Research Article

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Critical fractional Schrödinger-Poisson systems with lower perturbations: the existence and concentration behavior of ground state solutions

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Abstract: In this article, we study the following fractional Schrödinger-Poisson system:

\[
\begin{align*}
\epsilon^{2s}(-\Delta)^s u + V(x)u + \phi u &= f(u) + |u|^{2^*_s-2}u, & \text{in } \mathbb{R}^3, \\
\epsilon^{2t}(-\Delta)^t \phi &= u^2, & \text{in } \mathbb{R}^3, 
\end{align*}
\]

where \( \epsilon > 0 \) is a small parameter, \( 0 < s, t < 1, 2s + 2t > 3 \), and \( 2^*_s = \frac{6}{3-2s} \) is the critical Sobolev exponent in dimension 3. By assuming that \( V \) is weakly differentiable and \( f \in C(\mathbb{R}, \mathbb{R}) \) satisfies some lower order perturbations, we show that there exists a constant \( \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0) \), the above system has a semiclassical Nehari-Pohozaev-type ground state solution \( \hat{u}_\epsilon \). Moreover, the decay estimate and asymptotic behavior of \( \{\hat{u}_\epsilon\} \) are also investigated as \( \epsilon \to 0 \). Our results generalize and improve the ones in Liu and Zhang and Ambrosio, and some other relevant literatures.

Keywords: fractional Schrödinger-Poisson, Nehari-Pohozaev manifold, critical problem, ground state solution, semiclassical

MSC 2020: 35R11, 35A15, 35B33

1 Introduction and main results

In this article, we investigate the existence and concentration of solutions to the following fractional Schrödinger-Poisson system:

\[
\begin{align*}
\epsilon^{2s}(-\Delta)^s u + V(x)u + \phi u &= f(u) + |u|^{2^*_s-2}u, & \text{in } \mathbb{R}^3, \\
\epsilon^{2t}(-\Delta)^t \phi &= u^2, & \text{in } \mathbb{R}^3, 
\end{align*}
\] (1.1)

where \( \epsilon > 0 \) is a small parameter, \( 0 < t, s < 1, 2t + 2s > 3 \), \( 2^*_s = \frac{6}{3-2s} \) is fractional critical exponent in \( \mathbb{R}^3 \), and \((-\Delta)^a\), with \( a = s \) is the fractional Laplacian operator, which can be defined as, for any \( u : \mathbb{R}^3 \to \mathbb{R} \) belonging to the Schwartz class,
\[
(-\Delta)^s u(x) = C(3, s) \text{P.V.} \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3+2s}} \, dy,
\]

where \(\text{P.V.}\) represents the Cauchy principal value and \(C(3, s)\) stands for a normalizing constant; see [16].

When \(s = t = 1\), system (1.1) reduces to the following classical Schrödinger-Poisson system:

\[
\begin{cases}
-\varepsilon^2 \Delta u + V(x)u + \mu \phi u = f(u), & \text{in } \mathbb{R}^3, \\
-\varepsilon^2 \Delta \phi = u^2, & \text{in } \mathbb{R}^3.
\end{cases}
\]  

(1.2)

It characterizes systems of identical charged particle systems interacting with one another in the event that magnetic field effects are negligible. In particular, a standing wave is the solution for such a system. We cite [9] for a more comprehensive explanation of this system. For Schrödinger-Poisson systems, concerning existence, nonexistence, and multiplicity for both bound states and ground states, we cite [7,19,35,45,48]. For equation (1.2), with \(f(u) = |u|^{p-2}u (2 < p < 6)\) and some intervals contain \(\mu\), Ruiz [35] demonstrated that equation (1.2) admits a positive radial solution by building a constrained minimization on a new manifold based on the Nehari manifold and the Pohozaev identity. In circumstances where \(-u + f(u)\) satisfies Berestycki-Lions condition, Azzollini et al. [7] used the penalization method and the Ljusternik-Schnirelmann theory to show that equation (1.2) admits nontrivial solutions. Lately, there are some studies on the semiclassical state of the system (1.2) under various potential \(V\) and nonlinearity \(f\) conditions. As an illustration, He [20] investigated the multiplicity and concentration of positive solutions and showed that positive solutions concentrate around the global minimum of the potential \(V\) in the semiclassical limit, but the nonlinearity term satisfies subcritical growth. For the critical case, Chen et al. [12] employed a constrained minimization on a Nehari-Pohozaev manifold to show that system (1.2) admits a semiclassical ground state solution and study properties of these ground state solution, such as convergence and decay estimate. He and Zou [21] proved that system (1.2) admits a positive ground state solution concentrating around the global minimum of the potential \(V\), and they also investigated the exponential decay of ground state solutions. In [8,23,25,26,33,34,36,39,49] and the references therein, additional results regarding semiclassical states and variational method on elliptic equations are provided.

If \(\phi(x) = 0\), system (1.1) becomes the fractional Schrödinger equation like

\[
\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x, u), \quad \text{in } \mathbb{R}^3.
\]

(1.3)

The fractional Schrödinger equation has standing wave solutions in the form of equation (1.3):

\[
\frac{\partial \psi}{\partial t} = i\varepsilon^{2s}(-\Delta)^s \psi + V(x)\psi - f(x, |\psi|), \quad \text{in } \mathbb{R}^3,
\]

i.e., solutions of the form \(\psi(x, t) = e^{-iEt/\varepsilon} u(x)\), where \(E\) is a constant and \(u(x)\) is a solution of equation (1.3). A key equation in fractional quantum mechanics is the fractional Schrödinger equation. In the last two decades, many authors widely investigated problem (1.3), we quote [4,13,17,37,38,40] and its references for the readers’ information. In the case where \(V \equiv 1\) and \(f\) satisfies subcritical growth and Ambrosetti-Rabinowitz condition, Felmer et al. [17] investigated the existence, regularity, and symmetry of positive solutions to equation (1.3). Secchi [40] investigated equation (1.3) under reasonable assumptions on the behavior of the potential \(V\) at infinity and the nonlinearity \(f\) is superlinear with subcritical growth, but it does not satisfies the Ambrosetti-Rabinowitz condition. The existence and multiplicity results of equation (1.3) were shown by Shang et al. [37] with critical growth and needing the following global assumption on the potential \(V\) introduced by Rabinowitz [30]:

\[
V_0 = \liminf_{|x| \to \infty} V(x) > V_0 = \inf_{x \in \mathbb{R}^3} V(x) > 0.
\]

By using the penalization technique and the extension method [11], Alves and Miyagaki [1] proved equation (1.3) admits positive solutions, and studied their concentration behavior when \(V\) verifies \((V_1)\) and \((V_2)\) and \(f\) satisfies subcritical growth. Later, Ambrosio [5] used penalization method and the Ljusternik-Schnirelmann
theory to study the multiplicity of positive solutions of equation (1.3) with critical nonlinearities. See also [10,41] for critical problems in bounded domains and [2,3] for critical fractional periodic problems.

Now, to show our results, we make the following hypotheses on $V$:

(V1) $V \in C^{0}(\mathbb{R}^{3}, \mathbb{R}^{+})$ and $V_{0} = \liminf_{|x| \to \infty} V(x) > V_{0} = \inf_{x \in \mathbb{R}^{3}} V(x) > 0$;

(V2) $V(x)$ is weakly differentiable, the function $\tau \mapsto \tau^{2}[2s + 2t - 3)V(\tau x) - (\nabla V(\tau x), \tau x)]$ is increasing on $(0, +\infty)$, and for some $\rho_{0} > 0$, such that

$$2sV(x) + (\nabla V(x), x) \geq \rho_{0} \quad \text{a.e. } x \in \mathbb{R}^{3}.$$  

For the nonlinearity $f$, we assume the following conditions:

(F1) $f \in C(\mathbb{R}, \mathbb{R})$, $f(\tau) = 0$ for all $\tau \leq 0$ and $\lim_{\tau \to 0} \frac{f(\tau)}{\tau} = 0$ and $\lim_{\tau \to +\infty} \frac{f(\tau)}{\tau^{2s-1}} = 0$;

(F2) $\frac{s + 2f(\tau) - 3f(\tau)}{\tau^{2s-1}}$ is increasing on $(0, +\infty)$, where $F(\tau) = \int_{0}^{\tau} f(\theta) d\theta$;

(F3) there exist $\mu, \mu_{1} > 0$ and $\frac{4t + 2t}{s + t} < p < 2^{*}_{s}$ such that $f(\tau) \geq \mu p^{p-1} - \mu_{1} \tau^{2}$ for all $\tau \geq 0$.

Note that Gao et al. [18] first introduced (V2) and (F2) to study the fractional Schrödinger-Poisson system. However, in [18], the authors only showed the existence of ground state solutions for equation (1.1) with $\varepsilon = 1$ and subcritical growth under

(F4) $\lim_{|\tau| \to 0} \frac{f(\tau)}{\tau} = 0$, and there exists $q \in (2, 2^{*})$ and a constant $C_{0} > 0$ such that

$$|f(\tau)| \leq C_{0}(1 + |\tau|^{q-1})$$

and

(F5) $\lim_{|\tau| \to +\infty} \frac{f(\tau)}{\tau^{2s-1}} = +\infty$.

In fact, (F4) is weaker than (F2), and (F5) is weaker than (F3) when $\frac{4s + 2t}{s + t} < p \leq 3$ in (F3), namely, $s \leq t$.

As far as we know, less research has been done on systems like (1.1) except for works [6,28, 29,44,47]. Yu et al. in [47] considered the fractional Schrödinger-Poisson system:

$$\begin{align*}
\epsilon^{2s}(-\Delta)^{s} u + V(x) u + \phi u &= K(x)|u|^{p-2} u, & \text{in } \mathbb{R}^{3}, \\
\epsilon^{2s}(-\Delta)^{s} \phi &= u^{2}, & \text{in } \mathbb{R}^{3},
\end{align*}$$

(1.4)

where $s \in \left[\frac{3}{4}, 1\right]$, $4 < p < 2^{*}_{s} = \frac{6}{3-2s}$, and $K(x) \in C(\mathbb{R}^{3}) \cap L^{\infty}(\mathbb{R}^{3})$ has a global maximum and is positive, and they showed the existence of a positive ground state solution and studied the concentration position of these ground state solutions as $\epsilon \to 0$. When the $f$ is critical nonlinearity, Liu and Zhang [28] and Ambrosio [6] considered the critical problem:

$$\begin{align*}
\epsilon^{2s}(-\Delta)^{s} u + V(x) u + \phi u &= f(u) + |u|^{2^{*}_{s}-2} u, & \text{in } \mathbb{R}^{3}, \\
\epsilon^{2s}(-\Delta)^{s} \phi &= u^{2}, & \text{in } \mathbb{R}^{3},
\end{align*}$$

(1.5)

where $s \in \left[\frac{3}{4}, 1\right]$, $t \in (0, 1)$. In [28], the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is of $C^{1}$ class and satisfies the following conditions:

(f1) $f(\tau) = 0$ for all $\tau < 0$ and $f(\tau) = o(\tau^{q})$ as $\tau \to 0$;

(f2) there exists $4 < q < 2^{*}_{s} = \frac{6}{3-2s}$ such that

$$\lim_{u \to +\infty} \frac{f(\tau)}{\tau^{q-1}} = 0;$$

(f3) function $\tau \to \frac{f(\tau)}{\tau}$ is increasing in $(0, \infty)$;

(f4) $f(\tau) \geq \rho \tau^{\sigma}$ for all $\tau > 0$ with some $\rho > 0$ and $\sigma \in (3, q - 1)$. 

Obviously, it follows from \((f_1)-(f_3)\) that
\[
0 \leq 3F(u) \leq f'(u)u, \quad 0 \leq 4F(u) \leq f(u)u \quad \forall u \in \mathbb{R}.
\]

By using minimax theorems and Ljusternik-Schnirelmann theory, they shown the multiplicity and concentration of solutions for system (1.5). Later, Ambrosio [6] improved the conditions of nonlinearity \(f, f'\) is no longer \(C^1\)-class, and satisfies the following conditions:
\[
\begin{align*}
(f_1') & \quad f(t) = o(t^\sigma) \text{ as } t \to 0; \\
(f_2') & \quad \text{there exist } q, \sigma \in (4, 2^*_s), C_0 > 0 \text{ such that } f(t) \geq C_0 t^{q+1} \quad \forall t > 0, \quad \lim_{t \to -\infty} \frac{f(t)}{t^{q+1}} = 0; \\
(f_3') & \quad \text{there exists } \vartheta \in (4, 2^*_s) \text{ such that } 0 < \vartheta F(t) \leq tf(t) \text{ for all } t > 0; \\
(f_4') & \quad \text{the function } t \mapsto \frac{f(t)}{t^{q+1}} \text{ is increasing in } (0, \infty).
\end{align*}
\]

They investigated the relationship between the number of positive solutions and the topology of the set where the potential reaches its minimum value utilizing penalization techniques and Ljusternik-Schnirelmann theory.

Based on the aforementioned facts, we prove in this study that semiclassical ground state solutions exist for equation (1.1) with critical growth and more general subcritical perturbation. In contrast to [28], we concentrate on the analysis of equation (1.1) using a more general subcritical perturbation \(f\) with \((F_1)-(F_3)\). First, \(f\) is a continuous function rather than of \(C^1\)-class, which leads to Nehari-Pohozaev manifold not being a \(C^1\)-manifold. Second, \(f\) no longer satisfies monotonicity condition \((f_3)\), which plays a crucial role in using the Nehari manifold method. Finally, we use condition \((F_3)\), which is weaker than \((f_1')\) and \((f_2')\). Compared with [6], \(f\) no longer satisfies Ambrosetti-Rabinowitz condition \((f_3')\) and monotonicity condition \((f_4')\), so we have great difficulty in using Nehari manifold method; moreover, the condition \((F_3)\) is more general than \((f_1')\) and \((f_2')\). In a manner, our results generalize and improve the works of [6,18,28].

When we studying system (1.1) under more general subcritical perturbation \(f\) with \((F_1)-(F_3)\), there are three main difficulties. First, system (1.1) has two nonlocal terms makes our analysis more complicated and intriguing. Second, the nonlinearity \(f\) is not of \(C^1\)-class, and this leads to Nehari-Pohozaev manifold not being a \(C^1\)-manifold; we draw attention to the proofs in [35,44], which are on the basis of minimizing the associated functional restricted to a fitting manifold which is \(C^1\). As a result, the arguments provided by [35,44] cannot be employed in this article, hence new strategies will be established. Third, the lack of compactness of the embedding \(H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), p \in (2, 2^*_s)\) due to the domain \(\mathbb{R}^N\) and the critical Sobolev exponent. We use the Concentration-Compactness Lemma for the fractional Laplacian, see [31]. Consequently, a more thorough analysis is required, which originates from [12].

Here, we will list our main results.

**Theorem 1.1.** Assume that \((V_1), (V_2), \text{ and } (F_1)-(F_3)\) hold and \(2s + 2t > 3\), Then, there has a number \(\varepsilon_*>0\) determined by \(V\) and \(f\) such that, for all \(\varepsilon \in (0, \varepsilon_*]\), system (1.1) has a positive ground state solution if one of the following conditions is satisfied:
\[
\begin{align*}
(\tilde{C}_1) & \quad s > \frac{3}{4}, \quad \frac{4s}{3 - 2s} < p < 2^*_s \quad \text{and any } \mu > 0; \\
(\tilde{C}_2) & \quad s > \frac{3}{4}, \quad \frac{4s + 2t}{s + t} < p \leq \frac{4s}{3 - 2s} \quad \text{and } \mu > 0 \text{ sufficiently large}; \\
(\tilde{C}_3) & \quad \frac{1}{2} < s \leq \frac{3}{4}, \quad \frac{4s + 2t}{s + t} < p < 2^*_s \quad \text{and any } \mu > 0.
\end{align*}
\]

Next, we establish the concentration behavior of ground state solutions as follows.

**Theorem 1.2.** Assume that Theorem 1.1 is true, then there exists a number \(\varepsilon_0 > 0\) determined by \(V\) and \(f\), then system (1.1) has a positive ground state solution \(\hat{\psi}_\varepsilon \in H(\mathbb{R}^N)\), where \(\hat{\psi}_\varepsilon\) satisfies the following statements:
(i) For \(\varepsilon \in (0, \varepsilon_0]\), the function \(|\hat{\psi}_\varepsilon|\) achieves its maximum at a point \(x_\varepsilon\), which satisfies

\[
\int_{B_{2r}(x_\varepsilon)} |\hat{\psi}_\varepsilon|^2 \, dx = \lambda_1 \int_{B_{2r}(x_\varepsilon)} |\hat{\psi}_\varepsilon|^2 \, dx
\]
\[ \lim_{\epsilon \to 0} V(x_\epsilon) = V_0 = \min_{x \in \mathbb{R}^3} V(x). \]

(ii) There exist \( C > 0 \) independent of \( \epsilon \in (0, \epsilon_0] \) such that the maximum point \( x_\epsilon \) of \( |\hat{\psi}| \) satisfies the inequality

\[ |\hat{\psi}| \leq \frac{C e^{3+2s}}{\epsilon^{3+2s} + |x - x_\epsilon|^{3+2s}} \quad \text{for all } x \in \mathbb{R}^3 \text{ and } \epsilon \in (0, \epsilon_0]. \]

(iii) For any sequence \( \epsilon_n \to 0 \), the sequence \( \hat{\psi}_n(x + x_\epsilon) \) converges in \( H^s(\mathbb{R}^3) \) to a ground state solution \( u \) of the following autonomous equation:

\[ -(\Delta)^s u + V_0 u + \phi'_{\epsilon}(x) u = f(u) + u^{2s-1}, \quad (1.6) \]

In order to overcome the lack of compactness of the embedding \( H^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3), p \in (2, 2^*_s] \), we need show the existence of positive ground state solutions to the limit equation associated with equation (1.1)

\[ -(\Delta)^s u + au + \phi'_{\epsilon}(x) u = f(u) + u^{2s-1}, \quad (1.7) \]

where \( a \) is a positive constant with \( 0 < a \leq V_{\max} = \sup_{x \in \mathbb{R}^3} V(x) \).

