Research Article

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Existence and uniqueness of solution for a singular elliptic differential equation

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Abstract: In this article, we are concerned about the existence, uniqueness, and nonexistence of the positive solution for:
\[\begin{align*}
        -\Delta u - \frac{1}{2}(x \cdot \nabla u) &= \mu h(x)u^{q-1} + \lambda u - u^p, \quad x \in \mathbb{R}^N, \\
        u(x) &\to 0, \quad \text{as } |x| \to +\infty,
\end{align*}\]
where $N \geq 3$, $0 < q < 1$, $\lambda > 0$, $p > 1$, $\mu > 0$ is a parameter and the function $h(x)$ satisfies certain conditions. To start with, based on the variational argument and perturbation method, we obtain the existence and uniqueness of the positive solution for the aforementioned singular elliptic differential equation as $\lambda > \frac{N}{2}$. In addition, there is no solution as $\lambda \leq \frac{N}{2}$. Later, from an experimental point of view, we give the numerical solution of the aforementioned singular elliptic differential equation by means of a neural network in some special cases, which enrich the theoretical results. Our conclusions partially extend the results corresponding to the non-singular case.

Keywords: singular problem, variational method, existence, uniqueness, neural network

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1 Introduction

The one-parameter elliptic partial differential equation:
\[\begin{align*}
        -\Delta u &= \lambda a(x)u - b(x)u^p, \quad x \in \mathbb{R}^N
\end{align*}\]
arose in mathematical biology and Riemannian geometry, where $p > 1$, $N > 2$, $\lambda \in \mathbb{R}$ is a parameter, $a(\cdot)$ and $b(\cdot)$ are both smooth functions in $\mathbb{R}^N$. Equation (1) has important research value in mathematics and applications, and there is a vast literature concerned with this kind of problem. In the context of mathematical biology, it was studied in [2,12,35]. For example, if $b(x) \equiv 1$, $p = 2$, $a^+(x) = \max\{a(x), 0\} \leq \frac{k}{|x|^p}$ for some $k > 0$, $\delta > 0$ and all $x \in \mathbb{R}^N$, Afrouzi and Brown [2] proved that equation (1) has a unique positive solution when $\lambda > \lambda_1$ and has no positive solution in other cases by constructing the sub- and supersolutions, where $\lambda_1$ is the principal eigenvalue of the equation:
\[\begin{align*}
        -\Delta u &= \lambda a(x)u(x), \quad x \in \mathbb{R}^N, \\
        u(x) &\to 0, \quad \text{as } |x| \to +\infty.
\end{align*}\]
In addition, in [12], Du and Ma supposed that \( a(x) \to a > 0 \) and \( b(x) \to b > 0 \) as \(|x| \to +\infty\); they discussed the existence, uniqueness, and nonexistence of (1) again. Moreover, Du and Ma further obtained the properties of the solutions of (1) with other conditions in [13]. It is easy to see that the results remain valid if \( \Delta \) is replaced by a uniformly elliptic operator. In the area of Riemannian geometry, for example, Pino [11] proved that there exists a unique solution of equation (1) on a compact Riemannian manifold under certain conditions. In other various cases, one can refer to [4,21,22,24,28].

When the perturbation term is included, many articles also studied the existence, uniqueness, and nonexistence of the positive solution for Problem (1). For example, in 2019, Delgado et al. [9] added perturbation to (1) as \( a(x) = b(x) = K(x) \), namely, they obtained the existence of positive solutions for the following equation:

\[
-\Delta u + u = K(x)u \left( \lambda - u^p + a \int_{\mathbb{R}^N} M(x, y)g(u(y))dy \right), \quad x \in \mathbb{R}^N,
\]

where \( N \geq 1, \lambda, a \in \mathbb{R}, p > 0, g(\cdot), \) and \( K(\cdot) \) are the functions satisfying some conditions, \( M \in L^1(\mathbb{R}^N \times \mathbb{R}^N) \) is positive. Then, in 2021, under more general conditions of \( a(x) \) and \( b(x) \), Delgado et al. [10] discussed the following perturbation equation:

\[
-\Delta u = u \left( \lambda a(x) - b(x)u^p + a \int_{\mathbb{R}^N} M(x, y)u(x')dy \right), \quad x \in \mathbb{R}^N,
\]

\[
u \in D^{1,2}(\mathbb{R}^N), \quad u > 0,
\]

where \( N \geq 3, \lambda, a \in \mathbb{R}, p, r > 0, \) and \( M(x, y) \) satisfies certain conditions. In addition, using bifurcation theory, they discussed the properties of the solutions under various conditions about \( a, p, \) and \( r \).

On the other hand, singular elliptic differential equations are an important part of PDEs. It is a significant tool to describe natural phenomena and explain natural laws. The study of singular elliptic differential equations originated in the middle of the last century [1,7,20,23,25,30,32]. It aroused the interests of many mathematicians, and they did the groundworks, such as [8,18,33]. From a practical point of view, its applications involve reaction-diffusion, heat conduction, fluid dynamics, non-Newtonian fluids, and many other areas [7,20,23,25,30,32]. From a mathematical point of view, the study of singular elliptic equations can not only enrich the theory of differential equations but also promote the development of other mathematical and application branches. Up until now, much attention has been focused on singular elliptic differential problems.

It is natural to ask whether one adds a singular perturbation term to (1), does the result in [13] still hold?

With respect to this question, partial answers are given in some special cases. For example, assuming \( a(x) = 1 \) and \( b(x) = 0 \), Hai [19] studied the regularity of the solutions for a singular perturbation problem of (1). If \( a(x) \) is a nonnegative function belonging to a suitable Lebesgue space and \( b(x) = 0 \), Durastanti and Oliva [14] obtained the existence and uniqueness of positive solution when adding a singular term to (1). The conclusions in [19] and [14] are both discussed on a bounded domain \( \Omega \subset \mathbb{R}^N \) with \( u(x) = 0 \) on the boundary of \( \Omega \).

Inspired by the aforementioned references, in this article, we consider equation (1) again with a more general singular perturbation, and the Laplacian operator is replaced by a uniformly elliptic operator \( -\Delta u - \frac{1}{2}(x \cdot \nabla u) \), namely, one pays attention to the existence, uniqueness, and nonexistence of the positive solution for the following singular elliptic differential equation:

\[
-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu h(x)u^{q-1} + \lambda u - u^p, \quad x \in \mathbb{R}^N,
\]

\[
u(x) \to 0, \quad \text{as} \ |x| \to +\infty,
\]

where \( N \geq 3, p > 1, 0 < q < 1, \lambda > 0, \mu > 0 \) is a small parameter, \( (x \cdot \nabla u) \) equals to \( x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} + \cdots + x_n \frac{\partial u}{\partial x_n} \) and \( h > 0 \) satisfies

\[
h \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N),
\]

where \( L^1(\mathbb{R}^N) \) and \( L^2(\mathbb{R}^N) \) are defined in Section 2. In fact, the elliptic equation involving the uniformly elliptic operator \( -\Delta u - \frac{1}{2}(x \cdot \nabla u) \) has a long research history. As observed by Escobedo and Kavian in [15], when one deals with the nonlinear heat equation:
where \( \varepsilon = \pm 1 \) and \( 2 < p \leq 2^* = \frac{2N}{(N-2)} \). It is well known that if \( u \neq 0 \) is a solution, then for \( \lambda > 0 \), one can define a family of solutions \( (u_\lambda)_\lambda \) to (4) by:

\[
u_\lambda(t, x) = \lambda^{2/(p-1)} u(\lambda^2 t, \lambda x),\]

and one can look for solutions \( u \) of (4) that are invariant under the “similarity” operation defined by (5), i.e., solutions \( u \) such that \( \forall \lambda > 0, u_\lambda = u \). When looking for these so-called “self-similar” solutions, one finds that if we denote \( f(x) = u(1, x) \), then

\[
u(t, x) = t^{-1/(p-1)} f\left(\frac{x}{\sqrt{t}}\right) \quad \text{on } (0, +\infty) \times \mathbb{R}^N, \quad \text{and } f \neq 0
\]

has to satisfy a related elliptic equation, namely,

\[-\Delta u - \frac{1}{2} (x \cdot \nabla f) + \varepsilon |f|^{p-1} f = \lambda f \quad \text{on } \mathbb{R}^N \left\{ \lambda = \frac{1}{(p-1)} \right\}.
\]

