positive solution of
\[
-\Delta u = g(u), \quad x \in \mathbb{R}^3.
\]

Remark 1.3. As far as we know, there are few papers studying the tendency of the positive solution from the Klein-Gordon-Maxwell system to the Schrödinger equation.

Finally, we discuss the multiple solutions for system (1.1).

Theorem 1.4. Assume that \(g\) is odd and satisfies \((g_1) - (g_3)\). Then for any fixed \(k \in \mathbb{N}\), there exists \(\omega_k > 0\) such that system (1.1) has at least \(k\) pairs \(\pm u_1, \pm u_2, \ldots, \pm u_k\) solutions for \(0 < \omega < \omega_k\).

Remark 1.4. For all we know, the monotonicity trick method due to Struwe in [31, 32] and Jeanjean in [22] cannot obtain the existence of multiple solutions. Thus, how to obtain the boundedness of Palais-Smale sequences has become one of our difficulties again. To overcome this difficulty, we borrow a different method instead of using the monotonicity trick method. A key of the method is to look for a Palais-Smale sequence with an extra property related to the Pohozaev-Nehari type identity.

Remark 1.5. Truncation technique is also used in the proof of Theorem 1.4. Let \(I_\omega\) be the energy functional corresponding to system (1.1) and \(I_\omega^T\) be the energy functional about \(I_\omega\) with a cut off function. With the help of [9, 20], the existence of infinitely many solutions for \(I_\omega^T\) can be proved. In our proof in Section 5, one easily sees that the energy of the solutions for \(I_\omega^T\) tends to infinity. If we claim that the solutions for \(I_\omega^T\) are the ones for \(I_\omega\), the solutions’ value of \(L^p(\mathbb{R}^3)\) norm is needed to be finite. Hence, we can only obtain the existence of \(k\) pairs solutions instead of infinitely many solutions for our problem.

This article is organized as follows. In Section 2, we give the variational framework for system (1.1) and some preliminary lemmas. In Section 3, we research the existence of a nontrivial positive solution. And the properties of decay estimates at infinity and asymptotic behavior for positive solutions are considered in Section 4. Finally, Section 5 studies the existence of multiple solutions.

Throughout the article, the following notations will be used.

- \(H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}\).
- \(H^1_x(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}\).
- \(D^{1,2}(\mathbb{R}^3) = \{u \in L^2_x(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}\).
- \(D^{1,2}_\omega(\mathbb{R}^3) = \{u \in D^{1,2}(\mathbb{R}^3) : u(x) = u(|x|)\}\).

- \(\|u\| = \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}}\).
- \(\|u\|_p = \left( \int_{\mathbb{R}^3} |u|^p dx \right)^{\frac{1}{p}}\), for all \(p \in (1, 6)\).
- \(\|u\|_{D^{1,2}_\omega(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} (|\nabla u|^2) dx \right)^{\frac{1}{2}}\).
S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$ and

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)} \frac{\|\nabla u\|_2}{\|u\|_6}, \quad \|u\|_6 \leq S^{-1} \|\nabla u\|_2.$$ 

• $\langle \cdot, \cdot \rangle$ denotes the action of dual.
• $o(1)$ denotes a quantity that goes to zero.
• $C$ denotes various positive constants, which may change from line to line.

2 Preliminaries

Evidently, a weak solution $(u, \phi) \in H^1_t(\mathbb{R}^3) \times D^{1,2}_t(\mathbb{R}^3)$ for the system (1.1) being a critical point of the functional $I : H^1_t(\mathbb{R}^3) \times D^{1,2}_t(\mathbb{R}^3) \to \mathbb{R}$ can be defined by

$$I(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 - |\nabla \phi|^2 - (2\omega + \phi) u_3^2)dx - \int G(u)dx. \quad (2.1)$$

We recall the following facts to reduce the functional $I$ in the only variable $u$.

Lemma 2.1. [7] For any fixed $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$, which solves the following equation

$$-\Delta \phi + \phi u^3 = -\omega u^3. \quad (2.2)$$

Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \phi_u$ is continuously differentiable, and

$$-\omega \leq \phi_u \leq 0 \quad \text{on the set } \{x : u(x) \neq 0\}. \quad (2.3)$$

Multiplying (2.2) by $\phi_u$ and integrating by parts, one sees

$$\int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx = -\int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx. \quad (2.4)$$

Combining (2.1) and (2.4), one can rewrite $I$ as the functional $I : H^1_t(\mathbb{R}^3) \to \mathbb{R}$ as follows:

$$I_0(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} G(u)dx. \quad (2.5)$$

By using $(g_u)$ and $(g_2)$, $I$ belongs to $C(H^1_t(\mathbb{R}^3), \mathbb{R})$. For all $u, v \in H^1_t(\mathbb{R}^3)$, its derivative form can be written as follows:

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v - (2\omega + \phi_u) \phi_u u v)dx - \int_{\mathbb{R}^3} g(u)vdx. \quad (2.6)$$

Lemma 2.2. [7] If $u$ is radially symmetric, then $\phi_u$ is radially symmetric.

Lemma 2.3. If $u_n \to u$ in $H^1_t(\mathbb{R}^3)$, then $\phi_{u_n} \to \phi_u$ in $D^{1,2}_t(\mathbb{R}^3)$ and $\phi_{u_n} \to \phi_u$ in $L^6(\mathbb{R}^3)$. As a consequence, $I(u_n) \to I(u)$ in the sense of distributions.

Proof. As the proof of Lemma 3.2 in [13] and Lemma 2.7 in [4], we omit it. 

Here, we give a new property for the Klein-Gordon-Maxwell system on nonlocal terms $\phi_u$, which plays an important role in the full text.
Lemma 2.4. For any $\alpha \in (2, 3)$, there exist $\beta > \alpha$ and $C > 0$ such that
\[ \int_{\mathbb{R}^3} |\phi_u u|^2 \, dx \leq C \|u\|_\alpha^\beta. \]

Proof. Let $2 < \lambda < 3$. By the second equation of system (1.1), one has
\[ \int_{\mathbb{R}^3} |\nabla \phi_u|^2 \, dx = - \int_{\mathbb{R}^3} (\omega + \phi_u) \phi_u u^2 \, dx \leq \omega^2 \int_{\mathbb{R}^3} |\nabla \phi_u|^2 \, dx \leq \omega^2 \int_{\mathbb{R}^3} |\phi_u|^2 \, dx \frac{8 - \lambda}{(4 - \lambda)} \left( \int_{\mathbb{R}^3} |u|^{\frac{12}{3 - \lambda}} \, dx \right)^\frac{8 - \lambda}{3(4 - \lambda)}. \] (2.7)

Thus,
\[ \int_{\mathbb{R}^3} |\nabla \phi_u|^2 \, dx \leq \omega^2 S \left( \int_{\mathbb{R}^3} |u|^{\frac{12}{3 - \lambda}} \, dx \right)^\frac{8 - \lambda}{3(4 - \lambda)}. \] (2.8)

By combining (2.7) and (2.8), one sees
\[ \int_{\mathbb{R}^3} |\phi_u u|^2 \, dx \leq \omega^2 S \left( \int_{\mathbb{R}^3} |u|^{\frac{12}{3 - \lambda}} \, dx \right)^\frac{8 - \lambda}{3(4 - \lambda)}. \]

Let $\alpha = \frac{12}{8 - \lambda}$ and $\beta = \frac{4}{4 - \lambda}$, we obtain the conclusion. \qed

To achieve our goal, here we make use of a cut-off function $\chi \in C^\infty (\mathbb{R}^+, \mathbb{R})$ satisfying
\[
\begin{cases}
\chi(s) = 1 & \text{for } s \in [0, 1], \\
0 \leq \chi(s) \leq 1 & \text{for } s \in (1, 2), \\
\chi(s) = 0 & \text{for } s \in [2, +\infty), \\
\|\chi\|_{C^2} \leq 2,
\end{cases}
\]
and we change the functional $I_\omega$ to $I^T_\omega : H^1_\omega (\mathbb{R}^3) \to \mathbb{R}$
\[ I^T_\omega(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{2} \chi \left( \frac{\|u\|_\alpha^\beta}{T^\beta} \right) \int_{\mathbb{R}^3} \omega \phi_u u^2 \, dx - \int_{\mathbb{R}^3} G(u) \, dx. \] (2.9)

Then, we can obtain for all $v \in H^1_\omega (\mathbb{R}^3)$,
\[
\langle (I^T_\omega)'(u), v \rangle \int_{\mathbb{R}^3} \nabla u \nabla v \, dx - \chi \left( \frac{\|u\|_\alpha^\beta}{T^\beta} \right) \int_{\mathbb{R}^3} (2\omega + \phi_u) \phi_u u v \, dx - \frac{\beta \omega}{2T^\beta \chi} \left( \frac{\|u\|_\alpha^\beta}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_u u^2 \, dx \left( \int_{\mathbb{R}^3} |u|^{\alpha^2} \, dx \right)^{\frac{8 - \lambda}{2 - 1}} \int_{\mathbb{R}^3} |u|^{\alpha^2 - 2} u v \, dx - \int_{\mathbb{R}^3} G(u) v \, dx. \] (2.10)

By the definition of $\chi$ and $I^T_\omega$, it easily sees that $u$ is a critical point of $I_\omega$ if $u$ is a critical point of $I^T_\omega$ with $\|u\|_\alpha \leq T$.

Here, we study the Pohozaev identity for $I^T_\omega$, which is pivotal to prove the existence of solutions.
Lemma 2.5. Any critical point \( u \) of \( I_{\omega}^T \) satisfies the following Pohozaev equality:

\[
\int_{\mathbb{R}^3} |\nabla u|^2 \, dx - 5\omega \chi \left( \frac{\|u\|_\beta^p}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_x u^2 \, dx - 2\chi \left( \frac{\|u\|_\alpha^\beta}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_{xx} u^2 \, dx
\]

\[- \frac{3\beta \omega}{2\alpha T^\beta} \chi \left( \frac{\|u\|_\alpha^\beta}{T^\beta} \right) \cdot \int_{\mathbb{R}^3} \phi u^2 \, dx \cdot \left( \int_{\mathbb{R}^3} |u|^a \, dx \right)^{\frac{\beta}{a}} = 6 \int_{\mathbb{R}^3} G(u) \, dx.
\]  

(2.11)

Proof. If \( u \) is a critical point of \( I_{\omega}^T \), then \( (I_{\omega}^T)'(u) = 0 \), i.e., \( u, \phi \in H^1_{\text{loc}}(\mathbb{R}^3) \) solve

\[
\begin{cases}
-\Delta u - \chi \left( \frac{\|u\|_\beta^p}{T^\beta} \right) (2\omega + \phi) u - \frac{\beta \omega}{2T^\beta} h_\tau(u)|u|^{a-2} u = g(u), & \text{in } \mathbb{R}^3, \\
\Delta \phi = (\omega + \phi) u^2, & \text{in } \mathbb{R}^3,
\end{cases}
\]

(2.12)

where \( h_\tau(u) = \chi \left( \frac{\|u\|_\beta^p}{T^\beta} \right) \int_{\mathbb{R}^3} \phi u^2 \, dx \left( \int_{\mathbb{R}^3} |u|^a \, dx \right)^{\frac{\beta}{a}} \). By using Lemma 3.1 in [18], one sees that

\[
\int_{\mathbb{R}^3} -\Delta u \cdot \nabla u \, dx = -\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{R} \int_{\partial B_R} |x \cdot \nabla u|^2 \, d\sigma + \frac{R}{2} \int_{\partial B_R} |\nabla u|^2 \, d\sigma,
\]

(2.13)

\[
\int_{B_R} (2\omega + \phi) \phi u x \cdot \nabla u \, dx = -\int_{B_R} (\omega + \phi) u^2 x \cdot \nabla \phi \, dx - \frac{3}{2} \int_{B_R} (2\omega + \phi) \phi u^2 \, dx + \frac{R}{2} \int_{\partial B_R} (2\omega + \phi) \phi u^2 \, d\sigma,
\]

(2.14)

\[
\int_{B_R} |u|^{a-2} u x \cdot \nabla u \, dx = -\frac{3}{\alpha} \int_{B_R} |u|^a \, dx + \frac{R}{\alpha} \int_{\partial B_R} |u|^a \, d\sigma,
\]

(2.15)

\[
\int_{B_R} g(x) x \cdot \nabla u \, dx = -3 \int_{B_R} G(x) \, dx - R \int_{\partial B_R} G(u) \, d\sigma,
\]