**Theorem 1.3.** Assume that \( 2s + 2t > 3 \) and \( f \) satisfies (F1)–(F3). If one of \( (\tilde{C}_1), (\tilde{C}_2), \) or \( (\tilde{C}_3) \) holds in Theorem 1.1, then (1.7) has a positive ground state solution.

The structure of this article is as follows: In Section 2, we provide some preliminary lemmas which will be used later; in Section 3, we prove the autonomous equation (1.7) and conclude the proof of Theorem 1.3; in Section 4, we show the existence of semiclassical ground state solutions to equation (1.1) for all \( \epsilon \in (0, \epsilon_0] \) and give the proof of Theorem 1.1; in Sections 5 and 6, we study the concentration phenomenon and convergence of ground state solutions. In particular, we obtain the decay estimate of solution, which complete the proof of Theorem 1.2.

### 2 Preliminaries

Throughout this article, we denote \( |\cdot|_p \) the usual norm of the space \( L^p(\mathbb{R}^3), 1 \leq p < \infty \), \( |\cdot|_\infty \) denotes the norm of the space \( L^\infty(\mathbb{R}^3) \), \( C \) or \( C_i (i = 1, 2, \ldots) \) denotes some positive constants which may change from line to line. First, we introduce the space \( H^s(\mathbb{R}^3) \), which is defined as follows:

\[ H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) \mid (-\Delta)^s u \in L^2(\mathbb{R}^3) \right\}. \]

The inner product and the norm are defined, respectively, as follows:

\[ (u, v) = \int_{\mathbb{R}^3} (-\Delta)^s u(-\Delta)^s v dx + \int_{\mathbb{R}^3} uv dx, \]

and

\[ ||u||^2 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \text{d}x \text{d}y + \int_{\mathbb{R}^3} u^2 dx = ||u||^2_H + ||u||^2. \]

For convenience, we set

\[ ||u||^2_H = ||u||^2_L = \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \text{d}x \text{d}y. \]

From [14], we know that for any \( a \in (0, 1) \), \( D^{a,2}(\mathbb{R}^3) \) is continuously embedded into \( L^{2a}(\mathbb{R}^3) \) and define
where \( D^{s,2}(\mathbb{R}^3) \) is defined by

\[
D^{s,2}(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) | (-\Delta)^s u \in L^2(\mathbb{R}^3) \right\}.
\]

According to the Lax-Milgram Theorem, for \( u \in H^s \), there is a unique \( \phi^s_u \in D^{s,2}(\mathbb{R}^3) \) that satisfies:

\[
(-\Delta)^s \phi^s_u = u^2, \quad x \in \mathbb{R}^3.
\]

Moreover,

\[
\phi^s_u(x) = C(t) \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|^{1-2s}} dy, \quad x \in \mathbb{R}^3,
\]

where

\[
C(t) = \pi^{-\frac{1}{2} - 2s} \frac{\Gamma\left(\frac{3-2s}{2}\right)}{\Gamma(t)}.
\]

Observe that for any \( \varepsilon > 0 \),

\[
\phi^s_{\varepsilon u}(x) = C(t) \int_{\mathbb{R}^3} \frac{v^2(y)}{|x - y|^{1-2s}} dy = e^{-\varepsilon^2 t} \phi^s_u(x)
\]

is a weak solution of \(-\varepsilon^2 \Delta \phi = v^2\). Substituting \( \phi^s_{\varepsilon u} \) in equation (1.1), we obtain

\[
-e^{\varepsilon^2 t}(-\Delta)^s v + V(x)v + \phi^s_{\varepsilon u}(x)v = f(v) + v^{2s-1}.
\]

Making the scaling \( u(x) = v(\varepsilon x) \), we can rewrite the system (1.1) as the following equivalent system:

\[
(-\Delta)^s v + V(\varepsilon x)u + \phi^s_{\varepsilon u}(x)u = f(u) + u^{2s-1}.
\] (2.2)

Obviously, \( \{v, \phi^s_{\varepsilon u}\} \) is a solution of equation (1.1) if and only if \( u \) is a solution of equation (2.2). It is obvious that weak solutions to equation (2.2) are critical points of the following functional:

\[
I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^s u \right|^2 + \varepsilon^2 V(x)u^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \left| \phi^s_{\varepsilon u}(x)u \right|^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx + \frac{1}{2\varepsilon} \left| u_{2s} \right|^2 \, dx.
\] (2.3)

When \( V(x) = a \), we define the functional of equation (1.7) in \( H^s(\mathbb{R}^3) \) as follows:

\[
I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^s u \right|^2 + a u^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \left| \phi^s_{\varepsilon u}(x)u \right|^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx + \frac{1}{2\varepsilon} \left| u_{2s} \right|^2 \, dx.
\] (2.4)

We also study the counterpart results of the concentration phenomenon as \( \varepsilon \to 0 \) for ground state solutions to equation (1.1). For this reason, we define the Pohozaev-type functional \( \mathcal{P}_\varepsilon \) of equation (2.2) in \( H^s(\mathbb{R}^3) \) by

\[
\mathcal{P}_\varepsilon(u) = \frac{3 - 2s}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^s u \right|^2 + V(\varepsilon x)u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \left| \nabla V(\varepsilon x)(\varepsilon x) \right| u^2 \, dx + \frac{3 + 2t}{4} \int_{\mathbb{R}^3} \phi^s_{\varepsilon u}(x)u^2 \, dx - 3 \int_{\mathbb{R}^3} F(u) \, dx - \frac{3}{2\varepsilon^2} \left| u_{2s} \right|^2,
\]

and the Pohozaev-type functional \( \mathcal{P}_a \) of equation (1.7) in \( H^s(\mathbb{R}^3) \) by
\[
\mathcal{P}_d(u) = \frac{3 - 2s}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^\frac{s}{2} u \right|^2 \, dx + \frac{3}{2} \int_{\mathbb{R}^3} a u^2 \, dx + \frac{3 + 2t}{4} \int_{\mathbb{R}^3} \phi_0'(x) u^2 \, dx - 3 \int_{\mathbb{R}^3} F(u) \, dx - \frac{3}{2^*} \|u\|_{2^*}^{2^*}.
\]

According to [15], any solution \( u \) of equation (1.6) satisfies \( \mathcal{P}_d(u) = 0 \). Inspired by this fact and by the work of Ruiz [35], we introduce the following functional on \( H^s(\mathbb{R}^3) \):

\[
\mathcal{J}(u) = (s + t)\langle I_d(u), u \rangle - \mathcal{P}_d(u)
\]

\[
= \frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^\frac{s}{2} u \right|^2 \, dx + \frac{2s + 2t - 3}{2} \int_{\mathbb{R}^3} V(x) u^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \nabla V(x) (ex) u^2 \, dx
\]

\[
+ \frac{4s + 2t - 3}{4} \int_{\mathbb{R}^3} \phi_0'(x) u^2 \, dx - \int_{\mathbb{R}^3} [(s + t) f(u) u - 3F(u)] \, dx - \frac{4s + 2t - 3}{2} \|u\|_{2^*}^{2^*},
\]

(2.5)

and the constant potential case by

\[
\mathcal{J}(u) = (s + t)\langle I_d(u), u \rangle - \mathcal{P}_d(u)
\]

\[
= \frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^\frac{s}{2} u \right|^2 \, dx + \frac{2s + 2t - 3}{2} \int_{\mathbb{R}^3} a u^2 \, dx + \frac{4s + 2t - 3}{4} \int_{\mathbb{R}^3} \phi_0'(x) u^2 \, dx
\]

\[
- \int_{\mathbb{R}^3} [(s + t) f(u) u - 3F(u)] \, dx - \frac{4s + 2t - 3}{2} \|u\|_{2^*}^{2^*}.
\]

(2.6)

We define the Nehari-Pohozaev manifold of \( I_e \) by

\[
\mathcal{M}_e = \{ u \in H^s(\mathbb{R}^3) \setminus \{0\} : \mathcal{J}(u) = 0 \},
\]

(2.7)

and the constant potential case

\[
\mathcal{M}_a = \{ u \in H^s(\mathbb{R}^3) \setminus \{0\} : \mathcal{J}(u) = 0 \}.
\]

(2.8)

For any \( u \in \mathcal{M}_e \), we say that

\[
m_e = \inf_{u \in \mathcal{M}_e} \mathcal{J}(u),
\]

and

\[
m_a = \inf_{u \in \mathcal{M}_a} I_d(u).
\]

Then, every nontrivial solution of equation (2.2) is contained in \( \mathcal{M}_e \). In particular, we call a nontrivial solution \( \hat{u} \) of equation (2.2) to be a ground state solution of Nehari-Pohozaev-type if \( I_e(\hat{u}) = \inf_{u \in \mathcal{M}_e} I_e(u) \). Similarly, every nontrivial solution of equation (1.7) is contained in \( \mathcal{M}_a \). In particular, we call a nontrivial solution \( \tilde{u} \) of equation (1.7) to be a ground state solution of Nehari-Pohozaev-type if \( I_a(\tilde{u}) = \inf_{u \in \mathcal{M}_a} I_a(u) \).

By simple calculation, we obtain the following lemmas.

**Lemma 2.1.** Assume that (V1) and (V2) hold. Then, for all \( x \in \mathbb{R}^3 \) and \( \tau > 0 \),

\[
\beta(x, \tau) = [V(x) - \tau^{2s + 2t - 3}V(\tau^{-1}x)] - \frac{4s + 2t - 3}{4s + 2t - 3} [(2s + 2t - 3)V(x) - (\nabla V(x), x)]
\]

\[
> 0 \quad \forall \tau \in [0, 1) \cup (1, +\infty).
\]

(2.9)

**Proof.** We know that

\[
\frac{d}{d\tau} [\tau^{2s + 2t - 3}V(\tau^{-1}x)] = \tau^{2s + 2t - 4} [(2s + 2t - 3)V(\tau^{-1}x) - (\nabla V(\tau^{-1}x), \tau^{-1}x)],
\]

using (V2), implies
\[
[V(x) - \tau^{2s+2t-3}V(\tau^{-1}x)] - \frac{1 - \tau^{4s+2t-3}}{4s + 2t - 3}[(2s + 2t - 3)V(x) - (\nabla V(x), x)]
\]
\[
= \int_{\tau}^{1} \left[ \frac{d}{dv}[v^{2s+2t-3}V(v^{-1}x)] - v^{4s+2t-4}[(2s + 2t - 3)V(x) - (\nabla V(x), x)] \right] dv
\]
\[
= \int_{\tau}^{1} v^{4s+2t-4}[v^{-2s}(2s + 2t - 3)V(v^{-1}x) - (\nabla v^{-1}x), v^{-1}x] - [(2s + 2t - 3)V(x) - (\nabla V(x), x)] dv
\]
> 0 \ \forall \tau \in [0, 1) \cup (1, +\infty).

This concludes the proof. \(\square\)

Lemma 2.2. It is easy to check that (V₃) implies, for all \(x \in \mathbb{R}^3\), that
\[-2sV(x) + \rho_0 \leq (\nabla V(x), x) \leq (2s + 2t - 3)V(x) \ \forall x \in \mathbb{R}^3.\] (2.10)

Lemma 2.3. [44] For any \(u \in H^s(\mathbb{R}^3)\), if \(4s + 2t > 3\), then
(1) there exist \(C > 0\) such that
\[\int_{\mathbb{R}^3} \phi_u \tilde{u} dx \leq S^2 \|\tilde{u}\|_{L^2} \quad \forall u \in H^s(\mathbb{R}^3);\]
(2) \(\phi_u \tilde{u} > 0 \ \forall u \in H^s(\mathbb{R}^3);\)
(3) \(\phi_u \tilde{u} = \tau^s \phi_u \tilde{u} \ \forall \tau > 0 \text{ and } u \in H^s(\mathbb{R}^3);\)
(4) If \(u_n \rightharpoonup u\) in \(H^s(\mathbb{R}^3)\), then \(\phi_{u_n} \tilde{u} \rightharpoonup \phi_u \tilde{u}\) in \(D^{s,2}(\mathbb{R}^3)\).

Lemma 2.4. Assume \((F_2)\) holds, then
\[\frac{(s + t)(1 - \tau^{4s+2t-3})}{4s + 2t - 3} f(\theta)\theta - \frac{4s + 2t - 3\tau^{4s+2t-3}}{4s + 2t - 3} F(\theta) + \tau^{-3}F(\tau^{s+t}) \geq 0 \ \forall \tau \geq 0, \theta \in \mathbb{R}.\]

Proof. Without loss of generality, we can assume that \(\tau \neq 0\) and set
\[h(\tau) = \frac{(s + t)(1 - \tau^{4s+2t-3})}{4s + 2t - 3} f(\theta)\theta - \frac{4s + 2t - 3\tau^{4s+2t-3}}{4s + 2t - 3} F(\theta) + \tau^{-3}F(\tau^{s+t}).\]
By a direct computation, we have
\[h'(\tau) = -(s + t)\tau^{4s+2t-4}f(\theta)\theta + 3\tau^{4s+2t-4}F(\theta) - 3\tau^{-4}F(\tau^{s+t}) + (s + t)\tau^{s+t-4}f(\tau^{s+t})\theta
\]
\[= \tau^{4s+2t-4} \frac{4s+2t}{\theta^{3s+t}} \left[ (s + t) f(\tau^{s+t}) \tau^{-s+t} - 3F(\tau^{s+t}) \right] - (s + t)f(\theta)\theta - 3F(\theta)\theta\theta\theta^{3s+t}.\]
According to \((F_2)\), \(h(\tau) \geq h(1) = 0 \ \forall \theta \geq 0.\) \(\square\)

Lemma 2.5. It is simple to verify that if \((F_2)\) holds, then
\[(s + t)f(\theta)\theta - (4s + 2t)F(\theta) \geq 0, \text{ for all } \theta \in \mathbb{R}.\] (2.11)

3 Constant potential case

Lemma 3.1. Assume that \((F_1)\) and \((F_2)\) hold and \(4s + 2t > 3\). Then,
\[I_d(u) \geq I_d(u_\ast) + \frac{1 - \tau^{4s+2t-3}}{4s + 2t - 3} J_d(u) + \tilde{C}(\tau) \int_{\mathbb{R}^3} u^{2} dx \quad \forall u \in H^s(\mathbb{R}^3), \tau \geq 0,
\]
where \(u_\ast(x) = \tau^{s+t}u(\tau x)\) and \(\tilde{C}(\tau) \geq 0.\)
Proof. By Lemma 2.4, it is simple to verify that
\[
I_d(u) - I_d(u_*) = \frac{1}{2} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} |u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} au^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_0(x) u^2 dx
\]
- \left\{ \begin{array}{l}
F(u) - \frac{1}{\tau} F(\tau^{s+1} u) dx + \frac{\tau^{(s+1)\frac{2^*_s}{2^*_s} - 3}}{2^*_s} ||u||_{2^*_s}^{2^*_s} \\
+ \int_{\mathbb{R}^3} \left[ \frac{1}{\tau} F(\tau^{s+1} u) dx + \frac{3 \tau^{s+1} \tau^{2^*_s} - (4s + 2t) F(u) + \frac{1}{\tau^3} F(\tau^{s+1} u) dx \right] \\
\end{array} \right.
\]
= \frac{1 - \tau^{4s + 2t - 3}}{4s + 2t - 3} J_d(u) + \hat{\mathcal{C}}(\tau) \int \au^2 dx
\]
for 0 < \tau = \tau_0 \leq \infty be such that \tau_0 is unique. In fact, we just assume that there are two points \tau_1 and \tau_2 for \tau \in \mathbb{R}^3 such that \tau_1 and \tau_2 is achieved at a point \tau^* \in \mathbb{R}^3.

Hence, we have \tau_1 < \tau_2 and \tau_1 > \tau_2 for 0 < \tau = \tau_0 \leq \infty be such that \tau_0 is unique. In fact, we just assume that there are two points \tau_1 and \tau_2 for \tau \in \mathbb{R}^3 such that \tau_1 and \tau_2 is achieved at a point \tau^* \in \mathbb{R}^3.

Lemma 3.2. Assume that (F_1) and (F_2) hold and \( s + t \geq 2^*_s > 4s + 2t \). Then, for any \( u \in H^s(\mathbb{R}^3) \setminus \{0\} \), there is a unique constant \( \tau_0 > 0 \) such that \( u_0 \in M_\tau \). Moreover, \( I_d(u_0) = \max_{\tau > 0} I_d(u_\tau) \).

Proof. Set \( u \in H^s(\mathbb{R}^3) \setminus \{0\} \) be fixed and define a function \( \zeta_i(\tau) = I_d(u_\tau) \) on \( [0, \infty) \). Thus, we have
\[
\zeta_i'(\tau) = 0 \Leftrightarrow \left\{ \begin{array}{l}
\frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} |u|^2 dx + \frac{2s + 2t - 3}{2} \int \au^2 dx \\
+ \frac{4s + 2t - 3}{4s + 2t - 3} \int \phi_0(x) u^2 dx + \frac{1}{\tau^3} \int f(\tau^{s+1} u) dx - \frac{s + t}{\tau} \int \au^{s+1} u dx \\
- \frac{1}{\tau} \int \au^{s+1} u dx - \frac{((s + t)2^*_s - 3) \tau^{(s+1)\frac{2^*_s}{2^*_s} - 3}}{2^*_s} ||u||_{2^*_s}^{2^*_s} = 0 \\
\end{array} \right.
\]
\[
\Leftrightarrow I_d(u_\tau) = 0 \Leftrightarrow u_\tau \in M_\tau.
\]
Clearly, as a result of (F_1) and (F_2), \( \zeta_i(0) = 0 \) and \( \zeta_i(\tau) > 0 \) for \( \tau > 0 \) small and \( \zeta_i(\tau) < 0 \) for \( \tau \) large. Hence, \( \max_{\tau \in [0, \infty)} \zeta_i(\tau) \) is achieved at a \( \tau_0 = \tau_0(u) > 0 \) so that \( \zeta_i(\tau_0) = 0 \) and \( u_0 \in M_\tau \).

