On the Symmetry of the Ground States of Nonlinear Schrödinger Equation with Potential

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Abstract

We investigate the minimizers of the energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V|u|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} b|u|^{p+1} \, dx$$

under the constraint of the $L^2$-norm. We show that for the case $L^2$-norm is small, the minimizer is unique and for the case $L^2$-norm is large, the minimizer concentrate at the maximum point of $b$ and decays exponentially. By this result, we can show that if $V$ and $b$ are radially symmetric but $b$ does not attain its maximum at the origin, then the symmetry breaking occurs as the $L^2$-norm increases. Further, we show that for the case $b$ has several maximum points, the minimizer concentrates at a point which minimizes a function which is defined by $b$, $V$ and the unique positive radial solution of $-\Delta \phi + \phi - \phi^p = 0$. For the case when $V$ and $b$ are radially symmetric, we show that if the minimizer concentrates at the origin, then the minimizer is radially symmetric. Further, we construct an energy functional such that the minimizer breaks its symmetry once but after that it recovers to be symmetric as the $L^2$-norm increases.

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1 Introduction

In this paper, we consider the symmetry and the uniqueness of the standing waves of the nonlinear Schrödinger equation with potential and attractive nonlinearity,

\[ \begin{align*}
    iu_t &= -\Delta u + V(x)u - f(x, |u|)u, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^N, \\
    u &= \varphi_\omega(x).
\end{align*} \tag{1.1} \]

where standing waves are the solutions of (1.1) in the form \( u(t, x) = e^{i\omega t} \varphi_\omega(x) \). In this case \( \varphi_\omega \) satisfies the following elliptic partial differential equation.

\[ -\Delta \varphi_\omega + (V + \omega) \varphi_\omega - f(x, |\varphi_\omega|) \varphi_\omega = 0. \tag{1.2} \]

Under a suitable condition for \( V \) and \( f \), the flow of equation (1.1) conserves the \( L^2 \)-norm and the following energy functional,

\[ E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V|u|^2 \, dx - \int_{\mathbb{R}^N} F(x, |u|) \, dx, \]

where \( F(x, s) := \int_0^s f(x, \xi) \, d\xi \). One observes that, by the Lagrange multiplier method, the minimizer of \( E \) under the constraint \( \|u\|_{L^2} = \alpha > 0 \), satisfies (1.2) for some \( \omega \in \mathbb{R} \). Such solutions of Eq. (1.2) belonging to \( H^1(\mathbb{R}^N) \) are called the ground states.

Eq. (1.1) appears in Bose-Einstein condensation (BEC) and nonlinear optics. In the context of BEC, the ground states are considered to describe the physical properties of Bose gas at low temperature. Further, from the mathematical point of view, ground states are stable solutions of Eq. (1.1), i.e., a solution which has initial data near the ground states remains near the ground states globally in time. Therefore, it is important to investigate the ground states to understand the dynamics of Eq. (1.1).

We mainly consider here the relation of symmetry between \( V, f \) and the ground states. For example, if \( V \) and \( f \) are radially symmetric, then are the ground states radially symmetric? In the case \( f(x, |u|) = |u|^{p-1} \) and \( V(x) \equiv 0 \), all the positive solutions of Eq. (1.2) are radially symmetric ([12]). Since the ground states are positive, in this case we have an affirmative answer to our question. Further, for the power type nonlinearity, if \( V \) is radially symmetric and monotonically increasing with respect to \( |x| \), then by symmetric decreasing rearrangement we see that the ground states must be radially symmetric.

However, for the case \( V \) is radially symmetric but not monotonically increasing with respect to \( |x| \), there are cases that the ground states are not radially symmetric. That is, for the case \( f(x, |u|) = |x|^{-1} * |u|^2 \) where \( * \) is a convolution, Aschbacher, Frölich, Graf, Schnee and Troyer [3] showed that if the constraint \( \|u\|_{L^2} = \alpha \) is sufficiently large, then the ground state concentrates its \( L^2 \)-norm around one minimum point of \( V \). By this result one immediately sees that if \( V \) does not attain its minimum point at the origin, then for sufficiently large \( \alpha \), the ground state is not radially symmetric even if \( V \) is radially symmetric. They also showed that for the case \( f = W * |u|^2 \), where \( W \in L^1 \cap L^\infty \), for small constraint \( \|u\|_{L^2} = \alpha \), the ground states are unique up to constant phase. Further, for the case \( f(x, |u|) = b(x)(W * |u|^2) \),
where $W$ and $b$ are symmetric with respect to the hyperplane $\{x_1 = 0\}$ and $V$ is a double well potential, Kirr, Kevrekidis, Shlizerman and Weinstein [17] showed that there is a bifurcation from a symmetric state to an asymmetric state.

Now, we state our results. We consider the nonlinearity $f(x, |u|) = b(x)|u|^{p-1}$, for $1 < p < 1 + 4/N$. We show the uniqueness of the ground states when the $L^2$-norm is small and the concentration and exponential decay around the maximum point of $b$ (or minimum point of $V$ in the case $b \equiv 1$) when the $L^2$-norm is large. Thus, for the case $b$ and $V$ is radially symmetric and $b$ does not attain its maximum point at the origin (or $b \equiv 1$ and $V$ does not attain its minimum point at the origin), we show that the symmetry breaks when we enlarge the $L^2$-norm. Further, by using the concentration and exponential decay result we show that in the case $b$ has several maximum points, the ground state concentrate around a point which minimizes a function defined by $b$, $V$ and the minimizer of

$$
\tilde{J}_1(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{p + 1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx,
$$

under the constraint $||u||_{L^2} = 1$. We denote this minimizer as $\phi_1$ all through this paper.

The concentration results seem to be similar to the results for semiclassical limit of nonlinear Schrödinger equations which Floer and Weinstein [11] started to study. However, since the parameter is different, we need a different technique to obtain our results. For the case $b \equiv 1$, Wang [24] showed that for the semiclassical case, the ground states (which the definition is slightly different from ours) concentrate at the global minimum of $V$. Further, Grossi and Pistoia [15] showed that if $V$ has several global minimum points, then the ground state concentrates where $V$ is flattest. For the case $b \not\equiv 1$, Wang and Zeng [25] showed that the minimizer concentrates at the minimum point of $g(x) := V(2p+2+N-Np)/(2p-2)(x)b^{-2/(p-1)}(x)$. Therefore, since in our result, the ground state concentrates at the maximum point of $b$ and the effect of $V$ appears only when $b$ has several maximum points, we see that different phenomena occur in our case.

We showed that the symmetry breaking occurs because the concentration point is not at the origin. So next, we wish to know what happens when the ground state concentrates at the origin. In this case, we show combining the method of Grossi [14] with our estimate which we have obtained in the proof of the concentration result, that if $b$ attains a maximum point at the origin, and $b$ and $V$ satisfies some additional conditions, then the ground states are radially symmetric. Using this result and the result by Kirr et al. [17], we can construct an energy functional such that there exist $\alpha_1$, $\alpha_2$ and $\alpha_3$ such that for $\alpha \in (0, \alpha_1] \cup (\alpha_3, \infty)$ the ground states are symmetric and for $\alpha \in (\alpha_1, \alpha_2)$ the ground states are not symmetric. Therefore, in this case, the symmetry breaking once occurs but after that, the symmetry recovers.

For the Hartree type nonlinearity case, which is the case $f(x, |u|) = |x|^{-1} * |u|^2$, we can show the same results completely in the same way as the power type case.