(2.16)

where \( B_R \) is the ball of \( \mathbb{R}^3 \) centered in the origin and with radius \( R \). Multiplying the first equation of system (2.12) by \( x \cdot \nabla u \) and integrating on \( B_R \), from (2.13)–(2.16), it gets that

\[
-\frac{1}{2} \int_{B_R} |\nabla u|^2 \, dx + \chi \left( \frac{\|u\|_\beta^p}{T^\beta} \right) \int_{B_R} (\omega + \phi) u^2(x \cdot \nabla \phi) \, dx + \frac{3}{2} \chi \left( \frac{\|u\|_\alpha^\beta}{T^\beta} \right) \int_{B_R} (2\omega + \phi) \phi u^2 \, dx
\]

\[+ \frac{3\beta \omega}{2\alpha T^\beta} \chi \left( \frac{\|u\|_\alpha^\beta}{T^\beta} \right) \cdot \int_{\mathbb{R}^3} \phi u^2 \, dx \cdot \left( \int_{\mathbb{R}^3} |u|^a \, dx \right)^{\frac{\beta}{a}} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + 3 \int_{B_R} G(u) \, dx
\]

\[= \frac{1}{R} \int_{\partial B_R} |x \cdot \nabla u|^2 \, d\sigma - \frac{R}{2} \int_{\partial B_R} |\nabla u|^2 \, d\sigma + \frac{R}{2} \chi \left( \frac{\|u\|_\beta^p}{T^\beta} \right) \int_{\partial B_R} (2\omega + \phi) \phi u^2 \, d\sigma
\]

\[+ \frac{R\beta \omega}{2\alpha T^\beta} \chi \left( \frac{\|u\|_\alpha^\beta}{T^\beta} \right) \cdot \int_{\mathbb{R}^3} \phi u^2 \, dx \cdot \left( \int_{\mathbb{R}^3} |u|^a \, dx \right)^{\frac{\beta}{a}} \int_{\partial B_R} |u|^a \, d\sigma + R \int_{\partial B_R} G(u) \, d\sigma.
\]

(2.17)

Multiplying the first equation of system (2.12) by \( x \cdot \nabla \phi \) and integrating on \( B_R \), we see that

\[
\int_{B_R} (\omega + \phi) u^2 x \cdot \nabla \phi \, dx = \int_{B_R} \Delta \phi(x \cdot \nabla \phi) \, dx
\]

\[= \frac{1}{2} \int_{B_R} |\nabla \phi|^2 \, dx + \frac{1}{R} \int_{\partial B_R} |x \cdot \nabla \phi|^2 \, d\sigma - \frac{R}{2} \int_{\partial B_R} |\nabla \phi|^2 \, d\sigma.
\]

(2.18)
By combining (2.17) and (2.18), one has

\[-\frac{1}{2}\int_{\Omega_{R}}|\nabla u|^2\,dx + \frac{1}{2}\int_{\Omega_{R}}(|\nabla \phi|^2\,dx + \frac{3}{2\beta}\int_{\Omega_{R}} \left(\frac{|u|^{\beta}}{T_{\beta}}\right)\int_{\Omega_{R}} (2\omega + \phi)\phi u^2\,dx
\]

\[+ \frac{3\beta\omega}{2\alpha T_{\beta}}X\left(\frac{|u|^{\beta}}{T_{\beta}}\right)\int_{\Omega_{R}} \phi u^2\,dx \cdot \left(\int_{\Omega_{R}} |u|^\beta\,dx\right) \frac{\beta-1}{\beta} \int_{\Omega_{R}} |u|^\beta\,dx + 3 \int_{\Omega_{R}} G(u)\,dx\]

\[= \frac{1}{R}\int_{\partial\Omega_{R}} |x \cdot \nabla u|^2\,d\sigma - \frac{R}{2}\int_{\partial\Omega_{R}} |\nabla u|^2\,d\sigma + \frac{R}{2}\int_{\partial\Omega_{R}} \left(\frac{|u|^{\beta}}{T_{\beta}}\right)\int_{\partial\Omega_{R}} (2\omega + \phi)\phi u^2\,d\sigma
\]

\[+ \frac{R\beta\omega}{2\alpha T_{\beta}}X\left(\frac{|u|^{\beta}}{T_{\beta}}\right)\int_{\partial\Omega_{R}} \phi u^2\,dx \cdot \left(\int_{\partial\Omega_{R}} |u|^\beta\,d\sigma\right) \frac{\beta-1}{\beta} \int_{\partial\Omega_{R}} |u|^\beta\,d\sigma + R \int_{\partial\Omega_{R}} G(u)\,d\sigma
\]

\[\frac{1}{R^2}\int_{\Omega_{R}} |x \cdot \nabla \phi|^2\,d\sigma + \frac{R}{2}\int_{\Omega_{R}} \left(\frac{|u|^{\beta}}{T_{\beta}}\right)\int_{\Omega_{R}} |\nabla \phi|^2\,d\sigma.\]

(2.19)

Similar to [18], the right-hand side of (2.19) goes to zero as \(R \to +\infty\), and we obtain that

\[-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla \phi|^2\,dx + \frac{3}{2\beta}\int_{\mathbb{R}^3} \left(\frac{|u|^{\beta}}{T_{\beta}}\right)\int_{\mathbb{R}^3} (2\omega + \phi)\phi u^2\,dx
\]

\[+ \frac{3\beta\omega}{2\alpha T_{\beta}}X\left(\frac{|u|^{\beta}}{T_{\beta}}\right)\int_{\mathbb{R}^3} \phi u^2\,dx \cdot \left(\int_{\mathbb{R}^3} |u|^\beta\,dx\right) \frac{\beta-1}{\beta} \int_{\mathbb{R}^3} |u|^\beta\,dx + 3 \int_{\mathbb{R}^3} G(u)\,dx = 0.\]

(2.20)

Note that (2.4) holds. From (2.4) and (2.20), we can easily complete the proof. \(\square\)

### 3 Existence of positive solutions

To obtain a positive solution, let’s deal with the nonlinear term here. Following the idea in [8], we can define \(s_0 := \min\{s \in [\zeta, +\infty)\} \) \((s_0 = +\infty\) if \(g(s) \neq 0\) for any \(s \geq \zeta)\) and set \(\overline{g} : \mathbb{R} \to \mathbb{R}\) as follows:

\[
\overline{g}(s) = \begin{cases} 
  g(s) & s \in [0, s_0], \\
  0 & s \in \mathbb{R}, \{0, s_0\}, \\
  (g(-s) - ms) & s \in \mathbb{R}_{\leq 0}.
\end{cases}
\]

(3.1)

Then we obtain that \(0 \leq u \leq s_0\) if \(u\) is a nontrivial solution of (1.1) with \(\overline{g}\) taking the place of \(g\). Indeed, one can easily see that

\(\overline{g}(s) \geq -ms\), \(\quad\) for all \(s \in \mathbb{R}_{\leq 0}\).

If \(u\) is a nontrivial solution of (1.1), then \(u\) satisfies

\[-\Delta u - (2\omega + \phi)\phi u = \overline{g}(u), \quad \text{in } \mathbb{R}^3.\]

Multiplying the aforementioned formula by \(u^r\) and integrating by parts, one sees

\[\int_{\mathbb{R}^3} |\nabla u|^2\,dx - \int_{\mathbb{R}^3} (2\omega + \phi)\phi u^2\,dx = \int_{\mathbb{R}^3} \overline{g}(u)u^r\,dx = 0.\]

With the fact that \(-\omega \leq \phi \leq 0\) (see [7]), we know that \(u^r \equiv 0\) and then

\(u \geq 0.\)
On the other hand, set $\Omega = \{ x \in \mathbb{R}^3 | u(x) > s_0 \}$ and $w(x) = u(x) - s_0$. Then for $x \in \Omega$,

$$-\Delta w(x) = -\Delta u(x) = (2 \omega + \phi)u + \bar{g}(u) \leq 0;$$

$w(x) = u(x) - s_0 = 0$, for $x \in \partial \Omega$. From the maximum principle, $w(x) \equiv 0$ for $x \in \Omega$ and then $u(x) \equiv s_0$ for $x \in \Omega$. That is,

$$u(x) \leq s_0.$$

By using the strong maximum principle, it sees that $0 < u < s_0$, and so it is a positive solution of (1.1). Thus, for convenience, we can suppose that $g$ is defined as $\bar{g}$ in (3.1). Also, the condition $(g_2)$ can be replaced as follows:

$$(g'_2) \limsup_{s \to +\infty} \frac{g(s)}{s^2} = 0.$$

In this section, we use an indirect approach called the monotonicity trick method developed by Jeanjean in [21] as follows.

**Theorem 3.1.** Let $(X, \| \cdot \|)$ be a Banach space and $J \in \mathbb{R}$, an interval. Consider the family of $C^1$ functionals on $X$

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J,$$

with $B$ nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$ and such that $I_0(0) = 0$.

For any $\lambda \in J$, we set

$$\Gamma_\lambda = \{ y \in C([0, 1], X) | y(0) = 0, I_0(y(1)) < 0 \}.$$

If for every $\lambda \in J$, the set $\Gamma_\lambda$ is nonempty and

$$c_\lambda = \inf_{y \in \Gamma_\lambda} \max_{t \in [0, 1]} I_\lambda(y(t)) > 0,$$

then for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that

(i) $\{v_n\}$ is bounded;

(ii) $I_\lambda(v_n) \to c_\lambda$;

(iii) $(I_\lambda)'(v_n) \to 0$ in the dual $X^{-1}$ of $X$.

Set

$$g_\lambda(s) = \begin{cases} (g(s) + ms)^+, & \text{if } s \geq 0, \\ 0, & \text{if } s < 0. \end{cases} \quad (3.2)$$

$$g_\lambda(s) = g(s) - g(s), \quad \text{for all } s \in \mathbb{R}. \quad (3.3)$$

Then we can define

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2} \left( \frac{\|u\|^2}{T^2} \right) \int_{\mathbb{R}^3} \omega \phi_\lambda u^2 dx + \int_{\mathbb{R}^3} G_\lambda(u) dx,$$

$$B(u) = \int_{\mathbb{R}^3} G(u) dx,$$

and

$$I_{\omega, \lambda}^\prime(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2} \left( \frac{\|u\|^2}{T^2} \right) \int_{\mathbb{R}^3} \omega \phi_\lambda u^2 dx + \int_{\mathbb{R}^3} G_\lambda(u) dx - \lambda \int_{\mathbb{R}^3} G_\lambda(u) dx,$$

where $G_i(t) = \int_0^t g_i(s) ds, \quad i = 1, 2$. By $(g_3)$, there exists $z \in H_0^1(\mathbb{R}^3)$ such that
Indeed, as in [8], we define
\[
  w(R, x) = \begin{cases} 
  \zeta & \text{if } |x| \leq R, \\
  \zeta(R + 1 - |x|) & \text{if } R \leq |x| \leq R + 1, \\
  0 & \text{if } |x| \geq R.
  \end{cases}
\]

Obviously, \( w(R, x) \in H^j_R(\mathbb{R}^3) \). One can easily check that
\[
  \int_{\mathbb{R}^3} G(w(R, x)) dx \geq G(\zeta) \text{meas}(B_R) - \text{meas}(B_{R+1} - B_R) \left( \max_{s \in [0, \zeta]} |G(s)| \right),
\]
where \( \text{meas}(\cdot) \) denotes Lebesgue measure. Then there exist constants \( C, \overline{C} > 0 \) such that
\[
  \int_{\mathbb{R}^3} G(w(R, x)) dx \geq CR^3 - \overline{C}R^2,
\]
which implies that \( \int_{\mathbb{R}^3} G(w(\overline{R}, x)) dx > 0 \) for \( \overline{R} \) large enough. We take \( z = w(\overline{R}, x) \) and then (3.4) holds. Therefore, there exists \( 0 < \delta_0 < 1 \) such that
\[
  \delta_0 \int_{\mathbb{R}} G(z) dx - \int_{\mathbb{R}} G(z) dx > 0.
\]
Hence, we can take \( J = [\delta_0, 1] \) in Theorem 3.1.

**Lemma 3.1.** Assume that \((g_1), (g_2)\) and \((g_3)\) hold. Then for almost \( \lambda \in J \), there exists a bounded sequence \( \{u_n\} \subset H^j_R(\mathbb{R}^3) \) such that
\[
  I_{\lambda, \alpha} u_n \to c_\lambda \quad \text{and} \quad (I_{\lambda, \alpha})' u_n \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof.** First, we can easily see that \( A(u) \to \infty \) as \( \|u\| \to \infty \). Next, we only need to prove the following two claims, and then the proof can be finished by using Theorem 3.1.