Moreover, we assert that the critical point of \( \tau_0 \) is unique. In fact, we just assume that there are two points \( \tau_1, \tau_2 > 0 \) such that \( \zeta_i(u_\tau) = 0 \) for \( i = 1, 2 \). We can deduce that
\[
I_d(u_0) \geq I_d(u_\tau) + \frac{1 - \left( \frac{\tau_1}{\tau_2} \right)^{4s + 2t - 3}}{4s + 2t - 3} J_d(u_\tau) + \hat{\mathcal{C}}(\tau_1) \int \au^{4s + 2t - 3} dx
\]
= \( I_d(u_\tau) + \hat{\mathcal{C}}(\tau_1) \int \au^{4s + 2t - 3} dx \)
\]
and
\[
I_d(u_0) \geq I_d(u_\tau) + \frac{1 - \left( \frac{\tau_1}{\tau_2} \right)^{4s + 2t - 3}}{4s + 2t - 3} J_d(u_\tau) + \hat{\mathcal{C}}(\tau_1) \int \au^{4s + 2t - 3} dx
\]
= \( I_d(u_\tau) + \hat{\mathcal{C}}(\tau_1) \int \au^{4s + 2t - 3} dx \)
\]
so \( \tau_1 = \tau_2 \).

**Lemma 3.3.** Assume that \( (F_1) \) and \( (F_2) \) hold. Then,
\[
\inf_{u \in \mathcal{M}_d} I_d(u) = m_d = \inf_{u \in H^s(\mathbb{R}^3)} \max_{r > 0} I_d(u_r)
\]
(3.4)

**Lemma 3.4.** Assume that \( (F_1) \) and \( (F_2) \) hold and \( (s + t)2^s > 4s + 2t \). It holds that
\[
m_a = \inf_{u \in H^s(\mathbb{R}^3)} \max_{r > 0} I_d(u_r) > 0.
\]

**Proof.** It follows from Lemma 3.3 that
\[
m_a = \inf_{u \in H^s(\mathbb{R}^3)} \max_{r > 0} I_d(u_r).
\]

Next, we will prove \( m_a > 0 \). Indeed, it follows from \( (F_1) \) that there exists a constant \( C \) such that
\[
|F(u)| \leq \frac{1}{2} u^2 + C |u|^{2s} \quad \forall u \in H^s(\mathbb{R}^3),
\]
(3.5)

thus for any \( u \in \mathcal{M}_a \), by equation (3.5), Lemma 3.2, and Sobolev inequality, we obtain
\[
I_d(u) \geq I_d(u_r)
\]
\[
\geq \frac{\tau^{4s+2t-3} - 2}{2} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u^2 dx + \frac{\tau^{2s+2t-3}}{2} \int_{\mathbb{R}^3} au^2 dx - \tau^{-3} \int_{\mathbb{R}^3} F(x^s u) dx - \frac{\tau^{(s+1)2^s-3}}{2^s} \int_{\mathbb{R}^3} |u|^{2^s} dx
\]
\[
\geq \frac{\tau^{4s+2t-3} - 2}{2} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u^2 dx - \left( C + \frac{1}{2^s} \right) \tau^{(s+1)2^s-3} \int_{\mathbb{R}^3} |u|^{2^s} dx
\]
\[
= \frac{2s}{3 - 2s} \tilde{C} \left( \frac{1}{2^s} \right) > 0,
\]
if we take
\[
\tau = \left[ \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u^2 dx \right]^{-\frac{2}{s+1}} \left[ \tau^{(s+1)2^s-3} \right]^{-\frac{2}{s+1}}
\]
and
\[
\tilde{C} = \left( C + \frac{1}{2^s} \right) S_s \frac{2^s}{3 - 2s}.
\]

Therefore, we complete the proof. \( \Box \)

**Lemma 3.5.** Assume that \( (F_1) \) holds and \( 2s + 2t > 3 \). If \( u_n \rightharpoonup u \) in \( H^s(\mathbb{R}^3) \) and \( u_n \rightarrow u \) a.e. in \( \mathbb{R}^3 \). Then
\[
I_d(u_n) = I_d(u) + I_d(u_n - u) + o_n(1),
\]
(3.6)
\[
\langle I'_d(u_n), u_n \rangle = \langle I'_d(u), u \rangle + \langle I'_d(u_n - u), u_n - u \rangle + o_n(1),
\]
(3.7)

and
\[
\mathcal{F}_d(u_n) = \mathcal{F}_d(u) + \mathcal{F}_d(u_n - u) + o_n(1).
\]
(3.8)

**Proof.** We set \( v_n = u_n - u \), then \( v_n \rightharpoonup 0 \) in \( H^s(\mathbb{R}^3) \). It follows from \( (F_1) \) and the Brézis-Lieb lemma that
\[
\|u_n\|^2 = \|u\|^2 + \|v_n\|^2 + o_n(1),
\]
(3.9)
\[ \int F(u_n)dx = \int F(u)dx + \int F(v_n)dx + o_n(1), \] (3.10)

and

\[ \|u_n\|_{L^2}^2 = \|u\|_{L^2}^2 + \|v_n\|_{L^2}^2 + o_n(1). \] (3.11)

Combining equations (3.9)–(3.11), equation (3.6) holds.

We use the similar argument of Lemma 2.7 in [43] to show that there exists a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), such that

\[ \sup_{\phi \in H^1(R^3), \|\phi\| \leq 1} \left| \int (f(u_n) - f(u) - f(v_n))\phi dx \right| = o_n(1). \] (3.12)

Thus, one has

\[ \left| \int (f(u_n) - f(u) - f(v_n))u_n dx \right| \leq \|u_n\| \sup_{\phi \in H^1(R^3), \|\phi\| \leq 1} \left| \int (f(u_n) - f(u) - f(v_n))\phi dx \right| = o_n(1). \]

So, we obtain

\[ \int f(u_n)u_n dx = \int f(u)u dx + \int f(v_n)v_n dx + \int f(v_n)u dx + \int (f(u_n) - f(u) - f(v_n))u_n dx \]
\[ = \int f(u)u dx + \int f(v_n)v_n dx + o_n(1). \] (3.13)

Combining equations (3.9), (3.11), and (3.13), we know equation (3.7) holds. Finally, we note that

\[ J_d(u) = \frac{4s + 2t - 3}{2} (I'_d(u), u) - s \int au^2 dx - \frac{4s + 2t - 3}{4} \int \phi'_t u^2 dx \]
\[ - \int \left[ \frac{3 - 2s}{2} f(u)u - 3F(u) \right] dx. \]

Combining equations (3.7), (3.10), and (3.13), we can obtain equation (3.8). \qed

**Lemma 3.6.** Assume that \((F_1)\)–\((F_3)\) hold. It holds that \( m_n < \frac{s}{2} S_{2s}^{\frac{2}{s}} \) if one of \((C_1)\), \((C_2)\), or \((C_3)\) is satisfied.

**Proof.** Setting \( \psi \in C^0_0(R^3) \) be a cut-off function such that \( \psi(x) = 1 \) if \(|x| \leq r \), and \( \psi(x) = 0 \) if \(|x| \geq 2r \). For \( \epsilon > 0 \), we define

\[ u_\epsilon(x) = \psi(x)U_\epsilon(x), \quad x \in R^3, \]

where \( U_\epsilon(x) = e^{-\frac{\epsilon^2}{2}} u^R \left( \frac{x}{\epsilon} \right) \), \( u^R(x) = \left[ \frac{|x|/S^R_{2s}}{2|\nu|S^R_{2s}} \right]^\frac{1}{2}, \)

and

\[ U(x) = \kappa (r^2 + |x - x_0|^2)^{\frac{3-2s}{2}}, \]

with \( \kappa \in R \setminus \{0\}, \tau > 0, \) and \( x_0 \in R^3 \). Based on [42], we know that this is true

\[ \int \frac{1}{2} \left| (-\Delta)^s u_\epsilon \right|^2 dx \leq S_{2s}^3 + O(\epsilon^{3-2s}), \]
\[ \int \left| u_\epsilon \right|^2 dx = S_{2s}^3 + O(\epsilon^3), \]
and

\[
\int_{\mathbb{R}^3} |u_0|^q \, dx = \begin{cases} 
O\left(e^{3-\frac{3m}{2d}}\right), & \text{for } q > \frac{3}{3-2s}, \\
O\left(e^{\frac{|\log e|}{2s}}\right), & \text{for } q = \frac{3}{3-2s}, \\
O\left(e^{\frac{1-2s}{2s}}\right), & \text{for } q < \frac{3}{3-2s}.
\end{cases}
\]

From Lemmas 3.3 and 3.4, there exists a \( \tau_0 > 0 \) such that

\[
0 < m_0 \leq \max_{\tau \geq 0} I_d((u_0)_\tau) = I_d((u_0)_\tau_0).
\]  

(3.14)

Next, we will prove that there exist two constants \( \tau_0, \tau^* > 0 \) such that \( \tau_0 \leq \tau \leq \tau^* \). First, we claim that \( \tau_0 \) is bounded from below by a positive constant. Otherwise, we could find a sequence \( \varepsilon_n \to 0 \) such that \( \varepsilon_n \to 0 \), so we conclude that \( (u_{\varepsilon_n})_{\varepsilon_n} = 0 \) in \( H^1(\mathbb{R}^3) \). So we have

\[
0 < m_0 \leq I_d((u_{\varepsilon_n})_{\varepsilon_n}) \to I_d(0) = 0,
\]

this is clearly a contradiction. On the other hand, by (F3), we obtain that

\[
0 \leq I_d((u_\tau)_\tau) \leq \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{3}{2}} u_\tau \right|^2 \, dx + \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} a u_\tau^2 \, dx + \frac{\tau_0^{2s+22-3}}{4} \int_{\mathbb{R}^3} a \phi u_\tau^2 \, dx + \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} \mu u_\tau^2 \, dx
\]

\[
+ \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} \phi u_\tau^2 \, dx - \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} u_\tau^2 \, dx - \frac{\tau_0^{2s+22-3}}{4} \int_{\mathbb{R}^3} |u_\tau|^p \, dx + \frac{\tau_0^{2s+22-3}}{4} \int_{\mathbb{R}^3} |u_\tau|^p \, dx,
\]

which implies that there exists \( \tau^* > 0 \) such that \( \tau \leq \tau^* \). Therefore, the claim is proved. Thus, we conclude that

\[
I_d((u_\tau)_\tau) \leq \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{3}{2}} u_\tau \right|^2 \, dx + \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} a u_\tau^2 \, dx + \frac{\tau_0^{2s+22-3}}{4} \int_{\mathbb{R}^3} a \phi u_\tau^2 \, dx + \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} \mu u_\tau^2 \, dx
\]

\[
+ \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} \phi u_\tau^2 \, dx - \frac{\tau_0^{2s+22-3}}{2} \int_{\mathbb{R}^3} u_\tau^2 \, dx - \frac{\tau_0^{2s+22-3}}{4} \int_{\mathbb{R}^3} |u_\tau|^p \, dx + \frac{\tau_0^{2s+22-3}}{4} \int_{\mathbb{R}^3} |u_\tau|^p \, dx
\]

\[
\leq \left( \frac{\tau_0^{2s+22-3}}{2} - \frac{\tau_0^{2s+22-3}}{4} \right) S_{\frac{3}{5}}^{\frac{3}{2}} \left( O(3-\frac{2s}{2d}) + C \int_{\mathbb{R}^3} u_\tau^2 \, dx + C \int_{\mathbb{R}^3} \left| u_\tau \right|^\frac{12}{3-2d} \, dx \right) ^{\frac{3-2d}{12}} - C \int_{\mathbb{R}^3} |u_\tau|^p \, dx
\]

\[
\leq \frac{S_3^{\frac{3}{5}}}{3} + O(3-\frac{2s}{2d}) + C \int_{\mathbb{R}^3} u_\tau^2 \, dx + C \int_{\mathbb{R}^3} \left| u_\tau \right|^\frac{12}{3-2d} \, dx \right) ^{\frac{3-2d}{12}} - C \int_{\mathbb{R}^3} |u_\tau|^p \, dx.
\]

Next, we separate three cases:

**Case 1:** \( 2 < \frac{3}{3-2s} \Leftrightarrow s > \frac{3}{4} \).

\[
I_d((u_\tau)_\tau) \leq \frac{S_3^{\frac{3}{5}}}{3} + O(3-\frac{2s}{2d}) + C \int_{\mathbb{R}^3} \left| u_\tau \right|^\frac{12}{3-2d} \, dx \right) ^{\frac{3-2d}{12}} - C \int_{\mathbb{R}^3} |u_\tau|^p \, dx.
\]

**Case 2:** \( 2 = \frac{3}{3-2s} \Leftrightarrow s = \frac{3}{4} \).

\[
I_d((u_\tau)_\tau) \leq \frac{S_3^{\frac{3}{5}}}{3} + O(2|\log e|) + C \int_{\mathbb{R}^3} \left| u_\tau \right|^\frac{12}{3-2d} \, dx \right) ^{\frac{3-2d}{12}} - C \int_{\mathbb{R}^3} |u_\tau|^p \, dx.
\]
Case 3: $2 > \frac{3}{3 - 2s} \Leftrightarrow s < \frac{3}{4}$.

$$I_d(u_\epsilon) \leq \frac{S}{S^3} + O(\epsilon^{2s}) + C \epsilon^{\frac{12}{3 - 2s}} - C \epsilon \int |u_\epsilon|^p dx.$$  

By computations, we obtain that $\frac{3s + t}{s + t} < \frac{2s}{3 - 2s} = 1$ for any $s > \frac{3}{4}$ and $\frac{3s + t}{s + t} \geq \frac{2s}{3 - 2s} = \frac{3}{3 - 2s} - 1$ for any $s \leq \frac{3}{4}$.

**Case 1:** $s > \frac{3}{4}$.

$$\lim_{\epsilon \to 0} \frac{\int_{\Omega} |u_\epsilon|^{\frac{12}{3 - 2s}} dx}{\epsilon^{3 - 2s}} \leq \left\{ \begin{array}{ll} \lim_{\epsilon \to 0} O(\epsilon^{\frac{2s + 4s - 3}{3 - 2s}}) = 0, & \frac{12}{3 + 2t} > \frac{3}{3 - 2s}, \\ \lim_{\epsilon \to 0} O(\epsilon^{\frac{2s + 4s - 3}{3 - 2s}} |\log\epsilon|^{\frac{1}{3 - 2s}}) = 0, & \frac{12}{3 + 2t} = \frac{3}{3 - 2s}, \\ \lim_{\epsilon \to 0} O(\epsilon^{\frac{2s + 4s - 3}{3 - 2s}}) = 0, & \frac{12}{3 + 2t} < \frac{3}{3 - 2s}, \end{array} \right.$$  

and noting that $2s - \frac{3 - 2s}{2} p < 0$ if $\frac{4s}{3 - 2s} < p < \frac{6}{3 - 2s}$, we have

$$\lim_{\epsilon \to 0} \frac{\int_{\Omega} |u_\epsilon|^p dx}{\epsilon^{3 - 2s}} = \left\{ \begin{array}{ll} \lim_{\epsilon \to 0} O(\epsilon^{\frac{3 - 2s}{2s} p}) = +\infty, & \frac{4s}{3 - 2s} < p < \frac{6}{3 - 2s}, \\ \lim_{\epsilon \to 0} O(\epsilon^{\frac{3 - 2s}{2s} p}) = +\infty, & \frac{4s}{3 - 2s} < p < \frac{6}{3 - 2s}. \end{array} \right.$$  

Choosing $\mu$ large enough such that the above three limits equal to $+\infty$, e.g., $\mu = e^{-2s}$.

**Case 2:** $s = \frac{3}{4}$. In view of $\frac{12}{3 + 2t} > 2 = \frac{3}{3 - 2s}$, we have

$$\lim_{\epsilon \to 0} \frac{\int_{\Omega} |u_\epsilon|^{\frac{12}{3 - 2s}} dx}{\epsilon^{3 - 2s}} \leq \lim_{\epsilon \to 0} O(\epsilon^{\frac{2s + 4s - 3}{3 - 2s}} |\log\epsilon|^{\frac{1}{3 - 2s}}) = 0, \quad \frac{12}{3 + 2t} > \frac{3}{3 - 2s} = 2,$$  

and since $\frac{3}{3 - 2s} = 2 < \frac{3s + t}{s + t} + 1 < p$, for any $\mu > 0$, we have that

$$\lim_{\epsilon \to 0} \frac{\int_{\Omega} |u_\epsilon|^p dx}{\epsilon^{3 - 2s}} = \lim_{\epsilon \to 0} \frac{\mu}{\epsilon^{3 - 2s} |\log\epsilon|^{\frac{1}{3 - 2s}}} = +\infty, \quad \frac{4s + 2t}{s + t} < p < \frac{6}{3 - 2s}.$$

**Case 3:** $s < \frac{3}{4}$. Owing to $\frac{3}{3 - 2s} \in \left( \frac{3}{4}, 2 \right)$, thus $\frac{12}{3 + 2t} > \frac{3}{3 - 2s}$ and $\frac{3}{3 - 2s} < \frac{4s + 2t}{s + t} < p < \frac{6}{3 - 2s}$. Hence,

$$\lim_{\epsilon \to 0} \frac{\int_{\Omega} |u_\epsilon|^{\frac{12}{3 - 2s}} dx}{\epsilon^{3 - 2s}} \leq \lim_{\epsilon \to 0} O(\epsilon^{\frac{2s + 4s - 3}{3 - 2s}} |\log\epsilon|^{\frac{1}{3 - 2s}}) = 0, \quad \frac{12}{3 + 2t} > \frac{3}{3 - 2s},$$  

and for any $\mu > 0$, we have

$$\lim_{\epsilon \to 0} \frac{\mu}{\epsilon^{3 - 2s}} = \lim_{\epsilon \to 0} \frac{\mu}{\epsilon^{3 - 2s} |\log\epsilon|^{\frac{1}{3 - 2s}}} = +\infty, \quad \frac{4s + 2t}{s + t} < p < \frac{6}{3 - 2s}.$$
From the above inequalities, we conclude that

\[ m_a \leq I_d((u_n)_n) \leq \frac{s}{3} S_{\beta}^3. \]

Thus, we complete the proof. \qed

**Lemma 3.7.** Assume that \((F_1)-(F_3)\) hold and that \(2s + 2t > 3\). Then, \(m_a > 0\) is achieved at some \(\bar{u} \in M_a\).

