Research Article

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Multiple positive solutions for a class of concave-convex Schrödinger-Poisson-Slater equations with critical exponent

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Abstract: In this article, we consider the multiplicity of positive solutions for a static Schrödinger-Poisson-Slater equation of the type

$$-\Delta u + \left( u^2 \cdot \frac{1}{|x|} \right) u = \mu f(x)|u|^{p-2}u + g(x)|u|^4u \quad \text{in} \quad \mathbb{R}^3,$$

where $\mu > 0$, $1 < p < 2$, $f \in L^6(\mathbb{R}^3)$, and $f, g \in C(\mathbb{R}^3, \mathbb{R})$. Using Ekeland’s variational principle and a measure representation concentration-compactness of Lions, when $g$ has one local maximum point, we obtain two positive solutions for $\mu > 0$ small; while $g$ has $k$ strict local maximum points, we prove that the equation has at least $k + 1$ distinct positive solutions for $\mu > 0$ small by the Nehari manifold. Moreover, we show that one of the solutions is a ground state solution.

Keywords: Schrödinger-Poisson-Slater equations, critical exponent, multiple solutions, Nehari manifold, concentration-compactness principle

MSC 2020: 35B33, 35J20, 35Q55

1 Introduction

In this article, we consider the following static Schrödinger-Poisson-Slater equation:

$$-\Delta u + \left( u^2 \cdot \frac{1}{|x|} \right) u = \mu f(x)|u|^{p-2}u + g(x)|u|^4u \quad \text{in} \quad \mathbb{R}^3,$$  \hspace{1cm} (1.1)

where $\mu > 0$, $1 < p < 2$ (6 is the critical exponent), and $f, g : \mathbb{R}^3 \to \mathbb{R}$ is continuous function satisfying:

(Q1) $f \in L^r(\mathbb{R}^3)$, where $r = \frac{6}{6-p}$ and $f(x) > 0$ for all $x \in \mathbb{R}^3$; $\lim_{|x| \to +\infty} g(x) = g_{\infty} \in (0, +\infty)$ and $g(x) \geq g_{\infty}$ for all $x \in \mathbb{R}^3$.

(Q2) There exists $a_0 \in \mathbb{R}^3$ such that $g(a_0') = \max_{a \in \mathbb{R}^3} g(x) = 1$ and $g(x) - g(a_0') = o(|x - a_0'|^{\frac{6}{6-p}})$ as $x \to a_0$.

(Q3) There exist $k$ points $a_i, a_2, \ldots, a_k$ in $\mathbb{R}^3$ such that

$$g(a_i') = \max_{x \in \mathbb{R}^3} g(x) = 1 \quad \text{for} \quad 1 \leq i \leq k$$

and

$$g(x) - g(a_i') = o(|x - a_i'|^{\frac{6}{6-p}})$$

as $x \to a_i$ uniformly $i$.

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(Qₜ) Choosing \( \rho_0 > 0 \) such that 
\[
B_{\rho_0}(a^i) \cap B_{\rho_0}(a^j) = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad 1 \leq i, j \leq k
\]
and \( \bigcup_{i \neq j} B_{\rho_0}(a^i) \subset \mathbb{R}^3 \), where \( B_{\rho_0}(a^i) = \{ x \in \mathbb{R}^3 : |x - a^i| \leq \rho_0 \} \).

The nonlocal nonlinear Schrödinger equation, in natural units is
\[
\frac{i}{\partial t} \frac{\partial \psi}{\partial t} = -\Delta \psi + V(x)\psi + \left( \frac{1}{|4\pi \lambda|} \cdot |\psi|^2 \right) \psi - \mu |\psi|^{p-2} \psi, \quad (\psi, t) \in (\mathbb{R}^3, \mathbb{R}),
\]
and its stationary counterpart is
\[
-\Delta u + V(x)u + \left( u^2 \cdot \frac{1}{|4\pi \lambda|} \right) u = \mu |u|^{p-2} u, \quad \text{in } \mathbb{R}^3. \tag{1.2}
\]

The interest on this problem stems from the Slater approximation of the exchange term in the Hartree-Fock model, see [27]. Slater introduced the local term \( |u|^8 u \) with \( p = \frac{6}{7} \), and \( \mu \) is the so-called Slater constant (up to renormalization). For more information on these models and their deduction, see [5,11,22]. Recently, there has been much research on equation (1.2) by using variational methods in [5], the readers are referred to [1,2,19,23,24,26,29] and many others for detailed results.

If \( V(x) = 0 \) in equation (1.2), then the equation reduces to the following static case:
\[
-\Delta u + \left( u^2 \cdot \frac{1}{|4\pi \lambda|} \right) u = \mu |u|^{p-2} u, \quad \text{in } \mathbb{R}^3, \tag{1.3}
\]
which can be called a zero mass problem, see [6]. \( H^1(\mathbb{R}^3) \) is not the right space for problem (1.3) due to the absence of a phase term. Ruiz [25] introduced the following space:
\[
E = E(\mathbb{R}^3) = \{ u \in D^{1,2}(\mathbb{R}^3) : \iint_{\mathbb{R}^3 \times \mathbb{R}^3} u^2(x)u^2(y) \frac{dx dy}{|x - y|} < +\infty \},
\]
where the double integral expression is the so-called Coulomb energy of the wave and \( E(\mathbb{R}^3) \) is the space of functions in \( D^{1,2}(\mathbb{R}^3) \) such that the Coulomb energy of the charge is finite. It was shown in the study by Ruiz [25] that \( E \) is a uniformly convex separable Banach space and that \( E \to L^q(\mathbb{R}^3) \) continuously for \( q \in [3,6] \).

Ianni and Ruiz in [10] studied both the existence of ground and bound states, for \( p > 3 \) and proved that the problem has a radial solution, for \( p = 3 \). Furthermore, Liu et al. [20] considered the following type of the Schrödinger-Poisson-Slater equation with critical growth:
\[
-\Delta u + \left( u^2 \cdot \frac{1}{|4\pi \lambda|} \right) u = \mu |u|^{p-2} u + |u|^4 u \quad \text{in } \mathbb{R}^3, \tag{1.4}
\]
they proved the existence of positive solutions to equation (1.4) by using the novel perturbation approach, together with the well-known Mountain-Pass theorem, for \( p \in (3,6) \) and by using the truncation technique, for \( p \in (\frac{16}{7},3) \). Replacing \( \mu |u|^{p-2} u \) by \( \mu u^k(x)|u|^{p-2} u \) in equation (1.4), Yang and Liu [33] obtained infinitely many solutions for \( \mu > 0 \) small by a truncation technique and Krasnoselskii genus theory, where \( 1 < p < 2 \) and \( k \in L^{2/q}(\mathbb{R}^3) \). For more related research, we refer [3,12,34].

In this article, we will discuss the existence of positive ground state solutions and the multiplicity of positive solutions for equation (1.1). The multiplicity of positive solutions for equation (1.1) is inspired by Liao et al. [15], which studied the following concave-convex elliptic equation with critical exponent:
\[
\begin{align*}
-\Delta u &= g(x)|u|^{2^* - 2} u + \mu f(x)|u|^{q^* - 2} u, & x \in \Omega, \\
u &= 0, & x \in \partial \Omega,
\end{align*}
\tag{1.5}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is an open bounded domain with smooth boundary, \( \mu > 0, 1 < q < 2, \) and \( 2^* = \frac{2N}{N-2} \) is the critical Sobolev exponent, and the coefficient functions \( f \) and \( g \) satisfy the following conditions:

\( f \in L^r(\Omega) \) with \( r \geq 0 \) and \( f \neq 0 \), where \( r = \frac{2^*}{2^* - q} \).
(g_1) g is continuous on $\overline{\Omega}$ and $g > 0$.

(g_2) There exist $k$ points $a^1, a^2, \ldots, a^k$ in $\Omega$ satisfying
$$\{a^1, a^2, \ldots, a^k\} = \{x \in \Omega : g(x) = \max_{x \in \Omega} g(x) = 1\},$$
and moreover, $g(x) - g(a^i) = o(|x - a^i|^{|\frac{N+2}{2} - 1|})$ as $x \to a^i$ uniformly in $i \in \mathbb{N}$ and $1 \leq i \leq k$. By the Nehari method, they proved that equation (1.5) has at least $k + 1$ positive solutions for $\mu > 0$ small. For more problems on related results, see [7,9,13,14,17,21]. In particular, Cao and Chabrowski [7] considered the multiplicity of solutions of this type for the critical problem for the first time. They studied the multiplicity of positive solutions for the following semilinear elliptic equation with critical exponent:

$$\begin{cases}
-\Delta u = g(x)u^{2^*-1} + \mu f(x), & x \in \Omega, \\
u > 0, & x \in \Omega, \\
u = 0, & x \in \partial\Omega,
\end{cases}
(1.6)$$

where $\Omega \subset \mathbb{R}^N(N \geq 3)$ is an open bounded domain with smooth boundary, $f \in L^2(\Omega)$ is nonzero and nonnegative, and $g \in C(\overline{\Omega})$ is positive which satisfies the following condition:

(Q) There exist $k$ strict local maxima points $a^1, a^2, \ldots, a^k$ in $\Omega$ such that
$$g(a^i) = \max_{x \in \Omega} g(x) \text{ for } 1 \leq i \leq k$$
and $g(x) - g(a^i) = o(|x - a^i|^{|\frac{N+2}{2} - 1|})$ as $x \to a^i$ uniformly $i$.

They obtained that equation (1.6) has at least $k$ positive solutions for $\mu > 0$ small.

Our main results are the following.

**Theorem 1.1.** Assume that (Q_1) and (Q_2) hold, then there exists $\mu_* > 0$ such that equation (1.1) has at least two positive solutions for all $\mu \in (0, \mu_*)$, and one of the solutions is a ground state solution.

**Theorem 1.2.** Assume that (Q_1), (Q_2), and (Q_3) hold, then there exists $\mu^* \in (0, \mu_*)$ small enough such that equation (1.1) has at least $k + 1$ distinct positive solutions for all $\mu \in (0, \mu^*)$.