**Claim 1.** \( \Gamma_\lambda \neq \emptyset \). Indeed, define \( z = z(\frac{x}{\theta}) \) and
\[
  y(t) = \begin{cases} 
  0 & \text{if } t = 0, \\
  z(\frac{x}{t}) & \text{if } 0 < t \leq 1.
  \end{cases}
\]

Evidently,
\[
  \left\| z\left(\frac{x}{\theta}\right)\right\|_a^\beta = \left( \int_{\mathbb{R}^3} z\left(\frac{x}{\theta}\right)^\beta dx \right)^\frac{\beta}{\theta} = \left( \theta^3 \int_{\mathbb{R}^3} |z|^\beta dx \right)^\frac{\beta}{\theta} = \theta^\beta \|z\|_a^\beta.
\]

Then we obtain that
\[
  I_{\lambda, \alpha}^T(y(1)) = \frac{1}{2} \int_{\mathbb{R}^3} \nabla z\left(\frac{x}{\theta}\right) \left(\frac{x}{\theta}\right) dx - \frac{1}{2} \lambda \left( \left(\frac{z\left(\frac{x}{\theta}\right)}{T^\beta} \right)^\beta \int_{\mathbb{R}^3} \omega \phi_\lambda^a(z) \left(\frac{x}{T^\beta}\right) dx + \int_{\mathbb{R}^3} G(z\left(\frac{x}{\theta}\right)) dx - \lambda \int_{\mathbb{R}^3} G(z\left(\frac{x}{\theta}\right)) dx \right)
\]
\[
  \leq \frac{\theta}{2} \int_{\mathbb{R}^3} |\nabla z|^2 dx + \frac{\omega A}{C} \theta^\beta \left( \left( \left(\frac{\theta^\beta \|z\|_a^\beta}{T^\beta} \right)^\beta \|z\|_a^\beta - \theta^3 \int_{\mathbb{R}^3} (\delta_0 G_1(z) - G_2(z)) dx \right) \right).
\]
For $\theta > 0$ large enough, we have $\chi\left(\frac{x}{\theta}\right) = 0$ and then $I_{0}^{T}\left(\frac{x}{\theta}\right) < 0$. Therefore, we can complete the proof by taking $e_{0} = z\left(\frac{x}{\theta_{0}}\right)$ with $\theta_{0} > 0$ large enough.

Claim 2. $c_{1} > 0$. Indeed, combining $(g_{i})$ and $(g_{3}^{0})$, there exist constants $L > 0$ and $C > 0$ such that

$$G(s) \leq -Ls^{2} + Cs^{6}, \quad \text{for all} \ s \in \mathbb{R}. \quad (3.6)$$

From Lemma 2.1, (3.6), and Sobolev inequality, we obtain that

$$I_{0}^{T}(u) \geq \frac{1}{2} \int_{\mathbb{R}^{3}} |\nabla u|^{2} dx - \frac{1}{2} \int_{\mathbb{R}^{3}} |u_{x}|^{2} dx + \int_{\mathbb{R}^{3}} G(u) dx - \int_{\mathbb{R}^{3}} G(u) dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^{3}} |\nabla u|^{2} dx - \int_{\mathbb{R}^{3}} G(u) dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^{3}} |\nabla u|^{2} dx + \frac{L}{2} \int_{\mathbb{R}^{3}} u^{2} dx - \frac{C}{6} \int_{\mathbb{R}^{3}} u^{6} dx$$

$$\geq \min\{1, L\} ||u||^{2} - C||u||^{6}.$$ 

Then there exist $r_{0} > 0$ small enough and $\rho_{0} > 0$ such that $I_{0}^{T}(u) \geq \rho_{0} > 0$ for $||u|| = r_{0}$. Fix $\lambda \in I$ and $y \in \Gamma_{1}$, since $y(0) = 0$ and $I_{0}^{T}(y(1)) < 0$, one sees that there exists $t_{y} \in (0, 1)$ such that $||y(t_{y})|| = \rho_{0}$. Thus, $c_{1} \geq \rho_{0} > 0$ for any $\lambda \in J$.

The following lemma is developed by Strauss in [30], which is useful to obtain the convergence of the Palais-Smale sequence for $I_{0}^{T}$.

**Theorem 3.2.** Let $P$ and $Q: \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying

$$\lim_{n \to \infty} P(s) = 0,$$

$\{v_{n}\}, v, and w$ be measurable functions from $\mathbb{R}^{N}$ to $\mathbb{R}$, with $z$ bounded, such that

$$\sup_{n} \int_{\mathbb{R}^{N}} |Q(v_{n}(x))||w| dx < +\infty,$$

$$P(v_{n}(x)) \to v(x) \ a.e. \ in \ \mathbb{R}^{N}.$$ 

Then $||P(v_{n}(x)) - v)w||_{L^{1}(B)} \to 0$, for any bounded Borel set $B$.

Moreover, if we have also

$$\lim_{n \to 0} P(s) = 0,$$

$$\lim_{n \to \infty} \sup_{n} |v_{n}(x)| = 0,$$

then $||P(v_{n}(x)) - v)w||_{L^{1}(\mathbb{R}^{N})} \to 0$.

We are now ready to give the convergence of the Palais-Smale sequence for $I_{0}^{T}$.

**Lemma 3.2.** Assume that $(g_{1})$, $(g_{2}^{0})$, and $(g_{3})$ hold. Then for almost $\lambda \in J$, each bounded Palais-Smale sequence for $I_{0}^{T}$ has a convergent subsequence. Moreover, there exists $u_{\lambda} \in H_{0}^{1}(\mathbb{R}^{3}) \setminus \{0\}$ such that $I_{0}^{T}(u_{\lambda}) = c_{1}$ and $(I_{0}^{T})'(u_{\lambda}) = 0$. 


Proof. Let \( \{ u_n \} \) be a bounded Palais-Smale sequence for \( I_{\omega, \lambda}^T \) in \( H_1^1(\mathbb{R}^3) \). Passing to a subsequence again if necessary, there exists \( u_1 \in H_1^1(\mathbb{R}^3) \) such that
- \( u_n \rightharpoonup u_1 \) in \( H_1^1(\mathbb{R}^3) \),
- \( u_n \to u_1 \) in \( L^p(\mathbb{R}^3), \ 2 < p < 6 \),
- \( u_n \to u_1 \) a.e. in \( \mathbb{R}^3 \).

By the definition of \( g_1(s) \) and \( g_2(s) \) in (3.2)–(3.3), one has that
\[
\lim_{s \to 0} \frac{g_1(s)}{s} = 0, \quad \lim_{s \to \infty} \frac{g_1(s)}{s^5} = 0, \quad g_1(s) \geq ms. \tag{3.7}
\]
Then one can easily obtain that, for all \( s \geq 0 \),
\[
g_1(s) \leq C_5 s^5 + \varepsilon g_2(s), \tag{3.8}
\]
and also for all \( s \in \mathbb{R} \),
\[
g_2(s) \leq \frac{C_6}{6} s^6 + \varepsilon G_2(s), \tag{3.9}
\]
\[
G_2(s) \geq \frac{m}{2} s^2. \tag{3.10}
\]
First, we can apply Theorem 3.2 for \( P(s) = g_1(s), Q(s) = |s|^\nu; \nu = g_1(u), i = 1, 2 \) and \( w \in C_0^\infty(\mathbb{R}^3) \), from \((g'_2)\) and \((3.7)\):
\[
\int_{\mathbb{R}^3} g_1(u_n) w \ dx \to \int_{\mathbb{R}^3} g_1(u_1) w \ dx, \quad i = 1, 2. \tag{3.11}
\]
Moreover, we have
\[
\chi\left(\frac{\|u_n\|_{T^\beta}}{T^\beta}\right) \int_{\mathbb{R}^3} (2\omega + \phi_{u_n}) \phi_{u_n} u_n w \ dx \to \chi\left(\frac{\|u_1\|_{T^\beta}}{T^\beta}\right) \int_{\mathbb{R}^3} (2\omega + \phi_{u_1}) \phi_{u_1} u_1 w \ dx, \tag{3.12}
\]
\[
\chi\left(\frac{\|u_n\|_{T^\beta}}{T^\beta}\right) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \ dx \left(\int_{\mathbb{R}^3} |u_n|^\alpha \ dx\right)^{\frac{\nu}{\alpha} - 1} \int_{\mathbb{R}^3} |u_n|^\nu - 2 u_n w \ dx \to \chi\left(\frac{\|u_1\|_{T^\beta}}{T^\beta}\right) \int_{\mathbb{R}^3} \phi_{u_1} u_1^2 \ dx \left(\int_{\mathbb{R}^3} |u_1|^\alpha \ dx\right)^{\frac{\nu}{\alpha} - 1} \int_{\mathbb{R}^3} |u_1|^\nu - 2 u_1 w \ dx. \tag{3.13}
\]
Indeed, for (3.12), obviously, we know
\[
\chi\left(\frac{\|u_n\|_{T^\beta}}{T^\beta}\right) \int_{\mathbb{R}^3} (2\omega + \phi_{u_n}) \phi_{u_n} u_n w \ dx - \chi\left(\frac{\|u_1\|_{T^\beta}}{T^\beta}\right) \int_{\mathbb{R}^3} (2\omega + \phi_{u_1}) \phi_{u_1} u_1 w \ dx
\]
\[
= \left[ \chi\left(\frac{\|u_n\|_{T^\beta}}{T^\beta}\right) - \chi\left(\frac{\|u_1\|_{T^\beta}}{T^\beta}\right) \right] \int_{\mathbb{R}^3} (2\omega + \phi_{u_n}) \phi_{u_n} u_n w \ dx
\]
\[
+ \chi\left(\frac{\|u_1\|_{T^\beta}}{T^\beta}\right) \left[ \int_{\mathbb{R}^3} (2\omega + \phi_{u_n}) \phi_{u_n} u_n w \ dx - \int_{\mathbb{R}^3} (2\omega + \phi_{u_1}) \phi_{u_1} u_1 w \ dx \right]. \tag{3.14}
\]
Since \( u_n \to u_1 \) in \( L^\nu(\mathbb{R}^3) \), we have that
\[
\left( \int_{R^3} |u_n|^\alpha \, dx \right)^{\frac{2}{\alpha}} \rightarrow \left( \int_{R^3} |u|^\alpha \, dx \right)^{\frac{2}{\alpha}}.
\]

(3.15)

Together with the fact that \( \chi \in C^\infty \), one sees that
\[
\chi \left( \frac{\|u_n\|_{L^\beta}^\beta}{T^{\beta}} \right) \rightarrow \chi \left( \frac{\|u\|_{L^\beta}^\beta}{T^{\beta}} \right).
\]

(3.16)

Moreover,
\[
\int_{R^3} \left( 2\omega + \phi_{\alpha n} \right) \phi_{\alpha n} u_\alpha \, dx - \int_{R^3} \left( 2\omega + \phi_{\alpha n} \right) \phi_{\alpha n} u_\alpha \, dx = 2\omega \int_{R^3} \left( \phi_{\alpha n} u_\alpha \, dx - \phi_{\alpha n} u_\alpha \right) \, dx + \int_{R^3} (\phi_{\alpha n}^2 u_\alpha \, dx - \phi_{\alpha n}^2 u_\alpha \, dx)
\]
\[
= 2\omega \int_{R^3} \left( \phi_{\alpha n} u_\alpha \, dx - \phi_{\alpha n} u_\alpha \right) \, dx + 2\omega \int_{R^3} (\phi_{\alpha n} u_\alpha \, dx - \phi_{\alpha n} u_\alpha \, dx)
\]
\[
+ \int_{R^3} (\phi_{\alpha n} u_\alpha \, dx - \phi_{\alpha n} u_\alpha \, dx) + \int_{R^3} (\phi_{\alpha n} u_\alpha \, dx - \phi_{\alpha n} u_\alpha \, dx)
\]
\[
\leq 2\omega \|\phi_{\alpha n}\|_{L^\beta} \|u_n - u_\alpha\|_{L^\beta} + 2\omega \|\phi_{\alpha n}\|_{L^\beta} \|u_n - u_\alpha\|_{L^\beta} + \|\phi_{\alpha n}\|_{L^\beta} \|u_n - u_\alpha\|_{L^\beta} + \|\phi_{\alpha n}\|_{L^\beta} \|u_n - u_\alpha\|_{L^\beta}.
\]

(3.17)

Combining with \( \phi_{\alpha n} \rightarrow \phi_{\alpha n} \) in \( L^6(R^3) \) and \( u_n \rightarrow u_\alpha \) in \( L^\infty(R^3) \), we see that
\[
\int_{R^3} \left( 2\omega + \phi_{\alpha n} \right) \phi_{\alpha n} u_\alpha \, dx - \int_{R^3} \left( 2\omega + \phi_{\alpha n} \right) \phi_{\alpha n} u_\alpha \, dx \rightarrow 0.
\]

(3.18)