So there have been numerous researchers who have studied it extensively and achieved a series of remarkable results [3,6,16,26,29].

Furthermore, to test the rationality of the theoretical results, we conduct some comparative experiments to validate the uniqueness of the positive solution for (2) according to the convergence of ground truth and results predicted by the method of neural networks on validation points. Nowadays, neural network has triggered changes in many fields. Due to the powerful fitting ability, neural network can not only learn complex nonlinear mapping in computer vision and other research interests, but also solve some mathematical problems, such as nonlinear differential equation [31].

The rest of this article is organized as follows. Section 2 introduces some preliminary results that will be applied to prove main results. Section 3 is devoted to the existence, uniqueness, and nonexistence of the positive solution for (2). Finally, we apply the method of neural network to illustrate the aforementioned conclusions under a certain condition.

## 2 Preliminary results

In this section, we give some preliminary results that will be applied to prove main conclusions.

In order to make equation (2) has a variational structure, one multiplies the two sides of equation (2) by:

\[K(x) = \exp(|x|^2/4),\]

then can obtain its equivalent equation:

\[
\begin{cases}
-\text{div}(K(x)\nabla u) = \mu K(x) h(x) u^{p-1} + \lambda K(x) u - K(x) u^p, & x \in \mathbb{R}^N, \\
u(x) \to 0, & \text{as } |x| \to +\infty.
\end{cases}
\]

Here, \( -\text{div}(K(x)\nabla u) = -\text{div}(K(x)\frac{\partial u}{\partial x_1}, K(x)\frac{\partial u}{\partial x_2}, ..., K(x)\frac{\partial u}{\partial x_N}) \).

We are going to work in the space \( X \), which is the completion of \( C_0^n(\mathbb{R}^N) \) with respect to the norm:

\[|u| = \left( \int_{\mathbb{R}^N} K(x)|\nabla u|^2 dx \right)^{1/2}.
\]

Through Propositions 1.1 and 1.12 in [15], it is clear that \( X \) is a Hilbert space and continuously embedded into the weighted Lebesgue space:
The spaces $L^p_k(\mathbb{R}^N)$ and $L^q_k(\mathbb{R}^N)$ in Introduction are defined, respectively, by:

$$L^p_k(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : ||u||_p = \left( \int_{\mathbb{R}^N} K(x)|u|^p \, dx \right)^{1/p} < +\infty \right\},$$

for any $p \in (1, 2^*)$. Furthermore, the embedding is compact if $p \in (1, 2^*)$.

The spaces $L^p_k(\mathbb{R}^N)$ and $L^q_k(\mathbb{R}^N)$ in Introduction are defined, respectively, by:

$$L^q_k(\mathbb{R}^N) = \left\{ u \in L^q(\mathbb{R}^N) : ||u||_q = \left( \int_{\mathbb{R}^N} K(x)|u|^q \, dx \right)^{1/q} < +\infty \right\},$$

and

$$L^q_k(\mathbb{R}^N) = \left\{ u \in L^q(\mathbb{R}^N) : ||u||_q = \left( \int_{\mathbb{R}^N} K(x)|u|^q \, dx \right)^{1/q} < +\infty \right\}.$$
\[ I_\mu(u) = I_0(u) - \frac{\mu}{d} \int_{\mathbb{R}^N} K(x) h(x)(u^\delta)^\delta \, dx, \quad u \in X. \]

It is clear that \( I_\mu \) is a continuous functional in \( X \).

Next, let us give the definition of the positive solution for equation (6).

**Definition 1.** It should be argued that \( u \in X \) is a (weak) positive solution of equation (6) if:

(i) \( u > 0 \) a.e. in \( X \);

(ii) for any \( \phi \in X \), one has \( h(x)u^{\delta-1}\phi \in L^1_0(\mathbb{R}^N) \) and

\[
\int_{\mathbb{R}^N} K(x)[(\nabla u \cdot \nabla \phi) - \mu h(x)u^{\delta-1}\phi - \lambda u\phi + u^\delta\phi] \, dx = 0. \tag{12}
\]

In the following, it tends to show that the condition \( \lambda > \lambda_1 = \frac{N}{2} \) is necessary for the existence of the positive solution to (6) as \( \mu \) is small enough.

**Lemma 1.** Assume that \( \mu \) is small enough, and if equation (6) has a positive solution \( u \) on \( X \), then \( \lambda > \frac{N}{2} \).

**Proof.** Multiplying equation (6) by \( u \in X \) and then integrating by parts, one obtains

\[
\int_{\mathbb{R}^N} K(x)|\nabla u|^2 \, dx = \mu \int_{\mathbb{R}^N} K(x) h(x)u^\delta \, dx + \lambda \int_{\mathbb{R}^N} K(x) u^2 \, dx - \int_{\mathbb{R}^N} K(x) u^{\delta+1} \, dx. \tag{13}
\]

Hence, combining (9) and (13), one arrives at:

\[
\lambda_1 \int_{\mathbb{R}^N} K(x)|u|^2 \, dx \leq \int_{\mathbb{R}^N} K(x)|\nabla u|^2 \, dx
\]
\[
= \mu \int_{\mathbb{R}^N} K(x) h(x) u^\delta \, dx + \lambda \int_{\mathbb{R}^N} K(x) u^2 \, dx - \int_{\mathbb{R}^N} K(x) u^{\delta+1} \, dx. \tag{14}
\]

By (11), it yields

\[
\lim_{\mu \to 0^+} \mu \int_{\mathbb{R}^N} K(x) h(x) u^\delta \, dx \to 0.
\]

Take the limit \( \mu \to 0^+ \) on both sides of (14), and based on the sign-preserving theorem of limit, it is widely accepted that

\[
\lambda_1 \int_{\mathbb{R}^N} K(x) u^2 \, dx \leq \lambda \int_{\mathbb{R}^N} K(x) u^2 \, dx - \int_{\mathbb{R}^N} K(x) u^{\delta+1} \, dx
\]

as \( \mu \) is small enough. Therefore,

\[
(\lambda_1 - \lambda) \int_{\mathbb{R}^N} K(x) u^2 \, dx \leq - \int_{\mathbb{R}^N} K(x) u^{\delta+1} \, dx < 0.
\]

It follows

\[
\lambda > \lambda_1 = \frac{N}{2}. \quad \square
\]
3 Theoretical proof

First, it will be shown that when \(|u| = R\) with \(R\) large enough, the functional \(I_\rho(u)\) is bounded below.