**Remark 1.1.** Since the lack of compactness of the Sobolev embedding $E \hookrightarrow L^s(\mathbb{R}^3), \forall s \in [3, 6]$, inspired by [20, 33], we adapt a measure representation concentration-compactness principle of Lions [18] to overcome the difficulties, and by exploring the parameter $\mu$, we show that the associated energy functional satisfies, in general, the Palais-Smale condition at some level for $\mu > 0$ small enough under our assumptions. Because the associated energy functional is not bounded below, we apply Ekeland’s variational principle to obtain that equation (1.1) has at least $k + 1$ positive solutions by the Nehari manifold.

**Notations:**

- The usual norm in $L^s(\mathbb{R}^3)$ will be denoted by $\| \cdot \|_s$.
- $C$ denotes (possible different) any positive constant.
- $B_R(x)$ denotes the open ball with center $x$ and radius $R$ in $\mathbb{R}^3$.

The article is organized as follows: in Section 2, we give preliminary results; in Section 3, we give the proof of Theorem 1.1; and in Section 4, we prove Theorem 1.2.

## 2 Preliminary

We denote $S$ the best Sobolev constant of $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ by
\[ S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3)} \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right\} > 0, \tag{2.1} \]

where \( \mathcal{D}^{1,2}(\mathbb{R}^3) = \{ u \in L^2(\mathbb{R}^3) : \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^3), i = 1, 2, 3 \} \) is equipped with the norm

\[ |u| = \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}. \]

Noting that the function

\[ U(x) = \frac{x^4}{(1 + |x|^2)^2} \]

is an extremal function for the minimum problem (2.1). For each \( \varepsilon > 0 \),

\[ U_\varepsilon(x) = \varepsilon^{-\frac{1}{2}} U \left( \frac{x}{\sqrt{\varepsilon}} \right) = \frac{(3\varepsilon)^{\frac{1}{2}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}} \]

is a positive solution of the critical problem

\[-\Delta u = |u|^4 u \quad \text{in } \mathbb{R}^3,\]

with

\[ |U_\varepsilon|^2 = \int_{\mathbb{R}^3} |U_\varepsilon|^2 = S^2. \tag{2.2} \]

Define the norm of \( E \) by

\[ ||u||_E = \left( ||u||^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x - y|} \, dx \, dy \right)^{\frac{1}{2}} \]

for \( u \in E \).

Then, we have the following properties.

**Proposition 2.1.** [10,20,25] (\( ||\cdot||_E, E \)) is a uniformly convex Banach space. Moreover, \( C_0^\infty(\mathbb{R}^3) \) is dense in \( E \).

\( E \hookrightarrow L^s(\mathbb{R}^3) \) continuously for \( s \in [3, 6] \) and \( E \hookrightarrow L^s(\Omega) \) compactly for \( s \in [1, 6] \) with bounded \( \Omega \subset \mathbb{R}^3 \).

Define \( \phi_u = \frac{1}{4\pi |x|} \ast u^2 \), then \( u \in E \) if and only if both \( u \) and \( \phi_u \) belong to \( \mathcal{D}^{1,2}(\mathbb{R}^3) \). In such case, equation (1.1) can be rewritten as a system in the following form:

\[ \begin{cases} -\Delta u + \phi u = \mu |u|^p - 2 u + g(x) |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{2.4} \]

Moreover,

\[ \int_{\mathbb{R}^3} |\nabla \phi_u(x)|^2 \, dx = \int_{\mathbb{R}^3} \phi_u(x) u^2(x) \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x - y|} \, dx \, dy. \]

Next, we define

\[ T : E \times E \times E \times E \to \mathbb{R}, \quad T(u, v, w, z) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x)v(x)w(y)z(y)}{4\pi|x - y|} \, dx \, dy, \]

then \( T \) is a continuous map, linear in each variable, and we have the following technical results in \( E \), see [10,20,25].
Proposition 2.2. Given a sequence \( \{u_n\} \) in \( E \),

(i) \( u_n \to u \) strongly in \( E \) if and only if \( u_n \to u \) and \( \phi_{u_n} \to \phi_u \) in \( \mathcal{D}^{1,2}(\mathbb{R}^3) \).

(ii) \( u_n \to u \) weakly in \( E \) if and only if \( u_n \to u \) in \( \mathcal{D}^{1,2}(\mathbb{R}^3) \) and \( \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx \) is bounded. In such case, \( \phi_{u_n} \to \phi_u \) in \( \mathcal{D}^{1,2}(\mathbb{R}^3) \).

(iii) Assume that we have three weakly convergent sequences in \( E \), \( u_n \to u \), \( v_n \to v \), \( w_n \to w \), and \( z \in E \). Then, \( T(u_n, v_n, w_n, z) \to T(u, v, w, z) \).

For the sake of brevity, let us define \( M : E \to \mathbb{R} \) as

\[
M(u) = ||u||^2 + \int_{\mathbb{R}^3} \phi_u u^2 \, dx.
\]

Similar to [10,20], we have the fact that for any \( u \in E \),

\[
\frac{1}{2} \frac{||u||}{||u||_E} \leq M(u) \leq ||u||_E \frac{||u||}{||u||_E} \quad \text{if either} \quad ||u||_E \leq 1 \quad \text{or} \quad M(u) \leq 1.
\] (2.5)

Moreover, we have

**Proposition 2.3.** [20,25,26] There exists \( C > 0 \) such that

\[
|u|^s \leq CM^2 \frac{||u||}{||u||_E}(u)
\]

for \( u \in E \) with \( s \in [3, 6] \).

The corresponding energy functional of equation (1.1) is defined as follows:

\[
J_\mu(u) = \frac{1}{2} ||u||^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \frac{\mu}{p} \int_{\mathbb{R}^3} f(x)||u|^p \, dx - \frac{1}{6} \int_{\mathbb{R}^3} g(x)||u|^6 \, dx
\] (2.6)

for all \( u \in E \). Moreover, \( J_\mu \) is well-defined and \( C^1 \) at \( E \) for \( p \in (1, 2) \) because \( f \in L^{2\frac{3}{4}}(\mathbb{R}^3) \). And the critical points of \( J_\mu \) are the weak solutions of equation (1.1). Since \( J_\mu \) is unbounded below on \( E \), we consider the functional on the Nehari manifold

\[
\mathcal{N}_\mu = \{ u \in E \setminus \{0\} : \langle J_\mu'(u), u \rangle = 0 \}
\]

\[
= \left\{ u \in E \setminus \{0\} : ||u||^2 + \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \mu \int_{\mathbb{R}^3} f(x)||u|^p \, dx - \int_{\mathbb{R}^3} g(x)||u|^6 \, dx = 0 \right\}.
\]

Let \( \Psi(u) = (J_\mu'(u), u) \), then for all \( u \in \mathcal{N}_\mu \), we deduce that

\[
\langle \Psi_\mu(u), u \rangle = 2||u||^2 + \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \mu \int_{\mathbb{R}^3} f(x)||u|^p \, dx - 6 \int_{\mathbb{R}^3} g(x)||u|^6 \, dx
\]

\[
= -4||u||^2 - 2 \int_{\mathbb{R}^3} \phi_u u^2 \, dx + \mu(6 - p) \int_{\mathbb{R}^3} f(x)||u|^p \, dx
\]

\[
= (2 - p)||u||^2 + (4 - p) \int_{\mathbb{R}^3} \phi_u u^2 \, dx + (p - 6) \int_{\mathbb{R}^3} g(x)||u|^6 \, dx.
\] (2.7)

Adopting the method used in [30], we split \( \mathcal{N}_\mu \) into three parts:

\[
\mathcal{N}_\mu^+ = \{ u \in \mathcal{N}_\mu : \langle \Psi_\mu(u), u \rangle > 0 \}
\]

\[
\mathcal{N}_\mu^0 = \{ u \in \mathcal{N}_\mu : \langle \Psi_\mu(u), u \rangle = 0 \}
\]

\[
\mathcal{N}_\mu^- = \{ u \in \mathcal{N}_\mu : \langle \Psi_\mu(u), u \rangle < 0 \}.
\]

Now, we give some conclusions of the energy functional \( J_\mu \) on \( \mathcal{N}_\mu \).
Lemma 2.1. $J_\mu$ is coercive and bounded from below on $\mathcal{N}_\mu$.

Proof. For $u \in \mathcal{N}_\mu$, by the Young and Hölder inequalities, we deduce from Proposition 2.3 that

$$
J_\mu(u) = J_\mu(u) - \frac{1}{6} (J_\mu(u), u) + \frac{1}{12} \int_{\mathbb{R}^3} \phi_\mu u^2 dx - \frac{\mu(6-p)}{6p} \int_{\mathbb{R}^3} f(x) |u|^p dx \\
\geq \frac{1}{12} M(u) - \frac{\mu(6-p)}{6p} \int_{\mathbb{R}^3} f(x) |u|^p dx \\
> - C \mu^{\frac{2}{7+p}},
$$

which implies that $J_\mu$ is coercive and bounded from below on $\mathcal{N}_\mu$ due to $1 < p < 2$. This completes the proof of Lemma 2.1.

Next, we define

$$
\mu^0 = \frac{4}{6-p} \left( \frac{2-p}{6-p} \right)^{\frac{2}{7+p}} |f_\mu|^{\frac{7}{2-p}}.
$$

(2.9)

For each $u \in E$ with $\int_{\mathbb{R}^3} g(x) |u|^p dx > 0$, we set

$$
t_{\max} = t_{\max}(u) = \left\{ \begin{array}{ll}
(4-p) \int_{\mathbb{R}^3} \phi_\mu u^2 dx + [(4-p) \int_{\mathbb{R}^3} \phi_\mu u^2 dx + 4(2-p)(6-p)||u||^2 \int_{\mathbb{R}^3} g(x) |u|^p dx]^{\frac{1}{2}} \right. \\
\left. \frac{2(6-p) \int_{\mathbb{R}^3} g(x) |u|^p dx}{2(6-p) \int_{\mathbb{R}^3} g(x) |u|^p dx} \right).
$$

Then, we have the following lemma.