To state our results precisely, we introduce several notations. We consider the
minimizing problem in the following Sobolev space.

\[ H^1_V := \{ u \in H^1(\mathbb{R}^N) \mid V|u|^2 \in L^1(\mathbb{R}^N) \} . \]

In \( H^1_V \), the energy functional
\[
\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} b(x)|u|^{p+1} \, dx,
\]
is well defined. We always assume the following conditions on \( V \in C^1(\mathbb{R}^N) \) and \( b \in C^1(\mathbb{R}^N) \).

(V1) \( 0 = \inf_{x \in \mathbb{R}^N} V(x) < \lim_{|x| \to \infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x) \leq \infty \).

(V2) \( |\nabla V(x)| \leq C(1 + V(x)) \) for a constant \( C > 0 \).

(b1) \( 0 < \inf_{x \in \mathbb{R}^N} b(x) = \lim_{|x| \to \infty} b(x) \leq \sup_{x \in \mathbb{R}^N} b(x) = 1 \).

Remark 1.1 The assumptions \( \inf V = 0, \sup b = 1 \) are not essential, one can just assume that \( V \) is bounded from below and \( b \) is bounded form above.

Remark 1.2 In the case \( V \) is bounded, we do not need (V2).

Now, we define the ground state as the positive minimizer of \( \mathcal{E} \) under the \( L^2 \) constraint.

Definition 1.1 Set
\[
\mathcal{G}_\alpha := \{ u \in H^1_V \mid \|u\|_{L^2} = \alpha, \mathcal{E}(u) = E_\alpha, \, u > 0 \},
\]
where
\[
E_\alpha := \inf\{\mathcal{E}(u) \mid u \in H^1_V, \|u\|_{L^2} = \alpha \}.
\]
We call the elements of \( \mathcal{G}_\alpha \), the ground states, which are the minimizers of \( \mathcal{E} \) under the constraint \( \|u\|_{L^2} = \alpha \).

The existence of the ground states are well known. See for example Proposition 8.3.6 of [5].

Proposition 1.1 Let \( 1 < p < 1 + 4/N \). Assume (V1) and (b1). Then, for all \( \alpha > 0 \), we have \( \mathcal{G}_\alpha \neq \emptyset \).

We first show the uniqueness of the ground states when the constraint \( \|u\|_{L^2} = \alpha \) is small.

Theorem 1.1 Assume (V1), (V2) and (b1). Further assume that \( \lambda_0 := \inf \sigma(-\Delta + V) \) is an eigenvalue. Let \( p \geq 2 \). Then for sufficiently small \( \alpha > 0 \), the ground states are unique.

Remark 1.3 For the case \( \lim_{|x| \to \infty} V = \infty \), \( \lambda_0 := \inf \sigma(-\Delta + V) \) is always an eigenvalue.
We next show the concentration of the ground state for large $L^2$-norm.

**Theorem 1.2** Assume $(V_1)$, $(V_2)$ and $(b_1)$. Then for sufficiently large $\alpha > 0$, every ground state $u \in \mathcal{G}_\alpha$ has only one local maximum point $y_{\alpha,u}$ (which is a global maximum point), where $\inf_{y \in b^{-1}(\{1\})} |y_{\alpha,u} - y| \to 0$ as $\alpha \to \infty$ (or $\inf_{y \in V^{-1}(\{0\})} |y_{\alpha,u} - y| \to 0$ for the case $b \equiv 1$). Further, we have

$$u(x) \leq C_1 \alpha^{-\frac{4}{4-p-1}} \exp \left( -C_2 \alpha^{\frac{2(p-1)}{4}} |x - y_{\alpha,u}| \right),$$

where $C_1$ and $C_2$ are positive constants independent of $\alpha$.

By theorems 1.2, we immediately have the following symmetry breaking result.

**Corollary 1.1** Assume $b$ and $V$ satisfy the assumptions of Theorem 1.3. Let $V$, $b$ radially symmetric. Further, suppose $y_{\alpha,u}$ does not converge to $0$ as $\alpha \to 0$. Then, the ground states $u \in \mathcal{G}_\alpha$ is not radially symmetric. In particular if $b(0) < 1$ or if $b \equiv 1$ and $V(0) > 0$, then the symmetry breaking occurs.

We next investigate the case that $b$ has several maximum points. We set $\phi_1$ to be the positive radial minimizer of $\tilde{J}_1$ (see (1.3)) under the constraint $||u||_{L^2} = 1$.

**Theorem 1.3** Assume $(V_1)$, $(V_2)$ and $(b_1)$. Further, Assume that $V(x) \leq C(1 + |x|^r)$ for some $C > 0$ and $r > 0$. Let $y \in b^{-1}(\{1\})$. Assume there exists some real number $\beta := \beta(y)$ such that for some $r > 0$,

$$b(y + x) = 1 - Q_y(x) + R_y(x), \quad \forall x \in B(y, r),$$

where $B(y, r) := \{ x \in \mathbb{R}^N \mid |x - y| < r \}$, $Q_y$ satisfies

$$Q_y(tx) = t^{\beta} Q_y(x), \quad \forall t > 0, \quad \forall x \in \mathbb{R}^N$$

and $R_y$ satisfies

$$|R_y(x)| < C|x|^\beta + \varepsilon(y), \quad \forall x \in B(y, r),$$

for some constant $C > 0$ and $\varepsilon(y) > 0$. Let $u_\alpha \in \mathcal{G}_\alpha$ and $y_{\alpha,u_\alpha}$ as in Theorem 1.2. Set

$$\tilde{\beta} = \max_{y \in b^{-1}(\{1\})} \beta(y).$$

(i) If $\tilde{\beta} < 2$, then $y_{\alpha,u_\alpha}$ converges to some $y_0 \in b^{-1}(\{1\})$ as $\alpha \to \infty$ such that $\beta(y) = \tilde{\beta}$ and

$$\int_{\mathbb{R}^N} Q_{y_0}(x) \phi_1^{p+1} dx = \min_{\beta(y) = \tilde{\beta}} \int_{\mathbb{R}^N} Q_y(x) \phi_1^{p+1} dx.$$
(ii) If \( \tilde{\beta} \geq 2 \), then \( y_{\alpha,u_n} \) converges to some \( y_0 \in b^{-1}(\{1\}) \) as \( \alpha \to \infty \) such that \( \beta(y_0) \geq 2 \) and

\[
\sigma(\beta(y_0)) \int_{\mathbb{R}^N} Q_{y_0}(x) \phi^{p+1}_1 \, dx + \frac{1}{2} V(y_0) = \\
\min_{\beta(y_0) = \tilde{\beta}} \left( \frac{\sigma(\beta(y))}{p+1} \int_{\mathbb{R}^N} Q_y(x) \phi^{p+1}_1 \, dx + \frac{1}{2} V(y) \right),
\]

where \( \sigma(\beta) = 1 \) if \( \beta = 2 \) and \( \sigma(\beta) = 0 \) if \( \beta > 2 \).

We next show that when the ground states concentrate at the origin, then the ground states are symmetric.