**Proof.** Setting

\[
\Phi(u) = I_d(u) - \frac{1}{4s + 2t - 3} J_d(u)
\]

\[
= \frac{s}{4s + 2t - 3} \int_{\mathbb{R}^3} au^2dx + \frac{s + t}{4s + 2t - 3} \int_{\mathbb{R}^3} (f(u)u - \frac{4s + 2t}{s + t} F(u))dx + \frac{s}{3} \int_{\mathbb{R}^3} |u|^2dx.
\]

(3.15)

Let \([u_n] \subset M_a\) be a minimizing sequence for \(m_a\) such that

\[ I_d(u_n) \to m_a. \]

We divide the proof into three steps as follows:

**Step 1:** We prove \([u_n]\) is bounded in \(H^s(\mathbb{R}^3)\). It follows that by equations (2.11) and (3.15), we have that

\[ m_a + o_n(1) = I_d(u_n) - \Phi(u_n) \geq \frac{s}{4s + 2t - 3} \int_{\mathbb{R}^3} au_n^2dx + \frac{s}{3} \int_{\mathbb{R}^3} |u_n|^2dx. \]

(3.16)

Moreover, from \((F_1),(3.16),\) and \(J_d(u_n) = 0\), we obtain that

\[
\frac{2s + 2t - 3}{2} \|u_n\|^2 \leq \int_{\mathbb{R}^3} ((s + t)f(u_n)u_n - 3F(u_n))dx + \frac{2^*_s}{2s} \int_{\mathbb{R}^3} |u_n|^2dx
\]

\[
\leq C \int_{\mathbb{R}^3} u_n^2dx + C \int_{\mathbb{R}^3} |u_n|^2dx \leq C.
\]

Thus, \([u_n]\) is bounded in \(H^s(\mathbb{R}^3)\).

**Step 2:** There exist a sequence \([y_n] \subset \mathbb{R}^3\) and constants \(R, \beta > 0\) such that

\[
\liminf_{n \to \infty} \int_{B_\beta(y_n)} u_n^2dx \geq \beta > 0.
\]

(3.17)

Suppose, by contradiction, that for all \(R > 0\),

\[
\limsup_{n \to \infty} \int_{B_R(y)} u_n^2dx = 0,
\]

we can conclude that

\[ u_n \to 0 \quad \text{in} \quad L^q(\mathbb{R}^3), \quad 2 < q < 2^*_s. \]

(3.18)

It follows from equation (3.18) that

\[
\int_{\mathbb{R}^3} \phi_{u_n}^\pm u_n^2dx \to 0.
\]

(3.19)

Since \(J_d(u_n) = 0\), by equations (3.18) and (3.19), we have

\[
\frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} (\Delta u_n)^2dx \leq \frac{2s + 2t - 3}{2} \int_{\mathbb{R}^3} au_n^2dx + \frac{(s + t)2^*_s - 3}{2s} \int_{\mathbb{R}^3} |u_n|^2dx = o_n(1).
\]

(3.20)

From equation (2.1), we have
\[
\frac{4s + 2t - 3}{2} S_n \int_{\mathbb{R}^3} |u_n|^2 \, dx \leq \frac{(s + t)2^* - 3}{2^*} \int_{\mathbb{R}^3} |u_n|^2 \, dx + o_n(1). \tag{3.21}
\]

Without loss of generality, we may assume that
\[
\int_{\mathbb{R}^3} |u_n|^2 \, dx \to l \geq 0.
\]

We can conclude that \( l > 0 \); otherwise, \( \|u_n\| \to 0 \) as \( n \to \infty \), which contradicts \( m_n > 0 \). Letting \( n \to \infty \) in equation (3.21), we obtain that \( l \geq \frac{1}{S^{\frac{2}{\beta}}} \). Moreover, since \( I_d(u_n) \to m_n \), it follows from equations (3.18)–(3.20) that
\[
m_n = \frac{1}{2} \int_{\mathbb{R}^3} \left( -\Delta \right) \frac{1}{2} u_n \, dx + \frac{1}{2} \int_{\mathbb{R}^3} a u_n^2 \, dx - \frac{1}{2^*} \int_{\mathbb{R}^3} |u_n|^2 \, dx + o_n(1)
\]
\[
\geq \frac{1}{2(4s + 2t - 3)} \left( \int_{\mathbb{R}^3} \left( -\Delta \right) \frac{1}{2} u_n \, dx + (2s + 2t - 3) \int_{\mathbb{R}^3} a u_n^2 \, dx \right)
\]
\[
- \frac{1}{2^*} \int_{\mathbb{R}^3} |u_n|^2 \, dx + o_n(1)
\]
\[
= \frac{(s + t)2^* - 3}{2^*(4s + 2t - 3)} l - \frac{1}{2^*} l
\]
\[
= \frac{s}{3} l \geq \frac{s}{3} S^{\frac{2}{\beta}}.
\]

**Step 3:** \( m_n \) is achieved. Letting \( \tilde{u}_n(x) = u_n(x + y_n) \), then \( \tilde{u}_n \in M_{m_n} \) and \( \{ \tilde{u}_n \} \) is still a bounded minimizing sequence for \( m_n \). Up to a subsequence, we can assume that there is a \( \tilde{u} \in H^s(\mathbb{R}^3) \) such that
\[
\begin{align*}
\tilde{u}_n & \rightharpoonup \tilde{u} & \text{in } H^s(\mathbb{R}^3), \\
\tilde{u}_n & \to \tilde{u} & \text{in } L_r^r(\mathbb{R}^3), 1 \leq r < 2^*, \\
\tilde{u}_n & \to \tilde{u} & \text{a.e. in } \mathbb{R}^3.
\end{align*} \tag{3.22}
\]

From equation (3.17), we know that there exist \( R, \beta > 0 \) such that
\[
\int_{B_R(0)} \tilde{u}_n^2 \, dx \geq \beta > 0,
\]
which means that \( \tilde{u} \neq 0 \). Set \( \tilde{v}_n = \tilde{u}_n - \tilde{u} \). By using Lemma 3.5 and equations (3.19) and (3.21), we obtain that
\[
\Phi(\tilde{u}_n) = \Phi(\tilde{u}) + \Phi(\tilde{v}_n) + o_n(1), \tag{3.23}
\]
and
\[
J_d(\tilde{u}_n) = J_d(\tilde{u}) + J_d(\tilde{v}_n) + o_n(1). \tag{3.24}
\]

So we have
\[
m - \Phi(\tilde{u}) = \Phi(\tilde{v}_n) + o_n(1) \quad \text{and} \quad J_d(\tilde{v}_n) + o_n(1) = - J_d(\tilde{u}). \tag{3.25}
\]

Without loss of generality, we can assume that \( \tilde{v}_n \neq 0 \). From Lemma 3.3, there exists \( \tau_n > 0 \) such that \( (\tilde{v}_n)_{\tau_n} \in M_{m_n} \) for any \( n \). Now we assert that \( J_d(\tilde{u}) \leq 0 \). If \( J_d(\tilde{u}) > 0 \). By (3.22) we know that \( J_d(\tilde{v}_n) + o_n(1) < 0 \). It follows from Lemma 3.2 and equation (3.23) that
\[
m_n - \Phi(\tilde{u}) = \Phi(\tilde{v}_n) + o_n(1)
\]
\[
= I_d(\tilde{v}_n) - \frac{1}{4s + 2t - 3} J_d(\tilde{v}_n) + o_n(1)
\]
\[
\geq I_d(\tilde{v}_n) - \frac{1}{4s + 2t - 3} J_d(\tilde{v}_n) + o_n(1)
\]
\[
\geq m_n + o_n(1),
\]
which is clearly a contradiction, since \( \Phi(\bar{u}) > 0 \). From Lemma 3.3, we know that there exists \( \tilde{\tau} > 0 \) such that \( \bar{u}_{\tilde{\tau}} \in M_a \). Combining Lemma 3.2 and Fatou’s lemma, we obtain that

\[
m_a = \liminf_{n \to \infty} \left[ I_d(\bar{u}_n) - \frac{1}{4s + 2t - 3} J_d(\bar{u}_n) \right] = \liminf_{n \to \infty} \left[ \frac{s}{4s + 2t - 3} \int_{\mathbb{R}^3} a\bar{u}_n^2 dx + \frac{s + t}{4s + 2t - 3} \int_{\mathbb{R}^3} \left( f(\bar{u}_n)\bar{u}_n - \frac{4s + 2t}{s + t} F(\bar{u}_n) \right) dx + \frac{s}{3} \int_{\mathbb{R}^3} |\bar{a}|^2 dx \right] \geq \frac{s}{4s + 2t - 3} J_d(\bar{u}) \geq m_a.
\]

Therefore, we conclude that \( I_d(\bar{u}) = m_a \) and \( J_d(\bar{u}) = 0 \).

**Lemma 3.8.** Assume that \( (F_1) - (F_3) \) hold and \( 4s + 2t > 3 \). If \( I_d(u) = m_a \) for \( u \in M_a \), then \( u \) is a critical point of \( I_a \).

**Proof.** By contradiction, suppose \( I_d'(u) \neq 0 \), then \( \rho, \delta > 0 \) such that

\[
||I_d'(v)||_{H^s(\mathbb{R}^3)} \geq \rho \quad \text{if} \quad ||u - v|| \leq 3\delta, \forall v \in H^s(\mathbb{R}^3).
\]

We first claim that

\[
\lim_{\tau \to 1} ||u_\tau - u|| = 0. \quad (3.26)
\]

By contradiction, suppose that there exist \( \varepsilon_0 > 0 \) and a sequence \( \{\tau_n\} \) such that

\[
||u_{\tau_n} - u||^2 \geq \varepsilon_0 \quad \text{as} \quad \tau_n \to 1. \quad (3.27)
\]

Then, there exist two functions \( U_1 \) and \( U_2 \in C_0(\mathbb{R}^3) \) such that

\[
\int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u_{\tau_n} - U_1 \right|^2 dx < \frac{\varepsilon_0}{20} \quad \text{and} \quad \int_{\mathbb{R}^3} |u - U_2|^2 dx < \frac{\varepsilon_0}{20}.
\]

So we have that

\[
\int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} (u_{\tau_n} - u) \right|^2 dx \leq 2 \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u_{\tau_n} - U_1 \right|^2 dx + 2 \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u - U_1 \right|^2 dx \leq 6\tau_n^{4s+2t-3} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u - U_1 \right|^2 dx + 6\tau_n^{4s+2t} \int_{\mathbb{R}^3} U_1(\tau_n x) - U_1(x))^2 dx + 6(\tau_n^{2s+t} - 1)^2 \int_{\mathbb{R}^3} U_1^2 dx + 2 \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u - U_1 \right|^2 dx \leq \frac{2\varepsilon_0}{5} + o_\theta(1),
\]

and
\[
\int_{\mathbb{R}^3} |u_{n_{\tau}} - u|^2 \ dx \leq 2 \int_{\mathbb{R}^3} |u_{\tau_{n_{\tau}}} - U_{\tau_{n_{\tau}}}|^2 \ dx + 2 \int_{\mathbb{R}^3} |u - U_{\tau_{n_{\tau}}}|^2 \ dx
\]
\[
\leq 6 \tau_{n_{\tau}}^{2 + 2 \tau - 3} \int_{\mathbb{R}^3} |u - U_{\tau_{n_{\tau}}}|^2 \ dx + 6 \tau_{n_{\tau}}^{2 + 2 \tau} \int_{\mathbb{R}^3} |U_{\tau_{n_{\tau}}} - U_{\tau}(x)|^2 \ dx
\]
\[
+ 6 (\tau_{n_{\tau}}^{2 + \tau} - 1)^2 \int_{\mathbb{R}^3} U_{\tau_{n_{\tau}}}^2 \ dx + 2 \int_{\mathbb{R}^3} |u - U_{\tau}|^2 \ dx
\]
\[
\leq \frac{2\varepsilon_0}{5} + o_n(1).
\]
Therefore,
\[
\|u_{n_{\tau}} - u\|^2 = \int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{s}{2}}(u_{n_{\tau}} - u)\right|^2 \ dx + \int_{\mathbb{R}^3} |u_{n_{\tau}} - u|^2 \ dx \leq \frac{4}{5} \varepsilon_0 + o_n(1),
\]
which contradicts equation (3.27). According to equation (3.26), there exists \(\delta_1 > 0\) such that
\[
\|u_\tau - u\| \leq \delta \quad \text{if} \quad |\tau - 1| < \delta_1.
\] (3.28)
It follows from Lemma 3.2 that
\[
I_\delta(u_\tau) \leq I_\delta(u) - \hat{C}(\tau) \int_{\mathbb{R}^3} a u^2 \ dx = m_a - \hat{C}(\tau) \int_{\mathbb{R}^3} a u^2 \ dx \quad \forall \tau \geq 0.
\] (3.29)
Let \(\varepsilon = \min \left\{ \frac{3}{2} \min \left( \frac{C(\frac{1}{2})}{2}, \frac{C(\frac{3}{2})}{2} \right), \frac{\delta_0 \varepsilon}{8} \right\} \) and \(S = \{ v \in (\mathbb{R}^3) \ | \ |v|_{H^s} < \delta \}. \) According to Lemma 2.3 in [46], there exists a deformation \(\eta \in C([0, 1] \times H^s(\mathbb{R}^3), H^s(\mathbb{R}^3))\) such that

1. \(\eta(1, v) = v\) if \(v \not\in I_a^3(\{m_a - 2\varepsilon, m_a + 2\varepsilon\}) \cap S_{\delta_{\varepsilon}};\)
2. \(\eta(1, I_a^{m_a+\varepsilon} \cap S) \subset I_a^{m_a-\varepsilon};\)
3. \(I_\delta(\eta(1, v)) \leq I_\delta(v)\) for all \(v \in H^s(\mathbb{R}^3);\)
4. \(\eta(1, v)\) is a homeomorphism of \(H^s(\mathbb{R}^3).\)

By Lemma 3.2, we know that \(I_\delta(u_\tau) \leq I_\delta(u) = m_a\) for \(\tau \geq 0\), Combining equation (3.29) and (2), we have
\[
I_\delta(\eta(1, u_\tau)) \leq m_a - \varepsilon \quad \text{if} \quad |\tau - 1| < \delta_1.
\] (3.30)
On the other side, it follows from equation (3.26) and (3) that
\[
I_\delta(\eta(1, u_\tau)) \leq I_\delta(u_\tau) \leq m_a - \hat{C}(\tau) \int_{\mathbb{R}^3} a u^2 \ dx \quad \text{if} \quad |\tau - 1| \geq \delta_1.
\] (3.31)
By definition of \(\hat{C}(\tau) > 0\), we obtain that \(\hat{C}(\tau) > 0\) if \(|\tau - 1| \geq \delta_1\). Thus,
\[
\max_{\tau \in \left[ \frac{1}{2}, \frac{3}{2} \right]} I_\delta(\eta(1, u_\tau)) < m_a.
\] (3.32)
Next, we show that \(\eta(1, u_\tau) \cap M_a \neq \emptyset\) for some \(\tau \in \left[ \frac{1}{2}, \frac{3}{2} \right]\), which contradicts equation (3.32). In fact, we define
\[
\Phi_1(\tau) = J_\delta(u_\tau) \quad \text{and} \quad \Phi_\delta(\tau) = J_\delta(\eta(1, u_\tau)) \quad \forall \tau \geq 0.
\]
By Lemma 3.3 and the Brouwer degree, we obtain
\[
\deg \Phi_1 \left( \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}, 0 \right) = 1.
\]
Using equation (3.28) and (1), we obtain that \(\eta(1, u_\tau) = u_\tau\) for \(\tau = \frac{1}{2}\) and \(\tau = \frac{3}{2}\). Using (4) and the Brouwer degree, we have
\[
\deg(\Phi_1, \left[ \frac{1}{2}, \frac{3}{2} \right], 0) = \deg(\Phi_2, \left[ \frac{1}{2}, \frac{3}{2} \right], 0) = 1.
\]

So there exists \( \tau_0 \in \left[ \frac{1}{2}, \frac{3}{2} \right] \) such that \( \Phi(\tau_0) = 0 \), which means that \( \eta(1, u_{\tau_0}) \in M_a \). Thus, \( \eta(1, u_{\tau_0}) \cap M_a \neq \emptyset \). From Lemmas 3.7 and 3.8, we can easily obtain that \( I_a \) has a critical point \( u \in M_a \) such that \( I_a(u) = m_a > 0 \). \( \square \)

In summary, we complete the proof of Theorem 1.3.