**Lemma 2.** There exists \(\mu^* > 0\) such that, for any \(\mu \in (0, \mu^*)\), there holds

\[
I_\rho(u) \geq \rho, \quad \forall u \in \partial B_\rho(0)
\]

with \(\rho > 0, R > 0,\) and \(R\) large enough, where \(B_\rho(0) = \{u \in X, ||u|| < R\}\) and \(\partial B_\rho(0) = \{u \in X, ||u|| = R\}\).

**Proof.** By Corollary 1.11 in [15], it obtains that for a given \(\lambda\), there exists a constant \(C > 0\) such that

\[
\lambda \int_{\mathbb{R}^N} K(x)|u|^2 \, dx \leq C \left( \int_{\mathbb{R}^N} K(x)|u|^p \, dx \right)^{\frac{2}{p+1}} + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.
\]

So one can acquire

\[
I_\rho(u) = \frac{1}{2} ||u||^2 - \lambda \frac{1}{2} ||u^*||^2 + \frac{1}{p+1} ||u^*||_{p+1}^2
\]

\[
\geq \frac{1}{4} ||u||^2 - \frac{1}{4} ||u||^2 - \frac{C}{2} ||u^*||_{p+1}^2 + \frac{1}{p+1} ||u^*||_{p+1}^2
\]

\[
= \frac{1}{4} ||u||^2 + \frac{1}{p+1} ||u^*||_{p+1}^2 - \frac{C}{2} ||u^*||_{p+1}^2.
\]

Let

\[
g(t) = \frac{1}{p+1} t^{p+1} - \frac{C}{2} t^2.
\]

Hence, one arrives at:

\[
g_{\min} = \frac{1}{p+1} C_{p+1}^{p+1} - \frac{C}{2} C_{p+1}^{p+1} = \left( \frac{1}{p+1} - \frac{1}{2} \right) \frac{C_{p+1}^{p+1}}{C_{p+1}^{p+1}} < 0.
\]

Therefore,

\[
I_\rho(u) \geq \frac{1}{4} ||u||^2 + \frac{1}{p+1} - \frac{1}{2} C_{p+1}^{p+1}.
\]

By (15) and (11), one has

\[
I_\rho(u) = \frac{1}{2} ||u||^2 - \lambda \frac{1}{2} ||u^*||^2 + \frac{1}{p+1} ||u^*||_{p+1}^2 - \frac{\mu}{q} \int_{\mathbb{R}^N} K(x) h(x)(u^*)^q \, dx
\]

\[
\geq \frac{1}{4} ||u||^2 + \left( \frac{1}{p+1} - \frac{1}{2} \right) C_{p+1}^{p+1} - \mu C ||u||^q
\]

\[
\geq \rho = \frac{1}{4} R^2 + \left( \frac{1}{p+1} - \frac{1}{2} \right) C_{p+1}^{p+1} - \varepsilon > 0
\]

due to \(|u|| = R\) being large enough, where

\[
0 < \mu < \mu^* = \frac{\varepsilon}{C_2 R^2}, \quad \varepsilon > 0 \quad \text{small enough}.
\]

**Remark 1.** Through Lemma 2, it seems easy to know that the functional \(I_\rho(u)\) is bounded below and coercive. Thus,
is well defined, and hence, it gains the following lemma.

**Lemma 3.** For any \( \mu \in (0, \mu^*) \), \(-\infty < m_\mu < 0\).

**Proof.** By the fact that \( I_\mu \) maps bounded sets to bounded sets, it is easy to know \( m_\mu > -\infty \).

In order to prove \( m_\mu < 0 \), one chooses a nonnegative function \( \varphi \in C_c^0(\mathbb{R}^N) \setminus \{0\} \), then it has

\[
\lim_{t \to 0^-} \frac{I_\mu(t\varphi)}{t^q} = \lim_{t \to 0^-} \frac{1}{t^q} \left[ \frac{1}{2} \int_{\mathbb{R}^N} K(x)|\nabla(t\varphi)|^2dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} K(x)(t\varphi)^2dx \right] + \lim_{t \to 0^-} \frac{1}{t^q} \left[ \frac{1}{p+1} \int_{\mathbb{R}^N} K(x)(t\varphi)^{p+1}dx - \frac{\mu}{q} \int_{\mathbb{R}^N} K(x)h(x)(t\varphi)^qdx \right]
\]

\[
= -\frac{\mu}{q} \int_{\mathbb{R}^N} K(x)h(x)\varphi^qdx < 0.
\]

Therefore, choosing \( t_0 > 0 \) such that \( \|t_0\varphi\| \leq R \) and \( I_\mu(t_0\varphi) < 0 \), it is generally agreed that

\[
m_\mu = \inf_{u \in B_0(0)} I_\mu(u) < 0. \]

\[\square\]

**Remark 2.** The singular term causes difficulties even if the infimum of \( I_\mu(u) \) can be obtained in \( B_0(0) \). In fact, since \( 0 < q < 1 \), the term

\[
\int_{\mathbb{R}^N} K(x)h(x)(u^*)^qdx
\]

is continuous but not differentiable. So we do not know whether the minimizers of \( I_\mu(u) \) are the positive solutions of (6) or not. However, through the following lemma, one will give a positive answer to this question, namely, the minimizers of \( I_\mu(u) \) are still the positive solutions of (6).

**Lemma 4.** If \( u \in B_0(0) \) satisfies \( I_\mu(u) = m_\mu \), then \( u \) is a positive solution of equation (6).

**Proof.** Since \( u \in B_0(0) \), it is easy to know

\[ u^* \in B_0(0) \]

where \( u^* = \max\{u, 0\} \). In addition, \( u^- = u^* - u \).

By the definition of the functional \( I_\mu(u) \), it follows

\[
m_\mu \leq I_\mu(u^*) \leq I_\mu(u) = m_\mu.
\]

Hence, \( u^- \equiv 0 \), which means

\[ u = u^* \geq 0. \]

So \( u \) and \( u^* \) are interchangeable in the functionals \( I_\mu(u) \) and \( I_\mu(u) \).

We claim \( u > 0 \) a.e. on \( \mathbb{R}^N \).

Indeed, suppose, on the contrary, that the set

\[ \Omega_0 = \{x \in \mathbb{R}^N : u(x) = 0\} \]

has a positive measure. Let \( r > 0 \) such that
\( \Omega = \Omega_0 \cap B_r(0) \)

has a positive measure. Set a nonnegative function \( \psi \in X \) such that

\[ \text{supp}(\psi) \subset B_{2r}(0) \]

and

\[ \psi > 0 \quad \text{in} \quad B_r(0). \]

Since \( ||u|| < R \), it can be seen that

\[ ||u + t\psi|| < R, \]

for any \( t > 0 \) small enough. Since \( I_\delta(u) \) is the infimum in \( B_\delta(0) \), it yields

\[ I_\delta(u) \leq I_\delta(u + t\psi). \]

Using a direct calculation, it is found that

\[ \frac{\mu}{q} \int_{\Omega} K(x) h(x) (t\psi)^q dx \leq \frac{\mu}{q} \int_{\mathbb{R}^N} K(x) h(x) [(u + t\psi)^q - u^q] dx. \tag{16} \]

Dividing the two sides of (16) by \( t > 0 \) and combining the fact \( I_\delta(u) \leq I_\delta(u + t\psi) \), we can obtain

\[ \frac{\mu}{q} \int_{\Omega} K(x) h(x) (t\psi)^q dx \leq \frac{\mu}{q} \int_{\mathbb{R}^N} K(x) h(x) [(u + t\psi)^q - u^q] \frac{dx}{t}. \tag{17} \]

Passing to the limit, through Fatou’s lemma and (17), it follows

\[ +\infty = \frac{\mu}{q} \int_{\Omega} \liminf_{t \to 0^+} K(x) h(x) \frac{(t\psi)^q}{t} dx \leq \frac{\mu}{q} \liminf_{t \to 0^+} \int_{\mathbb{R}^N} K(x) h(x) \frac{(u + t\psi)^q - u^q}{t} dx \leq I_\delta(u)^* \psi, \]

which is impossible. Hence, \( \Omega_0 \) has zero measure and \( u > 0 \) a.e. in \( \mathbb{R}^N \). So Condition (i) of Definition 12 holds.