**Theorem 1.4** Let \( b, V \in C^1(\mathbb{R}^N) \) and radially symmetric. Assume \( (V1), (V2), (b1) \) and that \( V, |\nabla V|, |\nabla b| \leq C(1 + |x|)^r \) for some \( r \). Let \( V, b \) radially symmetric. Assume there exists some real number \( \beta, \gamma \) such that for some \( R > 0 \),

\[
b(x) = 1 - Q_b(x) + R_b(x), \\
V(x) = V(0) + Q_V(x) + R_V(x), \quad \forall x \in B(0, R)
\]

where \( Q_b, Q_V \in C^1(\mathbb{R}^N) \), \( Q_b, Q_V \not\equiv 0 \) satisfies
\[
Q_b(tx) = t^{\beta} Q_b(x), \\
Q_V(tx) = t^{\gamma} Q_V(x), \quad \forall t > 0, \quad \forall x \in \mathbb{R}^N
\]

and \( R_b, R_V \) satisfies
\[
|\nabla R_b(x)| \leq C |x|^{\gamma - \varepsilon_b - 1}, \\
|\nabla R_V(x)| \leq C |x|^{\gamma - \varepsilon_V - 1}, \quad \forall x \in B(0, R),
\]

for some \( \varepsilon_b, \varepsilon_V > 0 \). Further, assume \( \beta < \gamma + 2 \) or \( 0 \) is a local minimum point of \( V \). Then, if there exists \( u_{\alpha_n} \in G_{\alpha_n} \), \( \alpha_n \to \infty \) such that \( y_{\alpha_n,u_n} \to 0 \) as \( n \to \infty \), we have \( y_{\alpha_n,u_n} = 0 \) and the ground states \( u_n \) are radially symmetric for sufficiently large \( n \).

Combining the result of Kirr et al. [17] with Theorem 1.4, we can construct an example that the ground state breaks its symmetry but after that recovers its symmetry as the \( L^2 \)-norm increases.

**Theorem 1.5** Let \( N = 1 \). Assume that \( b \) is an even decreasing function of \( |x| \) which satisfies \((b1)\) and 0 is a nondegenerate maximum point of \( b \). Let \( v \in C_0^\infty \) be a nonnegative even function with \( \min_{x \in \mathbb{R}} v = 1 \). Set \( V_L(x) := 1 - v(x + L) - v(x - L) \).

Let
\[
E_L(u) := \frac{1}{2} \int_{\mathbb{R}} (|u'|^2 + V_L |u|^2) \, dx - \frac{1}{4} \int_{\mathbb{R}} b(x)|u|^4 \, dx.
\]

Then, for sufficiently large \( L \), there exist \( \alpha_1 < \alpha_2 < \alpha_3 \) such that for \( \alpha \in (0, \alpha_1) \cup (\alpha_3, \infty) \), the ground states are even and for \( \alpha \in (\alpha_1, \alpha_2) \), the ground states are not even.
Remark 1.4 For a double well potential in higher dimensions, we can also prove similar results.

Our strategy to prove Theorem 1.1 basically relies on [3]. However, we improved the results of [3], and our proof is applicable to more general nonlinearity.

The results of Theorems 1.2, 1.4 resemble the concentration results of the semiclassical nonlinear Schrödinger equation

\[ i\hbar u_t = -\hbar^2 \Delta u + Vu - |u|^{p-1}u. \]

For the Planck constant \( \hbar \) being small, Floer-Weinstein [11] first showed that there exists a single spike solution concentrating on a nondegenerate critical point of \( V \) in the one dimensional case. Later Oh [19] extended the results to higher dimensions. Next, Wang [24] showed that the concentration point must be a critical point of \( V \), and then Grossi and Pistoia [15] showed that the ground state must concentrate at the flattest minimum of \( V \). For the case that the critical point of \( V \) may be degenerate, there are many works in recent years. (See [1, 2, 6, 16, 8, 9, 10, 7, 21, 23, 26], and the references there in).

However, there are two differences between our results and the semiclassical results. First is that the parameters are different, so the results above for the semiclassical case are not directly applicable to our results. Indeed, in [6] Cid and Felmer also take the minimizer under the constraint of \( L^2 \)-norm and this process looks similar to our result but the way to show the exponential decay is different. To show Theorem 1.2, in Lemma 4.8, we establish an elliptic estimate (4.14) independent of \( \alpha \). By this estimate, we succeed in showing the exponential decay result in Theorem 1.2. The second is that the choice of the concentration point becomes different. In the semiclassical case the concentration point is the minimum point of 

\[ g(x) = V^{(2p+2+N-Np)/(2p-2)}(x)b^{-2/(p-1)}(x). \]

However, in our case, the concentration point is the maximum point of \( b \) and there is no effect of \( V \). The effect of \( V \) only appears in the case that \( b \) has several maximum points.

Theorem 1.4 is also a consequence of the estimate (4.14) combined to the method of Grossi [14]. The difference between the proof of Grossi [14] and the proof of Theorem 1.4 is as follows. Grossi counts the number of single peaked solutions which concentrates around a point. So, they assume that the solution has only one local maximum. On the other hand, we consider the ground states, so until we show Theorem 1.2, we do not know if the ground states are single peak. Therefore, Grossi uses the fact that the solution is single peaked in order to show the solution is sufficiently small outside a large ball. Using this Grossi shows an \( L^\infty \) convergence of a rescaled solution which is essential to show their results. In our case, on the contrary we use Lemma 4.8 to show that the ground states are sufficiently small out side a large ball. Using this, we show the \( L^\infty \) convergence of a rescaled ground state and the fact that the ground state is single peaked.

We conclude this introduction by mentioning the orbital stability of standing waves. We say that the ground states \( G_\alpha \) is orbitally stable if the solution of Eq. (1.1) with initial data near the ground states \( G_\alpha \) remains near to the ground states globally in time. Since we caught the ground states as an minimizer under the
constraint of $L^2$-norm of the energy, it can be directly shown using the method of Cazenave-Lions [4] that if (1.1) is locally well posed in $H^1_V$, then the ground states are stable.

2 Preliminaries

Before proving the theorems, we replace the minimizing problem in an equivalent form. Let $\varphi_\alpha$ be the ground state under the constraint $\|u\|_{L^2} = \alpha$. Set $u_\alpha = \alpha^{-1} \varphi_\alpha$. Then, $u_\alpha$ minimizes the following functional $J_\alpha$ under the constraint $\|u\|_{L^2} = 1$.

$$J_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V|u|^2) \, dx - \frac{\alpha^{p-1}}{p+1} \int_{\mathbb{R}^N} b(x)|u|^{p+1} \, dx, \quad 1 < p < 1 + 4/N.$$ 

Next, set

$$\lambda_\alpha := \inf_{\|u\|_{L^2} = 1} J_\alpha(u). \tag{2.4}$$

Then the set of ground states can be written as

$$G_\alpha := \{ \alpha u \in H^1_V | J_\alpha(u) = \lambda_\alpha, \|u\|_{L^2} = 1, \alpha > 0 \}.$$ 

So, from now on, we study the properties of the minimizers of $J_\alpha$ under the constraint $\|u\|_{L^2} = 1$. We set

$$J_\alpha := \{ u \in H^1_V | J_\alpha(u) = \lambda_\alpha, \|u\|_{L^2} = 1, \alpha > 0 \}.$$ 

Further, $u_\alpha \in J_\alpha$ satisfies

$$-\Delta u_\alpha + V(x)u_\alpha - \alpha^{p-1}b(x)|u_\alpha|^{p-1}u_\alpha = \nu_\alpha u_\alpha, \tag{2.5}$$

where $\nu_\alpha$ is the Lagrange multiplier.

3 Proof of Theorem 1.1

In this section we show the uniqueness of the ground state of $J_\alpha$ for small $\alpha > 0$ and $2 < p < 1 + 4/N$. By the condition $2 < p < 1 + 4/N$, we only need to consider the cases $N = 1, 2, 3$. First, set

$$Y_{\alpha^*} := \{ u \in H^1_V(\mathbb{R}^N) | \|u\|_{L^2} = 1, J_\alpha(u) = \lambda_\alpha \text{ for } \alpha \in [0, \alpha^*) \},$$

where $\lambda_\alpha = \inf_{\|u\|_{L^2} = 1} J_\alpha(u)$. We show $Y_{\alpha^*}$ is bounded in $L^\infty$.