### 4 Nonconstant potential case

In this section, we used method due to Jeanjean and Toland [24] to prove the existence of ground state solutions for equation (2.2). For this purpose, for \( \lambda \in [\frac{1}{2}, 1] \), we introduce the following two families of functionals on \( H^1(\mathbb{R}^3) \) defined by

\[
I_{\varepsilon, \lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ (-\Delta)^\varepsilon u^2 + V(x)u^2 \right] dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_0(x)u^2 dx - \lambda \int_{\mathbb{R}^3} F(u) + \frac{1}{2s} \|u\|^{2s}_s \] \tag{4.1}

and

\[
J_{\varepsilon, \lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ (-\Delta)^\varepsilon u^2 + V(x)u^2 \right] dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_0(x)u^2 dx - \lambda \int_{\mathbb{R}^3} F(u) + \frac{1}{2s} \|u\|^{2s}_s \] \tag{4.2}

Then, the following Pohozaev-type identity holds:

\[
\mathcal{P}_{\varepsilon, \lambda}(u) = \frac{3 - 2s}{2} \|u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} \left[ 3V(x) + \nabla V(x) \cdot (ex) \right] u^2 dx + \frac{3 + 2t}{4} \int_{\mathbb{R}^3} \phi_0(x)u^2 dx - 3\lambda \int_{\mathbb{R}^3} F(u) + \frac{1}{2s} \|u\|^{2s}_s \] \tag{4.3}

and

\[
J_{\varepsilon, \lambda}(u) = (s + t) \mathcal{J}_{\varepsilon, \lambda}(u) - \mathcal{P}_{\varepsilon, \lambda}(u)
\]

Similarly to equation (4.3), for all \( \lambda \in [\frac{1}{2}, 1] \) and \( u \in H^1(\mathbb{R}^3) \), we set

\[
\mathcal{M}_\lambda^\omega = \{ u \in H^1(\mathbb{R}^3) : \mathcal{J}_{\varepsilon, \lambda}(u) = 0 \}, \quad m_\lambda^\omega = \inf_{u \in \mathcal{M}_\lambda^\omega} \mathcal{J}_{\varepsilon, \lambda}(u). \tag{4.4}
\]
We note $I^*_1 = I_{V_1}$, $F^*_1 = f_{V_1}$, and $M^*_1 = M_{V_1}$. According to Theorem 1.3, if $(F_1)$–$(F_3)$ hold, problem (1.6) has a ground state solution $u^*$ on $M_1^*$, i.e.,

$$u^* \in M_1^*,$$

and $(I^*_1)'(u^*) = 0$ and $m_1^* = I^*_1(u^*)$.

For $\lambda \in [1 + 1, 1]$, we define the following functional in $H^s(\mathbb{R}^3)$:

$$I^*_\lambda(u) = \frac{1}{2} \left( \langle \nabla u \rangle^2 + V_{\text{max}}u^2 \right) + \frac{1}{4} \int_{\mathbb{R}^3} \phi_0^2 u^2 dx - \lambda \int_{\mathbb{R}^3} F(u) + \frac{1}{2\lambda^2} u^2 dx. \quad (4.5)$$

We can easily check that there exists $T > 1$ such that

$$I^*_\lambda(\tau^{s+1}(u^*)) < 0 \quad \text{for all } \tau \geq T.$$

**Lemma 4.1.** Under the assumptions of $(V_1)$–$(V_3)$, the following statement holds:

$$I_\tau(u) \geq I_\tau(u_\tau) + \frac{1}{4s + 2t - 3} J_\tau(u) + \frac{2\tau^{s+1}||u||^2_{L^2_\tau} - 2\tau^{s+1}2^{s+1} + 2s - 2}{2s + 2} ||u||^2_{L^2_\tau} \quad \text{for all } \tau > 0. \quad (4.6)$$

**Proof.**

$$I_\tau(u_\tau) = \frac{\tau^{4s+2t-3}}{2} \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u \, dx + \frac{\tau^{2s+2t-3}}{2} \int_{\mathbb{R}^3} V(\tau^\epsilon x) u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_0^2(\lambda) u dx - \frac{1}{2} \int_{\mathbb{R}^3} F(u) + \frac{1}{2s} ||u||^2_{L^2_\tau}.$$

By Lemma 2.4 and equation (2.9), we have

$$I_\tau(u_\tau) = \frac{1}{2s + 2t - 3} J_\tau(u_\tau) + \frac{1}{2} \int_{\mathbb{R}^3} \beta(\tau, \lambda) u^2 dx + \frac{2\tau^{s+1}||u||^2_{L^2_\tau} - 2\tau^{s+1}2^{s+1} + 2s - 2}{2s + 2} ||u||^2_{L^2_\tau}$$

$$\geq \frac{1}{4s + 2t - 3} J_\tau(u_\tau) + \frac{2\tau^{s+1}2^{s+1} - 2\tau^{s+1}2^{s+1} + 2s - 2}{2s + 2} ||u||^2_{L^2_\tau}. \quad \square$$

**Lemma 4.2.** Assume that $(V_1)$, $(V_2)$, $(F_1)$, and $(F_2)$ hold. Then, for $u \in M_\epsilon$

$$I_\tau(u) = \max_{\tau > 0} I_\tau(u_\tau).$$

**Lemma 4.3.** Assume that $(V_1)$, $(V_2)$, $(F_1)$, and $(F_2)$ hold. Then, for any $u \in H^s(\mathbb{R}^3)\setminus\{0\}$, there exists a unique $\tau_u > 0$ such that $u_\tau \in M_\epsilon$.

**Proof.** Set $u \in H^s(\mathbb{R}^3)\setminus\{0\}$ be fixed and define a function $\zeta(\tau) = I_\tau(u_\tau)$ on $[0, \infty)$. Thus, we have
Next, we will show the uniqueness of the \( \tau \). For any given \( \tau \in H^1(\mathbb{R}^3) \), suppose that there exist \( \tau_1, \tau_2 > 0 \) such that \( \zeta'(\tau_1) = \zeta'(\tau_2) = 0 \). Then, \( I_\delta(u_{\tau_1}) = I_\delta(u_{\tau_2}) = 0 \). Similar to the proof of Lemma 3.2, we have \( \tau_1 = \tau_2 \). \( \square \)

**Lemma 4.4.** Assume that \((V_1), (V_2), (F_1), \) and \((F_2)\) hold. Then, there exist constants \( \rho > 0 \) and \( \rho \) independent of \( \varepsilon \) such that \( \inf_{u \in M_\varepsilon} \|u\|_{L^2(\mathbb{R}^3)}^2 \geq \delta \) and \( m_\varepsilon = \inf_{u \in M_\varepsilon} I_\varepsilon(u) \geq \rho \).

**Proof.** By \((V_2)\), we obtain

\[
(\nabla V(x), x) \leq (2s + 2t - 3)V(x) \quad \forall x \in \mathbb{R}^3.
\]

Thus,

\[
\begin{align*}
\frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx & \leq \frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \, dx + \frac{1}{2} \int_{\mathbb{R}^3} (2s + 2t - 3)V(x) - (\nabla V(x), x)u^2 \, dx + \frac{4s + 2t - 3}{4} \int_{\mathbb{R}^3} \phi_0^2 u^2 \, dx \\
& = \int_{\mathbb{R}^3} [(s + t)f(u) - 3F(u)] \, dx + \frac{4s + 2t - 3}{2} \|u\|_{L^2(\mathbb{R}^3)}^2 \\
& \leq \frac{4s + 2t - 3}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx + S \frac{2^*}{2} \left(C_1 + \frac{s + 2t - 3}{2} \right) \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx,
\end{align*}
\]

where \( C_1 \) is a positive constant. This implies

\[
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \geq \rho_0 = \left\{ \begin{array}{ll}
\frac{4s + 2t - 3}{S^{\frac{2^*}{2}} \left(4C_1 + 8s + 4t - 6 \right)} & \forall u \in M_\varepsilon.
\end{array} \right.
\]

Next, we claim that \( m_\varepsilon > 0 \).

\[
I_\varepsilon(u) \geq I_\varepsilon(u_{\tau_1})
\]

\[
\geq \frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx - \frac{1}{\tau^3} \int_{\mathbb{R}^3} F(\tau^{s+1}u) \, dx - \frac{\tau^{(s+1)2^*-3}}{2^*} \|u\|_{L^2(\mathbb{R}^3)}^{2^*}
\]

\[
\geq \frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx - \left(C_1 + \frac{1}{2^*} \right) \tau^{(s+1)2^*-3} \int_{\mathbb{R}^3} |u|^{2^*} \, dx
\]

\[
\geq \frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx - \left(C_1 + \frac{1}{2^*} \right) S \frac{2^*}{2} \tau^{(s+1)2^*-3} \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \right)^{\frac{2^*}{2}}
\]

\[
= \frac{2s}{3 - 2s} \mathcal{C}(\frac{1}{2^*}) > 0,
\]

where \( \mathcal{C} \) is a positive constant.
if we take

\[ \tau = \left[ \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 dx \right]^{\frac{1}{1+s}} \] 

\[ \text{and} \quad \tilde{C} = \left( C_1 + \frac{1}{2^s} \right) S_1^{\frac{2}{1-s}}. \]

Therefore, the proof is completed. 

**Lemma 4.5.** Assume that \((V_1), (V_2), (F_1), \) and \((F_2)\) hold. Then,

(i) when \( T \) large enough, \( I_{\varepsilon, \lambda}(u^\varepsilon_T) < 0 \) for all \( \lambda \in [\frac{1}{2}, 1] \) and all \( \varepsilon \geq 0; \)

(ii) there is a positive constant \( \xi \), unrelated to \( \lambda \) and \( \varepsilon \), such that for all \( \lambda \in [\frac{1}{2}, 1] \) and all \( \varepsilon \geq 0, \)

\[ c_{\varepsilon, \lambda} = \inf_{y \in \text{Tr} \in [0,1]} \max I_{\varepsilon, \lambda}(y(\tau)) \geq \xi > \max \{ I_{\varepsilon, \lambda}(0), I_{\varepsilon, \lambda}(u^\varepsilon_T) \}, \]

where

\[ \Gamma = \{ y \in C([0,1], H^s(\mathbb{R}^3)) : y(0) = 0, y(1) = (u^\varepsilon)_T \}; \]

(iii) \( m^\varepsilon_\lambda \) is nonincreasing on \( \lambda \in [\frac{1}{2}, 1] \);

(iv) \( \limsup_{\lambda \to \lambda_0} c_{\varepsilon, \lambda} \leq c_{\varepsilon, \lambda_0} \) for all \( \lambda_0 \in (\frac{1}{2}, 1] \) and every \( \varepsilon \geq 0. \)

Since \( V \in C(\mathbb{R}^3, \mathbb{R}), \ V(0) < V_\infty \) and \( u^\varepsilon \in H^s(\mathbb{R}^3)[0], \) then there exist \( r_\varepsilon > 0 \) and \( R_\varepsilon > 0 \) such that

\[ V_\infty - V(x) > \frac{1}{4} (V_\infty - V(0)) \quad \text{for all } x \in \mathbb{R}^3, \ \text{with } |x| \leq r_\varepsilon, \]

and

\[ [2C_2 V_{\max} - (2C_2 - C_1) V_\infty - C_1 V(0)] \int_{|x| > R_\varepsilon} |u^\varepsilon|^2 dx \leq \frac{C_1}{2} (V_\infty - V(0)) \| u^\varepsilon \|_2^2. \]

**Lemma 4.6.** Assume \((V_1)-(V_3), (F_1), \) and \((F_2)\) hold. Then, there exists \( \lambda_\varepsilon \in [\frac{1}{2}, 1] \) such that \( c_{\varepsilon, \lambda} < m^\varepsilon_\lambda \) for \( \lambda \in (\lambda_\varepsilon, 1] \) and \( \varepsilon \in [0, \varepsilon_*] \), where \( \varepsilon_* = \frac{\lambda_\varepsilon}{2 \lambda_\varepsilon}. \)

**Proof.** For any \( \varepsilon \geq 0, \ I_{\varepsilon, \lambda}(u^\varepsilon_{\tau}) \) is continuous for \( \tau \in (0, \infty) \). Thus, for any \( \lambda \in [\frac{1}{2}, 1], \) choosing \( \tau_{\varepsilon, \lambda} \in (0, T) \) such that

\[ I_{\varepsilon, \lambda}(u^\varepsilon_{\tau_{\varepsilon, \lambda}}) = \max_{\tau \in (0, T)} I_{\varepsilon, \lambda}(u^\varepsilon_{\tau}). \]

We let

\[ y_0(\tau) = \begin{cases} (u^\varepsilon)_{(\tau T)}, & \text{for } \tau > 0, \\ 0, & \text{for } \tau = 0. \end{cases} \]

Then, \( y_0 \in \Gamma, \) where \( \Gamma \) is defined in equation (4.8). On the other hand,

\[ I_{\varepsilon, \lambda}(u^\varepsilon_{\tau_{\varepsilon, \lambda}}) = \max_{\tau \in [0,1]} I_{\varepsilon, \lambda}(y(\tau)) \geq c_{\varepsilon, \lambda}. \]

By \( F_2, \) the function \( \tau \mapsto \frac{F(u^\varepsilon_{\tau})}{(u^\varepsilon_{\tau})^{\frac{n-2}{2}}} \) is increasing in \((-\infty, 0)\) and \((0, \infty)\). Since \( \tau_{\varepsilon, \lambda} \in (0, T), \) then

\[ \frac{F(u^\varepsilon_{\tau_{\varepsilon, \lambda}})}{\left( u^\varepsilon_{\tau_{\varepsilon, \lambda}} \right)^{\frac{n-2}{2}}} \leq \frac{F(u^\varepsilon_{\tau})}{u^\varepsilon_{\tau}^{\frac{n-2}{2}}}. \]

Letting
We set \( w_n = u_n - u \). By equation (4.16), we obtain
\[
\lim_{n \to \infty} I_{\lambda, \beta}(w_n) = \lim_{n \to \infty} I_{\lambda, \beta}(u_n) - I_{\lambda, \beta}(u) \leq c_{\lambda, \beta}.
\] (4.17)
and
\[
\lim(I_{\lambda, \beta})'(w_n) = \lim(I_{\lambda, \beta})'(u_n) - (I_{\lambda, \beta})'(u) = 0.
\] (4.18)
We assert that as \( n \to \infty \)
\[
w_n = u_n - u \to 0.
\] (4.19)
On the contrary, if that $w_n \neq 0$ up to a subsequence. We claim that

$$\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} |w_n|^2 \, dx > 0 \quad (4.20)$$

If equation (4.20) is not true, we have that $w_n \to 0$ in $L^r(\mathbb{R}^3)$ for all $r \in (2, 2^*)$. So we have

$$0 = \lim_{n \to \infty} \langle (I_{x, a})'(w_n), w_n \rangle = \lim_{n \to \infty} \int_{\mathbb{R}^3} \left[ \|(-\Delta)^{\frac{s}{2}} w_n\|^2 + V(\epsilon x) w_n^2 \right] \, dx - \lambda \| w_n \|^2_{L^2}.$$

As $[w_n]$ is bounded in $H^s(\mathbb{R}^3)$, then up to a subsequence, as $n \to \infty$, we let

$$\int_{\mathbb{R}^3} \left[ \|(-\Delta)^{\frac{s}{2}} w_n\|^2 + V(\epsilon x) w_n^2 \right] \, dx \to l, \quad \lambda \| w_n \|^2_{L^2} \to l, \quad (4.21)$$

for some constant $l \geq 0$. By using Sobolev inequality, we obtain

$$\int_{\mathbb{R}^3} \left[ \|(-\Delta)^{\frac{s}{2}} w_n\|^2 + V(\epsilon x) w_n^2 \right] \, dx \geq \int_{\mathbb{R}^3} \left[ \|(-\Delta)^{\frac{s}{2}} w_n\|^2 \right] \, dx \geq S \| w_n \|^2_{L^2}.$$ Combining with equation (4.21), we obtain

$$S^{\frac{2}{s}} \leq \lambda^{\frac{2}{2s}} l \quad \text{for all } \lambda \in (\lambda_*, 1). \quad (4.22)$$

Moreover, by equation (4.17), we have

$$c_{\epsilon, \lambda} \geq \lim_{n \to \infty} I_{x, a}(w_n) = \lim_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^3} \left[ \|(-\Delta)^{\frac{s}{2}} w_n\|^2 + V(\epsilon x) w_n^2 \right] \, dx - \lambda \| w_n \|^2_{L^2}. \quad (4.23)$$

Similar to Lemma 3.6, we can show that $m_n^* < \frac{S^{\frac{2}{s}}}{\lambda^{\frac{2}{2s}}} S^{\frac{2}{s}}$. Combining equations (4.21)–(4.23) with Lemma 4.3, we have

$$S^{\frac{2}{s}} \leq \frac{S^{\frac{2}{s}}}{\lambda^{\frac{2}{2s}}} S^{\frac{2}{s}} \quad \text{for all } \lambda \in (\lambda_*, 1).$$

This is clearly a contradiction so equation (4.19) holds. So there exist $\delta > 0$ and $[y_n] \subset \mathbb{R}^3$ such that $\int_{B(\delta)} |w_n|^2 \, dx > \delta$. On the other hand, since $w_n \to 0$ in $H^s(\mathbb{R}^3)$, for every $\epsilon > 0$, as $n \to \infty$, we have

$$\int_{\mathbb{R}^3} |V(\epsilon x) - V_0| w_n^2 = o(1), \quad \int_{\mathbb{R}^3} \nabla V(\epsilon x)(\epsilon x) w_n^2 = o(1).$$

Combining with equations (4.17) and (4.18), we have

$$\lim_{n \to \infty} I_{x, a}(w_n) \leq c_{\epsilon, \lambda}, \quad \lim_{n \to \infty} I_{x, a}'(w_n) = 0. \quad (4.24)$$