Lemma 2.2. For each $u \in E$ with $\int_{\mathbb{R}^3} g(x) |u|^p dx > 0$, then

(i) Assume $\int_{\mathbb{R}^3} f(x) |u|^p dx = 0$, then there exists a unique $t^* = t^*(u) > t_{\max} > 0$ such that $t^* u \in \mathcal{N}_\mu$, and

$$
J_\mu(t^* u) = \sup_{t_{\max}} J_\mu(t u).
$$

(ii) Assume $\int_{\mathbb{R}^3} f(x) |u|^p dx > 0$, if $\mu \in (0, \mu^0)$, then there exist unique $t^+$ and $t^-$ with $0 < t^+ = t^*(u) < t_{\max} < t^-$ such that $t^* u \in \mathcal{N}_\mu$, $t^* u \in \mathcal{N}_{\mu^0}$, and

$$
J_\mu(t^+ u) = \inf_{0 < t < t_{\max}} J_\mu(t u), \quad J_\mu(t^- u) = \sup_{t < t^*} J_\mu(t u).
$$

Proof. For each $u \in E$ with $\int_{\mathbb{R}^3} g(x) |u|^p dx > 0$, let

$$
m(t) = t^{2-p}||u||^2 + t^{4-p} \int_{\mathbb{R}^3} \phi_\mu u^2 dx - t^{5-p} \int_{\mathbb{R}^3} g(x) |u|^p dx,
$$

for $t \geq 0$, obviously, $m(0) = 0$ and $m(t) \to -\infty$, as $t \to +\infty$. Moreover,

$$
m'(t) = (2-p) t^{1-p} ||u||^2 + (4-p) t^{3-p} \int_{\mathbb{R}^3} \phi_\mu u^2 dx - (6-p) t^{5-p} \int_{\mathbb{R}^3} g(x) |u|^p dx \\
= t^{1-p} \left( (2-p) ||u||^2 + (4-p) t^2 \int_{\mathbb{R}^3} \phi_\mu u^2 dx - (6-p) t^4 \int_{\mathbb{R}^3} g(x) |u|^p dx \right),
$$

and therefore, $m'(t_{\max}) = 0$ and $m'(t) > 0$ for $t \in (0, t_{\max})$ and $m'(t) < 0$ for $t \in (t_{\max}, +\infty)$, i.e., $m(t)$ achieves its maximum at $t_{\max}$ and $t_{\max}$ is unique. Let
Then, by the Sobolev embedding theorem, it holds that

\[
m(t_{\text{max}}) \geq m(t_{\text{max}}^0) \geq (t_{\text{max}}^0)^{2-p} \|u\|^2 - (t_{\text{max}}^0)^{6-p} \int g(x)|u|^6 \, dx \]

\[
\geq \left( \frac{2-p}{6-p} \int g(x)|u|^6 \, dx \right)^{\frac{2-p}{6-p}} \frac{4}{6-p} \|u\|^2 \]

\[
\geq \frac{4}{6-p} \left( \frac{2-p}{6-p} \right)^{\frac{2-p}{6-p}} \left( \int g(x)|u|^6 \, dx \right)^{\frac{2-p}{6-p}} S^{\frac{3(2-p)}{4-p}} \|u\|^p.
\]

**Case i.** If \( \int f(x)|u|^p \, dx = 0 \), then there exists a unique \( t^* = t^*(u) > t_{\text{max}} > 0 \) such that

\[
m(t^*) = \mu \int f(x)|u|^p \, dx = 0 \quad \text{and} \quad m'(t^*) < 0,
\]

and by simple computations, one has

\[
\langle f'(tu), tu \rangle = (t^*)^p(m(t^*) - \mu \int f(x)|u|^p \, dx) = 0,
\]

\[
\langle \Psi'_\mu(tu), tu \rangle = (t^*)^2m(t^*) < 0,
\]

\[
\frac{d}{dt} f'_\mu(tu) \bigg|_{t=t^*} = \frac{1}{t^*} \langle f'_\mu(tu), tu \rangle = 0,
\]

\[
\frac{d^2}{dt^2} f'_\mu(tu) \bigg|_{t=t^*} = \frac{1}{(t^*)^2} \langle \Psi''_\mu(tu), tu \rangle < 0,
\]

which implies that \( tu \in N^-_\mu \) and

\[
f'_\mu(tu) = \sup_{t \leq t_{\text{max}}} f'_\mu(tu).
\]

**Case ii.** If \( \int f(x)|u|^p \, dx > 0 \), then by the Sobolev embedding theorem and equation (2.10), for all \( \mu \in (0, \mu^0) \), it holds that

\[
m \int f(x)|u|^p \, dx \leq \mu \|f\|_{L^p} S^{\frac{2-p}{6-p}} \|u\|^p < \frac{4}{6-p} \left( \frac{2-p}{6-p} \right)^{\frac{2-p}{6-p}} S^{\frac{3(2-p)}{4-p}} \|u\|^p \leq m(t_{\text{max}}),
\]

and thus, there exist unique \( t^* \) and \( t^- \) with \( 0 < t^- = t^*(u) < t_{\text{max}} < t^* \) such that

\[
m(t^*) = \mu \int f(x)|u|^p \, dx = m(t^*) > 0 \quad \text{and} \quad m'(t^*) > 0 > m'(t^-).
\]

Similar to Case i, we can conclude that \( tu \in N^+_\mu \), \( tu \in N^-_\mu \) and

\[
f'_\mu(tu) = \inf_{0 < t \leq t_{\text{max}}} f'_\mu(tu), \quad f'_\mu(tu) = \sup_{t \geq t^-} f'_\mu(tu).
\]

Then, the proof of Lemma 2.2 is complete. \( \square \)
Lemma 2.3. If $\mu \in (0, \mu^0)$, then $N^-_{\mu} = \emptyset$ and $N^-_{\mu}$ is closed, where $\mu^0$ as in equation (2.9).

Proof. (i) Assuming the contrary, there exist $\mu_0 \in (0, \mu^0)$ and $0 \neq u_0 \in N^0_{\mu}$, and by the Sobolev embedding theorem and equation (2.8), one has

$$|u_0|^2 \leq \frac{6 - p}{2 - p} \int_{\mathbb{R}^2} g(x)|u_0|^pdx \leq \frac{6 - p}{2 - p} S^{-3}|u_0|^6,$$

i.e.,

$$|u_0|^2 \geq \left( \frac{2 - p}{6 - p} \right) \frac{1}{2} S^\frac{2}{3}.$$

(2.11)

On the other hand, it follows from equation (2.7) that

$$||u_0||^2 \leq \frac{(6 - p)\mu}{4} \int_{\mathbb{R}^2} f(x)|u_0|^pdx$$

$$\leq \frac{(6 - p)\mu}{4} |\int_{\mathbb{R}^2} S^{\frac{6}{2}}|u_0|^p$$

$$< \left( \frac{2 - p}{6 - p} \right) S^{\frac{6}{2}} S^{\frac{6}{2}} |u_0|^p, \quad \text{for all } \mu \in (0, \mu^0),$$

then

$$|u_0|^2 < \left( \frac{2 - p}{6 - p} \right) \frac{1}{3} S^\frac{2}{3}.$$

This contradicts equation (2.11). Thus, $N^0_{\mu} = \emptyset$ for all $\mu \in (0, \mu^0)$.

(ii) Suppose that $\{u_0\} \subset N^-_{\mu}$ and $u_n \to u \in E$, we need to show that $u \in N^-_{\mu}$. Due to $u_n \in N^-_{\mu}$, then

$$\langle f(u_n), u \rangle = 0 \quad \text{and} \quad \langle \Psi(u_n), u \rangle \leq 0,$$

so $u \in N^-_{\mu}$ or $u = 0$. Similar to equation (2.11), one has

$$||u_0||^2 > \left( \frac{2 - p}{6 - p} \right) \frac{1}{3} S^\frac{2}{3} > 0,$$

then it must have $u \in N^-_{\mu}$. Therefore, $N^-_{\mu}$ is closed, for all $\mu \in (0, \mu^0)$. Thus, the proof of Lemma 2.3 is completed. \qed

According to Lemma 2.3, $N_{\mu} = N^+_{\mu} \cup N^-_{\mu}$, for all $\mu \in (0, \mu^0)$. Define

$$a_{\mu} = \inf_{u \in N_{\mu}} f(u), \quad a_{\mu}^+ = \inf_{u \in N^+_{\mu}} f(u), \quad a_{\mu}^- = \inf_{u \in N^-_{\mu}} f(u).$$

Then, we have the following conclusion.

Lemma 2.4.

(i) $a_{\mu} \leq a_{\mu}^+ < 0$ for all $\mu \in (0, \mu^0)$, where $\mu^0$ as in equation (2.9).