**Lemma 3.1** Let $N = 1, 2, 3$. Let $1 < p < 1 + 4/N$. Assume (V1)-(V2) and (b1). Then for $\alpha^* < \infty$,

$$\sup_{u \in Y_{\alpha^*}} \|u\|_{L^\infty} = C < \infty.$$
Proof. Note that for $u \in Y_{\alpha^*}$ there exists an $\alpha \in [0, \alpha^*)$ such that $J_\alpha(u) = \lambda_\alpha$. So, $u$ is a minimizer of $J_\alpha$. Let $\nu_\alpha$ be the Lagrange multiplier of equation (2.5). Since $\alpha$ moves only in $[0, \alpha^*)$, one sees that $\nu_\alpha$ is bounded.

Now, since $V$ satisfies (V2), we have by [20] that
\[ u = (-\Delta + V + \xi)^{-1}v \in W^{2,q}(\mathbb{R}^N) \]
for $v \in L^q$ and $\xi > 0$. So, taking $\xi = 1$, $v = (1 + \nu_\alpha)u + \alpha^{p-1}b(x)|u|^{p-1}u$ and using the bootstrap argument, we have an uniform bound of $Y_{\alpha^*}$ in $L^\infty$-norm.

Lemma 3.1 is the key estimate for the proof of the uniqueness. The proof of the uniqueness is based on the way of Aschbacher et al. ([3]). However, we improved their results by using the contraction mapping method on only $Y_{\alpha^*}$ (in [3] the contraction mapping is used on $S := \{||u||_{L^2} = 1\}$) and using Lemma 3.1.

Proof. [Proof of Theorem 1.1] Fix $\alpha^* > 0$. We give $\alpha^*$ later in the proof. For $u \in Y_{\alpha^*}$, we set
\[ H^u(\alpha) := -\Delta + V - \alpha^{p-1}b|u|^{p-1}. \]

Then, since
\[ \langle v, |u|^{p-1}v \rangle \leq ||u||_{L^\infty}^p||v||_{L^2}^2, \]
we see that the operator norm of $b(x)|u|^{p-1}$ is bounded by $C_{p-1}$ where $C := \sup_{u \in Y_{\alpha^*}} ||u||_{L^\infty}$. So, we see that $H^u(\alpha)$ is an analytic family in the sense of Kato. By the Kato–Rellich theorem and its proof, we have the following.

(i) Let $\Gamma$ be a circle around $\lambda_0 := \inf \sigma(-\Delta + V)$ with radius $r$. Then if $r$ is small enough, no other point of $\sigma(H^u(0))$ is encircled. Further,
\[ M := \sup \{||H^u(\alpha) - z||^{-1} | z \in \Gamma, 0 \leq \alpha \leq \alpha^*\} < \infty \] (3.6)
if $\alpha^*$ is small enough, where $|| \cdot ||$ is the operator norm.

(ii) Provided $\alpha^*$ sufficiently small, there exists precisely one nondegenerate eigenvalue of $H^u(\alpha)$ inside $\Gamma$, for $\alpha \in [0, \alpha^*]$ and $u \in Y_{\alpha^*}$. It is the minimum of the spectrum of $H^u(\alpha)$, and its eigenprojection is
\[ P^u(\alpha) = -(2\pi i)^{-1} \oint_{\Gamma} dz (H^u(\alpha) - z)^{-1}. \]

By these results, we see that if $u$ is a positive ground state of $J_\alpha$, then $u$ is a fixed point of the map
\[ T_\alpha : Y_{\alpha^*} \to \{||u||_{L^2} = 1\}, \quad u \mapsto \frac{P^u(\alpha)\phi_0}{||P^u(\alpha)\phi_0||_{L^2}}, \]
where $\phi_0$ is an eigenfunction of $-\Delta + V$ with $(-\Delta + V)\phi_0 = \lambda_0\phi_0$, $||\phi_0||_{L^2} = 1$. We prove the theorem by contradiction. Assume $\alpha_n \to 0$ and there exist positive
ground states \( u_{\alpha_n}, v_{\alpha_n} \) of \( J_{\alpha_n} \) with \( u_{\alpha_n} \neq v_{\alpha_n} \). Since \( u_{\alpha_n}, v_{\alpha_n} \) are fixed points of \( T_{\alpha_n} \), it suffices to show
\[
\| T_{\alpha_n}(u_{\alpha_n}) - T_{\alpha_n}(v_{\alpha_n}) \|_{L^2} \leq c\| u_{\alpha_n} - v_{\alpha_n} \|_{L^2}
\]
for \( c < 1 \) and sufficiently large \( n \). First, we have
\[
P^{u_{\alpha_n}}(\alpha_n) - P^{v_{\alpha_n}}(\alpha_n)
= \frac{1}{2\pi i} \int_{\Gamma} dz (H^{u_{\alpha_n}}(\alpha_n) - z)^{-1} \alpha_n^{-1} b(u_{\alpha_n})^{p-1} - |v_{\alpha_n}|^{p-1})(H^{v_{\alpha_n}}(\alpha_n) - z)^{-1}.
\]
We next estimate \( \| u_{\alpha_n} \|^{p-1} - |v_{\alpha_n}|^{p-1} \|_L^\infty \). Let \( \nu_{\alpha_n}, \tilde{\nu}_{\alpha_n} \) be the Lagrange multiplier of \( u_{\alpha_n}, v_{\alpha_n} \) respectively. Then, we have
\[
\nu_{\alpha_n} = 2J_{\alpha_n}(u_{\alpha_n}) - \alpha_n^{-1} P_{\alpha_n}^{p-1} \int_{\mathbb{R}^N} b(x)u_{\alpha_n}^{p+1} dx,
\]
\[
\tilde{\nu}_{\alpha_n} = 2J_{\alpha_n}(v_{\alpha_n}) - \alpha_n^{-1} P_{\alpha_n}^{p-1} \int_{\mathbb{R}^N} b(x)v_{\alpha_n}^{p+1} dx.
\]
Since \( J_{\alpha_n}(u_{\alpha_n}) = J_{\alpha_n}(v_{\alpha_n}) = \lambda_{\alpha_n} \), we have
\[
|\nu_{\alpha_n} - \tilde{\nu}_{\alpha_n}| \leq \alpha_n^{p+1} P_{\alpha_n}^{p-1} \int_{\mathbb{R}^N} |u_{\alpha_n}^{p+1} - v_{\alpha_n}^{p+1}| dx
\]
\[
\leq \alpha_n^{p+1} P_{\alpha_n}^{p-1} \| u_{\alpha_n} - v_{\alpha_n} \|_{L^2} \left( \int_{\mathbb{R}^N} (p+1)(u_{\alpha_n}^{p} + v_{\alpha_n}^{p})^2 dx \right)^{1/2}
\]
\[
\leq C \alpha_n^{p-1} \| u_{\alpha_n} - v_{\alpha_n} \|_{L^2}.
\]
Here, we used \( |u_{\alpha_n}^{p+1} - v_{\alpha_n}^{p+1}| \leq (p+1)(u_{\alpha_n}^{p} + v_{\alpha_n}^{p})(u_{\alpha_n} - v_{\alpha_n}) \) at the second inequality. Next, subtracting the equations which \( u_{\alpha_n} \) and \( v_{\alpha_n} \) satisfy, we have
\[
(-\Delta + V + 1)(u_{\alpha_n} - v_{\alpha_n}) = (1 + \nu_{\alpha_n})(u_{\alpha_n} - v_{\alpha_n}) + (\nu_{\alpha_n} - \tilde{\nu}_{\alpha_n})v_{\alpha_n} + \alpha_n^{-1} b(u_{\alpha_n} - v_{\alpha_n}).
\]
Therefore, by bootstrap argument, we have
\[
\| u_{\alpha_n} - v_{\alpha_n} \|_L^\infty \leq C\| u_{\alpha_n} - v_{\alpha_n} \|_{L^2}.
\]
Now, since we assumed \( p \geq 2 \), we have
\[
\| u_{\alpha_n}^{p-1} - v_{\alpha_n}^{p-1} \|_L^\infty \leq C \left( \| u_{\alpha_n} \|_{L^\infty}^{p-2} + \| v_{\alpha_n} \|_{L^\infty}^{p-2} \right)\| u_{\alpha_n} - v_{\alpha_n} \|_{L^\infty}.
\]
So, we have
\[
\| u_{\alpha_n}^{p-1} - v_{\alpha_n}^{p-1} \|_L^\infty \leq C\| u_{\alpha_n} - v_{\alpha_n} \|_{L^2}. \tag{3.7}
\]
So, by (3.6) and (3.7), we have
\[
\| P^{u_{\alpha_n}}(\alpha_n) - P^{v_{\alpha_n}}(\alpha_n) \| \leq 2\alpha_n^{p-1} M^2 C^{2(p-1)}\| u_{\alpha_n} - v_{\alpha_n} \|_{L^2}.
\]
Further, since \( P^0(\alpha_n)\phi_0 = \phi_0 \), we have