It follows from (3.14), (3.16), and (3.18) that (3.12) holds. For (3.13), clearly one obtains that
\[
\chi \left( \frac{\|u_n\|_{L^\beta}^\beta}{T^{\beta}} \right) \int_{R^3} \phi_{\alpha n} u_\alpha \, dx \left( \int_{R^3} |u_n|^\alpha \, dx \right)^{\frac{2}{\alpha}} \int_{R^3} |u_n|^{\alpha-2} u_n \, dx
\]
\[-\chi \left( \frac{\|u\|_{L^\beta}^\beta}{T^{\beta}} \right) \int_{R^3} \phi_{\alpha n} u_\alpha \, dx \left( \int_{R^3} |u|^\alpha \, dx \right)^{\frac{2}{\alpha}} \int_{R^3} |u|^{\alpha-2} u \, dx
\]
\[
= \left[ \chi \left( \frac{\|u_n\|_{L^\beta}^\beta}{T^{\beta}} \right) \right] \int_{R^3} \phi_{\alpha n} u_\alpha \, dx \left( \int_{R^3} |u_n|^\alpha \, dx \right)^{\frac{2}{\alpha}} \int_{R^3} |u_n|^{\alpha-2} u_n \, dx
\]
\[
+ \chi \left( \frac{\|u\|_{L^\beta}^\beta}{T^{\beta}} \right) \int_{R^3} \phi_{\alpha n} u_\alpha \, dx - \int_{R^3} \phi_{\alpha n} u_\alpha \, dx \left( \int_{R^3} |u_n|^\alpha \, dx \right)^{\frac{2}{\alpha}} \int_{R^3} |u_n|^{\alpha-2} u_n \, dx
\]
\[
+ \chi \left( \frac{\|u\|_{L^\beta}^\beta}{T^{\beta}} \right) \int_{R^3} \phi_{\alpha n} u_\alpha \, dx \left( \int_{R^3} |u_n|^\alpha \, dx \right)^{\frac{2}{\alpha}} \int_{R^3} |u_n|^{\alpha-2} u_n \, dx
\]
\[
+ \chi \left( \frac{\|u\|_{L^\beta}^\beta}{T^{\beta}} \right) \int_{R^3} \phi_{\alpha n} u_\alpha \, dx \left( \int_{R^3} |u_n|^\alpha \, dx \right)^{\frac{2}{\alpha}} \int_{R^3} |u_n|^{\alpha-2} u_n \, dx - \int_{R^3} |u_n|^{\alpha-2} u_n \, dx\right).
\]

(3.19)

Since
\[
\int_{\mathbb{R}^3} \phi_m u_m^2 dx - \int_{\mathbb{R}^3} \phi_m u_l^2 dx = \int_{\mathbb{R}^3} (\phi_m - \phi_l) u_m^2 dx + \int_{\mathbb{R}^3} \phi_m (u_m^2 - u_l^2) dx \\
\leq \|\phi_m - \phi_l\|_\infty \|u_m\|_2^2 + \|\phi_m\|_\infty \|u_m - u_l\|_2^2,
\]

together with \( \chi \in C^\infty \), \( w \in C_0^\infty (\mathbb{R}^3) \), (3.15), (3.18), and \( u_n \rightarrow u_1 \) in \( L^p(\mathbb{R}^3) \) (2 < \( p < 6 \)), it follows from (3.19) that (3.13) holds. Hence, we can obtain that \((I_{\omega_1})'(u_1) = 0\) and
\[
\int |\nabla u_1|^2 dx - \chi \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \int (2\omega + \phi_{u_1}) \phi_{u_1} u_1^2 dx \\
- \frac{\beta \omega}{2T^\beta} \chi' \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \int \phi_{u_1} u_1^2 dx \cdot \|u_1\|_{T^\beta}^p = \lambda \int g(u_1) u_1 dx - \int g_1(u_1) u_1 dx.
\] (3.20)

Next we apply Theorem 3.2 for \( P(s) = g(s) \), \( Q(s) = s^2 + s^\epsilon \), \( \{v_m\} = \{u_n\} \), \( \nu = g(u_1) u \) and \( w = 1 \), and we obtain
\[
\lim_{n \to \infty} \int g(u_n) u_n dx \to \int g(u_1) u_1 dx.
\] (3.21)

Moreover, by Fatou’s lemma,
\[
\int g_1(u_1) u_1 dx \leq \liminf_{n \to \infty} \int g_1(u_n) u_n dx.
\] (3.22)

Similar to the proof of (3.12) and (3.13), we also have that
\[
\chi \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \int (2\omega + \phi_{u_1}) \phi_{u_1} u_1^2 dx \to \chi \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \int (2\omega + \phi_{u_1}) \phi_{u_1} u_1^2 dx,
\] (3.23)
\[
\chi' \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \int \phi_{u_1} u_1^2 dx \cdot \|u_1\|_{T^\beta}^p \to \chi' \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \int \phi_{u_1} u_1^2 dx \cdot \|u_1\|_{T^\beta}^p.
\] (3.24)

Combining (3.20), (3.21), (3.22), (3.23), (3.24), and \((I_{\omega_1})'(u_1), u_n \to 0\), one knows
\[
\limsup_{n \to \infty} \int |\nabla u_n|^2 dx = \limsup_{n \to \infty} \chi \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \int (2\omega + \phi_{u_1}) \phi_{u_1} u_1^2 dx + \frac{\beta \omega}{2T^\beta} \chi' \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \\
\cdot \int \phi_{u_1} u_1^2 dx \cdot \|u_1\|_{T^\beta}^p + \lambda \int g(u_1) u_1 dx - \int g_1(u_1) u_1 dx \\
\leq \chi \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \int (2\omega + \phi_{u_1}) \phi_{u_1} u_1^2 dx + \frac{\beta \omega}{2T^\beta} \chi' \left( \frac{\|u_1\|_{T^\beta}^p}{T^\beta} \right) \\
\cdot \int \phi_{u_1} u_1^2 dx \cdot \|u_1\|_{T^\beta}^p + \lambda \int g(u_1) u_1 dx - \int g_1(u_1) u_1 dx \\
= \int |\nabla u_1|^2 dx.
\] (3.25)

From weak lower semicontinuity, we have
\[
\int |\nabla u_1|^2 dx \leq \liminf_{n \to \infty} \int |\nabla u_n|^2 dx.
\] (3.26)

By combining (3.25) and (3.26), one sees
\[
\int |\nabla u_h|^2 \, dx = \liminf_{n \to \infty} \int |\nabla u_n|^2 \, dx
\]
(3.27)

And
\[
\lim_{n \to \infty} \int g_2(u_n)u_n \, dx = \int g_2(u)u \, dx.
\]

From \(g_2(s) \geq ms \) in (3.7), we see that \(g_2(s) = ms + h(s)\), where \(h\) is a positive and continuous function and then
\[
\lim_{n \to \infty} \int g_2(u_n)u_n \, dx = \lim_{n \to \infty} \int (mu_n^2 + h(u_n))u_n \, dx.
\]
(3.28)

By Fatou’s lemma, we have
\[
\int h(u_n)u_n \, dx \leq \liminf_{n \to \infty} \int h(u_n)u_n \, dx,
\]
(3.29)

\[
\int u_n^2 \, dx \leq \liminf_{n \to \infty} \int u_n^2 \, dx.
\]
(3.30)

From (3.28), (3.29), and (3.30), one obtains that
\[
\lim_{n \to \infty} \int u_n^2 \, dx = \int u_0^2 \, dx.
\]
(3.31)

Hence, it follows from (3.27) and (3.31) that \(u_n \to u_0\) in \(H^1_0(\mathbb{R}^3)\), which deduces that \(I_{\omega,\lambda}^T(u_0) = c_1\) and \(I_{\omega,\lambda}^T)'(u_0) = 0\).

Lemma 3.3. Assume that \((g_1), (g_2)\) and \((g_3)\) hold and \(u_{\lambda_n}(\lambda_n \in I)\) is a critical point for \(I_{\omega,\lambda_n}^T\). Then for \(T > 0\) sufficiently large, there exists \(\omega_0 = \omega(T)\) such that \(|u_{\lambda_n}| \leq T\) for any \(0 < \omega < \omega_0\).

Proof. Since \(u_{\lambda_n}\) is a critical point for \(I_{\omega,\lambda_n}^T\), by Lemma 2.5 and \(I_{\omega,\lambda_n}^T(u_{\lambda_n}) = c_{\lambda_n}\), we know that
\[
\int |\nabla u_{\lambda_n}|^2 \, dx - 5\omega \chi \left( \frac{\|u_{\lambda_n}\|_\alpha^p}{T^p} \right) \int \phi_{u_{\lambda_n}} u_{\lambda_n}^2 \, dx - 2\chi \left( \frac{\|u_{\lambda_n}\|_{\alpha}^p}{T^p} \right) \int \phi_{u_{\lambda_n}}^2 u_{\lambda_n}^2 \, dx
\]
\[
- \frac{3\beta \omega}{\alpha T^p} \chi \left( \frac{\|u_{\lambda_n}\|_{\alpha}^p}{T^p} \right) \int \phi_{u_{\lambda_n}} u_{\lambda_n}^2 \, dx \cdot \left( \int |u_{\lambda_n}|^p \, dx \right)^\frac{p}{2} \int (\lambda_n G_1(u_{\lambda_n}) - G_3(u_{\lambda_n})) \, dx
\]
(3.32)

and
\[
3 \int |\nabla u_{\lambda_n}|^2 \, dx - 3\omega \chi \left( \frac{\|u_{\lambda_n}\|_{\alpha}^p}{T^p} \right) \int \phi_{u_{\lambda_n}} u_{\lambda_n}^2 \, dx + 6 \int (G_2(u_{\lambda_n}) - \lambda_n G_1(u_{\lambda_n})) \, dx = 6c_{\lambda_n}.
\]
(3.33)

By combining (3.32) and (3.33), we obtain
\[
\int_{\mathbb{R}^3} |\nabla u_{\lambda h}|^2 \, dx = 3c_{\lambda h} - \omega x \left( \frac{||u_{\lambda h}||^2}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_{\lambda h}} u_{\lambda h}^2 \, dx - \chi \left( \frac{||u_{\lambda h}||^2}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_{\lambda h}} u_{\lambda h}^2 \, dx \\
- \frac{3\beta \omega}{2aT^p} \chi \left( \frac{||u_{\lambda h}||^2}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_{\lambda h}} u_{\lambda h}^2 \, dx \cdot ||u_{\lambda h}||^2 \\
\leq 3c_{\lambda h} - \omega x \left( \frac{||u_{\lambda h}||^2}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_{\lambda h}} u_{\lambda h}^2 \, dx \\
- \frac{3\beta \omega}{2aT^p} \chi \left( \frac{||u_{\lambda h}||^2}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_{\lambda h}} u_{\lambda h}^2 \, dx \cdot ||u_{\lambda h}||^2.
\]

(3.34)

By the definition of \( c_{\lambda h} \) and (3.5), there exists a constant \( A_1 > 0 \) such that

\[
c_{\lambda h} \leq \max_{t \in [0,1]} \left[ \left( \frac{x}{t} \right) \right] \leq \max_{t \in [0,1]} \left\{ \frac{t}{2} \int_{\mathbb{R}^3} |\nabla z|^2 \, dx - t^3 \int_{\mathbb{R}^3} G(z) \, dx \right\} + \max_{t \in [0,1]} \left\{ \frac{\omega}{2} \cdot t^{\frac{p}{2}} \cdot \chi \left( \frac{t^{\frac{p}{2}} |z||^2}{T^p} \right) \right\} \leq A_1 + A_2(T),
\]

(3.35)

where

\[
A_2(T) = \frac{\omega}{2} C \cdot t^{\frac{p}{2}} \cdot \chi \left( \frac{t^{\frac{p}{2}} |z||^2}{T^p} \right) ||z||^2.
\]

If \( t^{\frac{p}{2}} |z||^2 \geq 2T^p \), by the definition of \( \chi \), we know that \( A_2(T) = 0 \). If \( t^{\frac{p}{2}} |z||^2 < 2T^p \), then

\[
A_2(T) = \frac{\omega}{2} C \cdot t^{\frac{p}{2}} \cdot \chi \left( \frac{t^{\frac{p}{2}} |z||^2}{T^p} \right) ||z||^2 \leq \frac{\omega}{2} C \cdot t^{\frac{p}{2}} |z||^2 \leq C \omega T^p.
\]

(3.36)

Combining with (3.35), one has

\[
c_{\lambda h} \leq A_1 + C \omega T^p.
\]

(3.37)

From the definition of \( \chi \) and Lemma 2.4, we can also obtain

\[
\omega x \left( \frac{||u_{\lambda h}||^2}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_{\lambda h}} u_{\lambda h}^2 \, dx \leq C\omega x \left( \frac{||u_{\lambda h}||^2}{T^p} \right) ||u_{\lambda h}||^2 \leq 2C \omega T^p
\]

(3.38)

and

\[
\frac{3\beta \omega}{2aT^p} \chi \left( \frac{||u_{\lambda h}||^2}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_{\lambda h}} u_{\lambda h}^2 \, dx \cdot ||u_{\lambda h}||^2 \leq \frac{3\beta \omega}{2aT^p} \chi \left( \frac{||u_{\lambda h}||^2}{T^p} \right) C ||u_{\lambda h}||^2 \leq \frac{12CB}{\alpha} \omega T^p.
\]

(3.39)

Therefore, from (3.34), (3.37), (3.38), and (3.39), there exist constants \( C_1, C_2 > 0 \) such that

\[
\int_{\mathbb{R}^3} |\nabla u_{\lambda h}|^2 \, dx \leq C_1 + C_2 \omega T^p.
\]

(3.40)