We set $\bar{w}_n(x) = w_n(x + y_n)$. By equation (4.23), we obtain

$$\lim_{n \to \infty} I_{x, a}(\bar{w}_n) \leq c_{\epsilon, \lambda}, \quad \lim_{n \to \infty} I_{x, a}'(\bar{w}_n) = 0. \quad (4.25)$$

Moreover, for all $n \in \mathbb{N}$, $\int_{B(\delta)} |w_n|^2 \, dx > \delta$. So there exists a function $\bar{w} \in H^s(\mathbb{R}^3)[0]$ such that, passing eventually to a subsequence, $\bar{w}_n \rightharpoonup \bar{w}$ in $H^s(\mathbb{R}^3)$. Moreover, $(I_{x, a})'(\bar{w}) = 0$, and so $J_{x, a}(\bar{w}) = 0$ and $I_{x, a}(\bar{w}) \geq m_n^*$. By Lemma 4.5 and Fatou’s lemma, it is easy for us to obtain

$$m_n^* \geq c_{\epsilon, \lambda} \geq \lim_{n \to \infty} \left[ I_{x, a}(\bar{w}_n) - \frac{1}{4s + 2t - 3} J_{x, a}(\bar{w}_n) \right] \geq I_{x, a}(\bar{w}) - \frac{1}{4s + 2t - 3} J_{x, a}(\bar{w}) \geq m_n^*.$$

So equation (4.19) holds, this completes the proof.$\square$

For $\epsilon \in (0, \epsilon_*)$, for convenience, we set
Lemma 4.8. Assume \((V_1), (V_2), \) and \((F_1)-(F_3)\) hold. Then, \(\mathcal{N}_\varepsilon \neq \emptyset\). Moreover, for all \(u \in \mathcal{N}_\varepsilon\), there exists a constant \(\rho > 0\), independent of \(\varepsilon\) such that
\[
|u| \geq \rho.
\]

**Proof.** From Lemma 4.3, for any fixed \(\varepsilon \in (0, \varepsilon_*]\), we know that there exist \(\{\lambda_n\} \subseteq (\lambda_*, 1] \) and \(\{u_{\varepsilon, \lambda_n}\} \subset H^s(\mathbb{R}^3)[0]\), we still denote the latter by \(u_n\) such that
\[
\lambda_n \to 1, \quad c_{\varepsilon, \lambda_n} \to c^*_\varepsilon \in (0, c_{\varepsilon, 1}] \quad \text{as} \quad n \to \infty, \quad (I_{\varepsilon, \lambda_n})'(u_n) = 0, \quad I_{\varepsilon, \lambda_n}(u_n) = c_{\varepsilon, \lambda_n} \quad \text{for all} \quad n \in \mathbb{N}. \tag{4.27}
\]
Combining equations (2.11) and (4.27) and Lemma 4.5 (iv) that as \(n \to \infty\), we have
\[
c^*_\varepsilon + o(1) = c_{\varepsilon, \lambda_n} = I_{\varepsilon, \lambda_n}(u_n) = -\frac{1}{4s + 2t - 3} \int_{\mathbb{R}^3} [2sV(\varepsilon x) + \nabla V(\varepsilon x)(\varepsilon x)]|u|^2 \, dx
\]
\[
+ \frac{\lambda}{4s + 2t - 3} \int_{\mathbb{R}^3} \left[ f(u)u - (4s + 2t)F(u) \right] |u|^2 \, dx + \frac{s^3}{3} \|u\|^{2^*_\varepsilon}_s \geq 0
\]
\[
\geq \frac{\|u_n\|^{2^*_\varepsilon}_s}{6}.
\]
By using \((I_{\varepsilon, \lambda_n})'(u_n), u_n) = 0\) for all \(n \in \mathbb{N}\) and, equation (4.28), we have
\[
\int_{\mathbb{R}^3} [(\nabla \cdot u_n)^2 + V_0 |u_n|^2] \, dx \leq \int_{\mathbb{R}^3} \left[ (\nabla \cdot u_n)^2 + V(\varepsilon x) |u_n|^2 \right] \, dx + \int_{\mathbb{R}^3} \phi_n^\varepsilon |u_n|^2 \, dx
\]
\[
= \lambda_n \int_{\mathbb{R}^3} f(u_n)u_n \, dx + \lambda_n \|u_n\|^{2^*_\varepsilon}_s \tag{4.29}
\]
\[
\leq \frac{1}{2} V_0 \|u_n\|^{2}_2 + C_1 \|u_n\|^{2^*_\varepsilon}_s \leq \frac{1}{2} V_0 \|u_n\|^{2}_2 + \frac{6}{s} C_1 c^*_\varepsilon + o(1),
\]
so \(\{u_n\}\) is bounded in \(H^s(\mathbb{R}^3)\), we have that for any fixed \(\varepsilon \in (0, \varepsilon_*]\), there exists \(u_\varepsilon \in H^s(\mathbb{R}^3)[0]\) such that
\[
I'_\varepsilon(u_\varepsilon) = 0, \quad I_\varepsilon(u_\varepsilon) \leq c_{\varepsilon, 1}. \tag{4.30}
\]
Hence, we prove \(\mathcal{N}_\varepsilon \neq \emptyset\). So for all \(u \in \mathcal{N}_\varepsilon\), we have
\[
\int_{\mathbb{R}^3} [(\nabla \cdot u)^2 + V(0) |u|^2] \, dx \leq \int_{\mathbb{R}^3} \left[ (\nabla \cdot u)^2 + V(\varepsilon x) |u|^2 \right] \, dx + \int_{\mathbb{R}^3} \phi_n^\varepsilon u^2 \, dx
\]
\[
= \int_{\mathbb{R}^3} f(u)u \, dx + \|u\|^{2^*_\varepsilon}_s
\]
\[
\leq \frac{1}{2} V_0 \|u\|^{2}_2 + C_1 \|u\|^{2^*_\varepsilon}_s
\]
\[
\leq \frac{1}{2} V_0 \|u\|^{2}_2 + C_1 S_s^{3\frac{1}{3\cdot 2s}} \left( \int_{\mathbb{R}^3} [(\nabla \cdot u)^2] \, dx \right)^{\frac{3}{2}},
\]
Therefore, equation (4.26) holds. \(\square\)

**Proof of Theorem 1.1.** We know that \(m^*_\varepsilon \geq 0\) from equation (4.15). For every \(\varepsilon \in (0, \varepsilon_*]\), let \(\{u_n\} \subset \mathcal{K}_\varepsilon\) be such that
\[ I'_c(u_n) = 0 \quad \text{for all } n \in \mathbb{N}, \quad I'(u_n) \to m^*_c \quad \text{as } n \to \infty. \]

By equation (4.28), we have that \( \{u_n\} \) is bounded in \( H^s(\mathbb{R}^3) \). Hence, there exists a function \( u_e \in H^s(\mathbb{R}^3) \) such that, up to a subsequence, \( u_n \rightharpoonup u_e \) in \( H^s(\mathbb{R}^3) \). We can easily obtain \( m^*_c \leq c_{s,1} < m^*_c \) by Lemma 4.6 and equation (4.30). Arguing as in the proof of equation (4.19), we prove that \( u_n \to u_e \) in \( H^s(\mathbb{R}^3) \). Due to \( \epsilon_n \), from equation (3.26), we have that
\[
\rho \lim_{\epsilon \to 0} \|u\| = \max_{\epsilon \in [0, \epsilon_0]} \|u\|, \quad \text{and hence, for every } \epsilon \in (0, \epsilon_0], \text{ we prove that}
\]
\[ (u_n, \epsilon_n) \to (u_e, \epsilon_0) \quad \text{is a ground state solution of equation (2.2), and } \hat{\psi}_e(x) = u_e(2x) \text{ is a ground state solution for equation (1.1) with positive energy.} \]

5 Existence of ground state solutions of Nehari-Pohozaev-type

In this section, we will prove the existence of ground state solutions of the Nehari-Pohozaev-type for equation (2.2).

We know \( I_0 = I_{u_0}, J_0 = f_{u_0}, \text{ and } M_0 = \bar{M}_{u_0}. \) Let
\[ \bar{\psi} = \frac{1}{2}(V_0 + V) = \frac{1}{2}(V_0 + V(0)). \]

By Theorem 1.3, there exist \( \bar{u}_0 \in M_0 \) and \( \bar{u} \in \bar{M}_{\bar{\psi}} \) such that
\[ I'_0(\bar{u}_0) = 0, \quad I'_0(\bar{u}) = m_0 = \inf_{u \in M_0} I'_0(u) = \inf_{u \in H^s(\mathbb{R}^3)} \max_{\epsilon > 0} I'_0((u)_\epsilon) > 0, \]
and
\[ I'_0(\bar{u}) = 0, \quad I'_0(\bar{\psi}) = m_\psi = \inf_{u \in \mathcal{M}_\bar{\psi}} I'_0(u) = \inf_{u \in H^s(\mathbb{R}^3)} \max_{\epsilon > 0} I'_0((u)_\epsilon) > 0. \]

According to Lemma 3.2, there exists \( \tau_0 > 0 \) such that
\[ (\bar{u})_{\tau_0} \in M_0, \quad I'_0((\bar{u})_{\tau_0}) \geq m_0. \]

We can easily prove that there exists \( T_0 > 1 \) such that
\[ I'_0((\bar{u})_{\tau_0}) < 0 \quad \text{for all } \tau \geq T_0. \]

By equation (5.4) and Lemma 3.2, for any \( \epsilon > 0 \), there exists \( \tau_\epsilon \in (0, T_0) \) such that
\[ (\bar{u}_0)_\epsilon \in M_{c_{s,1}}, \quad I'_0((\bar{u}_0)_\epsilon) \geq m_\epsilon. \]

Lemma 5.1. \( m_\psi \geq m_0 + \delta_0, \) where \( \delta_0 = \frac{(V_0 - V_0)\epsilon_0^{2s_{2r-3}-3}\|u_0\|^2}{4} > 0 \) is independent of \( \epsilon > 0. \)

Proof. In view of equations (5.2) and (5.3), we deduce that
\[ m_\psi = I'_0(\bar{u}) \geq I'_0((\bar{u})_{\tau_0}) = I'_0((\bar{u})_{\tau_0}) + \frac{\bar{\psi} - V_0}{2} \int_{\mathbb{R}^3} |\bar{u}|^2 dx \]
\[ \geq m_0 + \frac{V_0 - V_0}{4} \int_{\mathbb{R}^3} |\bar{u}|^2 dx = m_0 + \delta_0. \]

Now, choosing \( R_0 > 0 \) sufficiently large, for all \( x, \) with \( |x| > R_0, \)
\[ V(x) \geq \bar{\psi}, \quad \frac{4 + 4T_0^{2s_{2r-3}}}{4s + 2r - 3} V_{\max} \int_{|x| > R_0} |\bar{u}|^2 dx \leq \frac{3}{8} \delta_0. \]

It follows from Lemma 5.1, \( V_1 \) and \( V_2, \) there is a number \( \epsilon_0 > 0 \) so small that for all \( \epsilon \in [0, \epsilon_0], \)
Lemma 5.2. $m_0 \geq m_e - \frac{3\delta_0}{4}$ for every $\varepsilon \in [0, \varepsilon_0]$, where $\delta_0$ is given in Lemma 5.1.

Proof. By $(V_2)$, there exist constants $\rho_0 > 0$ such that
\[ \rho_0 \leq (\nabla V(x), x) + 2sV(x); \quad (\nabla V(x), x) \leq (2s + 2t - 3)V(x). \] (5.8)
So we obtain
\[
m_0 = I_\varepsilon(\bar{u}_0) = I_\varepsilon(\bar{u}_0) + \frac{1}{2} \int_{\mathbb{R}^3} [V(0) - V(\varepsilon x)] \bar{u}_0^2 \, dx
\geq I_\varepsilon((\bar{u}_0)_\varepsilon) + \frac{1}{8s + 4t - 3} J_\varepsilon(\bar{u}_0) + \frac{1}{2} \int_{\mathbb{R}^3} [V(0) - V(\varepsilon x)] \bar{u}_0^2 \, dx
= I_\varepsilon((\bar{u}_0)_\varepsilon) + \frac{1}{8s + 4t - 3} J_\varepsilon(\bar{u}_0) + \frac{1}{2} \int_{\mathbb{R}^3} [V(0) - V(\varepsilon x)] \bar{u}_0^2 \, dx
+ \frac{1 - \tau_0^{4s+2t-3}}{8s + 4t - 6} \int_{\mathbb{R}^3} (2s + 2t - 3)(V(\varepsilon x) - V(0)) - \nabla V(\varepsilon x) \cdot \varepsilon x) \bar{u}_0^2 \, dx
= I_\varepsilon((\bar{u}_0)_\varepsilon) + \frac{1}{8s + 4t - 6} \int_{\mathbb{R}^3} [3V(\varepsilon x) - 3V(0) - 2\nabla V(\varepsilon x)] \bar{u}_0^2 \, dx
\geq m_e - \frac{1}{8s + 4t - 3} \int_{\{|x| \leq R_0\}} (3 + 3\tau_0^{4s+2t-3}) V(\varepsilon x) - V(0) \bar{u}_0^2 \, dx
- \frac{1}{4s + 2t - 3} \int_{\{|x| \leq R_0\}} (1 + \tau_0^{4s+2t-3}) \nabla V(\varepsilon x) \cdot \varepsilon x \bar{u}_0^2 \, dx - \frac{4 + 4\tau_0^{4s+2t-3}}{4s + 2t - 3} V_{\max} \int_{\{|x| > R_0\}} \bar{u}_0^2 \, dx
\geq m_e - \frac{3\delta_0}{8}. \]

Lemma 5.3. $m_e$ is achieved for all $\varepsilon \in (0, \varepsilon_0]$.

Proof.
\[
\Psi_\varepsilon(u) = I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{4s + 2t - 3} J_\varepsilon(u) = \frac{1}{8s + 4t - 6} \int_{\mathbb{R}^3} [2sV(\varepsilon x) + \nabla V(\varepsilon x) \cdot \varepsilon x] u^2 \, dx
+ \frac{1}{4s + 2t - 3} \int_{\mathbb{R}^3} [(s + t)f(u)u - (4s + 2t)F(u)] \, dx + \frac{5}{3} \frac{|u|_{L_3^s}^{2s}}{\rho_0}
\geq \frac{\rho_0}{8s + 4t - 6} |u|_{L_3^s}^{2s} + \frac{5}{3} \frac{|u|_{L_3^s}^{2s}}{\rho_0}. \]
(5.9)
Since $J_\varepsilon(u_n) = 0$, then it follows from $F_3$ and equations (5.8) and (5.9) that as $n \to \infty$,
\[
m_e + o(1) = I_\varepsilon(u_n) \geq \frac{1}{8s + t - 6} \rho_0 |u_n|_{L_3^s}^{2s} + \frac{5}{3} \frac{|u_n|_{L_3^s}^{2s}}{\rho_0}. \]
Moreover, it follows from $(F_1)$, and equations (5.8) and (5.9), that as $n \to \infty$, we have
where $\delta$ is given by Lemma 4.5. This is clearly a contradiction. So equation (5.13) holds. According to (5.12), (5.13), as well as the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^3)$, we have that

$$
\frac{4}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx + \frac{2s + 2t - 3}{2} \int_{\mathbb{R}^3} V(\varepsilon x) |u_n|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \nabla V(\varepsilon x) \nabla |u|^2 \, dx + \frac{4}{4} \int_{\mathbb{R}^3} \phi^3(x) |u|^2 \, dx
$$

$$
= \int_{\mathbb{R}^3} [(s + t) f'(u) u - 3F(u)] \, dx + \frac{4s + 2t - 3}{2} \|u\|^2_{2^*}
\leq \|u_n\|^2_{2^*} + C_j \|u_n\|^2_{2^*} \leq Cm_2 + o(1).
$$

Therefore, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Then, there exists a subsequence $\hat{u} \in H^s(\mathbb{R}^3)$ such that $u_n \to \hat{u}$ in $H^s(\mathbb{R}^3)$, $u_n \to \hat{u}$ in $L^2_{\text{loc}}(\mathbb{R}^3)$ for $2 \leq r < 2^*$, and $u_n \to \hat{u}$ a.e. in $\mathbb{R}^3$.