(ii) There exists a constant $c_0 = c_0(p, S, |f|, \mu) > 0$ such that $a_{\mu}^- \geq c_0 > 0$, for all $\mu \in (0, \tilde{\mu}_0)$, where $\tilde{\mu}_0 = \frac{p}{2}\mu^0$. 
Proof. (i) For each \( u \in \mathcal{N}_{\mu}^+ \), we deduce from equation (2.8) that
\[
J_\mu(u) = J_\mu(u) - \frac{1}{p} (J'_\mu(u), u)
\]
\[
= \frac{p - 2}{2p} ||u||^2 + \frac{p - 4}{4p} \int_{\mathbb{R}^3} \phi_u u^2 dx + \frac{6 - p}{6p} \int_{\mathbb{R}^3} g(x) |u|^6 dx
\]
\[
< \frac{1}{6p} \left( (p - 2)||u||^2 + (p - 4) \int_{\mathbb{R}^3} \phi_u u^2 dx + (6 - p) \int_{\mathbb{R}^3} g(x) |u|^6 dx \right) < 0,
\]
then \( \alpha_\mu^* < 0 \). By the definitions of \( \alpha_\mu \) and \( \alpha_\mu^* \), we have \( \alpha_\mu \leq \alpha_\mu^* < 0 \), for all \( \mu \in (0, \mu^*) \).
(ii) For each \( u \in \mathcal{N}_{\mu}^- \), similar to equation (2.11), it holds that
\[
\left( p - 2 \right) ||u||^2 + \left( p - 4 \right) \int_{\mathbb{R}^3} \phi_u u^2 dx + \left( 6 - p \right) \int_{\mathbb{R}^3} g(x) |u|^6 dx > 0.
\]
By the Sobolev embedding theorem and equation (2.8), we obtain
\[
J_\mu(u) = J_\mu(u) - \frac{1}{6} (J'_\mu(u), u)
\]
\[
\geq \frac{1}{3} ||u||^2 + \frac{1}{12} \int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{6 - p}{6p} \int_{\mathbb{R}^3} f(x) |u|^p dx
\]
\[
\geq \frac{1}{3} ||u||^2 - \frac{6 - p}{6p} \left| f \right|_{L^\frac{6}{6-p}} ||u||^p
\]
\[
= ||u||^p \left[ \frac{1}{3} ||u||^{2-p} - \frac{6 - p}{6p} \left| f \right|_{L^\frac{6}{6-p}} \right]
\]
\[
> \left( \frac{2 - p}{6 - p} \right) \left\{ S^{\frac{p}{6-p}} \left[ \frac{1}{3} \left( \frac{2 - p}{6 - p} \right) S^{\frac{32 - p}{6 - p}} - \frac{6 - p}{6p} \left| f \right|_{L^\frac{6}{6-p}} \right] \right\} = c_0
\]
\[
> 0 \quad \text{for all} \quad \mu \in (0, \bar{\mu}_0).
\]
Thus, there exists a constant \( c_0 = c_0(p, S, \left| f \right|_{L^\frac{6}{6-p}}, \mu) > 0 \) such that \( \alpha_\mu^- \geq c_0 > 0 \), for all \( \mu \in (0, \bar{\mu}_0) \). This completes the proof of Lemma 2.4. \( \square \)

3 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. We need to show that \( J_\mu \) satisfies the Palais-Smale condition at some level, for \( \mu \in (0, \mu_*) \) with some \( \mu_* > 0 \). To get compactness of the bounded (PS)-sequence in \( E \), we recall the well-known concentration-compactness principle of Lions [18].

Lemma 3.1. [18] Let \( \{u_n\} \) be a sequence weakly converging to \( u \) in \( \mathcal{D}^{1,2}(\mathbb{R}^3) \). Then, up to subsequences,
(i) \( |\nabla u_n|^2 \) weakly converges in \( M(\mathbb{R}^3) \) to a nonnegative measure \( \tilde{\mu} \),
(ii) \( |u_n|^6 \) weakly converges in \( \mathcal{M}(\mathbb{R}^3) \) to a nonnegative measure \( \nu \),

and there exist an at most countable index set \( I \), a family \( \{x_j : j \in I\} \) of distinct points of \( \mathbb{R}^3 \), and families \( \{\nu_j : j \in I\} \) of positive numbers such that
\[
\mu \geq |\nabla u|^2 dx + \sum_{j \in I} \mu_j \delta_{x_j}, \quad \nu = |u|^6 dx + \sum_{j \in I} \nu_j \delta_{x_j}
\]

and for all \( j \in I \), \( \frac{1}{2} \nu_j \leq \tilde{\mu}_j \), where \( \delta_{x_j} \) is the Dirac measure at point \( x_j \).
To study the concentration at infinity of the sequence, we recall the following quantities:

**Lemma 3.2.** [18] Let \( \{u_n\} \) be a sequence weakly converging to \( u \) in \( D^{1,2}(\mathbb{R}^3) \) and define

\[
\nu_n = \lim_{R \to +\infty} \limsup_{n \to +\infty} \int_{|x| > R} |u_n|^6 \, dx, \quad \mu_n = \lim_{R \to +\infty} \limsup_{n \to +\infty} \int_{|x| > R} |\nabla u_n|^2 \, dx.
\]

Then, the quantities \( \nu_n \) and \( \mu_n \) are well defined and satisfy

\[
\nu_n + \mu_n = \lim\sup_{n \to +\infty} \int_{|x| > R} |u_n|^6 \, dx, \quad \mu_n = \lim\sup_{n \to +\infty} \int_{|x| > R} |\nabla u_n|^2 \, dx,
\]

and \( S \nu_n \mu_n \leq \mu_n \), where \( \nu \) and \( \mu \) are defined in Lemma 3.1.

For detailed proofs of Lemmas 3.1 and 3.2, see [18, Lemmas I.1-I.2] and [31, Lemma 1.40]. Then, we have the following \( \text{PS}_{c} \)-condition for \( J_{\mu} \) on \( E \).

**Lemma 3.3.** There exists \( \mu_* \in (0, \bar{\mu}) \) such that \( J_{\mu} \) satisfies \( \text{PS}_{c} \)-condition with \( c \in (\infty, a_\mu^* + \frac{1}{3}S^2) \), for all \( \mu \in (0, \mu_* \) \), where \( \bar{\mu} \) is defined in Lemma 2.4.

**Proof.** Let \( \{u_n\} \subset E \) be a \( \text{PS}_{c} \)-sequence with \( c \in (\infty, a_\mu^* + \frac{1}{3}S^2) \). We could deduce from Lemma 2.1 that \( \{u_n\} \) is bounded in \( E \), then there is a constant \( C_0 > 0 \) such that \( \|u_n\| \leq C_0 \) for all \( n \). Hence, there exists \( \mu_* \in (0, \bar{\mu}) \) small enough such that \( 6 - \frac{p}{6p} \mu |f|, S^\mu C_0^p \leq -a_\mu^* \) for all \( \mu \in (0, \mu_* \).

Applying Lemmas 3.1 and 3.2, the following proof is almost identical to that of Lemma 2.6 of [33] and is omitted here for brevity. Then, the proof of Lemma 3.3 is complete. \( \square \)

By Ekeland’s variational principle [31] and using the same argument as in [8], we have the following lemma.

**Lemma 3.4.**

(i) There exists \( \mu_* \in (0, \bar{\mu}) \) such that \( J_{\mu} \) satisfies \( \text{PS}_{c} \)-condition with \( c \in (\infty, a_\mu^* + \frac{1}{3}S^2) \), for all \( \mu \in (0, \mu_* \).

(ii) There exists a \( \text{PS}_{c} \)-sequence \( \{u_n\} \subset E \) for all \( \mu \in (0, \bar{\mu}) \).

**Proposition 3.1.** Let \( \mu \in (0, \mu_* \), then there exists \( u_\mu \in N^*_\mu \) such that

(i) \( J_{\mu}(u_\mu) = a_\mu^* = a_\mu \),

(ii) \( u_\mu \) is a positive ground state solution of problem (1.1),

(iii) \( J_{\mu}(u_\mu) \to 0 \) and \( \|u_\mu\| \to 0 \), as \( \mu \to 0^* \),

where \( \mu_* \) is defined in Lemma 3.3.

**Proof.** (i) According to Lemma 3.4(i), there is a \( \text{PS}_{c} \)-sequence \( \{u_n\} \subset N_{\mu} \) of \( J_{\mu} \) in \( E \) and \( a_\mu < a_\mu^* + \frac{1}{3}S^2 \), then applying Lemma 3.3, there exist a subsequence \( \{u_n\} \) (still denote by itself) and \( u_\mu \in E \) such that \( u_n \to u_\mu \) in \( E \). Hence,

\[
J_{\mu}(u_\mu) = \lim_{n \to +\infty} J_{\mu}(u_n) = a_\mu, \quad u_\mu \neq 0 \quad \text{and} \quad J_{\mu}(u_\mu) = 0.
\]

i.e., \( u_\mu \) is a ground state solution of equation (1.1). By Lemmas 2.3 and 2.4, one has \( u_\mu \in N_{\mu} = N^*_\mu \cup N^-_{\mu} \) and \( a_\mu^* = c_0 > 0 > a_\mu^* \geq a_\mu \), for all \( \mu \in (0, \mu_* \). If \( u_\mu \in N^-_{\mu} \), then

\[
0 < a_\mu^* \leq J_{\mu}(u_\mu) = a_\mu < 0,
\]

which is impossible. We obtain that \( u_\mu \in N^*_\mu \) and then \( J_{\mu}(u_\mu) = a_\mu^* = a_\mu^* = a_\mu \), and it follows that \( J_{\mu}(u_\mu) = a_\mu^* = a_\mu \).
(ii) We need to show that $u_\mu > 0$. Suppose that $u_\mu^* \neq 0$, where $u_\mu^* = \max\{u_\mu, 0\}$ and $u_\mu^- = \max\{-u_\mu, 0\}$, it is easy to see that $u_\mu^* \in \mathcal{N}_\mu$, then

$$a_\mu = J_\mu(u_\mu) = J_\mu(u_\mu^*) + J_\mu(u_\mu^-) \geq 2a_\mu,$$

which contradicts $a_\mu < 0$ for $\mu \in (0, \mu_*)$. Since $J_\mu$ is even, we can assume that $u_\mu \geq 0$, applying strong maximum principle, $u_\mu$ is a positive ground state solution of equation (1.1).