\[
\| P^{\alpha_n}(\alpha_n)\phi_0 \|_{L^2} \geq \| P^0(\alpha_n)\phi_0 \|_{L^2} - \| (P^{\alpha_n}(\alpha_n) - P^0(\alpha_n))\phi_0 \|_{L^2} > \frac{1}{2}
\]

for \( n \) sufficiently large. Now,

\[
T_{\alpha_n}u_{\alpha_n} - T_{\alpha_n}v_{\alpha_n} = \| P^{\alpha_n}(\alpha_n)\phi_0 \|_{L^2}^{-1}(P^{\alpha_n}(\alpha_n) - P^0(\alpha_n))\phi_0
\]

\[
+ \| P^{\alpha_n}(\alpha_n)\phi_0 \|_{L^2}^{-1}|| P^{\alpha_n}(\alpha_n)\phi_0 \|_{L^2}^{-1}(|| P^{\alpha_n}(\alpha_n)\phi_0 \|_{L^2} - || P^0(\alpha_n)\phi_0 \|_{L^2})
\times P^{\alpha_n}(\alpha_n)\phi_0.
\]

Therefore,

\[
\| T_{\alpha_n}(u_{\alpha_n}) - T_{\alpha_n}(v_{\alpha_n}) \|_{L^2} \leq CA^{p-1}_{\alpha} \| u_{\alpha_n} - v_{\alpha_n} \|_{L^2},
\]

and we have the contradiction for sufficiently large \( n \).

## 4 Proofs of Concentration results

We first show a concentration result which is a weak version of Theorem 1.2.

**Theorem 4.1** Assume (V1) and (b1). Then for any \( \varepsilon > 0 \), there exists an \( \alpha_\varepsilon > 0 \) such that for all \( \alpha \geq \alpha_\varepsilon \) and all \( u_{\alpha_n} \in J_{\alpha_n} \), there exists \( y_{\alpha,u_{\alpha_n}} \in \mathbb{R}^N \), such that

\[
\int_{|x-y_{\alpha,u_{\alpha_n}}| > \varepsilon} |u_{\alpha_n}|^2 \, dx < \varepsilon^2.
\]  

(4.8)

The concentration point \( y_{\alpha,u_{\alpha_n}} \) converges to the maximum point of \( b \) as \( \alpha \to \infty \), that is, for any \( \delta > 0 \), there exists an \( \varepsilon > 0 \) such that for all \( \alpha \geq \alpha_\varepsilon \) and every \( u_{\alpha_n} \in J_{\alpha_n} \), we have \( y_{\alpha,u_{\alpha_n}} \in \{ x \in \mathbb{R}^N \mid |b(x) - (1 - \delta, 1)| \} \), where \( \alpha_\varepsilon \) is the same constant appeared above. Further, in the case \( b \equiv 1 \), for any \( y_{\alpha,u_{\alpha_n}} \) converges to the minimum point of \( V \).

### 4.1 Proof of Theorem 4.1 in the case \( b \equiv 1 \)

We first prove Theorem 4.1 for the case \( b \equiv 1 \). We denote the energy functional without potential by \( \tilde{J}_{\alpha} \), that is

\[
\tilde{J}_{\alpha}(u) := \frac{1}{2}|| \nabla u ||_{L^2}^2 - \frac{\alpha^{p-1}}{p+1}|| u ||_{L^{p+1}}^{p+1}.
\]  

(4.9)

In addition, as the case with potential, we denote the minimal energy as

\[
\tilde{\lambda}_{\alpha} := \inf_{||u||_{L^2} = 1} \tilde{J}_{\alpha}(u),
\]

and the set of ground states of \( \tilde{J}_{\alpha} \) as

\[
\tilde{J}_{\alpha} := \{ u \in H^1 \mid \tilde{J}_{\alpha}(u) = \tilde{\lambda}_{\alpha}, ||u||_{L^2} = 1, u > 0 \}.
\]

It is well known that there exists a ground state of \( \tilde{J}_{\alpha} \) (See Proposition 8.3.6 of [5]).
Proposition 4.1 Let $1 < p < 1 + 4/N$.

(i) There exists a radially symmetric positive function $\phi_\alpha \in H^1(\mathbb{R}^N)$ such that

$$\tilde{J}_\alpha = \{ \phi_\alpha(\cdot - y) \mid y \in \mathbb{R}^N \}.$$  \hfill (4.10)

(ii) Suppose $\{u_n\}, \|u_n\|_{L^2} = 1$ is a minimizing sequence of $\tilde{J}_\alpha$. Then there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $\{y_k\} \subset \mathbb{R}^N$ such that $u_{n_k}(\cdot - y_k)$ converges to $\phi_\alpha$ in $H^1$.

For the proof, see [12] and [18]. Henceforth we denote the positive minimizer centralized at $0$ as $\phi_\alpha$.

By a simple calculation we see the following scaling property of $\tilde{\phi}_\alpha$.

Lemma 4.1 Let $1 < p < 1 + 4/N$. Suppose $\phi_\alpha$ is the positive radial ground state of $\tilde{J}_\alpha$. Further, for $u \in H^1(\mathbb{R}^N)$, set

$$u_\mu(x) = \mu^{N/2}u(\mu x).$$

Then, we have

(i) $\tilde{J}_\alpha(u_{\mu_\alpha}) = \mu_\alpha^2 \tilde{J}_1(u)$,

(ii) $\phi_\alpha = (\phi_1)_{\mu_\alpha}$,

where $\mu_\alpha = \alpha^{\frac{(p-1)}{2(N-p+1)}}$.

Next, we show that the minimal energy of $J_{1,\alpha}$ goes to the minimal energy of $\tilde{J}_\alpha$ which is the energy functional without potential.

Lemma 4.2 Let $1 < p < 1 + 4/N$. Assume (V1). Then we have

$$\lambda_\alpha - \tilde{\lambda}_\alpha \to 0 \quad \text{as} \quad \alpha \to \infty,$$

where $\lambda_\alpha$ is defined in (2.4).