Next, we will show that $\hat{u} \neq 0$. Contrary to that, we assume that $\hat{u} = 0$, then $u_n \to 0$ in $L^r_{\text{loc}}(\mathbb{R}^3)$ for $2 \leq r < 2^*$ and $u_n \to 0$ a.e. in $\mathbb{R}^3$. According to Lemma 3.2, there exists $\tau_n > 0$ such that $(u_n)_{\tau_n} \in \hat{M}_\varepsilon$ for every $n \in \mathbb{N}$. We claim that there are two positive numbers $\tau_0$, $\tau^*$, with $\tau_0 < \tau^*$, such that

$$
\tau_0 \leq \tau_n \leq \tau^*, \quad \text{for all } n \in \mathbb{N}.
$$

(5.11)

We suppose $\tau_n \to 0$ as $n \to \infty$, then we have

$$
0 < \tilde{m}_\varepsilon \leq \frac{\tau_n^{4s + 2r - 3}}{2} \int_{\mathbb{R}^3} \phi^3(x) |u_n|^2 \, dx - \frac{\tau_n^{4s + 2r - 3}}{2} \|u_n\|^2_{2^*} + \frac{\tau_n^{4s + 2r - 3}}{2} \int_{\mathbb{R}^3} \phi^3(x) |u_n|^2 \, dx - \frac{\tau_n^{4s + 2r - 3}}{2} \|u_n\|^2_{2^*} = o(1).
$$

This is clearly a contradiction because $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. So we have $\tau_n < \tau$. Moreover, by (F1) and (F3), we deduce that there exist $C_1$, $C_2 > 0$ such that

$$
(s + t) |f'(\tau)| |\tau| \geq (4s + 2t) F(\tau) \geq C_1 |\tau|^p - C_2 |\tau|^2
$$

for all $\tau \in \mathbb{R}$. (5.12)

Now let us prove that

$$
\liminf_{n \to m} \|u_n\|^2_{2^*} > 0.
$$

(5.13)

If equation (5.13) does not hold, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to 0$ in $L^q(\mathbb{R}^3)$, and so $u_{n_k} \to 0$ in $L^r(\mathbb{R}^3)$ for all $r \in (2, 2^*)$. By (F1), for every $\varepsilon > 0$ and some $q \in (2, 2^*)$, there exists $C_\varepsilon > 0$ such that

$$
|f(\tau)| + |F(\tau)| \leq \varepsilon (\tau^2 + \tau^2) + C_\varepsilon |\tau|^q
$$

for all $\tau \in \mathbb{R}$. Let $k \to \infty$, and we have

$$
0 = J(u_{n_k}) \geq \frac{4s + 2t - 3}{2} \|u_{n_k}\|^2_{2^*} + \frac{(2s + 2t - 3) a_1}{2} \|u_{n_k}\|^2_{2^*}
$$

$$
- \frac{(s + t) 2^* - 3}{4} \left( \|u_{n_k}\|^2_{2^*} + \|u_{n_k}\|^2_{2^*} \right) - C \|u_{n_k}\|^q - \frac{(s + t) 2^* - 3}{2^*} \|u_{n_k}\|^2_{2^*}
$$

$$
\geq \frac{4s + 2t - 3}{2} \|u_{n_k}\|^2_{2^*} + o(1) \geq \frac{4s + 2t - 3}{2} \delta + o(1),
$$

where $\delta$ is given by Lemma 4.5. This is clearly a contradiction. So equation (5.13) holds. According to (5.12), (5.13), as well as the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^3)$, we have that
so there exists $\tau^* > 0$ such that

$$I_{\tilde{\tau}}(u_n) < 0, \quad \text{for all } \tau > \tau^* \text{ and } n \in \mathbb{N}. \quad (5.14)$$

By equation (5.2), we have $I_{\tilde{\tau}}(u_n) \geq \tilde{m}_\Psi > 0$, so we have $\tau_n \leq \tau^*$ for all $n \in \mathbb{N}$. This shows that equation (5.11) holds. In view of equations (5.6) and (5.11) that as $n \to \infty$,

$$m_\varepsilon + o(1) = I_{\varepsilon}(u_n) \geq I_{\tilde{\tau}}(u_n) = I_{\tilde{\tau}}(u_n) + \frac{\tilde{m}_\Psi}{2} \int_{\mathbb{R}^d} |V(\tau_n^\tau\varepsilon x) - V|u_n^2 dx$$

$$\geq \tilde{m}_\Psi - \frac{\tilde{m}_\Psi}{2} \int_{|x| \leq \tau_n^\tau\varepsilon} u_n^2 dx = \tilde{m}_\Psi + o(1).$$

Combining with Lemmas 5.1 and 5.2, we have

$$m_\varepsilon \geq \tilde{m}_\Psi \geq m_0 + \delta_0 \geq m_\varepsilon + \frac{\delta_0}{4} > m_\varepsilon.$$

This is clearly a contradiction, thus, $\tilde{u} \neq 0$. Let $w_n = u_n - \tilde{u}$. By using Brézis-Lieb-type lemma, as $n \to \infty$, we deduce that

$$I_{\tilde{\tau}}(u_n) = I_{\tilde{\tau}}(\tilde{u}) + I_{\tilde{\tau}}(w_n) + o(1), \quad J_{\tilde{\tau}}(u_n) = J_{\tilde{\tau}}(\tilde{u}) + J_{\tilde{\tau}}(w_n) + o(1). \quad (5.15)$$

Moreover, in view of equations (5.9) and (5.15) that as $n \to \infty$,

$$\Psi_{\tilde{\tau}}(w_n) = m_\varepsilon - \Psi_{\tilde{\tau}}(\tilde{u}) + o(1), \quad J_{\tilde{\tau}}(w_n) = -J_{\tilde{\tau}}(\tilde{u}) + o(1). \quad (5.16)$$

If $\{w_n\}$ exists as a subsequence $\{w_n\}$ such that $w_n = 0$, then we have

$$I_{\tilde{\tau}}(\tilde{u}) = m_\varepsilon, \quad J_{\tilde{\tau}}(\tilde{u}) = 0. \quad (5.17)$$

So we assume that $w_n \neq 0$ for all $n \in \mathbb{N}$. Next we prove that $J_{\tilde{\tau}}(\tilde{u}) \leq 0$. By contradiction, we suppose $J_{\tilde{\tau}}(\tilde{u}) > 0$, from equation (5.16), we have $J_{\tilde{\tau}}(w_n) < 0$ for large $n$. By Lemma 3.6, there exists $\tau_{n} > 0$ such that $(w_n)_{\tau_{n}} \in \mathcal{M}_{\varepsilon}$. In view of equations (5.9) and (5.16), we obtain as $n \to \infty$

$$m_\varepsilon - \Psi_{\tilde{\tau}}(\tilde{u}) + o(1) \geq \Psi_{\tilde{\tau}}(w_n) = I_{\tilde{\tau}}(w_n) - \frac{1}{4s + 2t - 3} J_{\tilde{\tau}}(w_n) \geq I_{\tilde{\tau}}((w_n)_{\tau_{n}}) - \frac{1}{4s + 2t - 3} J_{\tilde{\tau}}((w_n)_{\tau_{n}}) \geq m_\varepsilon,$$

this is clearly a contradiction because $\Psi_{\tilde{\tau}}(\tilde{u}) > 0$. Hence, $J_{\tilde{\tau}}(\tilde{u}) \leq 0$. From Lemma 3.6, there exists $\tau > 0$ such that $(\tilde{u})_{\tau} \in \mathcal{M}_{\varepsilon}$. Combining $F_{\tau}$, equation (5.9), the weak semicontinuity of norm, and Fatou’s lemma yields

$$m_\varepsilon = \lim_{n \to \infty} \left[ I_{\tilde{\tau}}(u_n) - \frac{1}{4s + 2t - 3} J_{\tilde{\tau}}(u_n) \right] = \lim_{n \to \infty} \Psi_{\tilde{\tau}}(u_n)$$

$$\geq \Psi_{\tilde{\tau}}(\tilde{u}) = I_{\tilde{\tau}}(\tilde{u}) - \frac{1}{4s + 2t - 3} J_{\tilde{\tau}}(\tilde{u}) \geq I_{\tilde{\tau}}((\tilde{u})_{\tau}) - \frac{1}{4s + 2t - 3} J_{\tilde{\tau}}((\tilde{u})_{\tau}) \geq m_\varepsilon,$$
this is clearly a contradiction, so the proof is completed. □

**Lemma 5.4.** Assume that \((V_1)-(V_3)\) and \((F_1)-(F_3)\) hold. Then, for every \(\varepsilon \in (0, \varepsilon_0]\), equation (2.2) has a ground state solution \(\hat{u}_\varepsilon\) such that

\[
I_\Delta(\hat{u}_\varepsilon) = m_\varepsilon = \inf_{u \in H^\varepsilon(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} I_\varepsilon((u,t)) > 0. \tag{5.18}
\]

### 6 Concentration of ground state solutions for equation (2.2)

In this section, we are devoted to the concentration behavior of the ground state solutions for equation (2.2) as \(\varepsilon \to 0\). For this purpose, for every \(\varepsilon \in (0, \varepsilon_0]\) let \(\hat{u}_\varepsilon\) be a ground state solution of equation (2.2) obtained in the proof of Theorem 1.1, which satisfies equation (5.18). For convenience, we set

\[
\mathcal{N}_\varepsilon = \{ u \in H^\varepsilon(\mathbb{R}^3) \setminus \{0\} : I'_\varepsilon(u) = 0 \}, \quad \mathcal{L}_{m_\varepsilon} = \{ u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) = m_\varepsilon \}
\]

and set

\[
\Lambda = \{ u \in \mathcal{L}_{m_\varepsilon} : \varepsilon \in [0, \varepsilon_0] \}. \tag{6.1}
\]

**Lemma 6.1.** There exists a constant \(C_0 > 0\), independent of \(\varepsilon\), such that \(\|u\| \leq C_0\) for all \(u \in \Lambda\).

**Proof.** Fix \(\varepsilon \in [0, \varepsilon_0]\) and \(u_\varepsilon \in \mathcal{L}_{m_\varepsilon}\). Then, we easily obtain

\[
m_\varepsilon^m > m_\varepsilon = I_\varepsilon(u_\varepsilon) - \frac{1}{4s + 2t - 3} J_\varepsilon(u_\varepsilon)
\]

\[
= \frac{1}{8s + 4t - 6} \int_{\mathbb{R}^3} [2sV(\varepsilon x) + \nabla V(\varepsilon x) \cdot \varepsilon x ] |u|^2 \, dx
\]

\[
+ \frac{1}{4s + 2t - 3} \int_{\mathbb{R}^3} [(s + t)f(u) - (4s + 2t)F(u)] \, dx + s |u|^{3s}_\varepsilon
\]

\[
\geq \frac{s}{3} |u|^{3s}_\varepsilon.
\] \tag{6.2}

Since \(u \in \mathcal{L}_{m_\varepsilon}\), it follows from equation (6.2) that

\[
\|u_\varepsilon\|_1^2 + \|u_\varepsilon\|_2^2 \leq C_1 \|u_\varepsilon\|^{3s}_\varepsilon \leq \frac{3}{s} C_1 m_\varepsilon^m = C_0.
\]

\[
\limsup_{\varepsilon \to 0} m_\varepsilon = m_0.
\]

**Lemma 6.2.** Let \(\bar{\varepsilon} \in [0, \varepsilon_0]\), then \(\limsup_{\varepsilon \to \bar{\varepsilon}} m_\varepsilon \leq m_{\bar{\varepsilon}}\).

**Proof.** Fix \(\bar{\varepsilon} \in [0, \varepsilon_0]\). By contradiction, we suppose \(\limsup_{\varepsilon \to \bar{\varepsilon}} m_\varepsilon > m_{\bar{\varepsilon}}\). Let

\[
\varepsilon_0 = \limsup_{\varepsilon \to \bar{\varepsilon}} m_\varepsilon - m_{\bar{\varepsilon}} > 0.
\]

From Lemma 3.6, for any \(\varepsilon > 0\), there exists \(\bar{\tau}_\varepsilon > 0\) such that \((\hat{u}_\varepsilon)_{\bar{\tau}_\varepsilon} \in M_\varepsilon\), so we have

\[
I_\varepsilon((\hat{u}_\varepsilon)_{\bar{\tau}_\varepsilon}) \geq m_\varepsilon, \quad I_\varepsilon((\hat{u}_\varepsilon)_{\bar{\tau}_\varepsilon}) \geq I_\varepsilon((\hat{u}_\varepsilon)_{\bar{\tau}_\varepsilon}) \quad \text{for all} \quad \tau > 0. \tag{6.3}
\]

For any \(u \in H^\varepsilon(\mathbb{R}^3) \setminus \{0\}\), there exists a number \(\tau_u^* > 0\), independent of \(\varepsilon\) such that

\[
I_\varepsilon(u) \leq I_{\max}(u) < 0 \quad \text{for all} \quad \tau > \tau_u^*. \tag{6.4}
\]

By equations (6.3) and (6.4), we have \(0 < \bar{\tau}_\varepsilon \leq \tau^*\) for some number \(\tau^* = \tau_{\bar{\varepsilon}} > 0\). Moreover, for any bounded set \(\Omega \subset \mathbb{R}^3\), we have

\[
\lim \sup_{\varepsilon \to \bar{\varepsilon}} \left| \int \frac{|V(\varepsilon x) - V(\bar{\varepsilon} x)| + |\nabla V(\varepsilon x) - \nabla V(\bar{\varepsilon} x)| \, dx}{\varepsilon} \right| = 0. \tag{6.5}
\]
Choosing $R_1 > R_0$ such that

$$V_{\max} \left( \frac{6 + 6r^{4s+2t-3}}{8s + 4t - 6} \right) \int_{|x| \geq R_1} \hat{u}_x^2 \, dx \leq \frac{\varepsilon_0}{2}. \tag{6.6}$$

Combining equations (5.8), (6.3), (6.5), and (6.6), we have

\[
\frac{m_{\varepsilon} - 1 + \frac{\tau^{*4s+2t-3}}{4s + 2t - 3}}{6 + 6r^{4s+2t-3}} \int_{|x| \leq R_1} |V(\varepsilon x) - \varepsilon x| \hat{u}_x^2 \, dx - \frac{1}{2} \int_{|x| \leq R_1} |V(\varepsilon x) - \varepsilon x| \hat{u}_x^2 \, dx \leq \frac{m_{\varepsilon} - 1 + \frac{\tau^{*4s+2t-3}}{4s + 2t - 3}}{6 + 6r^{4s+2t-3}} \int_{|x| \leq R_1} \hat{u}_x^2 \, dx \leq \frac{m_{\varepsilon} - 1 + \frac{\tau^{*4s+2t-3}}{4s + 2t - 3}}{6 + 6r^{4s+2t-3}} \int_{|x| \leq R_1} \hat{u}_x^2 \, dx - \frac{1}{2} \int_{|x| \leq R_1} |V(\varepsilon x) - \varepsilon x| \hat{u}_x^2 \, dx
\]

So, we obtain

$$m_{\varepsilon} + \varepsilon_0 = \limsup_{\varepsilon \to \varepsilon_0} m_{\varepsilon} \leq m_{\varepsilon} + \frac{\varepsilon_0}{2}.$$ 

This is clearly a contradiction. So the proof is completed. \hfill \square

**Lemma 6.3.** If $u \in \Lambda$, then $u \in C(\mathbb{R}^3, \mathbb{R})$ and $\lim_{|x| = u(x)} = 0$. Moreover, there exists a constant $a_0$ for any $u \in \Lambda$. Choosing $R_1 > R_0$ such that

$$|u(x)| \leq a_0 \int_{B(x)} |u(y)| \, dy, \quad \text{for all } x \in \mathbb{R}^3. \tag{6.7}$$

**Proof.** For $r \geq 2$, it follows from Lemma 6.1 and the standard bootstrap argument (see [32]) that there exists $C_r > 0$ independent of $u \in \Lambda$ such that

$$u \in W^{1,r}(\mathbb{R}^3), \quad ||u||_{W^{1,r}(\mathbb{R}^3)} \leq C_r, \quad \text{for all } u \in \Lambda.$$ 

By using the Sobolev embedding theorem, we deduce that there exists $C_\infty > 0$, independent of $u \in \Lambda$, such that

$$|u|_\infty \leq C_\infty \text{ for all } u \in \Lambda. \tag{6.8}$$

In view of (F2), we know that there exists a constant $a_0 > V_0$ such that

$$|f(t) + \tau^{2r-1}| \leq a_0 |t| \quad \text{for all } \tau, \text{ with } |t| \leq C_\infty. \tag{6.9}$$

By equations (6.8) and (6.9), we have $u \in C(\mathbb{R}^3, \mathbb{R})$ and $\lim_{|x| = u(x)} = 0$. Since $u \in L_{m_0}$, from equation (6.9) and Lemma (6.2), we have that
\[(\Delta)^{\gamma}|u| = \frac{u \ (\Delta)^{\gamma} u}{|u|} = \frac{1}{|u|} [V_\gamma(x) u^2 + \phi_u' u^2 - f(u) u - u^2] \tag{6.10}\]

This implies that \(|u|\) is a sub-solution of equation \((-\Delta)^{\gamma} + V_\gamma - a_0) u = 0\), so equation (6.7) holds. \(\square\)

**Lemma 6.4.** For every \(\varepsilon \in [0, \varepsilon_0]\), there exists \(\gamma \in \mathbb{R}^3\) such that \(|u(\gamma)| = \max_{x \in \mathbb{R}^3} |u(x)|\) for all \(u \in L_{m_\varepsilon} \subset \Lambda\).

Let \(\bar{u}_\varepsilon(x) = u_\varepsilon(x + \gamma)\) and \(u_{\varepsilon x} \in L_{m_{\varepsilon x}} \subset \Lambda\), with \(\limsup_{n \to \infty} \varepsilon_n = \bar{\varepsilon}\) and \(\bar{\varepsilon} \in [0, \varepsilon_0]\).

(i) If \(\bar{\varepsilon} > 0\), then \(\{u_{\varepsilon x}\}\) has a convergence subsequence, whose limit belongs to \(\Lambda\);

(ii) If \(\bar{\varepsilon} = 0\), then \(\{\bar{u}_{\varepsilon x}\}\) has a convergence subsequence, whose limit is not zero.