(iii) By the Sobolev embedding theorem and Hölder inequality, it holds that

$$\frac{1}{2} |\nabla f(x)|u_\mu|^{p-2}u_\mu \leq \frac{\mu(6-p)}{6p} |f| \int_{\mathbb{R}^3} \mathcal{M}^2(u_\mu),$$

thus $f_\mu(u_\mu) \to 0$, as $\mu \to 0^*$. Since $u_\mu \in \mathcal{N}_\mu^*$, by the Hölder’s inequality, Proposition 2.3, and equation (2.7), we deduce that

$$2M(u_\mu) < (6-p) \int_{\mathbb{R}^3} |\nabla f(x)|u_\mu|^{p-2}u_\mu \leq \frac{\mu(6-p)}{6p} |f| \int_{\mathbb{R}^3} \mathcal{M}^2(u_\mu),$$

then $M(u_\mu) \to 0$ as $\mu \to 0^*$ due to $1 < p < 2$. Moreover, from equation (2.5), $\frac{1}{2} ||u_\mu||_6^2 \leq M(u_\mu) \to 0$ as $\mu \to 0^*$, and therefore, $||u_\mu||_6 \to 0$ as $\mu \to 0^*$. The proof is over. \qed

For convenience of the proof of Theorem 1.2, we assume that $(Q_2)$ and $(Q_3)$ hold in Lemmas 3.1 and 3.2. Now, for $0 \leq i \leq k$, let $\eta_i \in C_0^\infty(\mathbb{R}^3)$ be a radially symmetric function with $0 \leq \eta_i \leq 1$, $|\nabla \eta_i| \leq C$, and

$$\eta_i(x) = \begin{cases} 1, & |x| \leq \frac{\rho_0}{2}, \\ 0, & |x| \geq \rho_0, \end{cases}$$

where $\rho_0$ is a positive constant, when $1 \leq i \leq k$ and $\rho_0$ is given in condition $(Q_2)$. For $0 \leq i \leq k$, we define $u_i^\epsilon(x) = \eta_i(x) U_\epsilon(x - a_\epsilon)$, for all $\epsilon > 0$, where $U_\epsilon$ is given in equation (2.2). From the argument in [20,28], one has

$$\int_{\mathbb{R}^3} |\nabla u_i^\epsilon|^2 \leq S_2 + O(\epsilon^2), \quad \int_{\mathbb{R}^3} |u_i^\epsilon|^2 \leq S_2 + O(\epsilon^2), \quad \int_{\mathbb{R}^3} |u_i^\epsilon|^3 \leq O(\epsilon^{1/2}), \quad \int_{\mathbb{R}^3} \phi_{u_i^\epsilon}(u_i^\epsilon)^2 \leq C |u_i^\epsilon|^3 \lesssim C \epsilon^{\frac{\mu}{2}},$$

uniformly in $i$, as $\epsilon \to 0^*$, the last inequality holds true because the classic sharp Hardy-Littlewood-Sobolev inequality in [16]. Then, we have the following results.

**Lemma 3.5.**

$$\int_{\mathbb{R}^3} g(x)|u_i^\epsilon|^p \leq S_2 + O(\epsilon), \quad \text{uniformly in } 0 \leq i \leq k.$$

**Proof.** By $(Q_2)$ and $(Q_3)$, for all $\eta > 0$, there exists $\rho > 0$ such that
\[ |g(x) - g(x')| < \eta |x - x'|^\frac{1}{2} \quad \text{for all } 0 < |x - x'| < \rho. \]

Consequently, we deduce that

\[
\begin{align*}
0 \leq & \int_{\mathbb{R}^3} |u_i|^{\frac{6}{2}} dx - \int_{\mathbb{R}^3} g(x)|u_i|^{\frac{6}{2}} dx \\
\leq & \int_{\{x \in \mathbb{R}^3 : |x - a'| \leq \rho_0\}} \frac{\epsilon^{\frac{3}{2}} |g(a') - g(x)|}{\epsilon + (x - a')^2} \ dx \\
< & \int_{\{x \in \mathbb{R}^3 : |x - a'| \leq \rho_0\}} \frac{\epsilon^{\frac{3}{2}} \eta |x - a'|^\frac{1}{2}}{\epsilon + (x - a')^2} \ dx + \int_{\{x \in \mathbb{R}^3 : |x - a'| \leq \rho_0\}} \frac{\epsilon^{\frac{3}{2}}}{\epsilon + (x - a')^2} \ dx \\
= & 3^2 \eta \int_{0}^{\rho_0} \frac{\rho_0^3}{(\epsilon + \rho_0^2)} dr + 3^2 \int_{0}^{\rho_0} \frac{\rho_0^{\frac{1}{2}}}{\epsilon + \rho_0^2} dr \\
= & 3^2 \eta \epsilon^\frac{1}{2} \int_{0}^{\rho_0} \frac{r^{\frac{3}{2}}}{(1 + r^2)^{\frac{3}{2}}} dr + 3^2 \int_{0}^{\rho_0} \frac{r^{\frac{1}{2}}}{(1 + r^2)^{\frac{3}{2}}} dr \\
\leq & C \eta \epsilon^\frac{1}{2} + C' \epsilon^\frac{3}{2}.
\end{align*}
\]

Thus, one has

\[
\left| \int_{\mathbb{R}^3} |u_i|^{\frac{6}{2}} dx - \int_{\mathbb{R}^3} g(x)|u_i|^{\frac{6}{2}} dx \right| \leq C \eta + C' \epsilon^\frac{3}{2},
\]

which implies that

\[
\limsup_{\epsilon \to 0} \left| \int_{\mathbb{R}^3} |u_i|^{\frac{6}{2}} dx - \int_{\mathbb{R}^3} g(x)|u_i|^{\frac{6}{2}} dx \right| \leq C \eta.
\]

For the arbitrariness of \( \eta \), combining with equation (3.2), one obtains

\[
\int_{\mathbb{R}^3} g(x)|u_i|^{\frac{6}{2}} dx = \int_{\mathbb{R}^3} |u_i|^{\frac{6}{2}} dx + o(\epsilon^\frac{1}{2}) = S_i + o(\epsilon^\frac{1}{2}).
\]

This completes the proof of Lemma 3.5.

**Lemma 3.6.** There exists \( \epsilon_0 > 0 \) small enough such that for \( 0 < \epsilon < \epsilon_0 \), we have

\[
\sup_{i \geq 0} \int_{\mathbb{R}^3} u_i^2 + tu_i^2) < a_i^* + \frac{1}{3} S_i^2, \quad \text{uniformly for } 0 \leq i \leq k, \quad \text{for all } \mu \in (0, \mu_0),
\]

where \( u_i^* \) and \( \mu_0 \) are given in Proposition 3.1 and Lemma 3.3, respectively.

**Proof.** By equation (2.4) and the Hölder’s inequality, one has

\[
\int_{\mathbb{R}^3} \phi(u_i + tu_i^2) dx \\
\leq \int_{\mathbb{R}^3} \phi(u_i + tu_i^2) dx + 4t \int_{\mathbb{R}^3} \phi(u_i + tu_i^2) dx + 6t^2 \int_{\mathbb{R}^3} \phi(u_i + tu_i^2) dx \int_{\mathbb{R}^3} \phi(u_i + tu_i^2) dx \int_{\mathbb{R}^3} \phi(u_i + tu_i^2) dx \\
+ 4t^3 \int_{\mathbb{R}^3} \phi(u_i + tu_i^2) dx + \int_{\mathbb{R}^3} \phi(u_i + tu_i^2) dx + t^4 \int_{\mathbb{R}^3} \phi(u_i + tu_i^2) dx.
\]
Let $H = \left(\int_{\mathbb{R}^3} \phi_{u^i} u^i_{x}^2 \, dx\right)^{\frac{1}{3}} \geq 0$ be a constant, then using the following two elementary inequalities:

\[(a + b)^y \geq a^y + y b^y, \quad \text{for } a, b > 0, 1 < y < 2;\]

\[(a + b)^m \geq a^m + b^m + ma^{m-1}b + \tilde{C}ab^{m-1}, \quad \text{for } m > 2, 0 \leq a \leq M_0, b \geq 1,\]

where $\tilde{C} = C(m)$ and $M_0$ are two positive constants. By basic calculation, for each $t \geq 0$, it holds that

\[
\begin{align*}
J_p(u^i + tu^i_0) &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla (u^i + tu^i_0)|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u^i+tu^i_0} (u^i + tu^i_0)^2 \, dx \\
&\quad - \frac{\mu}{p} \int_{\mathbb{R}^3} f(x)(u^i + tu^i_0)^p \, dx - \frac{1}{6} \int_{\mathbb{R}^3} g(x)(u^i + tu^i_0)^6 \, dx \\
&\leq \frac{1}{2} \left|\|u^i\|^2 + t^2|u^i_0|^2\right| + 2t \int_{\mathbb{R}^3} \nabla u^i \nabla u^i_0 \, dx - \frac{\mu}{p} \int_{\mathbb{R}^3} f(x)|u^i_0|^p + ptu^i_0 - u^i_0|^6 \, dx \\
&\quad + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u^i} u^i_0^2 \, dx + 4t \int_{\mathbb{R}^3} \phi_{u^i} u^i_0 u^i_0 \, dx + t^4 \int_{\mathbb{R}^3} \phi_{u^i} (u^i_0)^2 \, dx \\
&\quad + 6H^2 t^2 \left\|\phi_{u^i}(u^i_0)^2\right\| + 4H t^3 \left\|\phi_{u^i}(u^i_0)^2\right\|^2 \\
&\quad - \frac{1}{6} \int_{\mathbb{R}^3} g(x)|u^i_0|^6 + t^6|u^i_0|^6 + 6tu^i_0 u^i_0 + \tilde{C}t^2 u^i_0(u^i_0)^3 \, dx \\
&\leq J_p(u^i) + t^2 \left[\frac{1}{2} |\|u^i\|^2| + \frac{3H^2}{2} \left\|\phi_{u^i}(u^i_0)^2\right\|^2 \right] + t^3 H \left\|\phi_{u^i}(u^i_0)^2\right\|^3 \\
&\quad + \frac{t^4}{4} \int_{\mathbb{R}^3} \phi_{u^i}(u^i_0)^2 \, dx - \frac{\tilde{C} t^5}{6} \int_{\mathbb{R}^3} g(x)u^i_0(u^i_0)^5 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} g(x)(u^i_0)^6 \, dx.
\end{align*}
\]