From (3.8) and \( \langle (T_{\omega h}^{\frac{p}{2}}) (u_{\lambda h}, u_{\lambda h}) \rangle = 0 \), we know that

\[
\int_{\mathbb{R}^3} |\nabla u_{\lambda h}|^2 \, dx - \chi \left( \frac{||u_{\lambda h}||^2}{T^p} \right) \int_{\mathbb{R}^3} \left( 2\omega + \phi_{u_{\lambda h}} \right) u_{\lambda h}^2 \, dx - \frac{\beta \omega}{2T^p} \chi \left( \frac{||u_{\lambda h}||^2}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_{\lambda h}} u_{\lambda h}^2 \, dx \\
\cdot ||u_{\lambda h}||^2 + \int_{\mathbb{R}^3} g_{\delta}(u_{\lambda h}) \, dx = \lambda_n \int_{\mathbb{R}^3} g_{\delta}(u_{\lambda h}) \, dx \leq C ||u_{\lambda h}||_6^6 + \varepsilon \int_{\mathbb{R}^3} g_{\delta}(u_{\lambda h}) \, dx.
\]

(3.41)

Notice that
\[
\int_{\mathbb{R}^3} |\nabla u_{k_n}|^2 \, dx \geq 0 \quad \text{and} \quad -\chi \left( \frac{\|u_{k_n}\|_a^2}{T^2} \right) \int_{\mathbb{R}^3} (2\omega + \phi_{\omega_{k_n}}) \phi_{\omega_{k_n}} u^2_{k_n} \, dx \geq 0.
\]

It follows from (3.39), (3.40), and (3.41) that
\[
(1 - t) \int_{\mathbb{R}^3} g_t(u_{k_n}) u_{k_n} \, dx \leq C_1 \|u_{k_n}\|_a^6 + \frac{\beta \omega}{2T^2} \chi \left( \frac{\|u_{k_n}\|_a^2}{T^2} \right) \int_{\mathbb{R}^3} \phi_{\omega_{k_n}} u_{k_n}^2 \, dx \cdot \|u_{k_n}\|_a^6
\]
\[
\leq C S_a^3 \int_{\mathbb{R}^3} |\nabla u_{k_n}|^2 \, dx + \frac{\beta \omega}{2T^2} \chi \left( \frac{\|u_{k_n}\|_a^2}{T^2} \right) \int_{\mathbb{R}^3} \phi_{\omega_{k_n}} u_{k_n}^2 \, dx \cdot \|u_{k_n}\|_a^6
\]
\[
\leq C S_a^3 (C_1 + C_2 \omega T^3) + 4C_3 \beta \omega T^3.
\]

Combining with \( g_t(s) \geq m(s) \) in (3.7), there exists a constant \( C > 0 \) such that
\[
\int_{\mathbb{R}^3} u_{k_n}^2 \, dx \leq C (C_1 + C_2 \omega T^3)^3 + C \omega T^3.
\]

Therefore, by (3.40) and (3.42), it knows that
\[
\|u_{k_n}\|^2 = \int_{\mathbb{R}^3} |\nabla u_{k_n}|^2 \, dx + \int_{\mathbb{R}^3} u_{k_n}^2 \, dx \leq C_1 + C_2 \omega T^3 + C(C_1 + C_2 \omega T^3)^3 + C \omega T^3
\]
\[
\leq C + C \omega T^3 + C(C_1 + C_2 \omega T^3)^3.
\]

We suppose by contradiction that \( \|u_{k_n}\|_a > T \). By Sobolev inequality and (3.43), we have that
\[
T^2 < \|u_{k_n}\|_a^2 \leq C \|u_{k_n}\|^2 \leq C + C T^3 + C(C_1 + C_2 \omega T^3),
\]
which is contradiction with that \( T \) is large enough and \( \omega T^3 < 1 \).

\[\square\]

**Proof of Theorem 1.1.** By using Lemmas 3.1 and 3.2, we can assume that \( u_{k_n} \) be a critical point for \( I_{\omega_{k_n}} \) at level \( c_{\lambda_n} \). Since \( \lambda_n \to 1 \), we can see that \( \{u_{k_n}\} \) is a bounded Palais-Smale sequence for \( I_\omega \). Indeed, from Lemma 3.3, one may suppose that
\[
\|u_{k_n}\|_a \leq T,
\]
which implies that
\[
I_{\omega_{k_n}}'(u_{k_n}) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_{k_n}|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_{\omega_{k_n}} u_{k_n}^2 \, dx + \int_{\mathbb{R}^3} G_\omega(u_{k_n}) \, dx - \lambda_n \int_{\mathbb{R}^3} G_\lambda(u_{k_n}) \, dx
\]
\[
= I_{\omega_{k_n}}(u_{k_n}).
\]

Also,
\[
\langle I_\omega'(u_{k_n}), v \rangle = \langle (I_{\omega_{k_n}})'(u_{k_n}), v \rangle + (\lambda_n - 1) \int_{\mathbb{R}^3} G_\lambda(u_{k_n}) \, dx
\]
\[
= \langle (I_{\omega_{k_n}})'(u_{k_n}), v \rangle + (\lambda_n - 1) \int_{\mathbb{R}^3} G_\lambda(u_{k_n}) \, dx.
\]

Therefore, by using \( \lambda_n \to 1 \), we know that \( I_\omega(u_{k_n}) \to c_1 \) and \( I_{\omega_{k_n}}'(u_{k_n}) \to 0 \), together with Theorem 3.1, we can also know that \( \{u_{k_n}\} \) is a bounded Palais-Smale sequence for \( I_\omega \). By using Lemma 3.2, there exists \( u_\omega \in H^1(\mathbb{R}^3) \) such that \( u_{k_n} \rightharpoonup u_\omega \). Hence, \( I_\omega'(u_\omega) = 0 \) and \( I_\omega(u_\omega) = c_1 \). Then from Remark 1.4, \( u_\omega \) is a positive solution for system (1.1). \[\square\]
4 Properties of the positive solution

4.1 Decay rate of the positive solution

We recall Strauss inequalities that can be found as Radial Lemma A.II. and Radial Lemma A.III. in [8].

**Lemma 4.1.** If \( N \geq 2 \), there exists constants \( C_N, \overline{R} > 0 \) such that, for every \( u(x) = u(|x|) \in H^1(\mathbb{R}^N) \),
\[
|u(x)| \leq C_N|x|^{-\frac{N}{2}}, \quad \text{for} \quad x \geq \overline{R}.
\]

**Lemma 4.2.** If \( N \geq 3 \), there exists constants \( C_N, \overline{R} > 0 \) such that, for every \( u(x) = u(|x|) \in D^{1,2}(\mathbb{R}^N) \),
\[
|u(x)| \leq C_N|x|^{-\frac{N}{2}}||u||_{D^{1,2}(\mathbb{R}^N)}, \quad \text{for} \quad x \geq \overline{R}.
\]

**Lemma 4.3.** Let \( u_\omega \) be the positive solution obtained by Theorem 1.1. Then there exist constants \( M_0, \zeta, R_1 > 0 \) such that
\[
u_\omega \leq M_0 e^{-\zeta |x|}, \quad \text{for all} \quad |x| > R_1.
\]

**Proof.** By the expression of the system (1.1), we know that
\[
-\Delta u_\omega - \left( 2\omega + \phi_{u_\omega} \right) \phi_{u_\omega} u_\omega = g(u_\omega).
\] (4.1)

From \((g_1)\) and \((g'_1)\), there exists a constant \( C > 0 \) such that
\[
g(t) \leq -mt + Ct^5, \quad \text{for all} \quad t \geq 0.
\] (4.2)

By combining (4.1) and (4.2), one obtains
\[
-\Delta u_\omega - \left( 2\omega + \phi_{u_\omega} \right) \phi_{u_\omega} u_\omega + mu_\omega = g(u_\omega) + mu_\omega \leq Cu_\omega^5.
\] (4.3)

From Lemma 4.1, there exists a constant \( R_1 > 0 \) such that
\[
Cu_\omega^5 \leq \frac{m}{2} u_\omega, \quad \text{for all} \quad |x| \geq R_1.
\]

Therefore,
\[
-\Delta u_\omega - \left( 2\omega + \phi_{u_\omega} \right) \phi_{u_\omega} u_\omega + mu_\omega \leq \frac{m}{2} u_\omega, \quad \text{for all} \quad |x| \geq R_1.
\] (4.4)

It follows from Lemma 2.1 and (4.4) that
\[
-\Delta u_\omega + \frac{m}{2} u_\omega \leq 0.
\] (4.5)

Let \( \varphi(x) = M_0 e^{-\zeta |x|} \), where \( \zeta < \frac{m}{2} \) and \( M_0 e^{-\zeta R_1} \geq u_\omega(x) \) for all \( |x| = R_1 \). By a simple calculation, we can easily verify that
\[
\Delta \varphi(x) \leq \zeta^2 \varphi(x),
\]
which implies that
\[
-\Delta \varphi(x) + \frac{m}{2} \varphi(x) > 0.
\] (4.6)

Define \( \varphi_\xi(x) = \varphi(x) - u_\omega(x) \). By combining (4.5) and (4.6), we can see that
\[
-\Delta \varphi_\xi(x) + \frac{m}{2} \varphi_\xi(x) > 0, \quad \text{in} \quad |x| \geq R_1,
\]
\[
\varphi_\xi(x) \geq 0, \quad \text{in} \quad |x| = R_1.
\]
\[ \lim_{|x| \to \infty} \phi(x) = 0. \]

Hence, by maximum principle, we know that \( \phi(x) \geq 0 \) in \( |x| \geq R \). That is, \( u_\omega \leq M_0 e^{-|x|} \) for \( |x| > R \) and \( \omega \in (0, \omega_0) \).

Before proving Theorem 1.2, we make use of the ODE’s (ODE is the short for Ordinary Differential Equation) techniques to obtain the following result.

**Lemma 4.4.** If \( \omega \in (0, \omega_0) \) and \((g_1), (g_2), (g_3)\) hold, then \( u_\omega \in C^2(\mathbb{R}^3) \).

**Proof.** We rewrite (4.1) as follows:

\[-\Delta u_\omega = (2\omega + \phi_{u_\omega}) u_\omega + g(u_\omega). \tag{4.7} \]

Define

\[ q_\omega(x) = (2\omega + \phi_{u_\omega}) u_\omega + \frac{g(u_\omega)}{u_\omega}. \]

Then (4.7) can be transformed into

\[-\Delta u_\omega = q_\omega(x) u_\omega. \]

Since \( u_\omega \in H^1(\mathbb{R}^3) \) and \( \phi_{u_\omega} \in D^{1,2}(\mathbb{R}^3) \), we easily see that \( q_\omega \in L^3(\mathbb{R}^3) \). By using a result of Brezis and Kato in [10], we have that \( u_\omega \in L^p_{loc}(\mathbb{R}^3) \) for \( 1 \leq p < \infty \). And then a classical bootstrap argument on balls implies that \( u_\omega \in C^{1,0}(\mathbb{R}^3) \) for \( 0 < \alpha < 1 \). As \( u_\omega \in H^1(\mathbb{R}^3) \), we know that

\[ -(u_\omega)_r - \frac{N-1}{r}(u_\omega)_r = (2\omega + \phi_{u_\omega}) u_\omega + g(u_\omega), \quad \text{for} \quad r \in (0, +\infty). \tag{4.8} \]

We need to discuss that \( (u_\omega)_r \) is continuous at \( r = 0 \), and we complete the proof. Rewrite (4.8) as follows:

\[ -\frac{d}{dr} (r^{N-1}(u_\omega)_r) = r^{N-1} \left[ (2\omega + \phi_{u_\omega(r)}) u_\omega(r) + g(u_\omega(r)) \right]. \tag{4.9} \]

Integrating (4.9) from 0 to \( r \), one obtains that

\[ r^{N-1}(u_\omega)_r = -r^{N-1} \int_0^r s^{N-1} \left[ (2\omega + \phi_{u_\omega(s)}) u_\omega(s) + g(u_\omega(s)) \right] ds. \tag{4.10} \]

By using a change of variables to (4.10), we obtain

\[ \frac{(u_\omega)_r}{r} = -\int_0^r t^{N-1} \left[ (2\omega + \phi_{u_\omega(t)}) u_\omega(t) + g(u_\omega(t)) \right] dt. \]

Evidently, as \( r \to 0 \), the integral \( \int_0^r t^{N-1} \left[ (2\omega + \phi_{u_\omega(t)}) u_\omega(t) + g(u_\omega(t)) \right] dt \) exists. Combining with (4.8), \( (u_\omega)_r(0) \) exists, and then \( u_\omega \in C^2(\mathbb{R}^3) \).