Next, one will prove that Condition (ii) of Definition 12 is also true.

Taking an arbitrary nonnegative function \( \psi \in X \) and arguing as (17), one can arrive at:

\[ I_\delta(u)^* \psi \geq \frac{\mu}{q} \liminf_{t \to 0^+} \int_{\mathbb{R}^N} K(x) h(x) \frac{(u + t\psi)^q - u^q}{t} dx \]

\[ = \frac{\mu}{q} \liminf_{t \to 0^+} \int_{[\psi > 0]} K(x) h(x) \frac{(u + t\psi)^q - u^q}{t} dx \]

\[ + \frac{\mu}{q} \liminf_{t \to 0^+} \int_{[\psi > 0]} K(x) h(x) \frac{(u + t\psi)^q - u^q}{t} dx \]

\[ = \frac{\mu}{q} \liminf_{t \to 0^+} \int_{[\psi > 0]} K(x) h(x) \frac{(u + t\psi)^q - u^q}{t} dx. \tag{18} \]

Since \( \psi \geq 0 \), one has

\[ u + t\psi \geq u, \]

for any \( t > 0 \). Then, using (18) and Fatou’s lemma, it can derive

\[ I_\delta(u)^* \psi - \frac{\mu}{R} \int_{\mathbb{R}^N} K(x) h(x) u^{q-1} dx \geq 0, \quad \forall \psi \in X, \psi \geq 0. \tag{19} \]

Since \( ||u|| \neq 0 \), one can define

\[ s_0 = \frac{R}{||u||} - 1 > 0. \]
By this, it is easy to know

$$
\| (1 + s_0)u \| = R.
$$

Then, for any $s \in (-1, s_0)$, it easily obtains

$$
\| (1 + s)u \| < R,
$$
denoted by:

$$
T(s) = I_p((1 + s)u), \quad s \in (-1, s_0).
$$

By the assumption of $m_\mu = I_p(u)$, one has

$$
m_\mu \leq \inf_{s \in (-1, s_0)} T(s) \leq T(0) = I_p(u) = m_\mu
$$
since $-1 < 0 < s_0$. Hence, $T$ attains its minimum at $s = 0$. By direct computation,

$$
T(s) = \frac{(1 + s)^2}{2} \left[ \| u \|^2 - \lambda \| u \|_p^2 \right] + \frac{(1 + s)^{p+1}}{p+1} \| u \|_p^{p+1} - \frac{\mu (1 + s)^q}{q} \int \mathbb{R}^N K(x) h(x) u^q \, dx.
$$

It follows that $T$ is differentiable for $s \in (-1, s_0)$. Therefore,

$$
T'(0) = I'_p(u)u - \mu \int \mathbb{R}^N K(x) h(x) u^q \, dx = I'_p(u)u = 0.
$$

On the other hand, for any $\phi \in X$, $\delta > 0$, define

$$
\Omega_\delta^+ = \{ x \in \mathbb{R}^N \mid 0 < \delta \phi(x) < \delta, x \in \mathbb{R}^N \}.
$$

If $\Omega_\delta^+ = \emptyset$, the following inequality will not be affected. Through (19) with $\psi = (u + \delta \phi)^+$ and a simple calculation, it yields

$$
0 \leq I'_p(u) \psi - \mu \int \mathbb{R}^N K(x) h(x) u^{q-1} \psi \, dx
$$

$$
= I'_p(u)(u + \delta \phi)^+ - \mu \int \mathbb{R}^N K(x) h(x) u^{q-1} (u + \delta \phi)^+ \, dx
$$

$$
= I'_p(u) [(u + \delta \phi)^+ + (u + \delta \phi)^-] - \mu \int \mathbb{R}^N K(x) h(x) u^{q-1} [(u + \delta \phi) + (u + \delta \phi)^-] \, dx
$$

$$
= I'_p(u)u - \mu \int \mathbb{R}^N K(x) h(x) u^q \, dx + \delta I'_p(u) \phi - \delta \mu \int \mathbb{R}^N K(x) h(x) u^{q-1} \phi \, dx
$$

$$
+ I'_p(u)(u + \delta \phi)^- - \mu \int \mathbb{R}^N K(x) h(x) u^{q-1} (u + \delta \phi)^- \, dx,
$$
where

$$
I'_p(u)(u + \delta \phi)^- - \mu \int \mathbb{R}^N K(x) h(x) u^{q-1} (u + \delta \phi)^- \, dx
$$

$$
= \int_{\mathbb{R}^N \setminus \Omega_\delta^+} K(x) [\nabla u \cdot \nabla (u + \delta \phi)^-] \, dx + \int_{\mathbb{R}^N \setminus \Omega_\delta^+} K(x) (u + \delta \phi)^- u^q \, dx
$$

$$
= - \int_{\mathbb{R}^N \setminus \Omega_\delta^+} K(x) (u + \delta \phi^-) [\lambda u + \mu h(x) u^{q-1}] \, dx
$$

$$
+ \int_{\Omega_\delta^+} K(x) [\nabla u \cdot \nabla (u + \delta \phi)^-] \, dx + \int_{\Omega_\delta^+} K(x) (u + \delta \phi^-) u^q \, dx
$$

$$
= 0 - \int_{\Omega_\delta^+} K(x) [\nabla u \cdot \nabla (u + \delta \phi)^-] \, dx - \int_{\Omega_\delta^+} K(x) (u + \delta \phi) u^q \, dx
$$

$$
+ \int_{\Omega_\delta^+} K(x) (u + \delta \phi)^- [\lambda u + \mu h(x) u^{q-1}] \, dx.
$$

(21)
Combining (20) and (21), it should be argued that

\[
0 \leq I''(u)u - \mu \int_{\mathbb{R}^N} K(x)h(x)u^q dx \\
+ \delta I''(u)\phi - \delta \mu \int_{\mathbb{R}^N} K(x)h(x)u^{q-1}\phi dx - \int_{\Omega_\epsilon} K(x)[\nabla u \cdot \nabla (u + \delta \phi)] dx - \int_{\Omega_\epsilon} K(x)(u + \delta \phi)u^q dx \\
+ \int_{\Omega_\epsilon} K(x)(u + \delta \phi)[\lambda u + \mu h(x)u^{q-1}] dx.
\]

Recalling (3), it obtains

\[
I'(u)u - \mu \int_{\mathbb{R}^N} K(x)h(x)u^q dx = 0.
\]