**Proof.** For \(\varepsilon_n \subset [0, \varepsilon_0]\) and \(u_{\varepsilon n} \in L_{m_{\varepsilon n}}\), we know that \(\{u_{\varepsilon n}\}\) is bounded in \(H^3(\mathbb{R}^3)\) by Lemma 6.1. From Lemma 4.5, we have \(0 < m_{\varepsilon n} < \ell_\varepsilon < \frac{3}{3} \mathcal{S}_{\varepsilon}^\gamma\), as in the proof of equation (3.17), we obtain

\[
\limsup_{n \to \infty} \int_{y \in \mathbb{R}^3} |u_{\varepsilon n}|^2 \, dx > 0. \tag{6.11}\]

From Lemma 6.3, there exists \(\gamma \in \mathbb{R}^3\) such that \(|u(\gamma)| = \max_{x \in \mathbb{R}^3} |u(x)|\), and by equation (6.11), we obtain

\[
\limsup_{n \to \infty} |u_{\varepsilon n}(\gamma_n)|^2 \geq \frac{3}{4\pi} \limsup_{n \to \infty} \int_{y \in \mathbb{R}^3} |u_{\varepsilon n}|^2 \, dx > 0. \tag{6.12}\]

(i) If \(0 < \varepsilon \leq \varepsilon_0\), then passing to a subsequence, we may assume that as \(n \to \infty\)

\[
\varepsilon_n \to \bar{\varepsilon} \in (0, \varepsilon_0), \quad u_{\varepsilon n} \to \bar{u} \text{ in } H^3(\mathbb{R}^3).
\]

Similar to the proof for Lemma 4.5, as \(n \to \infty\), we have

\[

u_{\varepsilon n} \to \bar{u} \text{ in } H^3(\mathbb{R}^3), \quad I_{\bar{\varepsilon}}^{\gamma}(\bar{u}) = 0, \quad I_{\bar{\varepsilon}}^{\gamma}(\bar{u}) = \lim_{n \to \infty} I_{\varepsilon n}^{\gamma}(u_{\varepsilon n}) = m_{\bar{\varepsilon}}. \tag{6.13}\]

This implies that \(\bar{u} \in L_{m_{\bar{\varepsilon}}} \subset \Lambda\).

(ii) If \(\bar{\varepsilon} = 0\), then passing to a subsequence, we may assume that as \(n \to \infty\)

\[

\varepsilon_n \to 0, \quad \bar{u}_{\varepsilon n} \to \bar{u}_0 \text{ in } H^3(\mathbb{R}^3).
\]

By equation (6.12), we have that \(\bar{u}_0 \neq 0\). Since \(V\) is bounded, we assume that

\[

\lim_{n \to \infty} V(\varepsilon_n y_{\varepsilon n}) = \beta. \tag{6.14}\]

Similar to the proof for Lemma 6.2, we can prove \(\limsup_{n \to \infty} m_{\varepsilon n} \leq \tilde{m}_\beta\). Note that

\[
m_{\varepsilon n} = I_{\varepsilon n}^{\gamma}(u_{\varepsilon n}) = \frac{1}{2} \|\bar{u}_{\varepsilon n}\|^2 + \int_{\mathbb{R}^3} V(\varepsilon_n (x + y_{\varepsilon n}))|\bar{u}_{\varepsilon n}|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{\varepsilon n}' |\bar{u}_{\varepsilon n}|^2 \, dx

\quad - \int_{\mathbb{R}^3} F(\bar{u}_{\varepsilon n}) + \frac{1}{2\beta} |\bar{u}_{\varepsilon n}|^2 \, dx. \tag{6.15}\]

By a standard argument, we easily prove that \(I_{\bar{\varepsilon}}^{\gamma}(\bar{u}_0) = 0\), and so \(I_{\bar{\varepsilon}}^{\gamma}(\bar{u}_0) = 0\). Hence, we have
It follows from the above and Lemma 6.2 that
\[ I_{\beta}'(\bar{u}_0) = 0, \quad \bar{m}_\beta = I_{\beta}'(\bar{u}_0) = \lim_{n \to \infty} I_{\epsilon_n}(u_{\epsilon_n}) \leq m_0. \tag{6.16} \]

**Lemma 6.5.** \(\inf\{||u||_\infty : u \in \Lambda\} = \delta_0 > 0.\)

**Proof.** By contradiction, we suppose \(\delta_0 = 0.\) Then, there is a sequence \(\{u_{\epsilon_n}\} \subseteq \Lambda\) such that \(\lim_{n \to \infty} ||u_{\epsilon_n}||_\infty = 0.\) Let \(u_\delta \in L_{m_0}.\) So there are two cases.

If \(0 < \limsup_{n \to \infty} \epsilon_n \leq \epsilon_0.\) By Lemma 6.4, there exists \(u_* \in \Lambda\) such that \(u_* \to u_{\epsilon_n}\) in \(H^s(\mathbb{R}^3).\) By using Hölder inequality, for all \(x \in \mathbb{R}^3,\)
\[
\int_{B(x)} |u_{n}(y)| dy \leq \int_{B(x)} |u_n(y) - u_{\epsilon_n}(y)| dy + \int_{B(x)} |u_{\epsilon_n}(y)| dy
\]
\[
\leq \frac{2\sqrt{\pi}}{\sqrt{3}} ||u_n - u_{\epsilon_n}||_2 + \frac{4\pi}{3} ||u_{\epsilon_n}||_{\infty} = o(1),
\]
as \(n \to \infty.\) This implies that \(u_* = 0,\) which is a contradiction. So we obtain that \(\delta_0 > 0\) as \(n \to \infty.\) This implies that \(u_* = 0,\) which is clearly a contradiction. Therefore, \(\delta_0 > 0.\)

If \(\lim_{n \to \infty} \epsilon_n = 0.\) By Lemma 6.4, there exists \(\bar{u}_* \in H^s(\mathbb{R}^3)\setminus\{0\}\) such that \(\bar{u}_n \to \bar{u}_*\) in \(H^s(\mathbb{R}^3).\) By using Hölder inequality, for all \(x \in \mathbb{R}^3,\)
\[
\int_{B(x)} |\bar{u}_n(y)| dy \leq \int_{B(x)} |\bar{u}_n(y) - \bar{u}_*(y)| dy + \int_{B(x)} |\bar{u}_*(y)| dy
\]
\[
\leq \frac{2\sqrt{\pi}}{\sqrt{3}} ||\bar{u}_n - \bar{u}_*||_2 + \frac{4\pi}{3} ||\bar{u}_*||_{\infty} = o(1) + \frac{4\pi}{3} ||u_{\epsilon_n}||_{\infty} = o(1).
\]
as \(n \to \infty.\) This implies that \(\bar{u}_* = 0,\) which is also a contradiction. \(\square\)

**Lemma 6.6.** There exists \(C > 0\) such that for any \(v_n \in \Lambda,\)
\[
v_n(x) \leq \frac{C}{1 + |x|^{3 + 2s}} \quad \forall x \in \mathbb{R}^3. \tag{6.17}
\]

**Proof.** According to ([17], Lemma 4.3), there exists a continuous function \(\bar{u}\) such that
\[
0 < \bar{u}(x) \leq \frac{C}{1 + |x|^{3 + 2s}}, \tag{6.18}
\]
and
for some suitable $\bar{R} > 0$. Thanks to Lemma 6.3, we have that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n$. Therefore, for some large $R_i > 0$, we obtain

$$(-\Delta)^\kappa v_n + \frac{V_{\min}}{2} u = (-\Delta)^\kappa v_n + V(\varepsilon_n(x + y_n))v_n - \left(V(\varepsilon_n(x + y_n)) - \frac{V_{\min}}{2}\right)v_n$$

$$= -\phi_{v_n}^* v_n + |v_n|^2 \phi_{v_n}^* v_n + f(v_n) - \left(V(\varepsilon_n(x + y_n)) - \frac{V_{\min}}{2}\right)v_n$$

$$\leq \left(|v_n|^2 \phi_{v_n}^* + f(v_n) - \frac{V_{\min}}{2}\right)v_n$$

$$\leq 0$$

for $x \in \mathbb{R}^3 \setminus B_{\bar{R}}(0)$. Now we take $R_0 = \max\{\bar{R}, R_i\}$ and set

$$z_n := (m + 1)\bar{u} - bv_n,$$

where $m = \sup_{n \in \mathbb{N}} \|v_n\|_{\infty} < \infty$ and $b = \min_{\mathbb{R}^3 \setminus B_{\bar{R}}(0)} \bar{u} > 0$. We next show that $z_n \geq 0$ in $\mathbb{R}^3$. For this, we suppose by contradiction that there is a sequence $\{x_n\}$ such that

$$\inf_{x \in \mathbb{R}^3} z_n(x) = \lim_{j \to \infty} z_n(x_n^j) < 0.$$  

(6.22)

Note that

$$\lim_{|x| \to \infty} \bar{u}(x) = 0.$$  

Jointly with Lemma 6.3, we obtain

$$\lim_{|x| \to \infty} z_n(x) = 0$$

uniformly in $n \in \mathbb{N}$. Consequently, the sequence $\{x_n\}$ is bounded, and therefore, up to a subsequence, we may assume that $x_n^j \to x_n^* \in \mathbb{R}^3$. Hence, equation (6.22) becomes

$$z_n(x_n^*) = \inf_{x \in \mathbb{R}^3} z_n(x) < 0.$$  

(6.23)

From equation (6.23), we have

$$(-\Delta)^\kappa z_n(x_n^*) = -\frac{C(s)}{2} \int_{\mathbb{R}^3} \frac{z_n(x_n^* + y) + z_n(x_n^* - y) - 2z_n(x_n^*)}{|y|^{3+2s}} dy \leq 0.$$  

(6.24)

By equation (6.22), we obtain

$$z_n(x) \geq mb + \bar{u} - mb > 0, \text{ in } B(0, R_0).$$

Therefore, combining this with equation (6.23), we see that

$$x_n^* \in \mathbb{R}^3 \setminus B_{R_0}(0).$$  

(6.25)

From equations (6.19) and (6.20), we conclude that

$$(-\Delta)^\kappa z_n + \frac{V_{\min}}{2} z_n \geq 0, \text{ in } \mathbb{R}^3 \setminus B_{\bar{R}}(0).$$

(6.26)

Thinks to equation (6.24), we can evaluate equation (6.26) at the point $x_n^*$, and recall equations (6.23) and (6.24), we conclude that

$$0 \leq (-\Delta)^\kappa z_n(x_n^*) + \frac{V_{\min}}{2} z_n(x_n^*) < 0,$$
which is a contradiction, so \( z_0(x) \geq 0 \) in \( \mathbb{R}^3 \). That is to say, \( v_n \leq (m + 1)b^{-1}\tilde{u} \), which together with equation (6.19) implies that

\[
v_n(x) \leq \frac{C}{1 + |x|^{3s + 2s}} \quad \forall x \in \mathbb{R}^3.
\]

Then, the proof is completed. \( \square \)

**Lemma 6.7.** Let \( u_\varepsilon \in \mathcal{L}_{m_\varepsilon} \) for \( \varepsilon \in (0, \varepsilon_0) \) and let \( y_\varepsilon \in \mathbb{R}^3 \) be a global maximum point of \( u_\varepsilon \). Then,

(i) \( \sup_{\varepsilon \in (0, \varepsilon_0)} |dy_\varepsilon| < \infty; \)

(ii) as \( \varepsilon_n \to 0^+ \), up to a subsequence, \( \tilde{u}_{\varepsilon_n} = u_{\varepsilon_n}(\cdot + y_{\varepsilon_n}) \) converges in \( H^s(\mathbb{R}^3) \) to a ground state solution of equation (1.6).

**Proof.** (i) By contradiction, we suppose that there exists a sequence \( \varepsilon_n \subset [0, \varepsilon_0] \) such that \( \varepsilon_n|y_{\varepsilon_n}| \to \infty \). There are two possible cases.

If \( \limsup_{\varepsilon_n \to 0} \varepsilon_n = \bar{\varepsilon} \in (0, \varepsilon_0) \). By Lemma 6.4 (i), there exists \( \varepsilon_0 \in \Lambda \) such that, up to a subsequence, one has \( |y_{\varepsilon_n}| \to \infty \) and \( u_{\varepsilon_n} \to u_\varepsilon \) in \( H^s(\mathbb{R}^3) \). Hence, it follows from equation (6.7), Lemma 6.5, and the Hölder inequality that as \( n \to \infty \).

\[
\int\int\int |u_{\varepsilon_n}(y)|dy \leq a_\varepsilon \int |u_{\varepsilon_n}(y)|\,dy + a_\varepsilon \int |u_\varepsilon(y)|\,dy \leq C_\varepsilon \|u_{\varepsilon_n} - u_\varepsilon\|_2 + a_\varepsilon \int |u_\varepsilon(y)|\,dy = o(1).
\]

This proves the assertion.

If \( \lim_{n \to \varepsilon_n} \varepsilon_n = 0 \), then by equation (6.16), we have

\[
I_{p_\varepsilon}(\tilde{u}_0) = \limsup_{n \to \infty} I_{p_\varepsilon}(u_{\varepsilon_n}) \leq m_0,
\]  

(6.27)

where the definitions of \( \beta \) and \( \tilde{u}_0 \) are given in Lemma 6.4 (ii). From 4.2, there exists \( \tilde{\varepsilon} > 0 \) such that \( (\tilde{u}_0)_\varepsilon \in \mathcal{M}_0 \), and so \( I_{p_\varepsilon}(\tilde{u}_0)_{\varepsilon} \geq m_0 \). Since \( \varepsilon_n y_{\varepsilon_n} \to \infty \), then by \( (V) \) and equation (6.14), going to a subsequence if necessary, we have

\[
V(0) = \inf_{x \in \mathbb{R}^3} V(x) < V_\infty = \lim_{n \to \infty} V(\varepsilon_n y_{\varepsilon_n}) = \beta.
\]  

(6.28)

We using Theorem 1.3 to \( I_{p_\varepsilon} \), we derive from equations (6.27) and (6.28) that

\[
m_0 \geq I_{p_\varepsilon}(\tilde{u}_0) \geq I_{p_\varepsilon}((\tilde{u}_0)_{\varepsilon}) = I_{p_\varepsilon}((\tilde{u}_0)_{\varepsilon}) + \frac{\beta - V(0)}{2} \xi^{2s + 2r - 3} \|\tilde{u}_0\|^2_L
\]

\[
\geq m_0 + \frac{\beta - V(0)}{2} \xi^{2s + 2r - 3} \|\tilde{u}_0\|^2_L > m_0.
\]

This obviously proves (i).

(ii) From Lemma 6.1, the sequence \( \{u_{\varepsilon_n}\} \) is bounded in \( H^s(\mathbb{R}^3) \), and so \( \{\tilde{u}_{\varepsilon_n}\} \) is bounded in \( H^s(\mathbb{R}^3) \). After extracting a subsequence, we may assume that \( \tilde{u}_{\varepsilon_n} \to \tilde{u} \) in \( H^s(\mathbb{R}^3) \), \( \tilde{u}_{\varepsilon_n} \to \tilde{u} \) in \( L^r_{\text{loc}}(\mathbb{R}^3) \) for any \( s \), with \( 2 \leq r < 2^*_s \), and \( \tilde{u}_{\varepsilon_n} \to \tilde{u} \) a.e. on \( \mathbb{R}^3 \). We claim that \( \tilde{u} \neq 0 \). Indeed, (6.7) and Lemma 6.6 yield

\[
\delta_0 \leq |\tilde{u}_{\varepsilon_n}(0)| = |u_{\varepsilon_n}(y_{\varepsilon_n})| \leq a_\varepsilon \int |u_{\varepsilon_n}(y)|\,dy
\]

\[
= a_\varepsilon \int |\tilde{u}_{\varepsilon_n}(y)|\,dy = a_\varepsilon \int |\tilde{u}(y)|\,dy + o(1).
\]
This shows that \( \bar{u} \neq 0 \). Moreover, by the conclusion (i), there exists \( \bar{y} \in \mathbb{R}^3 \) such that, up to a subsequence, \( \varepsilon_0 y_{\varepsilon_0} \to \bar{y} \).

Finally, we prove that \( \bar{u}_{\varepsilon_0} \to \bar{u} \) in \( H^s(\mathbb{R}^3) \), \( V(\bar{y}) = V_0 \), and \( \bar{u} \) is a ground state solution of equation (1.6). Let \( \bar{V} = V(\bar{y}) \). Since \( V_0 \leq \bar{V} \) and \( \bar{m}_a \) is nondecreasing for \( a \in (0, V_{\text{max}}] \), it follows from Lemma 6.2 that
\[
\limsup_{n \to \infty} m_{\varepsilon_0} \leq m_0 \leq \bar{m}.\]

As in the proof of equation (6.16), we deduce that as \( n \to \infty \),
\[
I_\varepsilon(\bar{u}) = 0, \quad I_\varepsilon(\bar{u}) = \bar{m} = \lim_{n \to \infty} I_\varepsilon(u_{\varepsilon_0}).
\]

So we obtain
\[
\bar{m} = I_\varepsilon(\bar{u}) \leq m_0 \leq \bar{m}.
\]

This implies that
\[
\lim_{\varepsilon \to 0} V(\varepsilon y_{\varepsilon}) = V(\bar{y}) = V_0,
\]
and \( \bar{u} \) is a ground state solution of equation (1.6), as asserted. □

**Proof of Theorem 1.2.** In view of Theorem 1.1, the function \( \tilde{v}_{\varepsilon_0}(x) = \tilde{u}_{\varepsilon_0}(e^{-\varepsilon y}(x - x_0)) \) is a ground state solution of equation (1.1) for every \( \varepsilon \in (0, \varepsilon_0] \). Let \( x_{\varepsilon_0} = x_0 + \varepsilon_0 y_{\varepsilon_0} \), then (i) follows from equation (6.29). Moreover, 6.7 imply the validity of (iii).

Let \( \tilde{v}_{\varepsilon_0}(x) = u_{\varepsilon_0} \left( \frac{x}{\varepsilon_0} \right) \) and by Lemma 6.6, we have
\[
u_{\varepsilon_0} \left( \frac{x}{\varepsilon_0} \right) = \tilde{u}_{\varepsilon_0} \left( \frac{x}{\varepsilon_0} - y_n \right) \leq \frac{C}{1 + \left| \frac{x}{\varepsilon_0} - y_n \right|^{3s+2s}}
\]
\[
\leq \frac{C \varepsilon_0^{3s+2s}}{\varepsilon_0^{3s+2s} + |x - y_n|^{3s+2s}}
\]
\[
= \frac{C \varepsilon_0^{3s+2s}}{\varepsilon_0^{3s+2s} + |x - x_{\varepsilon_0}|^{3s+2s}} \quad \forall x \in \mathbb{R}^3.
\]

Hence, the proof of Theorem 1.2 is completed. □

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