Proof. Since $V \geq 0$, we have $J_\alpha(u) \geq \tilde{J}_\alpha(u)$ for any $u \in H^1(\mathbb{R}^N)$.

Now, for the case

$$\int_{\mathbb{R}^N} V(x)|\phi_\alpha(x - x_0)|^2 \, dx < \infty,$$

for all $\alpha > 0$, where $x_0 \in \{x \in \mathbb{R}^N \mid V(x) = 0\}$. Then we have $J_\alpha(\phi_\alpha(\cdot - x_0)) - \tilde{\lambda}_\alpha \to 0$ ($\alpha \to \infty$). Indeed,

$$J_\alpha(\phi_\alpha(\cdot - x_0)) - \tilde{\lambda}_\alpha = \frac{1}{2} \int_{\mathbb{R}^N} V(x)|\phi_\alpha(x - x_0)|^2 \, dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^N} V(z/\mu_\alpha + x_0)|\phi_1(z)|^2 \, dz$$

$$\to 0.$$
Since \( J_\alpha (\phi (-x_0)) > \lambda_\alpha > \tilde{\lambda}_\alpha \), we have the conclusion.

Next, we consider the case \( V[\phi_\alpha]^2 \notin L^1(\mathbb{R}^N) \). Take \( \chi \in C_c^\infty (\mathbb{R}^N) \) to satisfy \( 0 \leq \chi \leq 1 \) and \( \chi(x) = 1 \) for \( |x| \leq 1 \). Set \( \chi_R(x) := \chi(x/R) \) and \( \phi_{\alpha,R,x_0}(x) := \chi_R(x-x_0)\phi_\alpha(x-x_0) \). Then, we have

\[
J_\alpha (\phi_{\alpha,R,x_0}) - \tilde{\lambda}_\alpha \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \chi_R|^2 |\phi_\alpha|^2 \, dx + \int_{\mathbb{R}^N} |\nabla \chi_R||\phi_\alpha||\nabla \phi_\alpha| \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |1 - \chi_R^2||\nabla \phi_\alpha|^2 \, dx + \frac{1}{p+1} \int_{\mathbb{R}^N} |1 - \chi_R^{p+1}| |\phi_\alpha|^{p+1} \, dx
\]

Now, since \( \phi_1 \) and \( \nabla \phi_1 \) decays exponentially and \( \phi_\alpha = \mu_\alpha^{N/2} \phi_1(\mu_\alpha x) \), we have

\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla \chi_R|^2 |\phi_\alpha|^2 \, dx + \int_{\mathbb{R}^N} |\nabla \chi_R||\phi_\alpha||\nabla \phi_\alpha| \, dx \leq CR^{-1}
\]

Next, we consider the case \( \phi_{\alpha,R,x_0}(x) \). Then, we have

\[
\frac{1}{2} \int_{\mathbb{R}^N} |1 - \chi_R^2||\nabla \phi_\alpha|^2 \, dx + \frac{1}{p+1} \int_{\mathbb{R}^N} |1 - \chi_R^{p+1}| |\phi_\alpha|^{p+1} \, dx
\]

Therefore, taking arbitrary \( \varepsilon > 0 \), there exists \( R_\varepsilon > 0 \) such that

\[
J_\alpha (\phi_{\alpha,R,x_0}) - \tilde{\lambda}_\alpha \leq \varepsilon,
\]

for sufficiently large \( \alpha > 0 \). This gives us the conclusion.

Now for showing the concentration, we define “approximate minimizers”.

**Definition 4.1** For \( \alpha, \delta > 0 \), set

\[
X(\alpha, \delta) := \{ u \in H^1(\mathbb{R}^N) \mid J_\alpha (u) - \tilde{\lambda}_\alpha < \delta, \quad \|u\|_{L^2} = 1 \}.
\]

Functions \( u \in X(\alpha, \delta) \) are called \( \delta \)-approximate minimizers of \( J_\alpha \).

We state another scaling properties of the approximate minimizers of \( J_\alpha \).

**Lemma 4.3** Let \( 1 < p < 1 + 4/N \). Suppose \( \alpha_1, \alpha_0 > 0 \) and \( u \in H^1 \) then we have

\[
\begin{align*}
&u_{\mu_0} \in X(\alpha_1, \delta) \Rightarrow u_{\mu_{\alpha_0}} \in X(\alpha_0, (\alpha_0/\alpha_1)^2 \delta) \\
&\text{for any } \delta > 0, \text{ where } u_{\mu}(x) = \mu^{N/2}(\mu x) \text{ and } \mu_\alpha = \alpha^{2(p-1)/(N(p-1))}.
\end{align*}
\]
Proof. Suppose \( u_{\mu_{\alpha_1}} \in X(\alpha_1, \delta) \), then we have
\[
\delta > \tilde{J}_{\alpha_1}(u_{\mu_{\alpha_1}}) - \tilde{\lambda}_{\alpha_1} \\
= \mu_{\alpha_1}^2 (\tilde{J}_1(u) - \tilde{\lambda}_1) \\
= \left( \frac{\mu_{\alpha_1}}{\mu_{\alpha_0}} \right)^2 \mu_{\alpha_0}^2 (\tilde{J}_1(u) - \tilde{\lambda}_1) \\
= \left( \frac{\mu_{\alpha_1}}{\mu_{\alpha_0}} \right)^2 (\tilde{J}_{\alpha_0}(u_{\mu_{\alpha_0}}) - \tilde{\lambda}_{\alpha_0}).
\]

Therefore we have the conclusion. \( \blacksquare \)

We now show that \( \delta \)-approximate minimizers concentrate around one point in \( \mathbb{R}^N \). This is the key lemma to prove Theorem 4.1.

**Lemma 4.4** Let \( 1 < p < 1 + 4/N \). For any \( \varepsilon > 0 \), there exist positive constants \( \delta_{\varepsilon} \) and \( \alpha_{\varepsilon} \) such that for all \( u \in X(\alpha_{\varepsilon}, \delta_{\varepsilon}) \), there exists \( y_0 \in \mathbb{R}^N \) such that
\[
\int_{|x-y_0|>\varepsilon} (|\nabla u|^2 + |u|^2) \, dx < \varepsilon.
\]

Proof. We prove Lemma 4.4 by contradiction. Suppose there exists an \( \varepsilon_0 > 0 \) such that for any \( \delta > 0 \) and \( \alpha > 0 \), there exists \( u_{\alpha, \delta} \in X(\alpha, \delta) \) such that
\[
\inf_{y\in\mathbb{R}^N} \int_{|x-y|>\varepsilon_0} (|\nabla u_{\alpha, \delta}|^2 + |u_{\alpha, \delta}|^2) \, dx > \varepsilon_0. \tag{4.11}
\]

Now take \( \alpha_1 > 0 \) sufficiently large so that for \( \alpha \geq \alpha_1 \),
\[
\int_{|x|>\varepsilon_0} (|\nabla \phi_{\alpha}|^2 + |\phi_{\alpha}|^2) \, dx < \frac{\varepsilon_0}{2}. \tag{4.12}
\]

This is possible because of Lemma 4.1 and the fact that \( \phi_1 \) decays exponentially. Now for \( \delta_{n} \to 0 \), set \( u_n := u_{\alpha_{n}, \delta_{n}} \). Then by the definition of \( \delta \)-approximate minimizer, \( u_n \) is a minimizing sequence of \( \tilde{J}_n \). So, because of Proposition 4.1, there exists a subsequence \( \{u_{n_k}\} \subset \{u_n\} \) and \( \{y_k\} \subset \mathbb{R}^N \) such that \( u_{n_k} \cdot (\cdot - y_k) \to \phi_{\alpha} \) strongly in \( H^1(\mathbb{R}^N) \). However by assumption we have
\[
\int_{|x|>\varepsilon_0} (|\nabla u_{n_k}(x - y_k)|^2 + |u_{n_k}(x - y_k)|^2) \, dx > \varepsilon_0.
\]

This is a contradiction. \( \blacksquare \)

We now prove Theorem 4.1 for the case \( b \equiv 1 \).