By the fact of \( K(x) > 0, h(x) > 0, u > 0, \) and \( (u + \delta \phi) < 0 \) on \( \Omega^\epsilon \), it yields

\[
\int_{\Omega_\epsilon} K(x)(u + \delta \phi)[\lambda u + \mu h(x)u^{q-1}] dx < 0.
\]

Hence, it can be deduced that

\[
0 \leq \delta I''(u)\phi - \delta \mu \int_{\mathbb{R}^N} K(x)h(x)u^{q-1}\phi dx - \int_{\Omega_\epsilon} K(x)[\nabla u \cdot \nabla (u + \delta \phi)] dx - \int_{\Omega_\epsilon} K(x)(u + \delta \phi)u^q dx
\]

\[
\leq \delta \left[ I''(u)\phi - \mu \int_{\mathbb{R}^N} K(x)h(x)u^{q-1}\phi dx \right] - \delta \left[ \int_{\Omega_\epsilon} K(x)(\nabla u \cdot \nabla \phi) dx + \int_{\Omega_\epsilon} K(x)u^q \phi dx \right].
\]

Using \( u > 0 \) a.e. in \( \mathbb{R}^N \), we can obtain

\[
\lim_{\delta \to 0^+} 1_{\Omega^\epsilon}(x) = 0, \quad \text{a.e. in } \mathbb{R}^N,
\]

where \( 1_{\Omega^\epsilon} \) represents the characteristic function of the set \( \Omega^\epsilon \). Hence, dividing (22) by \( \delta > 0 \) and taking the limit as \( \delta \to 0^+ \),

\[
I'(u)\phi - \mu \int_{\mathbb{R}^N} K(x)h(x)u^{q-1}\phi dx \geq 0, \quad \forall \phi \in X
\]

holds by Lebesgue’s theorem. This inequality still holds with writing \(-\phi\) instead of \(\phi\), so it is generally agreed that \( u \in X \) satisfies (12). In addition, \( h(x)u^{q-1}\phi \in L_0^2(\mathbb{R}^N) \).

Note that the functional \( I_\mu \notin C^1 \), so one cannot apply the standard minimization arguments. To overcome this difficulty, we consider the perturbation argument. Specifically speaking, for each \( k \in \mathbb{N} \), define \( \mathcal{X}_k : \mathbb{R} \to \mathbb{R} \) as:

\[
\mathcal{X}_k = s^* + \frac{1}{k} |s^{\gamma-1}| d t = \frac{1}{q} \left[ |s^{*} + \frac{1}{k}| - \left( \frac{1}{k} \right) \right] - \left( \frac{1}{k} \right) s^{\gamma-1}.
\]

Now, one gives the auxiliary functional of equation (6), namely,

\[
I_{\mu,k}(u) = I_\mu(u) - \mu \int_{\mathbb{R}^N} K(x)h(x)\mathcal{X}_k(u) dx, \quad u \in X.
\]

It is clear that \( I_{\mu,k} \in C^1(X, \mathbb{R}) \) since \( \mathcal{X}_k \) is differentiable and

\[
\mathcal{X}_k'(s) = \left( s^* + \frac{1}{k} \right)^{\gamma-1}, \quad s \in \mathbb{R}.
\]
Remark 3. The technique for dealing with singular perturbation problems is mainly inspired by [16].

Next, it is positioned to show that $I_{\mu,k}$ attains its minimum at $u_k \in B_R(0)$ and the desired positive solution will be obtained by passing to the limit as $k \to +\infty$. Based on this idea, one shall obtain the following lemma.

Proposition 1. For any $0 < \mu < \mu^*, \frac{N}{2} < \lambda < +\infty$, there exists $u \in X$ such that $|u| \leq R$ and $I_{\mu}(u) = m_{\mu}$.

Proof. By the fact of $0 < q < 1$, it is widely accepted that

$$X_k(s) \leq \int_0^s (t^*)^{q-1} dt.$$ 

Hence,

$$I_{\mu,k}(u) \geq I_{\mu}(u)$$

holds for any $u \in X$ and $k \in \mathbb{N}$. Through Remark 1, one can define

$$m_{\mu,k} = \inf_{|u| \leq R} I_{\mu,k}(u).$$

Then, applying the Ekeland variational principle, there exists a sequence $(u_n,k)_{n \in \mathbb{N}} \subset B_R(0)$ such that

$$\lim_{n \to +\infty} I_{\mu,k}(u_n,k) = m_{\mu,k}, \quad \lim_{n \to +\infty} I'_{\mu,k}(u_n,k) = 0.$$ 

Since $X_k(s^*) \leq X_k(s)$, it follows

$$m_{\mu,k} \leq I_{\mu,k}(u_n^*,k) \leq I_{\mu,k}(u_n,k) = m_{\mu,k}.$$ 

So it has

$$I_{\mu,k}(u_n^*,k) = I_{\mu,k}(u_n,k),$$

which means $u_n^* \equiv 0$. In other words,

$$u_n,k = u_n^*, \quad \text{in } \mathbb{R}^N.$$ 

Up to a subsequence, one has

$$\begin{cases} 
  u_{n,k} \rightharpoonup u_k, & \text{in } X, \\
  u_{n,k} \to u_k, & \text{in } L^2_R(\mathbb{R}^N), \\
  u_{n,k}(x) \to u_k(x), & \text{a.e. in } \mathbb{R}^N, \\
  |u_{n,k}(x)| \leq g_{\gamma}(x), & \text{a.e. in } \mathbb{R}^N
\end{cases}$$

(23)

for any $1 < \gamma \leq 2^*$ and some $g_{\gamma} \in L^1(\mathbb{R}^N)$. For $s \geq 0$, the following inequality

$$|X_k(s)| = \int_0^s \left(\int_0^t t^* + \frac{1}{k} \right)^{q-1} dt 
\leq \int_0^s \left(\int_0^t + \frac{1}{k} \right)^{q-1} dt 
\leq \int_0^s \left(\frac{1}{k} \right)^{q-1} dt 
= \left(\frac{1}{k} \right)^{q-1} |s|$$

(24)

holds. Thus, based on (23) and (24), it tends to be
\[ |K(x)h(x)X_\delta(u_{n,k})| \leq \left( \frac{1}{K} \right)^{q-1} K(x)h(x)g_\delta(x) \]
a.e. in \( \mathbb{R}^N \). Considering
\[
\int_{\mathbb{R}^N} K(x)h(x)g_\delta(x)dx \leq \left( \int_{\mathbb{R}^N} (K(x)h(x))^2dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} g_\delta^2(x)dx \right)^{\frac{1}{2}} < +\infty,
\]
one has
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x)h(x)X_\delta(u_{n,k})dx = \int_{\mathbb{R}^N} K(x)h(x)X_\delta(u_k)dx \tag{25}
\]
using Lebesgue’s theorem.