**Proof of Theorem 1.2.** Lemma 4.3 obtains the exponential decay of \( u_\omega \). Next we show that the similar conclusion for \( (u_\omega)_r \) holds. From (4.2), Lemmas 2.1, 4.2, and 4.3, there exist constants \( m_2 \geq m_1 > 0 \) such that

\[ m_1 |u_\omega| \leq |(2\omega + \phi_{u_\omega}) u_\omega + g(u_\omega)| \leq m_2 |u_\omega|. \tag{4.11} \]

By integrating (4.9) from \( r \) to \( R \), and letting \( R \to \infty \), we obtains that \( \lim_{r \to \infty} r^{N-1}(u_\omega)_r = 0 \). Also, by integrating (4.9) from \( r \) to \( +\infty \), together with (4.11) and Lemma 4.3, one sees that
which shows that there exist constants $M_1, R_2 > 0$ such that
$$|Du_\omega| \leq M_2 e^{-\zeta|x|}, \quad \text{for all } |x| > R_2.$$ Combining with (4.8), there exist constants $M_3, R_3 > 0$ such that
$$|D^2u_\omega| \leq M_2 e^{-\zeta|x|}, \quad \text{for all } |x| > R_3.$$ Let $M = \max\{M_0, M_1, M_3\}$ and $R_0 = \max\{R_1, R_2, R_3\}$, then the conclusion of Theorem 1.2 is proved. $\square$

4.2 Asymptotic behavior of the positive solution

Proof of Theorem 1.3. Let $\{u_{\omega_n}\} \subset H^1_0(\mathbb{R}^3)$ be the solutions which are obtained in Theorem 1.1 for each $n \in \mathbb{N}$. Then for any $\varphi \in C^\infty_0(\mathbb{R}^3)$, we have that
$$\int_{\mathbb{R}^3} (\nabla u_{\omega_n} \nabla \varphi - (2\omega_n + \phi_{u_{\omega_n}}) \phi_{u_{\omega_n}} u_{\omega_n} \varphi) dx = \int_{\mathbb{R}^3} g(u_{\omega_n}) \varphi dx. \quad (4.13)$$

From the proof of Theorem 1.1, one sees that $\{u_{\omega_n}\}$ is bounded in $H^1_0(\mathbb{R}^3)$. Similar to the proof of Lemma 3.2, as $\omega_n \to 0$, there exists $u_\ast \in H^1_0(\mathbb{R}^3)$ such that
$$u_{\omega_n} \rightharpoonup u_\ast \quad \text{in} \quad H^1_0(\mathbb{R}^3). \quad (4.14)$$

Then Lemma 2.3 shows that
$$\phi_{u_{\omega_n}} \rightharpoonup \phi_{u_\ast} \quad \text{in} \quad D^{1,2}_0(\mathbb{R}^3). \quad (4.15)$$

Combining (3.17), (4.14), and (4.15), it has that
$$\int_{\mathbb{R}^3} \phi_{u_{\omega_n}} u_{\omega_n} \varphi dx \to \int_{\mathbb{R}^3} \phi_{u_\ast} u_\ast \varphi dx,$$
$$\int_{\mathbb{R}^3} \phi^2_{u_{\omega_n}} u_{\omega_n} \varphi dx \to \int_{\mathbb{R}^3} \phi^2_{u_\ast} u_\ast \varphi dx.$$ By Lemma 2.1, we know that $-\omega_n \leq \phi_{u_{\omega_n}} \leq 0$, then $\phi_{u_\ast} = 0$ as $\omega_n \to 0$. Therefore,
$$\int_{\mathbb{R}^3} (2\omega_n + \phi_{u_{\omega_n}}) \phi_{u_{\omega_n}} u_{\omega_n} \varphi dx = \int_{\mathbb{R}^3} 2\omega_n \phi_{u_{\omega_n}} u_{\omega_n} \varphi dx + \int_{\mathbb{R}^3} \phi^2_{u_{\omega_n}} u_{\omega_n} \varphi dx \to 0. \quad (4.16)$$

From (3.11) and (4.14), one sees that
$$\int_{\mathbb{R}^3} g(u_{\omega_n}) \varphi dx \to \int_{\mathbb{R}^3} g(u_\ast) \varphi dx, \quad (4.17)$$
$$\int_{\mathbb{R}^3} \nabla u_{\omega_n} \nabla \varphi dx \to \int_{\mathbb{R}^3} \nabla u_\ast \nabla \varphi dx. \quad (4.18)$$

Hence, together with (4.13), (4.16), (4.17), and (4.18), we arrive that
which implies that $u_*$ solves equation (1.4).

5 Existence of multiple solutions

To obtain the existence of multiple solutions for system (1.1), the energy functional corresponding to system (1.1) needs to be even. Therefore, we redefine $\overline{g}$ in (3.1) as follows:

$$\overline{g}(s) = \begin{cases} g(s) & s \in [0, s_0], \\ 0 & s \in \mathbb{R}_+ \setminus [0, s_0], \\ -g(-s) & s \in \mathbb{R}_-. \end{cases}$$

where $s_0$ is defined in Section 3. Similar to the discussion in Section 3, we can also suppose that $g$ is defined as $\overline{g}$ in (5.1) for convenience.

By using Lemma 2.5, in this section, we will use the notations of $P(u)$ and $H(u)$, where

$$P(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx - 5\alpha \chi \left( \frac{\|u\|_{L^\theta}^\beta}{T^\theta} \right) \int_{\mathbb{R}^3} \phi_\beta u^2 dx - 2\frac{\|u\|_{L^\theta}^\beta}{T^\theta} \int_{\mathbb{R}^3} \phi_\beta \overline{u}^2 dx$$

$$- \frac{3\beta \omega}{\alpha T^\beta} \chi \left( \frac{\|u\|_{L^\theta}^\beta}{T^\theta} \right) \cdot \int_{\mathbb{R}^3} \phi_\beta u^2 dx \cdot \left( \int_{\mathbb{R}^3} |u|^\alpha dx \right)^\frac{\theta}{\pi} - 6 \int_{\mathbb{R}^3} G(u) dx$$

and

$$H(u) = \langle (I^T_\omega \overline{J})(u), u \rangle - \frac{1}{2} P(u)$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\alpha \chi}{2} \left( \frac{\|u\|_{L^\theta}^\beta}{T^\theta} \right) \int_{\mathbb{R}^3} \phi_\beta u^2 dx - \frac{\beta \omega (\alpha - 3)}{2 \alpha T^\beta} \chi \left( \frac{\|u\|_{L^\theta}^\beta}{T^\theta} \right) \int_{\mathbb{R}^3} \phi_\beta u^2 dx$$

$$+ \left( \int_{\mathbb{R}^3} |u|^\alpha dx \right)^\frac{\theta}{\pi} + \int_{\mathbb{R}^3} (3G(u) - g(u)u) dx.$$ 

Clearly, $H(u) = 0$ for all $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ satisfying $(I^T_\omega \overline{J})(u) = 0$.

In order to prove Theorem 1.4, we first look for the existence of infinitely many solutions for $I^T_\omega$. Now we construct an auxiliary function $J$ to obtain the result. By the definition of $\chi$ and Lemma 2.1, one sees that

$$I^T_\omega(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2} \chi \left( \frac{\|u\|_{L^\theta}^\beta}{T^\theta} \right) \int_{\mathbb{R}^3} \phi_\beta u^2 dx - \int_{\mathbb{R}^3} G(u) dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} G(u) dx = J(u).$$

In the following lemma, we study the minimax geometry structure for $I^T_\omega$ and $J$ to obtain the minimax level values.

**Lemma 5.1.** Suppose that $(g_1)$, $(g_1')$ and $(g_2)$ hold.

(a) There exist constants $r_1, \rho_1 > 0$ such that $I^T_\omega(u) \geq J(u) \geq \rho_1$ for $u \in H^1(\mathbb{R}^3)$ with $\|u\| = r_1$. 
(b) For each $n \in \mathbb{N}$, there exists an odd continuous mapping $y_{\delta n} : S^{n-1} = \{ \delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}^n : |\delta| = 1 \} \rightarrow H^1_0(\mathbb{R}^3)$ such that

$$I(y_{\delta n}(\delta)) \leq I_{w}^r(y_{\delta n}(\delta)) < 0, \quad \text{for all } \delta \in S^{n-1}.$$  

**Proof.** (a) By Lemma 2.1, (3.6), and Sobolev inequality, we obtain the following:

$$I_{w}^r(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{2} \chi \left( \frac{|u|}{T^\beta} \right) \int_{\mathbb{R}^3} \omega \phi_x u^2 dx - \int_{\mathbb{R}^3} G(u) dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} G(u) dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{I}{2} \int_{\mathbb{R}^3} u^2 dx - \frac{C}{6} \int_{\mathbb{R}^3} u^4 dx$$

$$\geq \frac{\min\{1, I\}}{2} \|u\|^2 - C\|u\|^6.$$  

Hence, there exist $r_1 > 0$ small enough and $\rho_1 > 0$ such that $I_{w}^r(u) \geq \rho_1$ for $|u| = r_1$.

(b) From the discussion of (3.4), there exists a function $z \in H^1_0(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} G(z) dx > 0$. Arguing as Theorem 10 in [9], for any $n \in \mathbb{N}$, we can find an odd continuous mapping $\pi_n : S^{n-1} \rightarrow H^1_0(\mathbb{R}^3)$ such that

$$0 \notin \pi_n(S^{n-1}), \quad \int_{\mathbb{R}^3} G(\pi_n(\delta)) dx > 0, \quad \text{for all } \delta \in S^{n-1}.$$  

Set

$$\varsigma_n(\delta)(x) = \pi_n(\delta) \left( \frac{x}{t} \right) : S^{n-1} \rightarrow H^1_0(\mathbb{R}^3).$$

Notice that

$$\left\| \varsigma_n(\delta)(\frac{x}{t}) \right\|^{\beta} = \left( \int_{\mathbb{R}^3} \|\varsigma_n(\delta)(\frac{x}{t})\|^{\beta} dx \right)^{\frac{1}{\beta}} \leq \left( t^3 \int_{\mathbb{R}^3} \|\varsigma_n(\delta)(x)\|^\beta dx \right)^{\frac{1}{\beta}} = t^\frac{\beta}{\beta} \|\varsigma_n(\delta)\|^{\beta}. \quad (5.4)$$

By combining (5.4) and Lemma 2.4, we can obtain that

$$I_{w}^r \left( \varsigma_n(\delta)(\frac{x}{t}) \right) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \varsigma_n(\delta)(\frac{x}{t})|^2 dx - \frac{1}{2} \chi \left( \frac{\|\varsigma_n(\delta)(\frac{x}{t})\|^{\beta}}{T^\beta} \right) \int_{\mathbb{R}^3} \omega \phi_x (\varsigma_n(\delta)(\frac{x}{t}))^2 dx - \int_{\mathbb{R}^3} G(\varsigma_n(\delta)(\frac{x}{t})) dx$$

$$\leq \frac{t}{2} \int_{\mathbb{R}^3} |\nabla \varsigma_n(\delta)|^2 dx + \frac{\omega}{2} \frac{C}{t^\beta} \cdot \chi \left( \frac{\|\varsigma_n(\delta)\|^{\beta}}{T^\beta} \right) \|\varsigma_n(\delta)\|^{\beta} - t^\frac{\beta}{\beta} \int_{\mathbb{R}^3} G(\varsigma_n(\delta)) dx. \quad (5.5)$$

For $t > 0$ large enough, we have $\chi \left( \frac{\|\varsigma_n(\delta)\|^{\beta}}{T^\beta} \right) = 0$ and then $I_{w}^r \left( \varsigma_n(\delta)(\frac{x}{t}) \right) < 0$. Therefore, there exists $\tilde{t}_n > 0$ such that $I_{w}^r(y_{\delta n}(\delta)) < 0$ for $|y_{\delta n}(\delta)| = \left\| \varsigma_n(\delta)(\frac{x}{t}) \right\|.$
By using Chapter 9 of [28] and Lemma 5.1, the symmetric mountain pass values can be defined as follows:  
\[ c_n := \inf_{\gamma \in E_n} \max_{\delta \in E_n} I_n^\gamma(y(\delta)), \quad b_n := \inf_{\gamma \in E_n} \max_{\delta \in E_n} J(y(\delta)), \quad n \in \mathbb{N}. \]

Here,  
\[ E_n = \{ \delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}^n : |\delta| \leq 1 \} \]
and  
\[ \Gamma_n = \{ \gamma \in C(E_n, H_0^1(\mathbb{R}^3)) : \gamma(-\delta) = -\gamma(\delta), \quad \forall \delta \in E_n; \quad \gamma(\delta) = \gamma_{\text{on}}(\delta), \quad \forall \delta \in \partial E_n \}, \]
where \( \gamma_{\text{on}}(\delta) : \partial E_n = S^{n-1} \rightarrow H_0^1(\mathbb{R}^3) \) is given in Lemma 5.1. We remark that  
\[ \gamma(\delta) = \begin{cases} |\delta| \gamma_{\text{on}} \left( \frac{\delta}{|\delta|} \right) & \text{for } \delta \in E_n \setminus \{0\}, \\ 0 & \text{for } \delta = 0, \end{cases} \]
belongs to \( \Gamma_n \) and \( \Gamma_n \neq \emptyset \) for all \( n \in \mathbb{N} \). Obviously, for any \( \gamma \in \Gamma_n \), one sees that \( \gamma(E_n) \cap \{u \in H_0^1(\mathbb{R}^3) : \|u\| = r_1\} \neq \emptyset \). Therefore, from Lemma 5.1, we have that  
\[ c_n \geq b_n \geq \rho_1 > 0. \]