**Proof.** [Proof of Theorem 4.1 for the case \( b \equiv 1 \)] Fix \( \varepsilon > 0 \) arbitrary. Take \( \alpha_{\varepsilon} \) and \( \delta_{\varepsilon} \) as Lemma 4.4. Further take \( \alpha_1 > \alpha_{\varepsilon} \) sufficiently large so that \( |\lambda_{\alpha_1} - \lambda_{\alpha_{\varepsilon}}| < \delta_{\varepsilon} \). This is possible because of Lemma 4.2. Then, for a ground state \( u_{\alpha_1} \in J_{\alpha_1} \), we
have \( u_{\alpha_1} \in X(\alpha_1, \delta_\varepsilon) \). So, we have \((u_{\alpha_1})_{\mu_{\alpha_1}/\mu_1} \in X(\alpha_\varepsilon, (\mu_{\alpha_1}/\mu_1)^2 \delta_\varepsilon)\) by Lemma 4.3. Note that \((\mu_{\alpha_1}/\mu_1)^2\) is smaller than 1. Therefore, by Lemma 4.4, there exists \( y_0 \in \mathbb{R}^N \) such that

\[
\int_{|x-y_0|>\varepsilon} |(u_{\alpha_1})_{\mu_{\alpha_1}/\mu_1}|^2 dx < \varepsilon.
\]

Since,

\[
\int_{|x-y_0|>\varepsilon} |(u_{\alpha_1})_{\mu_{\alpha_1}/\mu_1}|^2 dx = \int_{|z-y_1|>(\mu_{\alpha_1}/\mu_1)\varepsilon} |u_{\alpha_1}|^2 dz
\]

for \( y_1 = (\mu_{\alpha_1}/\mu_1)y_0 \), we have the \( L^2 \)-concentration (4.8).

Finally we consider where \( y_\alpha \) goes. Suppose there exists \( \delta_0 > 0 \) and \( \alpha_n \to \infty \), such that there exists a ground state \( u_n \) of \( J_{\alpha_n} \) such that

\[
\inf_n V(y_{\alpha_n}, u_n) > \delta_0.
\]

Take \( \varepsilon > 0 \) sufficiently small, so that \( \inf_{|x-y_{\alpha_n, u_n}|<\varepsilon} V(x) > \delta_0/2 \) for all \( n \). Then, we have

\[
\int_{\mathbb{R}^N} V|u_n|^2 dx \geq \frac{1}{2} \delta_0 (1 - \varepsilon).
\]

This contradicts Lemma 4.2.

\[\square\]

### 4.2 Proof of Theorem 4.1 in the general case

In this section, we consider the general case. Let \( \tilde{J}_\alpha \) be the one defined in (4.9). We first estimate the difference of the minimum energy of \( J_\alpha \) and \( \tilde{J}_\alpha \).

**Lemma 4.5** Let \( 1 < p < 1 + 4/N \). Assume (V1) and (b1) Then, for any \( \varepsilon > 0 \), there exists an \( \alpha_\varepsilon \) such that for any \( \alpha \geq \alpha_\varepsilon \),

\[
\mu_{\alpha}^2 (\lambda_\alpha - \tilde{\lambda}_\alpha) \to 0, \quad (\alpha \to \infty),
\]

where \( \lambda_\alpha \) is defined in (2.4).

**Proof.** Let \( \phi_\alpha \) be the minimizer of \( \tilde{J}_\alpha \). Take \( x_0 \) to satisfy \( b(x_0) = 1 \). Then as Lemma 4.2, we see that

\[
J_\alpha(\phi_\alpha(-x_0)) - \tilde{\lambda}_\alpha = \mu_{\alpha}^2 \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla \phi_\alpha|^2 + \frac{1}{p+1} b(x/\mu_\alpha + x_0)|\phi_\alpha|^{p+1} \right) dx
\]

\[+ \int_{\mathbb{R}^N} V(x/\mu_\alpha + x_0)|\phi_\alpha|^2 dx.
\]

Therefore we have the conclusion. \[\square\]
Lemma 4.6 Let $1 < p < 1 + 4/N$. Let $u_\alpha \in \mathcal{J}_\alpha$ and let $\phi_1$ be the positive radial ground state of $\tilde{J}_1$. Let $\psi_\alpha = \mu_\alpha^{-N/2} u_\alpha (\mu_\alpha^{-1} x)$. Then for each $\alpha$, there exists a $z_\alpha \in \mathbb{R}^N$ such that $\psi_\alpha (\cdot + z_\alpha) \rightarrow \phi_1$ in $H^1$.

Proof. By Proposition 4.1 and Lemmas 4.3, 4.5, there exists a sequences $\{\alpha_k\}$, $z_{\alpha_k}$ such that $\psi_{\alpha_k} (\cdot + z_{\alpha_k}) \rightarrow \phi_1$ in $H^1$. Therefore, since arbitrary subsequence converges to the same function, we have the conclusion. 

Remark 4.1 By rescaling, one sees that $z_\alpha = \mu_\alpha y_{\alpha,u_n}$.

We now prove Theorem 4.1 for the general case. The strategy of the proof is same as the case $b = 1$.

Proof. [Proof of Theorem 4.1 for general case] Take $\varepsilon$ arbitrary. Take $\alpha_\varepsilon$, $\delta_\varepsilon$ as in Lemma 4.4. Take $\alpha > \alpha_\varepsilon$ sufficiently large so that

$$
(\mu_\alpha / \mu_\varepsilon)^2 (V(x_0) + 1) < \delta_\varepsilon
$$

and $u_\alpha \in X(\alpha, V(x_0) + 1)$, where $u_\alpha \in \mathcal{J}_\alpha$. Then we have $(u_\alpha (\cdot + z_{\alpha})) \in X(\alpha, \delta_\varepsilon)$. Therefore, by Lemma 4.4 and the proof of Theorem 4.1, we see that $u_\alpha$ concentrates its $L^2$-norm around a point $y_{\alpha,u_n} \in \mathbb{R}^N$. Note that by Lemma 4.6, we can take $y_{\alpha,u_n} = \mu_\alpha^{-1} z_\alpha$.

Next, suppose there exists a sequence $\delta_0 > 0$ and $\{\alpha_n\}$, $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$, such that there exists a ground state $u_n$ of $J_{\alpha_n}$ such that $\sup_n b(y_{\alpha_n,u_n}) \leq 1 - \delta_0$. Then by Lemma 4.6, we have $\psi_n (\cdot + z_{\alpha_n}) \rightarrow \phi_1$ in $H^1$. Therefore, since

$$
\mu_{\alpha_n}^{-2} J_{\alpha_n}(u_n) = \frac{1}{2} \|\nabla \psi_n\|_{L^2}^2 + \frac{1}{2} \mu_{\alpha_n}^{-2} \int_{\mathbb{R}^N} V(x/\mu_{\alpha_n} + y_{\alpha_n,u_n}) |\psi_n|^2 \, dx
$$

$$
= - \frac{1}{p+1} \int_{\mathbb{R}^N} b(x/\mu_{\alpha_n} + y_{\alpha_n,u_n}) |\psi_n|^{p+1} \, dx,
$$

we see that

$$
\liminf_{n \rightarrow \infty} \mu_{\alpha_n}^{-2} J_{\alpha_n}(u_n) \geq \frac{1}{2} \|\nabla \phi_1\|_{L^2}^2 - \frac{1 - \delta_0/2}{p+1} \int_{\mathbb{R}^N} |\phi_1|^{p+1} \, dx.
$$

On the other hand, by Lemmas 4.1, 4.5, we see that $\mu_{\alpha}^{-2} (J_{\alpha}(\phi_{\alpha})) \rightarrow \tilde{\lambda}_1$. This is a contradiction.