Let

\[
\psi(t) = A_t t^2 + B_t t^3 + C_t t^4 - D_t t^5 - E_t t^6 \quad \text{for } t \geq 0 \quad \text{and} \quad \varepsilon > 0,
\]

where $A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon > 0$ are bounded for all $\varepsilon > 0$ as follows:

\[
\begin{align*}
A_\varepsilon &= \frac{1}{2} |\|u\|^2| + \frac{3H^2}{2} \left\|\phi_{u}(u_0^i)^2\right\|^2 , \\
B_\varepsilon &= H \left\|\phi_{u}(u_0^i)^2\right\|^2 , \quad C_\varepsilon = \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u^i}(u^i_0)^2 \, dx, \\
D_\varepsilon &= \frac{\tilde{C}}{6} \int_{\mathbb{R}^3} g(x)u^i_0(u^i_0)^5 dx, \quad E_\varepsilon = \frac{1}{6} \int_{\mathbb{R}^3} g(x)(u^i_0)^6 \, dx,
\end{align*}
\]

and

\[
\psi'(t) = 2A_\varepsilon t + 3B_\varepsilon t^2 + 4C_\varepsilon t^3 - 5D_\varepsilon t^4 - 6E_\varepsilon t^5 = t^4 \left[\phi_{\varepsilon,i}(t) - \phi_{\varepsilon,i,c}(t)\right] \quad \text{for } t > 0,
\]

where

\[
\phi_{\varepsilon,i}(t) = 2A_\varepsilon t^3 + 3B_\varepsilon t^2 + 4C_\varepsilon t^1 \quad \text{and} \quad \phi_{\varepsilon,i,c}(t) = 5D_\varepsilon + 6E_\varepsilon t.
\]

Since $\psi_\varepsilon(0) = 0$, $\psi_\varepsilon(t) \to -\infty$, as $t \to +\infty$, and $\psi_\varepsilon(t) > 0$, as $t > 0$ small enough, uniformly for all $\varepsilon$ and $i$. Thus, there exists $T_\varepsilon > 0$ such that $\psi_\varepsilon'(T_\varepsilon) = 0$, i.e., $\phi_{\varepsilon,i}(T_\varepsilon) - \phi_{\varepsilon,i,c}(T_\varepsilon) = 0$. It is easy to see $\phi_{\varepsilon,i}(t)$ is strictly decreasing on
$(0, +\infty)$ and $\phi_{\varepsilon}(t)$ is strictly increasing on $(0, +\infty)$, then $\phi_{\varepsilon}(t) - \phi_{\varepsilon}(0)$ is strictly decreasing on $(0, +\infty)$. Thus, $T_\varepsilon^i$ is unique and $\psi(T_\varepsilon^i) = \max_{t \geq 0} \phi_0(t)$. We have the following two cases.

**Case i.** If $T_\varepsilon^i \to 0$, as $\varepsilon \to 0$, then $\psi(T_\varepsilon^i) \to 0$, as $\varepsilon \to 0$, and we can conclude that

$$\sup_{t \geq 0} J_\varepsilon(u_\mu + tu_i^\varepsilon) \leq J_\varepsilon(u_\mu) = a_\mu^* + a_i^* + \frac{1}{3}S_\varepsilon^3$$

for all $\varepsilon > 0$ small enough.

**Case ii.** If there is a constant $T_1 > 0$ (independent of $\varepsilon$ and $\mu$) such that $T_\varepsilon^i \geq T_1 > 0$, we suppose that $T_\varepsilon^i \to +\infty$, as $\varepsilon \to 0$ due to $T_\varepsilon^i(T_\varepsilon^i) = 0$, then

$$0 \leftarrow \frac{2A_\mu}{(T_\varepsilon^i)^3} + \frac{3B_\varepsilon}{(T_\varepsilon^i)^2} + \frac{4C_\varepsilon}{T_\varepsilon^i} = 5D_\varepsilon + 6E_\varepsilon T_\varepsilon^i \to +\infty,$$

which is impossible. Therefore, we choose $\varepsilon_1 > 0$ small enough, there exists a constant $0 < T_2 < +\infty$ (independent of $\varepsilon$ and $\mu$) such that $0 < T_1 \leq T_\varepsilon^i \leq T_2 < +\infty$ for $\varepsilon \in (0, \varepsilon_1)$. Applying $\max_{t \geq 0} \left( \frac{A_\mu}{(t^2 + \frac{n}{6} t^4)} \right) = \frac{1}{2} \left( \frac{A_\mu}{B_\varepsilon} \right)^{\frac{1}{2}}$, for any $A > 0$ and $B > 0$, from $u_\mu \in C^1(B_\rho(d))$, (Lemma 3.5), and equations (3.1)–(3.4), one deduces that

$$\sup_{t \geq 0} J_\varepsilon(u_\mu + tu_i^\varepsilon) \leq a_\mu^* + \frac{1}{3}S_\varepsilon^3 + Ce_\varepsilon^\varepsilon + Ce_\varepsilon^\varepsilon - C \int_{\{x \in \mathbb{R}^n : |x - u| < S_\varepsilon \}} (u_i^\varepsilon)^3 dx \leq a_\mu^* + \frac{1}{3}S_\varepsilon^3 + o\left(\varepsilon^3 \right) + Ce_\varepsilon^\varepsilon + Ce_\varepsilon^\varepsilon + Ce_\varepsilon^\varepsilon - Ce_\varepsilon^\varepsilon.$$

It follows that there exists a $\varepsilon_0 \in (0, \varepsilon_1)$ small enough such that

$$\sup_{t \geq 0} J_\varepsilon(u_\mu + tu_i^\varepsilon) < a_\mu^* + \frac{1}{3}S_\varepsilon^3,$$

uniformly for $0 \leq i \leq k$, for all $\varepsilon \in (0, \varepsilon_0)$.

The proof of Lemma 3.5 is finished.

**Lemma 3.7.** There exists $\tilde{\varepsilon}_0 \in (0, \varepsilon_0)$ small enough, if $\varepsilon \in (0, \tilde{\varepsilon}_0)$, then one has $(u_i^\varepsilon)^\gamma = t_\varepsilon(u_i^\varepsilon) > 0$ such that $u_\mu + (t_i^\varepsilon)^\gamma u_i^\varepsilon \in \mathcal{N}_\mu$, for $\mu \in (0, \mu_0)$ and $0 \leq i \leq k$. Moreover, $0 < a_\mu^* \leq a_i^* + \frac{1}{3}S_\varepsilon^3$, where $\varepsilon_0$ is given in Lemma 3.6.

**Proof.** According to Lemma 2.2, there exists $u \in E(\mu)$ and $\int_{\mathbb{R}^n} g(x) |u|^4 dx > 0$, consequently, there exists unique $t^* (u) > 0$ such that $t^* (u) u \in \mathcal{N}_\mu$. Set

$$C_1 = \left\{ u \in E(\mu) : ||u||_E < t^* \left( \frac{u}{||u||_E} \right) \right\} \cup \{0\},$$

and

$$C_2 = \left\{ u \in E(\mu) : ||u||_E > t^* \left( \frac{u}{||u||_E} \right) \right\}.$$

Then $\mathcal{N}_\mu^- = \left\{ u \in E(\mu) : ||u||_E = t^* \left( \frac{u}{||u||_E} \right) \right\}$, thus, $E \setminus \mathcal{N}_\mu^- = C_1 \cup C_2$ and $u_\mu \in \mathcal{N}_\mu^+ \subset C_1$. By simple computations, one has a suitable constant $A > 0$ such that $0 < t^* \left( \frac{u_\mu + tu_i^\varepsilon}{||u_\mu + tu_i^\varepsilon||_E} \right) < A$ for $t \geq 0$ (see [32, Lemma 3.6 (iii)]). Let $t_0^\varepsilon = \frac{1}{2} \left( \frac{||u_\mu||_E + \frac{1}{2}S_\varepsilon^3}{||u_i^\varepsilon||_E} \right) + 1$, then by $u_\mu \in C^1(B_\rho(d))$, (Lemma 3.7) and (3.3)–(3.4), one has
\[ |u_\mu + t_\mu u_\mu^i|^2 \geq |u_\mu|^2 + (t_\mu^i)^2|u_\mu^i|^2 + 2t_\mu \left[ \mu \int f(x)u_\mu^{p-1}u_\mu^i dx + \int g(x)u_\mu^{\frac{p-1}{2}}u_\mu^i dx - \int \phi_{u_\mu}u_\mu u_\mu^i dx \right] \]
\[ \geq |u_\mu|^2 + (t_\mu^i)^2|u_\mu^i|^2 + Ct_\mu^i \int_{x \in \mathbb{R}^3, x \neq \phi_{u_\mu}} u_\mu^i dx - \left( \int \phi_{u_\mu^i}(u_\mu^i)^2 dx \right) \]
\[ \geq |u_\mu|^2 + (t_\mu^i)^2|u_\mu^i|^2 + Ct_\mu^i(e^x - e^{-x}) \]
\[ > |u_\mu|^2 + (t_\mu^i)^2|u_\mu^i|^2 > A^2 \]
\[ \geq \left( \frac{r}{|u_\mu + t_\mu u_\mu^i|} \right)^2 \quad \text{for } \varepsilon > 0 \text{ small enough.} \]

Hence, there is an \( \tilde{\varepsilon}_0 \in (0, \varepsilon_0) \) such that \( u_\mu + t_\mu u_\mu^i \in C_2 \), for \( \varepsilon \in (0, \varepsilon_0) \). Define \( h_\varepsilon(t) = u_\mu + t_\mu u_\mu^i, \ t \in [0, 1] \). Since \( h_\varepsilon(0) = u_\mu \in \mathcal{N}_\mu^+ \subset C_1 \) and \( h_\varepsilon(1) = u_\mu + t_\mu u_\mu^i \in C_2 \), so there exists \( (t_\mu^i)^- \) with \( 0 < (t_\mu^i)^- < t_0^i \) such that \( u_\mu + (t_\mu^i)^- u_\mu^i \in \mathcal{N}_\mu^- \)

\[ 0 < a_\mu^- \leq f_\mu(u_\mu + (t_\mu^i)^- u_\mu^i) \leq \sup_{t \geq 0} f_{u_\mu}(u_\mu + t u_\mu^i) < a_\mu^+ + \frac{1}{3} S^2. \]

Then, the proof of Lemma 3.7 is complete. \( \square \)

**Proposition 3.2.** Equation (1.1) has a positive solution \( v_\mu \) with \( v_\mu \in \mathcal{N}_\mu^+ \) for all \( \mu \in (0, \mu_\mu) \), where \( \mu_\mu \) is given in Lemma 3.3.