With the help of [20], the following facts are true:  
\[ b_n(n \in \mathbb{N}) \text{ are critical values of } J(u), \quad \text{(5.6)} \]
\[ b_n \to \infty \text{ as } n \to \infty. \quad \text{(5.7)} \]

By using minimax theorems in [37], for any \( n \in \mathbb{N} \), there exist Palais-Smale sequences \( \{u_n^j\} \) at level \( c_n \), namely,  
\[ I_n^\gamma(u_n^j) \to c_n, \quad (I_{\text{on}}^\gamma)^{(j)}(u_n^j) \to 0, \quad \text{as } j \to \infty. \]

But the boundedness of Palais-Smale sequences for the functional \( I_n^\gamma(u) \) cannot be obtained directly. Hence, we need the special Palais-Smale sequences for the functional \( I_n^\gamma(u) \) to overcome the difficulty. In other words, we will find Palais-Smale sequences with an extra property related to Pohozaev-Nehari type identity to obtain the Palais-Smale conditions. Some prior knowledge is presented here to obtain the special Palais-Smale sequences. First, we consider a map \( \phi : \mathbb{R} \times H_0^1(\mathbb{R}^3) \rightarrow H_0^1(\mathbb{R}^3) \), which is defined by \( \phi(\theta, u) = e^{\theta}u(e^{\theta}x) \). The space \( \mathbb{R} \times H_0^1(\mathbb{R}^3) \) is equipped with a standard product norm \( \|\theta, u\|_{\mathbb{R} \times H_0^1(\mathbb{R}^3)} = (|\theta|^2 + \|u\|^2)^{\frac{1}{2}} \). Recall the fact that the map \( \phi \) satisfies the property as follows (see [13,18]):  
\[ \phi(\lambda u(\lambda x)) = [\phi(u)](\lambda x), \quad \text{for all } \lambda \in \mathbb{R}^+, \quad u \in H_0^1(\mathbb{R}^3), \]
which can deduce that  
\[ \phi_{\theta}(\theta, u)(e^{\theta}x) = \phi_{\theta}(e^{\theta}x)(e^{\theta}x), \quad \text{for all } \theta \in \mathbb{R}, \quad u \in H_0^1(\mathbb{R}^3). \]

Next, we introduce an auxiliary functional \( \tilde{I}_n^\gamma(\theta, u) \in \mathcal{C}^{\gamma}(\mathbb{R} \times H_0^1(\mathbb{R}^3), \mathbb{R}) \) as follows:  
\[ \tilde{I}_n^\gamma(\theta, u) = I_n^\gamma(\phi(\theta, u)) \]
\[ = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi(\theta, u)|^2 dx - \frac{\omega}{2} \left( \left\| \phi(\theta, u) \right\|_T^2 \right) \int_{\mathbb{R}^3} \phi_{\theta}(\theta, u) \phi(\theta, u) dx - \int_{\mathbb{R}^3} G(\phi(\theta, u)) dx \]
\[ = \frac{e^{\theta}}{2} \int_{\mathbb{R}^3} |u|^2 dx - \frac{\omega}{2e^{\theta}} \left( \frac{e^{\theta} - 1}{e^{\theta} - \left\| u \right\|_{T^\beta}^2} \right) \int_{\mathbb{R}^3} \phi_{\theta} u^2 dx - \frac{1}{e^{\theta}} \int_{\mathbb{R}^3} G(e^{\theta}u) dx. \]

And a minimax value \( \tilde{c}_n \) for \( \tilde{I}_n^\gamma \) is defined by
\[ \tilde{c}_n = \inf_{y \in \Gamma, \delta \in E_n} \max \overline{I}_n(y(\delta)), \]

where

\[ \tilde{\Gamma}_n := \{ \tilde{y}(\delta) \in C(E_n, \mathbb{R} \times H^1_0(\mathbb{R}^3)) : \tilde{y}(\delta) = (\theta(\delta), \zeta(\delta)) \text{ satisfies} \]

\[ (\theta(-\delta), \zeta(-\delta)) = (\theta(\delta), -\zeta(\delta)), \quad \forall \delta \in E_n; \quad (\theta(\delta), \zeta(\delta)) = (0, y_{0n}(\delta)), \quad \forall \delta \in \partial E_n. \]

Then one can obtain that

**Lemma 5.2.** \( \tilde{c}_n = c_n. \)

**Proof.** On one hand, since \( \overline{I}_n(0, u) = I_n^0(u), \) it obtains that

\[ \max_{\delta \in E_n} \overline{I}_n(0, y(\delta)) = \max_{\delta \in E_n} I_n^0(y(\delta)). \]

By the definitions of \( \Gamma_n \) and \( \tilde{\Gamma}_n, \) one sees that \( (0, y(\delta)) \in \tilde{\Gamma}_n \) for each fixed \( y(\delta) \in \Gamma_n. \) Then,

\[ \tilde{c}_n = \inf_{y \in \Gamma, \delta \in E_n} \max \overline{I}_n(y(\delta)) = \inf_{y \in \Gamma, \delta \in E_n} \max \overline{I}_n(0, y(\delta)) = \inf_{y \in \Gamma, \delta \in E_n} \max I_n^0(y(\delta)) = c_n. \]

On the other hand, for any fixed \( \tilde{y}(\delta) \in \tilde{\Gamma}_n, \) let \( y(\delta) = e^{i(\delta)}u(\delta)(e_{\theta(\delta)}x), \) and then we obtain \( y(\delta) \in \Gamma_n \) and

\[ \overline{I}_n(\tilde{y}(\delta)) = I_n^0(y(\delta)). \]

Therefore,

\[ \max_{\delta \in E_n} \overline{I}_n(\tilde{y}(\delta)) = \max_{\delta \in E_n} I_n^0(y(\delta)) \geq \inf_{y \in \Gamma \Delta E_n} \max_{\delta \in E_n} I_n^0(y(\delta)), \]

which shows that

\[ \inf_{y \in \Gamma \Delta E_n} \max_{\delta \in E_n} \overline{I}_n(\tilde{y}(\delta)) \geq \inf_{y \in \Gamma \Delta E_n} \max_{\delta \in E_n} I_n^0(y(\delta)). \]

That is, \( \tilde{c}_n \geq c_n. \)

Also, the following lemma is a proof assistant for the special Palais-Smale sequences.

**Lemma 5.3.** [37] Let \( X \) be a Banach space. Let \( M_0 \) be a closed subspace of the metric space \( M \) and \( \Gamma_0 \subset C(M_0, X). \) Define

\[ \Gamma := \{ y \in C(M, X) : y|_{M_0} \in \Gamma_0 \}. \]

If \( \Psi \in C^1(X, \mathbb{R}) \) satisfies

\[ \inf_{y \in \Gamma} \max_{M} \Psi(y(u)) > c = \sup_{y \in \Gamma, u \in M} \Psi(y(u)) = \sup_{y \in \Gamma, u \in M_0} \Psi(y_0(u)), \]

then, for every \( \varepsilon \in (0, \frac{\varepsilon_a}{2}), \delta > 0 \) and \( y \in \Gamma \) such that

\[ \sup_{M} \Psi \circ y \leq c + \varepsilon, \]

there exists \( u \in X \) such that

(a) \( c - 2\varepsilon \leq \Psi(u) \leq c + 2\varepsilon, \)

(b) \( \text{dist}(u, y(M)) \leq 2\delta, \)

(c) \( \|\Psi'(u)\| \leq \frac{2a}{\varepsilon}. \)

Now we can obtain the special Palais-Smale sequences for \( I_n^0. \)

**Lemma 5.4.** Assume that \( (g_1), (g_2), \) and \( (g_3). \) Then for each \( n \in \mathbb{N}, \) there exists a sequence \( \{u_n^0\} \subset H^1_0(\mathbb{R}^3) \) satisfying
\( I^L_\omega(u^n_j) \to c_n, \quad (I^L_\omega)'(u^n_j) \to 0 \) and \( H(u^n_j) \to 0 \), as \( j \to \infty \).

**Proof.** By taking partial derivatives of \( I^L_\omega(\theta, u) \) with respect to \( \theta \) and \( u \), we can obtain

\[
\partial_\theta I^L_\omega(\theta, u) = \frac{e^\theta}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{\omega}{2e^\theta} \left( \frac{e^{\theta(\alpha - 3)} \|u\|_T^\alpha}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_u u^2 \, dx

- \frac{\beta \omega (\alpha - 3)}{2T^\beta} e^{\theta(\alpha - 3) - \beta - \alpha} \chi \left( \frac{e^{\theta(\alpha - 3)} \|u\|_T^\alpha}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_u u^2 \, dx \cdot \left( \int_{\mathbb{R}^3} |u|^{\alpha} \, dx \right)^{\frac{\beta - 1}{\beta}}

+ \frac{1}{e^\theta} \int_{\mathbb{R}^3} (3G(e^\theta u) - g(e^\theta u)e^\theta u) \, dx = H(\varphi(\theta, u))
\]

and

\[
\langle \partial_u I^L_\omega(\theta, u), v \rangle = e^\theta \int_{\mathbb{R}^3} \nabla u \nabla v \, dx - \frac{1}{e^\theta} \left( \frac{e^{\theta(\alpha - 3)} \|u\|_T^\alpha}{T^\beta} \right) \int_{\mathbb{R}^3} (2\omega + \phi_u) \phi_u uv \, dx

- \frac{\omega \beta}{2T^\beta} e^{\theta(\alpha - 3) - \beta - \alpha} \chi \left( \frac{e^{\theta(\alpha - 3)} \|u\|_T^\alpha}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_u u^2 \, dx \cdot \left( \int_{\mathbb{R}^3} |u|^{\alpha} \, dx \right)^{\frac{\beta - 1}{\beta}}

\cdot \int_{\mathbb{R}^3} |u|^{-\beta} uv \, dx - \frac{1}{e^\theta} \int_{\mathbb{R}^3} g(e^\theta u)e^\theta v \, dx

= \langle (I^L_\omega)'(\varphi(\theta, u)), \varphi(\theta, v) \rangle, \quad \text{for any } v \in H^1_1(\mathbb{R}^3).
\]

For any \( n \in \mathbb{N} \), by the definition of \( c_n \), there exists \( \gamma_j \in \Gamma_n \) such that

\[
\max_{\delta \in E_n} I^L_\omega(\gamma_j(\delta)) \leq c_n + \frac{1}{j}.
\]

Notice that \( I^L_\omega(0, \gamma_0(\delta)) = I^L_\omega(\gamma_j(\delta)) \) and \( (0, \gamma_j(\delta)) \in \Gamma_n \) for any fixed \( j \in \Gamma_n \). By combining with Lemma 5.2, we know that

\[
\max_{\delta \in E_n} I^L_\omega(\gamma_j(\delta)) \leq c_n + \frac{1}{j}.
\]

By using Lemma 5.3 and \( \gamma_j(\delta) \subset \{0\} \times H^1_1(\mathbb{R}^3) \), as \( n \to \infty \), we can find a \( (\theta^n_j, w^n_j) \) such that

(1) \( c_n - \frac{1}{j} \leq I^L_\omega(\theta^n_j, w^n_j) \leq c_n + \frac{1}{j} \),

(2) \( \text{dist}_{\mathbb{R} \times H^1_1(\mathbb{R}^3)}((\theta^n_j, w^n_j), \gamma_j(\delta)) \leq \frac{2}{\sqrt{j}} \),

(3) \( \|I^L_\omega(\theta^n_j, w^n_j)\|_{\mathbb{R} \times H^1_1(\mathbb{R}^3)} \leq \frac{2}{\sqrt{j}} \).

From (2), we see that \( \theta^n_j \to 0 \) as \( n \to \infty \). For all \( (\tau, v) \in \mathbb{R} \times H^1_1(\mathbb{R}^3) \), we know that

\[
\langle (I^L_\omega)'(\theta^n_j, w^n_j), (\tau, v) \rangle = \langle (I^L_\omega)'(\varphi(\theta^n_j, w^n_j)), \varphi(\theta^n_j, v) \rangle + H(\varphi(\theta^n_j, w^n_j))\tau.
\]

Therefore, by combining (5.9) and (5.10), let \( \tau = 1 \) and \( v = 0 \), one obtains

\[
H(\varphi(\theta^n_j, w^n_j)) \to 0.
\]

Let \( u^n_j = \varphi(\theta^n_j, w^n_j) \), one can know that

\[
I^L_\omega(u^n_j) \to c_n, \quad (I^L_\omega)'(u^n_j) \to 0 \quad \text{and} \quad H(u^n_j) \to 0, \quad \text{as} \quad j \to \infty. \quad \square
\]
Lemma 5.5. Suppose that \((g_1), (g_2),\) and \((g_3)\) hold. Then there exists \(\omega_1 > 0\) such that for any \(0 < \omega < \omega_1\), each sequence \(\{u_n\}\) satisfying (5.8) is bounded in \(H^1(\mathbb{R}^3)\).