Set
\[ v_{n,k} = u_{n,k} - u_k. \]
Since \( v_{n,k} \to 0 \), using a simple calculation, it acquires
\[ \frac{1}{2} \| u_{n,k} \|^2 = \frac{1}{2} \| v_{n,k} \|^2 + \frac{1}{2} \| u_k \|^2. \]
Hence, by (23), (25), and Brezis-Lieb lemma [5, Theorem 1], it could be concluded that
\[
m_{\mu,k} = I_{\mu,k}(u_{n,k}) + o_\mu(1)
\]
\[
= \frac{1}{2} \| u_{n,k} \|^2 - \frac{\lambda}{2} \| u_{n,k}^* \|^2 + \frac{1}{p+1} \| u_{n,k}^* \|_{p+1}^{p+1} - \frac{\lambda}{2} \| u_k^* \|^2 - \frac{1}{2} \| u_k^* \|^2 - \frac{\lambda}{2} \int_{\mathbb{R}^N} K(x)h(x)X_\delta(u_{n,k})dx
\]
\[ + \frac{1}{p+1} \| v_{n,k}^* \|_{p+1}^{p+1} + \frac{1}{p+1} \| u_k^* \|_{p+1}^{p+1} + o_\mu(1) \]
\[ = \frac{1}{2} \| v_{n,k} \|^2 + \frac{1}{2} \| u_k \|^2 - \frac{\lambda}{2} \| u_k^* \|^2 - \frac{1}{2} \| u_k^* \|^2 - \mu \int_{\mathbb{R}^N} K(x)h(x)X_\delta(u_k)dx
\]
\[ + \frac{1}{p+1} \| v_{n,k}^* \|_{p+1}^{p+1} + \frac{1}{p+1} \| u_k^* \|_{p+1}^{p+1} + o_\mu(1), \tag{26}
\]
where \( o_\mu(1) \) denotes a quantity approaching zero as \( n \to +\infty \).

It is easy to know
\[ \frac{1}{2} \| v_{n,k} \|^2 + \frac{1}{p+1} \| v_{n,k}^* \|_{p+1}^{p+1} \geq 0, \]
which combines with (26) to imply
\[ m_{\mu,k} \geq I_{\mu,k}(u_k) + o_\mu(1). \]
Passing to the limit as \( n \to +\infty \) and knowing \( m_{\mu,k} \) is the infimum of \( I_{\mu,k}(u_k) \), it is concluded that
\[ m_{\mu,k} = I_{\mu,k}(u_k). \]
Moreover, since
\[ m_{\mu,k} \leq I_{\mu,k}(0) = 0 \]
and
\[ I_{\mu,k} \geq \rho > 0, \]
it follows that

$$|u_k| < R.$$  

For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, it seems that

$$\left| K(x)h(x)\left[u^*_{n,k}(x) + \frac{1}{k}\right]^{q-1}\varphi(x) \right| \leq \left( \frac{1}{k} \right)^{q-1} K(x)h(x)|\varphi(x)|,$$  

a.e. in $\mathbb{R}^N$ with $0 < q < 1$. By the assumptions of $h$ and $\varphi$, it follows that

$$\left( \frac{1}{k} \right)^{q-1} K(x)h(x)|\varphi(x)| \in L^1(\mathbb{R}^N).$$

Using pointwise convergence and Lebesgue's theorem, one concludes that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x)h(x)\left[u^*_{n,k} + \frac{1}{k}\right]^{q-1}\varphi(x)dx = \int_{\mathbb{R}^N} K(x)h(x)\left[u^*_k + \frac{1}{k}\right]^{q-1}\varphi(x)dx.$$  

In addition, by (23) and a standard density argument, it yields

$$\int_{\mathbb{R}^N} K(x)\nabla u_{n,k} \cdot \nabla \varphi dx < +\infty$$  

and

$$\lambda \int_{\mathbb{R}^N} K(x)u_{n,k}\varphi dx < +\infty.$$  

So, it should be argued that

$$\int_{\mathbb{R}^N} K(x)u^*_{n,k}\varphi dx = \mu \int_{\mathbb{R}^N} K(x)h(x)\left[u^*_{n,k} + \frac{1}{k}\right]^{q-1}\varphi(x)dx + \lambda \int_{\mathbb{R}^N} K(x)u_{n,k}\varphi dx + \int_{\mathbb{R}^N} K(x)\nabla u_{n,k} \cdot \nabla \varphi dx < +\infty.$$  

Hence, one has

$$I^*_\mu(u_k) = 0$$  

by $\lim_{n \to +\infty} I^*_{\mu,k}(u_{n,k}) = 0$.

Next, one declares

$$\lim_{k \to +\infty} I^*_{\mu,k}(u_k) = m_\mu.$$  

Since $I^*_\mu(u_k) \geq I_\mu(u_k) \geq m_\mu$, it is only necessary to verify that

$$\limsup_{k \to +\infty} I^*_\mu(u_k) \leq m_\mu.$$  

Let $(w_n) \subset B_\mu(0)$ such that

$$I_\mu(w_n) \to m_\mu$$  

as $n \to +\infty$. It is clear

$$w^*_n \subset B_\mu(0)$$  

and

$$I_\mu(w^*_n) \leq I_\mu(w_n).$$

So, we assume $w_n \geq 0$ and replace $(w_n)$ by $(w^*_n)$ if necessary. By direct calculation, it can be seen that
\[ I_\mu(w_n) = I_{\mu,k}(w_n) + \mu \int_{\mathbb{R}^N} K(x) h(x) X_k(w_n) \, dx - \frac{\mu}{q} \int_{\mathbb{R}^N} K(x) h(x)(w_n^+)^q \, dx \]
\[
\geq m_{\mu,k} + \mu \int_{\mathbb{R}^N} K(x) h(x) X_k(w_n) \, dx - \frac{\mu}{q} \int_{\mathbb{R}^N} K(x) h(x)(w_n^+)^q \, dx.
\]

(29)

Fixed \( n \in \mathbb{N} \), from the definition of \( X_k \) and \( w_n \geq 0 \), one receives

\[
\int_{\mathbb{R}^N} K(x) h(x) X_k(w_n) \, dx = \int_{\mathbb{R}^N} K(x) h(x) \left( \frac{w_n^+ + \frac{1}{k}}{q} - \frac{1}{k} \right)^q \, dx 
= \frac{1}{q} \int_{\mathbb{R}^N} K(x) h(x)(w_n^+)^q \, dx + o_k(1).
\]

Combining this formula with (29) and taking the limsup as \( k \to +\infty \), it yields

\[ I_\mu(w_n) \geq \limsup_{k \to +\infty} m_{\mu,k} = \limsup_{k \to +\infty} \mu_{\mu,k}(u_k). \]

Moreover, passing to the limit as \( n \to +\infty \), then it gains

\[ \limsup_{k \to +\infty} I_{\mu,k}(u_k) \leq m_\mu. \]

Hence,

\[ \lim_{k \to +\infty} I_{\mu,k}(u_k) = m_\mu \]

holds.

Now, it is positioned to prove that the infimun \( m_\mu \) can be attained.

Since \( (u_k) \) is a bounded sequence in \( X \), one takes a subsequence and re-denote it as \( (u_k) \). Thus,

\[ u_k \to u, \text{ in } X. \]

Similar to (25), it appears that

\[ \lim_{k \to +\infty} \int_{\mathbb{R}^N} K(x) h(x) X_k(u_k) \, dx = \frac{1}{q} \int_{\mathbb{R}^N} K(x) h(x)(u^*)^q \, dx. \]

Then, using (28) and a similar argument used to prove \( I_{\mu,k}(u_k) = m_{\mu,k} \) (now considering the limit on \( k \)), one can achieve

\[ I_\mu(u) = m_\mu. \]

\[ \square \]

**Theorem 1.** For any \( 0 < \mu < \mu^* \), equation (6) exists at least one positive solution on \( X \) when \( \lambda > \frac{N}{2} \).