**Proof.** According to Lemmas 3.4(ii) and 3.7, there is a (PS)\( a^- \)-sequence \( \{v_n\} \subset \mathcal{N}_\mu^- \) of \( f_\mu \) in \( E \) and \( a_\mu^- \leq a_\mu^+ + \frac{1}{3} S^2 \), then applying Lemma 3.3, there exist a subsequence \( \{v_n\} \) (still denote by itself) and \( v_\mu \in E \) such that \( v_n \rightarrow v_\mu \) in \( E \). Hence,

\[ f_\mu(v_n) = \lim_{n \rightarrow \infty} f_\mu(v_n) = a_\mu, \quad v_\mu \neq 0 \quad \text{and} \quad f_\mu'(v_\mu) = 0. \]

From Lemma 2.3, one obtains that \( \mathcal{N}_\mu^- \) is closed, then \( v_\mu \in \mathcal{N}_\mu^- \). Similar to Proposition 3.1, we can conclude that \( v_\mu \) is a positive solution of equation (1.1). This completes the proof of Proposition 3.2. \( \square \)

**Proof of Theorem 1.1.** Applying Propositions 3.1 and 3.2. The proof is complete. \( \square \)

## 4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We define \( K = \{a^i : 1 \leq i \leq k\} \) and \( K_{\mu_\mu} = \bigcup_{a^i \in K} B_{\mu_\mu}(a^i) \). Suppose \( \bigcup_{i=1}^k B_{\mu_\mu}(a^i) \subset B_{\mu_\mu}(0) \) for some \( r_0 > 0 \). Let \( Q : E[0] \rightarrow \mathbb{R}^3 \) be a barycenter map defined as follows:

\[ Q(u) = \frac{\int_{\mathbb{R}^3} X(x)|u|^6 dx}{\int_{\mathbb{R}^3} |u|^6 dx}, \]

where \( X : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) and

\[ X(x) = \begin{cases} x, & \text{for } |x| \leq r_0, \\ r_0 x / |x|, & \text{for } |x| > r_0. \end{cases} \]
For each $1 \leq i \leq k$, we define

$$Q^i_\mu = \{ u \in \mathcal{N}^-_{\mu} : |Q(u) - \alpha| < \rho_0 \},$$

$$\partial Q^i_\mu = \{ u \in \mathcal{N}^-_{\mu} : |Q(u) - \alpha| = \rho_0 \},$$

$$\beta^i_\mu = \inf_{u \in \partial Q^i_\mu} f_\mu(u) \quad \text{and} \quad \bar{\beta}^i_\mu = \inf_{u \in Q^i_\mu} f_\mu(u).$$

**Lemma 4.1.** There exists $\varepsilon_0 \in (0, \tilde{c}_0)$ such that for $0 < \varepsilon < \varepsilon_0$, then $Q(u_\mu + (t^i_\varepsilon)u^i_\mu) \subset K_{\varepsilon/2}$ for each $1 \leq i \leq k$, where $\tilde{c}_0$ is given in Lemma 3.7.

**Proof.** Since

$$Q(u_\mu + (t^i_\varepsilon)u^i_\mu) = \frac{\int_{\mathbb{R}^3} N(x)u_\mu + (t^i_\varepsilon)\eta(x)U(x - \alpha)^{\|\|} \mathrm{d}x}{\int_{\mathbb{R}^3} |u_\mu + (t^i_\varepsilon)\eta(x)U(x - \alpha)^{\|\|} \mathrm{d}x}$$

$$= \frac{\int_{\mathbb{R}^3} N(\sqrt{\varepsilon}x + \alpha)^{\|\|} \mathrm{d}x + (t^i_\varepsilon)\eta(\sqrt{\varepsilon}x + \alpha)U(x) \mathrm{d}x}{\int_{\mathbb{R}^3} \eta(\sqrt{\varepsilon}x + \alpha)U(x) \mathrm{d}x}$$

$$\rightarrow a^i, \quad \text{as} \; \varepsilon \rightarrow 0^*.$$ 

There exists $\varepsilon_0 \in (0, \tilde{c}_0)$ such that

$$Q(u_\mu + (t^i_\varepsilon)u^i_\mu) \subset K_{\varepsilon/2} \quad \text{for any} \; 0 < \varepsilon < \varepsilon_0 \quad \text{and each} \; 1 \leq i \leq k.$$

This completes the proof of Lemma 4.1. \hfill \Box

Let

$$f_0(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \frac{1}{6} \int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x,$$

and we have the following lemma.

**Lemma 4.2.**

$$a_0 = \inf_{u \in \mathcal{N}_0} f_0(u) = \frac{1}{3} S^2,$$

where $\mathcal{N}_0 = \{ u \in E \setminus \{0\} : \langle f_0'(u), u \rangle = 0 \}$.

**Proof.** Let $u \in \mathcal{N}_0$, then $\langle f_0'(u), u \rangle = 0$ and

$$\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x = \int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x.$$

Thus, we can assume that

$$\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x = l_1 > 0, \quad \int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x = l_2 > 0.$$

Obviously, $l_2 \geq l_1 > 0$. By the definition of $S$, we have $l_2 \geq S^{l_2}/2$, thus, $l_2 \geq S^{l_2}$ and $l_1 \geq S^{l_2}$. Then, we have

$$f_0(u) = \frac{1}{4} \langle f_0'(u), u \rangle = \frac{1}{4} l_1 + \frac{1}{12} l_2 \geq \frac{1}{3} S^2.$$

Thus, $a_0 = \inf_{u \in \mathcal{N}_0} f_0(u) \geq \frac{1}{3} S^2.$
On the other hand, similar to Lemma 2.2, there exists $t_{ε,0}^i > 0$ such that

$$\sup_{t \geq 0} f_0(t_{ε,0}^i u_t^i) = f_0(t_{ε,0}^i u_t^i) \quad \text{and} \quad t_{ε,0}^i u_t^i \in N_0.$$ 

Moreover, we also conclude that $t_{ε,0}^i$ is uniformly bounded as the proof of Lemma 3.6. Applying equations (3.1), (3.2), and (3.4) and Lemma 3.5, one has

$$\frac{1}{3} S_{\frac{7}{2}}^2 - \frac{1}{4} S_{\frac{7}{2}}^2 + O(\varepsilon^2) \rightarrow - \frac{1}{3} S_{\frac{7}{2}}^2, \quad \text{as} \ \varepsilon \to 0^+.$$ 

Therefore, we obtain $a_0 = \inf_{u \in N_0} f_0(u) = \frac{1}{3} S_{\frac{7}{2}}^2$. The proof is complete. □

**Lemma 4.3.** Let $\{v_n\} \subset E$ be a nonnegative function sequence with $|v_n|_6 = 1$ and $\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \to S$. Then, there exists a sequence $\{(y_n, \varepsilon_n)\} \subset \mathbb{R}^3 \times \mathbb{R}^*$ such that

$$v_n(x) = S^{-1} U_{\varepsilon_n}(x - y_n) + o(1).$$

Moreover, if $y_n \to y$, then $\varepsilon_n \to 0$ or it is unbounded.

**Proof.** Similar to ([4], Corollary 2.11) □

**Lemma 4.4.** There is a number $\delta_0 > 0$ such that if $u \in N_0$ and $f_0(u) \leq \frac{1}{3} S_{\frac{7}{2}}^2 + \delta_0$, then $Q(u) \in K_{\mathbb{R}^*}$.

**Proof.** Assuming the contrary, there is a sequence $\{u_n\} \subset N_0$ such that $f_0(u_n) = \frac{1}{3} S_{\frac{7}{2}}^2 + o(1)$ as $n \to +\infty$ and $Q(u_n) \notin K_{\mathbb{R}^*}$, for all $n \in \mathbb{N}$. First of all, for $\{u_n\} \subset N_0$, we have

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} \Phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} g(x)|u_n|^6 dx,$$

and it is easy to show that $\{u_n\}$ is bounded in $E$. Set

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \to l_1, \quad \int_{\mathbb{R}^3} g(x)|u_n|^6 dx \to l_2, \quad \text{as} \ n \to +\infty,$$

and similarly, we deduce that $l_1 \geq \frac{1}{3} S_{\frac{7}{2}}^2$ and $l_2 \geq \frac{1}{3} S_{\frac{7}{2}}^2$. Moreover, we can obtain

$$\frac{1}{3} S_{\frac{7}{2}}^2 + o(1) = f_0(u_n) - \frac{1}{4} f_0'(u_n, u_n)$$

$$= \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{12} \int_{\mathbb{R}^3} g(x)|u_n|^6 dx$$

$$= \frac{1}{4} l_1 + \frac{1}{12} l_2 + o(1)$$

$$\geq \frac{1}{3} S_{\frac{7}{2}}^2 + o(1),$$

it follows that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \to S_{\frac{7}{2}}^2 \quad \text{and} \quad \int_{\mathbb{R}^3} g(x)|u_n|^6 dx \to S_{\frac{7}{2}}^2, \quad \text{as} \ n \to +\infty. \ (4.1)$$

Since $f_0'(|u_n|)|u_n| = f_0'(u_n, u_n)$, i.e., $\{|u_n|\} \subset N_0$ and $f_0(|u_n|) = f_0(u_n) = \frac{1}{3} S_{\frac{7}{2}}^2 + o(1)$, as $n \to +\infty$, we can assume that $u_n \geq 0$. Define

$$v_n = \frac{u_n}{|u_n|} \geq 0,$$

we deduce that $|v_n|_6 = 1$ and $\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \to S$. Then, by Lemma 4.3, there exists a sequence $\{(y_n, \varepsilon_n)\} \subset \mathbb{R}^3 \times \mathbb{R}^*$ such that
\[ v_n(x) = S^{-\frac{1}{2}} U_n(x - y_n) + o(1). \]

Moreover, if \( y_n \to y \), then \( \varepsilon_n \to 0 \) or it is unbounded.

**Case i.** Suppose \(|y_n| \to +\infty \), by equations (2.3) and (4.1), one obtains
\[
1 = \int_{\mathbb{R}^3} g(x)|u_n|^6 \, dx + o(1)
= \int_{\mathbb{R}^3} g(x)v_n|^6 \, dx + o(1)
= S^{-\frac{1}{2}} \int_{\mathbb{R}^1} g(x)|U_n(x - y_n)|^6 \, dx + o(1)
= S^{-\frac{1}{2}} \int_{\mathbb{R}^1} g(y_n + \sqrt{\varepsilon_n} x)|U(x)|^6 \, dx + o(1)
= g(y_n) \text{ as } n \to +\infty,
\]
and this is impossible since \( g_{\infty} < 1 \).