Proof. From (2.9), (2.10), and (5.2), we can obtain that

\[
3c_n + o(1) \geq 3I_{\omega}^n(u_n^j) - \langle I_{\omega}^n(u_n^j), u_n^j \rangle + H(u_n^j)
\]

\[
= \int_{\mathbb{R}^3} |\nabla u_n^j|^2 dx + \omega \left( \frac{\|u_n^j\|_{H^1}}{T^\beta} \right)^2 \int_{\mathbb{R}^3} \phi_{u_n^j}(u_n^j)^2 dx + \chi \left( \frac{\|u_n^j\|_{H^1}}{T^\beta} \right)^2 \int_{\mathbb{R}^3} \phi_{u_n^j}(u_n^j)^2 dx
\]

\[
+ \frac{3\beta \omega}{2\alpha T^\beta} \frac{\|u_n^j\|_{H^1}}{T^\beta} \int_{\mathbb{R}^3} \phi_{u_n^j}(u_n^j)^2 dx \|u_n^j\|_{H^1}^\beta.
\]

(5.11)

By the definition of \(\chi\) and Lemma 2.4, one obtains that

\[
\omega \chi \left( \frac{\|u_n^j\|_{H^1}}{T^\beta} \right)^2 \int_{\mathbb{R}^3} \phi_{u_n^j}(u_n^j)^2 dx \leq C\omega \left( \frac{\|u_n^j\|_{H^1}}{T^\beta} \right) \|u_n^j\|_{H^1} \leq 2C\omega T^\beta,
\]

(5.12)

\[
\chi \left( \frac{\|u_n^j\|_{H^1}}{T^\beta} \right)^2 \int_{\mathbb{R}^3} \phi_{u_n^j}(u_n^j)^2 dx \leq \omega \left( \frac{\|u_n^j\|_{H^1}}{T^\beta} \right)^2 \int_{\mathbb{R}^3} \phi_{u_n^j}(u_n^j)^2 dx \leq 2C\omega T^\beta,
\]

(5.13)

\[
\frac{3\beta \omega}{2\alpha T^\beta} \frac{\|u_n^j\|_{H^1}}{T^\beta} \int_{\mathbb{R}^3} \phi_{u_n^j}(u_n^j)^2 dx \|u_n^j\|_{H^1}^\beta \leq \frac{3\beta \omega}{2\alpha T^\beta} \left( \frac{\|u_n^j\|_{H^1}}{T^\beta} \right) \|u_n^j\|_{H^1} \leq \frac{12CB}{\alpha} \omega T^\beta.
\]

(5.14)

Hence, combining (5.11), (5.12), (5.13), and (5.14), there exist \(\omega_1 > 0\) and \(C > 0\) such that for any \(0 < \omega < \omega_1\),

\[
\int_{\mathbb{R}^3} |\nabla u_n^j|^2 dx \leq C.
\]

(5.15)

Moreover, by (3.6), Sobolev inequality and \(I_{\omega}^{n,h}(u_n^j) \to c_n\) as \(n \to \infty\), we obtain that

\[
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n^j|^2 dx - \frac{1}{2} \chi \left( \frac{\|u_n^j\|_{H^1}}{T^\beta} \right)^2 \int_{\mathbb{R}^3} \omega \phi_{u_n^j}(u_n^j)^2 dx = \int_{\mathbb{R}^3} G(u_n^j) dx + c_n + o(1)
\]

\[
\leq -L\|u_n^j\|_2^2 + C\|u_n^j\|_{H^1}^\beta + c_n + o(1)
\]

\[
\leq -L\|u_n^j\|_2^2 + C \left( \int_{\mathbb{R}^3} |\nabla u_n^j|^2 dx \right) + c_n + o(1),
\]

(5.16)

together with (5.12) and (5.15), there exists \(C > 0\) such that \(\|u_n^j\|_2^2 \leq C\). By combining with (5.15), we obtain that \(\{u_n^j\}\) is bounded in \(H^1(\mathbb{R}^3)\).

Now we are ready to find the existence of infinitely many solutions for \(I_{\omega}^n\).

Lemma 5.6. Suppose that \((g_1), (g_2),\) and \((g_3)\) hold. Then there exist infinitely many solutions for \(I_{\omega}^n\) when \(0 < \omega < \omega_1\), where \(\omega_1\) is given in Lemma 5.5.

Proof. Similar to Lemma 3.2, there exists \(u_0^0 \in H^1(\mathbb{R}^3)\) such that \(u_n^j \to u_0^0\) in \(H^1(\mathbb{R}^3)\). That is, for each \(n \in \mathbb{N}\), \(u_0^0\) satisfies \(I_{\omega}^n(u_0^0) = c_n\) and \(\langle I_{\omega}^n(u_0^0), u_0^0 \rangle = 0\). Therefore, \(c_n \ (n \in \mathbb{N})\) is a critical value of \(I_{\omega}^n(u)\). Together with the fact that \(I_{\omega}^n(u) \geq J(u)\) and (5.6)–(5.7), we know that \(c_n \to \infty\), and then the existence of infinitely many solutions for \(I_{\omega}^n\) is obtained.
**Proof of Theorem 1.4.** For any fixed $k \in \mathbb{N}$, if
\[
\|u_k\|_{L^6} \leq T
\]  
(5.17)
is true, then $\chi \left( \frac{\|u_k\|_{L^6}}{T^\beta} \right) = 1$. Therefore, $u_k^0$ is a solution for system (1.1), and we complete the proof. Next we show that there exists $\omega_k(T) < \omega_1$ such that (5.17) holds for $0 < \omega \leq \omega_k(T)$, where $\omega_1$ is given in Lemma 5.5. Since $u_k^0$ is a solution for $I_{\omega k}^0$, together with Lemma 2.5 and $I_{\omega k}^0(u_k^0) = c_k$, we know that
\[
\int_{\mathbb{R}^3} |\nabla u_k^0|^3 \, dx - 5 \omega \chi \left( \frac{\|u_k^0\|_{L^6}}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_{u_k^0}(u_k^0)^3 \, dx - 2 \chi \left( \frac{\|u_k^0\|_{L^6}}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_{u_k^0}(u_k^0)^2 \, dx \\
- \frac{3 \beta \omega}{4 T^\beta} \chi \left( \frac{\|u_k^0\|_{L^6}}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_{u_k^0}(u_k^0)^2 \, dx \cdot \left( \int_{\mathbb{R}^3} |u_k^0|^2 \, dx \right)^{\frac{3}{2}} = 6 \int_{\mathbb{R}^3} G(u_k^0) \, dx
\]  
(5.18)
and
\[
3 \int_{\mathbb{R}^3} |\nabla u_k^0|^3 \, dx \geq 3 \omega \chi \left( \frac{\|u_k^0\|_{L^6}}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_{u_k^0}(u_k^0)^2 \, dx - \frac{3 \beta \omega}{4 T^\beta} \chi \left( \frac{\|u_k^0\|_{L^6}}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_{u_k^0}(u_k^0)^2 \, dx \cdot \|u_k^0\|_{L^6}^2.
\]  
(5.19)
Then we have that
\[
\int_{\mathbb{R}^3} |\nabla u_k^0|^3 \, dx \leq 3 c_k - \omega \chi \left( \frac{\|u_k^0\|_{L^6}}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_{u_k^0}(u_k^0)^2 \, dx - \frac{3 \beta \omega}{4 T^\beta} \chi \left( \frac{\|u_k^0\|_{L^6}}{T^\beta} \right) \int_{\mathbb{R}^3} \phi_{u_k^0}(u_k^0)^2 \, dx \cdot \|u_k^0\|_{L^6}^2.
\]  
(5.20)
It follows from (5.5) that there exists a constant $B_1 > 0$ such that
\[
c_k \leq \max_{\delta \in E_n} I_{\omega}(c_n(\delta)) \leq \max_{\delta \in E_n} \left( \frac{t}{2} \int_{\mathbb{R}^3} |\nabla c_n(\delta)|^2 \, dx - t^3 \int_{\mathbb{R}^3} G(c_n(\delta)) \, dx \right) \\
+ \max_{\delta \in E_n} \left( \frac{\omega}{2} \cdot t^\alpha \cdot \chi \left( \frac{t^\alpha \|c_n(\delta)\|_{a}^\beta}{T^\beta} \right) \|c_n(\delta)\|_{a}^\beta \right) \\
\leq B_1 + \max_{\delta \in E_n} \left( \frac{\omega}{2} \cdot t^\alpha \cdot \chi \left( \frac{t^\alpha \|c_n(\delta)\|_{a}^\beta}{T^\beta} \right) \|c_n(\delta)\|_{a}^\beta \right).
\]  
(5.21)
Supposing that $t^\alpha \|c_n(\delta)\|_{a}^\beta \geq 2 T^\beta$, one easily obtains that
\[
\frac{\omega}{2} \cdot t^\alpha \cdot \chi \left( \frac{t^\alpha \|c_n(\delta)\|_{a}^\beta}{T^\beta} \right) \|c_n(\delta)\|_{a}^\beta = 0.
\]
If $t^\alpha \|c_n(\delta)\|_{a}^\beta < 2 T^\beta$, then
\[
\frac{\omega}{2} \cdot t^\alpha \cdot \chi \left( \frac{t^\alpha \|c_n(\delta)\|_{a}^\beta}{T^\beta} \right) \|c_n(\delta)\|_{a}^\beta \leq \frac{\omega}{2} \cdot t^\alpha \|c_n(\delta)\|_{a}^\beta \leq C t^\alpha \|c_n(\delta)\|_{a}^\beta \leq C a T^\beta.
\]  
(5.22)
From (5.21) and (5.22), one obtains that
\[
c_k \leq B_1 + C a T^\beta.
\]  
(5.23)
Thus, by (5.12), (5.14), (5.20), and (5.23), there exist constants $C_1, C_4 > 0$ such that
\[
\int_{\mathbb{R}^3} |\nabla u_k^0|^3 \, dx \leq C_1 + C_4 a T^\beta.
\]  
(5.24)
With the fact that \(3.8\) and \(u_k^0\) \((k \in \mathbb{N})\) is a solution for \(I_\omega\), one also knows that
\[
\left[ \nabla u_k^0 \right]^2 \in \mathcal{H}^1_\omega(\mathbb{R}^3) \quad \text{for all } k \in \mathbb{N},
\]
and
\[
\left[ \nabla u_k^0 \right]^2 \in \mathcal{H}^1_\omega(\mathbb{R}^3) \quad \text{for all } k \in \mathbb{N}.
\]
By using (5.14), (5.24), and (5.25), one obtains that
\[
\begin{align*}
(1 - \epsilon) \int_{\mathbb{R}^3} |g_\epsilon(u_k^0)u_k^0|^2 \, dx &\leq C_{\epsilon}\|u_k^0\|_{\mathcal{H}}^3 + \frac{\beta \omega}{2T^p} \left( \frac{\|u_k^0\|_{\mathcal{H}}^p}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_k^0}(u_k^0)^2 \, dx \|u_k^0\|_{\mathcal{H}}^2 \\
&\leq C_{\epsilon} \left( \int_{\mathbb{R}^3} |\nabla u_k^0|^2 \, dx \right)^3 + \frac{\beta \omega}{2T^p} \left( \frac{\|u_k^0\|_{\mathcal{H}}^p}{T^p} \right) \int_{\mathbb{R}^3} \phi_{u_k^0}(u_k^0)^2 \, dx \|u_k^0\|_{\mathcal{H}}^2 \\
&\leq C_{\epsilon} S \left( C_3 + C_4\omega T^\beta \right)^3 + 4C\beta\omega T^\beta,
\end{align*}
\]
and together with (3.7), we can easily obtain the following:
\[
\int_{\mathbb{R}^3} (u_k^0)^2 \, dx \leq C(C_3 + C_4\omega T^\beta)^3 + C\omega T^\beta.
\]
Then combining (5.24) and (5.26), there exists a constant \(C > 0\) such that
\[
\|u_k^0\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^3} |\nabla u_k^0|^2 \, dx + \int_{\mathbb{R}^3} (u_k^0)^2 \, dx \leq C + C\omega T^\beta + C(C_3 + C_4\omega T^\beta)^3.
\]
We assume by contradiction that \(\|u_k^0\|_{\mathcal{H}} > T\). From (5.27) and Sobolev inequality, one obtains that
\[
T^2 < \|u_k^0\|_{\mathcal{H}}^2 \leq C\|u_k^0\|_{\mathcal{H}}^2 \leq C + C_\omega T^\beta + C(C_3 + C_4\omega T^\beta)^3.
\]
If \(T\) is large enough and \(\omega T^\beta < 1\), (5.28) is not true, and we obtain the conclusion.

**Acknowledgement:** The authors are very grateful to the referee and the handling editor for valuable suggestions, which helped us to improve our article greatly. This work was supported by National Natural Science Foundation of China (Grant No. 11971393), the China Postdoctoral Science Foundation (Grant No. 2020M683251), and the special subsidy from Chongqing human resources and Social Security Bureau.

**Conflict of interest:** The authors state no conflict of interest.

**References**