**Proof.** Through Lemmas 1 and 4 and Proposition 1, it is easy to obtain that (6) has at least one positive solution on the space \( X \) as \( \lambda > \frac{N}{2} \).

\[ \square \]

**Lemma 5.** If \( u \) is a positive solution of (6) with \( \lambda > 0 \), then \( u(x) \leq C||u|| \) for all \( x \in \mathbb{R}^N \), where constant \( C \) is independent of \( u \).

**Proof.** First, based on Sobolev’s inequality, it obtains

\[ ||u||_{L^2(\mathbb{R}^N)} \leq C_0 ||u||; \]

in other words, it derives \( u \in L^2(\mathbb{R}^N) \). Let

\[ Lu = -\text{div}(K(x)\nabla u) - \lambda K(x)u. \]
Then, from equation (6), one acquires

\[ Lu \leq \mu K(x) h(x) u^{q-1}. \]

By Theorem 8.17 of [17], it follows that

\[ 0 \leq u(x) \leq \sup_{B_r(x)} u \leq C_\gamma \|u\|_{L^q(B_r(x))}, \]

where \( B_r(x) = \{ y \in \mathbb{R}^N : |y - x| < r \} \), \( r > 1 \), and \( C_\gamma \) is a constant independent of \( x \) and \( u \). Clearly, it attains \( u(x) \leq C_\gamma \|u\| \) for all \( x \in \mathbb{R}^N \) according to the embedding theorem, where \( C_0 \) is another constant.

Next, let us show the uniqueness of the positive solution for equation (6).

**Theorem 2.** For any \( 0 < \mu < \mu^* \), there exists at most one positive solution of equation (6) on \( X \) for any \( 0 < \lambda < +\infty \).

**Proof.** Suppose, on the contrary, that both \( u_1 \) and \( u_2 \) are the positive solutions of equation (6) with \( u_1 \neq u_2 \), namely,

\[ -\text{div}(K(x)\nabla u_1) = \mu K(x) h(x) u_1^{q-1} + \lambda K(x) u_1 - K(x) u_1^\phi \] (30)

and

\[ -\text{div}(K(x)\nabla u_2) = \mu K(x) h(x) u_2^{q-1} + \lambda K(x) u_2 - K(x) u_2^\phi. \]

Let \( \varepsilon > 0 \) and \( \varepsilon_1 = \varepsilon \) and \( \varepsilon_2 = \frac{\varepsilon}{2} \). Denote

\[ v_i = \left[ (u_2 + \varepsilon_2)^2 - (u_1 + \varepsilon_1)^2 \right]^{\frac{1}{2}}, \quad i = 1, 2. \]

Using Lemma 5, it is found that \( v_i \in X \) and \( v_i \) have a compact support, where \( i = 1, 2 \).

If one multiples (30) by \( v_i \) and integrates by parts, then the following equality

\[ \int_{\mathbb{R}^N} K(x) \nabla u_1 \cdot \nabla v_1 dx = \mu \int_{\mathbb{R}^N} K(x) h(x) u_1^{q-1} v_1 dx + \lambda \int_{\mathbb{R}^N} K(x) u_1 v_1 dx - \int_{\mathbb{R}^N} K(x) u_1^\phi v_1 dx \] (31)

holds. Similarly, it could be concluded that

\[ \int_{\mathbb{R}^N} K(x) \nabla u_2 \cdot \nabla v_2 dx = \mu \int_{\mathbb{R}^N} K(x) h(x) u_2^{q-1} v_2 dx + \lambda \int_{\mathbb{R}^N} K(x) u_2 v_2 dx - \int_{\mathbb{R}^N} K(x) u_2^\phi v_2 dx. \] (32)

Then, (31) minus (32), one receives

\[ \int_{\mathbb{R}^N} K(x)(\nabla u_1 \cdot \nabla v_1 - \nabla u_2 \cdot \nabla v_2) dx = \mu \int_{\mathbb{R}^N} K(x) h(x)(u_1^{q-1} - u_2^{q-1}) v_1 dx + \lambda \int_{\mathbb{R}^N} K(x)(u_1 v_1 - u_2 v_2) dx \]

\[ = \int_{\mathbb{R}^N} K(x)(u_1 v_1 - u_2 v_2) dx + \mu \int_{\mathbb{R}^N} K(x) h(x)(u_1^{q-1} v_1 - u_2^{q-1} v_2) dx \]

\[ - \int_{\mathbb{R}^N} K(x)(u_1^\phi v_1 dx - u_2^\phi v_2 dx). \] (33)

Define

\[ \Omega(\varepsilon) = \{ x : u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1 \}. \]

According to the definition of \( \Omega(\varepsilon) \), (33) is reduced to:
\[
\int_{\Omega(\varepsilon)} K(x)(\nabla u_1 \cdot \nabla v_1 - \nabla u_2 \cdot \nabla v_2)\,dx
\]
\[
= \int_{\Omega(\varepsilon)} \lambda K(x)(u_1 v_1 - u_2 v_2)\,dx + \mu \int_{\Omega(\varepsilon)} K(x)h(x)(u_1^{q-1}v_1 - u_2^{q-1}v_2)\,dx
\]
\[
- \int_{\Omega(\varepsilon)} K(x)(u_1^q v_1 - u_2^q v_2)\,dx.
\]

On the one hand, using a simple calculation, the left-hand side of (34) is equal to:
\[
\int_{\Omega(\varepsilon)} K(x)(\nabla u_1 \cdot \nabla v_1 - \nabla u_2 \cdot \nabla v_2)\,dx
\]
\[
= \int_{\Omega(\varepsilon)} K(x)\left(\nabla u_1 \cdot \nabla \left[\frac{(u_2 + \varepsilon_2)^2 - (u_1 + \varepsilon_1)^2}{u_1 + \varepsilon_1} - \nabla u_2 \cdot \nabla \left[\frac{(u_2 + \varepsilon_2)^2 - (u_1 + \varepsilon_1)^2}{u_2 + \varepsilon_2}\right]\right)\,dx
\]
\[
= \int_{\Omega(\varepsilon)} \left(1 - \frac{(u_2 + \varepsilon_2)^2}{(u_1 + \varepsilon_1)^2}\right) |\nabla u_1|^2 + 2 \frac{(u_2 + \varepsilon_2)}{(u_1 + \varepsilon_1)} \nabla u_1 \nabla u_2\,dx
\]
\[
- \int_{\Omega(\varepsilon)} \left(1 + \frac{(u_2 + \varepsilon_2)^2}{(u_1 + \varepsilon_1)^2}\right) |\nabla u_2|^2 - 2 \frac{(u_1 + \varepsilon_1)}{(u_2 + \varepsilon_2)} \nabla u_2 \nabla u_1\,dx
\]
\[
= - \int_{\Omega(\varepsilon)} K(x)\left(\nabla u_2 - \frac{u_2 + \varepsilon_2}{u_1 + \varepsilon_1} \nabla u_1 + \nabla u_1 - \frac{u_1 + \varepsilon_1}{u_2 + \varepsilon_2} \nabla u_2\right)\,dx,
\]
which is nonpositive.

On the other hand, through a similar consideration with (27), we would have the convergence as follows.