**Case ii.** Suppose \( y_n \to y \) and \( \varepsilon_n \to 0 \), it holds that
\[
1 = \int_{\mathbb{R}^3} g(x)|u_n|^6 \, dx + o(1)
= \int_{\mathbb{R}^3} g(x)v_n|^6 \, dx + o(1)
= S^{-\frac{1}{2}} \int_{\mathbb{R}^1} g(x)|U_n(x - y_n)|^6 \, dx + o(1)
= S^{-\frac{1}{2}} \int_{\mathbb{R}^1} g(y_n + \sqrt{\varepsilon_n} x)|U(x)|^6 \, dx + o(1)
= g(y) \text{ as } n \to +\infty,
\]
which implies that \( y \in K \). Applying the general Lebesgue dominated convergence theorem, we deduce that
\[
Q(u_n) = \int_{\mathbb{R}^3} X(x)|u_n|^6 \, dx
= \int_{\mathbb{R}^3} X(x)v_n|^6 \, dx
= S^{-\frac{1}{2}} \int_{\mathbb{R}^3} X(x)|U_n(x - y_n)|^6 \, dx + o(1)
= S^{-\frac{1}{2}} \int_{\mathbb{R}^3} X(y_n + \sqrt{\varepsilon_n} x)|U(x)|^6 \, dx + o(1)
= y_n \text{ as } n \to +\infty,
\]
which contradicts our assumption. We complete the proof. \( \square \)

**Lemma 4.5.** If \( u \in \mathcal{N}_{\mu} \) and \( J_\mu(u) \leq \frac{1}{2} S^2 + \delta_0 \), then there exists \( \mu^* \in (0, \mu_*) \) small enough such that \( Q(u) \in K_\mu \) for all \( \mu \in (0, \mu^*) \), where \( \mu_* \) and \( \delta_0 \) are defined in Lemmas 3.3 and 4.4, respectively.
Proof. For $u \in \mathcal{N}_\mu^-$, it follows from equation (2.8) that
\[
\int_{\mathbb{R}^3} g(x)|u|^6 \, dx > \frac{2 - p}{6 - p} M(u) > 0.
\]
Similar to Lemma 2.2, there is $t_u > 0$ such that $t_u u \in \mathcal{N}_0$, i.e.,
\[
\int_{\mathbb{R}^3} |\nabla u|^2 \, dx + t_u^2 \int_{\mathbb{R}^3} \phi_{u} u^2 \, dx = t_u^4 \int_{\mathbb{R}^3} g(x)|u|^6 \, dx,
\]
we claim that $t_u < C_1$, for some $C_1 > 0$ (independent of $u$). If not, we suppose that there exists $\{t_{u_n}\} \to +\infty$, as $n \to +\infty$. Then,
\[
\int_{\mathbb{R}^3} g(x)|u_n|^6 \, dx = t_{u_n}^{-4} \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx + t_{u_n}^{-2} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx \to 0, \quad \text{as} \quad n \to +\infty.
\]
Since $u_n \in \mathcal{N}_\mu^-$, then
\[
\int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx = \mu \int_{\mathbb{R}^3} f(x)|u_n|^p \, dx + \int_{\mathbb{R}^3} g(x)|u_n|^6 \, dx 
\leq \mu |f|_p |u_n|^p + \int_{\mathbb{R}^3} g(x)|u_n|^6 \, dx \to 0
\]
as $n \to +\infty$. This contradicts $J_\mu(u_n) \geq a_\mu^- \geq c_0 > 0$ due to Lemma 2.4 (ii). Moreover, it holds that
\[
J_\mu(u) = \sup_{\tau \geq 0} J_\mu(t_u u) = J_0(t_u u) - \frac{t_u^p}{p} \int_{\mathbb{R}^3} f(x)|u|^p \, dx 
\geq J_0(t_u u) - \frac{c_1^p}{p} |f|_p |S^{-\frac{p}{2}}| |u||_{L^p}^p.
\]
One deduces from Lemma 2.1 that there exists a constant $C_2 > 0$ such that $||u||_{L^p} \leq C_2$, then
\[
J_0(t_u u) \leq \frac{1}{3} S^2 + \frac{\delta_0}{2} + \mu \frac{c_1^p C_2^p}{p} |f|_p |S^{-\frac{p}{2}}|,
\]
and there exists sufficiently small $\mu^* \in (0, \mu_\mu)$ such that $J_0(t_u u) \leq \frac{1}{3} S^2 + \delta_0$, for $\mu \in (0, \mu^*)$. Then, $Q(u) = Q(t_u u) \in K_{\mu_\mu}$ for all $\mu \in (0, \mu^*)$. The proof of Lemma 4.5 is finished.

Lemma 4.6. Given $u \in O_\mu^-$, then there exist $\tau > 0$ and a differentiable function $l: B(0; \tau) \subset E \to \mathbb{R}^+$ such that $l(0) = 1$ and $l(v)(u - v) \in O_\mu^-$ for any $||v||_E \in B(0; \tau)$ and
\[
\langle l'(0), \varphi \rangle = \frac{\langle \Psi'_\mu(u), \varphi \rangle}{\langle \Psi'_\mu(u), u \rangle} \quad \text{for any} \quad \varphi \in C_0^\infty(\mathbb{R}^3).
\] (4.2)

Proof. Similar to [15, Lemma 4.7], we omit it.

According to Lemma 4.1, one has $u_\mu + (t_\varepsilon^\mu) u_\varepsilon^\mu \in O_\mu^-$ for $0 < \varepsilon^0 \leq \bar{\varepsilon}_0$, then by Lemma 3.6,
\[
\beta_\mu^\mu \leq J_\mu(u_\mu + (t_\varepsilon^\mu) u_\varepsilon^\mu) < a_\mu^* + \frac{1}{3} S^2 \quad \text{for any} \quad \varepsilon \in (0, \varepsilon^0) \quad \text{and} \quad \mu \in (0, \mu_\mu).
\] (4.3)

From Lemma 4.5 and the definition of $\tilde{\beta}_\mu^\mu$, we obtain
\[
\tilde{\beta}_\mu^\mu > \frac{1}{3} S^2 + \frac{\delta_0}{2} \quad \text{for any} \quad \mu \in (0, \mu^*).
\] (4.4)
Thus, for each $1 \leq i \leq k$, by equations (4.3) and (4.4), we have
\[ \beta^i_\mu < \beta^j_\mu \quad \text{for any } \mu \in (0, \mu^*). \] (4.5)

Then,
\[ \beta^i_\mu = \inf_{u \in \Omega_\mu \cup \partial \Omega_\mu} J_\mu(u) \quad \text{for any } \mu \in (0, \mu^*). \]

Applying Ekeland’s variational principle, we have the following lemma.

**Lemma 4.7.** For each $1 \leq i \leq k$, there is a $(PS)_{\beta^i_\mu}$-sequence $\{u_n^i\} \subset O^i_\mu$ in $E$ for $J_\mu$, for all $\mu \in (0, \mu^*)$, where $\mu^* > 0$ is defined in Lemma 4.5.

**Proof.** Let $\{u_n^i\} \subset O^i_\mu \cup \partial \Omega_\mu^i$ be a minimizing sequence for $\beta^i_\mu$. Applying Ekeland’s variational principle, there is a subsequence $\{u_n^i\}$ such that $J_{\mu, \mu}(u_n^i) = \beta^i_\mu + \frac{1}{n}$ and
\[ J_\mu(u_n^i) \leq J_\mu(w) + \frac{||w - u_n^i||_E}{n} \quad \text{for all } w \in O^i_\mu \cup \partial \Omega_\mu^i. \] (4.6)

By equation (4.5), we can assume that $u_n^i \in O^i_\mu$ for sufficiently large $n$. From Lemma 4.6, there are $\tau_n^i > 0$ and a differentiable functional $l_n^i : B(0; \tau_n^i) \subset E \to \mathbb{R}^*$ such that $l_n^i(0) = 1$, $l_n^i(v)(u_n^i - v) \in O^i_\mu$ for $v \in B(0; \tau_n^i)$. Let $v_\varsigma = \varsigma v$ with $||v||_E = 1$ and $0 < \varsigma < \tau_n^i$. Then, $v_\varsigma \in B(0; \tau_n^i)$ and $w_n^i = l_n^i(v_\varsigma)(u_n^i - v_\varsigma) \in O^i_\mu$. By equation (4.6) and using the mean value theorem, we deduce that as $\varsigma \to 0$,
\[ ||w_n^i - u_n^i||_E \to 0 \quad \text{as } \varsigma \to 0. \]
Thus,
\[ \frac{||w_n^i - u_n^i||_E}{n} \to 0 \quad \text{as } \varsigma \to 0. \]

**Proof of Theorem 1.2.** By Lemma 4.7, there exists a $(PS)_{\beta^i_\mu}$-sequence $\{u_n^i\} \subset O^i_\mu$ for $J_\mu$ in $E$, for all $\mu \in (0, \mu^*)$, where $1 \leq i \leq k$. We deduce from equation (4.3) and Lemma 3.3 that $J_\mu$ has at least $k$ distinct nontrivial critical points in $\mathcal{N}_\mu$. If we consider
\[ f'_\mu(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_\mu u^2 \, dx - \frac{\mu}{8} \int_{\mathbb{R}^3} f(x)(u^*)^2 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} g(x)(u^*)^3 \, dx, \]  

(4.7)

where \( u^* = \max\{u, 0\} \). By repeating all the steps in the article to \( f'_\mu \), we obtain that \( f'_\mu \) has at least \( k \) nontrivial and nonnegative critical points in \( E \). Applying the maximum principle, equation (1.1) has at least \( k \) positive solutions in \( \mathcal{N}_\mu \). By Propositions 3.1, we obtain that equation (1.1) has at least \( k + 1 \) positive solutions in \( E \). The proof is complete.

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