The first term on the right-hand side of (34)
\[
\int_{\Omega(\varepsilon)} \lambda K(x)(u_1 v_1 - u_2 v_2)\,dx
\]
go\[
\text{ster to zero as } \varepsilon \to 0 \text{ since the integrand goes to zero on bounded sets. The second term on the right-hand side of (34) converges to:}\]
\[
\mu \int_{\Omega(0)} K(x)h(x)(u_1^{q-1}v_1 - u_2^{q-1}v_2)\,dx.
\]
Since 0 < q < 1 and \(u_1 < u_2\), it follows that
\[
u_1^{q-1}v_1 - u_2^{q-1}v_2 > 0.
\]
Hence,
\[
\mu \int_{\Omega(0)} K(x)h(x)(u_1^{q-2} - u_2^{q-2})(v_2^2 - v_1^2)\,dx > 0.
\]
By a similar consideration, it is found that the third term on the right-hand side of (34) converges to:
\[
- \int_{\Omega(0)} K(x)(u_1^{q-1} - u_2^{q-1})(v_2^2 - v_1^2)\,dx,
\]
which is positive. So, it is generally agreed that the right-hand side of (34) is positive, which is a contradiction.
Thus, \(u_2 \leq u_1\) on \(\mathbb{R}^N\). Similarly, it is said that \(u_1 \leq u_2\) on \(\mathbb{R}^N\). So, it concludes
\[
u_1 \equiv u_2 \text{ on } \mathbb{R}^N.
\]
Theorem 3. For any $0 < \mu < \mu^*$, equation (6) has no positive solution on $X$ as $0 < \lambda \leq \frac{N}{2}$.

Proof. Through Lemma 1, it is easy to know that equation (6) has no positive solution on $X$ as $0 < \lambda \leq \frac{N}{2}$. □

4 Numerical experiments

In this section, by applying the method of a neural network, one obtains the approximate solution of equation (2) under the condition of $N = 3$ and $p = 1.5$.

In the first step, one sets $\mu = 0$. For avoiding the trivial solution, it yields a regularizer of the solutions according to Theorem 3.12 in [15]. During this step, we also verify the influence of $\lambda$ on the existence of the positive solution by setting $\lambda = 1$ and $\lambda = 3$, respectively.

In the second step, one sets $\mu = 10^{-5}$, $q = 0.5$, and $h(x) = e^{-\|x\|_1^3}$ and generates the initial and boundary values by the network trained in the first step, then one trains the network as a physics-informed procedure [31]; in this process, we also provide examples to illustrate the approximate solution that can be optimized by different initial weights.

Figure 1: should be located in Section 4.

Figure 2: Demonstration of learned $u$ after the first step training ($\mu = 0$).
Figure 3: Demonstration of learned $u$ after the second step training ($\mu = 10^{-5}$).

Figure 4: Equation loss when $\lambda = 1$ and $\mu = 0$.

Figure 5: Equation loss when $\lambda = 3$ and $\mu = 0$. 
Figure 6: Equation loss when $\lambda = 3$ and $\mu = 10^{-5}$.

Figure 7: $1$-norm.

Figure 8: Infinite norm.
The neural network one uses is an eight-layer multilayer perceptron, with each hidden layer containing 32 neurons, and non-linear mapping is achieved by embedding sigmoid and tanh activation functions between the linear layers. Defining $\sigma$ as the activation function of the neural network, and the nonlinear mapping in the neural network as shown in Figure 1 could be expressed as:

$$x_i = \sigma(\text{Linear}(x_{i-1})) \otimes \text{Linear}(x_{i-1}),$$

where $i$ denotes the index of the hidden layer, $m$ and $n$ denote the dimension of vectors as input and output, respectively, $x_{i-1} \in \mathbb{R}^m$ denotes the input of the hidden layer, $x_i \in \mathbb{R}^n$ denotes the output of the hidden layer and $\otimes$ denotes the vector dot multiplication. From the first to the seventh layer, we use sigmoid as an activation function; owing to the objective being to find a positive solution, in the last hidden layer, one uses tanh as an activation function.

During the first step, without loss of generality, one samples 1,024 points uniformly each epoch in the open ball with a radius of 10 as a batch to feed the network. During the second step, we sample 1,024 points uniformly each epoch in the open ball with a radius of 5 as a batch to feed the network and take the initial and boundary values on the open ball with a radius of 5 generated by the neural network trained in the first stage as the initial and boundary conditions for equation (2).

Define $\phi$ as uniformly elliptic operator and $\mathcal{N}$ to be given by the right-hand term of (2), namely,

$$\mathcal{N}(u, h, \mu, \lambda, p, q) = \mu h(x)u_{t-1} + \lambda u - u^p.$$

The loss of the following loss consists of two parts:

$$\text{Loss} = \text{Loss}_{\text{PDE}} + k \times \text{regularizer},$$

where

$$\text{Loss}_{\text{PDE}} = \frac{1}{1,024} \sum_{i=1}^{1,024} |\phi(\text{net}(x_i^j)) - \mathcal{N}(\text{net}(x_i^j), h(x_i^j), \mu, \lambda, p, q)|^2$$

and

$$\text{regularizer} = \frac{1}{2} \sum_{i=1}^{1,024} \left| \frac{e^{-x_i^j}}{\text{net}(x_i^j)} \right|^2,$$

where $x_i^j$ denotes the $i$th random sample point at the $j$th epoch and $\text{net}(x_i^j)$ denotes the prediction of a neural network. Here, one takes $k = 1$, because as $k = 1$, the loss function has already converged. The results of uniformly elliptic operator $\phi(\text{net}(x_i^j))$ can be derived by applying the chain rule for differentiating compositions of functions using automatic differentiation.

Wang et al. [36] supposed that the commonly used $L^2$ loss is not suitable for training physics-informed neural network, while $L^\infty$ loss is a better choice on some high-dimensional PDEs. Therefore, for the sake of a balance between efficiency and numerical solution accuracy, the neural network is trained by minimizing the 8-norm with the Adam optimizer.

In the first step, the initial learning rate is set to 0.001 and halved every 10,000 epochs. With $k = 1$, the procedure converges after 50,000 epochs. In the second step, the initial learning rate is set to 0.01 and halved every 2,000 epochs. With $k = 0$, the procedure converges after 10,000 epochs. Specifically, considering the bivariate function $\text{net}(x_0, x_2, 0)$ and using a heatmap to show its function value in the domain of $x_1 \in [0, 5]$ and $x_2 \in [0, 5]$, it is found that networks trained with different initial weights produce an approximate function value (Figures 2 and 3).

One derives the function values and differential results for all grid points of the trained convergent neural network in this subset and examine the quality of the learned solution $u$ by visualizing point-wise error between the left and right sides of equation (2). The evaluation metric is the equation loss.

When $\lambda = 1$, the absolute error is more than 0.1 for most of the area, as shown in Figure 4. When $\lambda = 3$, it is less than 0.01 in the whole domain, as shown in Figures 5 and 6. It is clearly the case that the equation loss of equation (2) is smaller when $\lambda = 3$, which means the numerical solution one obtains is closer to the true solution.
In this special case, to further demonstrate the influence of parameter $\lambda$ on the existence of a positive solution of the equation, one sets $\lambda = 1$ and 3, respectively, and traces the 1-norm and infinite norm of each epoch during the first step of training. It can be found that the overall results of both are convergent, which shows that our algorithm is effective. In addition, as shown in Figures 7 and 8, it can be seen that when the parameter $\lambda = 1$, the vibration amplitude is obviously larger than that when $\lambda = 3$, and the equation loss is also larger. So, the convergence result of $\lambda = 1$ is obviously not as good as the result of $\lambda = 3$.

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