4.3 Proof of Theorem 1.2

In this section, we show the exponential decay of the ground states. First, we investigate the asymptotic of the Lagrange multiplier $\nu_{\alpha}$ in equation (2.5). Let $\tilde{J}_{\alpha}$ be the one defined in (4.9). Let $\tilde{\nu}_1 < 0$ be the constant such that

$$
- \Delta \phi_1 - \phi_1^p = \tilde{\nu}_1 \phi_1,
$$

where $\phi_1$ is the ground state of $\tilde{J}_1$. 

We now show that a rescaled ground state of $J_{\alpha}$ converges to $\phi_\alpha$ in $H^1$. 

We now prove Theorem 4.1 for the general case. The strategy of the proof is same as the case $b = 1$.

Proof. [Proof of Theorem 4.1 for general case] Take $\varepsilon$ arbitrary. Take $\alpha_\varepsilon$, $\delta_\varepsilon$ as in Lemma 4.4. Take $\alpha > \alpha_\varepsilon$ sufficiently large so that

$$
(\mu_\alpha / \mu_\varepsilon)^2 (V(x_0) + 1) < \delta_\varepsilon
$$

and $u_\alpha \in X(\alpha, V(x_0) + 1)$, where $u_\alpha \in \mathcal{J}_\alpha$. Then we have $(u_\alpha (\cdot + z_{\alpha})) \in X(\alpha, \delta_\varepsilon)$. Therefore, by Lemma 4.4 and the proof of Theorem 4.1, we see that $u_\alpha$ concentrates its $L^2$-norm around a point $y_{\alpha,u_n} \in \mathbb{R}^N$. Note that by Lemma 4.6, we can take $y_{\alpha,u_n} = \mu_\alpha^{-1} z_\alpha$.

Next, suppose there exists a sequence $\delta_0 > 0$ and $\{\alpha_n\}$, $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$, such that there exists a ground state $u_n$ of $J_{\alpha_n}$ such that $\sup_n b(y_{\alpha_n,u_n}) \leq 1 - \delta_0$. Then by Lemma 4.6, we have $\psi_n (\cdot + z_{\alpha_n}) \rightarrow \phi_1$ in $H^1$. Therefore, since

$$
\mu_{\alpha_n}^{-2} J_{\alpha_n}(u_n) = \frac{1}{2} \|\nabla \psi_n\|_{L^2}^2 + \frac{1}{2} \mu_{\alpha_n}^{-2} \int_{\mathbb{R}^N} V(x/\mu_{\alpha_n} + y_{\alpha_n,u_n}) |\psi_n|^2 \, dx
$$

$$
= - \frac{1}{p+1} \int_{\mathbb{R}^N} b(x/\mu_{\alpha_n} + y_{\alpha_n,u_n}) |\psi_n|^{p+1} \, dx,
$$

we see that

$$
\liminf_{n \rightarrow \infty} \mu_{\alpha_n}^{-2} J_{\alpha_n}(u_n) \geq \frac{1}{2} \|\nabla \phi_1\|_{L^2}^2 - \frac{1 - \delta_0/2}{p+1} \int_{\mathbb{R}^N} |\phi_1|^{p+1} \, dx.
$$

On the other hand, by Lemmas 4.1, 4.5, we see that $\mu_{\alpha}^{-2} (J_{\alpha}(\phi_{\alpha})) \rightarrow \tilde{\lambda}_1$. This is a contradiction.
Lemma 4.7 Let $1 < p < 1 + 4/N$. Assume (V1). Suppose $\nu_\alpha$ be the Lagrange multiplier $\nu_\alpha$ in equation (2.5) and let $\mu_\alpha = \alpha^{1-4/p}$. Then we have

$$
\mu_\alpha^{-2} \nu_\alpha \rightarrow \tilde{\nu}_1, \quad (\alpha \rightarrow \infty).
$$

Proof. Let $u_\alpha \in \mathcal{J}_\alpha$. Set $\psi_\alpha(x) := (u_\alpha)_{\alpha^{-1/2}}(x + z_\alpha) := \mu_\alpha^{-1/2} u_\alpha (\mu_\alpha^{-1} x + y_\alpha, u_\alpha)$, where $z_\alpha = \mu_\alpha y_\alpha, u_\alpha$ and $y_\alpha, u_\alpha \in \mathbb{R}^N$ is the concentration point of $u_\alpha$ given in Theorem 4.1. By simple calculation, we see that $\psi_\alpha$ satisfies the following equation.

$$
-\Delta \psi_\alpha + \mu_\alpha^{-2} V(x/\mu_\alpha + y_\alpha) \psi_\alpha - b(x/\mu_\alpha + y_\alpha) \psi_\alpha^{\mu_\alpha - 2} = \mu_\alpha^{-2} \nu_\alpha \psi_\alpha. \quad (4.13)
$$

Since $\| \phi_1 \|_{L^2} = 1$ and $\| \psi_\alpha \|_{L^2} = 1$, we have

$$
\begin{align*}
\tilde{\nu}_1 &= |\nabla \phi_1|_{L^2}^2 - \int_{\mathbb{R}^N} \phi_1^{p+1} \ dx \\
\mu_\alpha^{-2} \nu_\alpha &= |\nabla \psi_\alpha|_{L^2}^2 + \mu_\alpha^{-2} \int_{\mathbb{R}^N} V(x/\mu_\alpha + y_\alpha, u_\alpha) |\psi_\alpha|^2 \ dx \\
&\quad - \int_{\mathbb{R}^N} b(x/\mu_\alpha + y_\alpha, u_\alpha) \psi_\alpha^{p+1} \ dx.
\end{align*}
$$

On the other hand, by Lemma 4.6, $\psi_\alpha \rightarrow \phi_1$ as $\alpha \rightarrow \infty$ in $H^1$. Therefore we have the conclusion.

We now establish an elliptic estimate independent of $\alpha$.

Lemma 4.8 Assume (V1) and (V2). Let $q \in [1, \infty]$. Then, there exists an $\alpha_0 > 0$ such that

$$
\|(-\Delta + \mu_\alpha^{-2} V(\cdot/\mu_\alpha) - \mu_\alpha^{-2} \nu_\alpha)^{-1} u\|_{W^{2,q}} \leq C\|u\|_{L^q}, \quad (4.14)
$$

for any $\alpha \geq \alpha_0$ and $u \in L^q$, where $C$ is independent of $\alpha$.

Proof. By Lemma 4.7, we see that there exists an $\alpha_0 > 0$ such that, for $\alpha \geq \alpha_0$,

$$
-2\tilde{\nu}_1 > -\mu_\alpha^{-2} \nu_\alpha > -\tilde{\nu}_1/2 > 0. \quad \text{Since } \mu_\alpha^{-2} V(\cdot/\mu_\alpha) \text{ satisfies (V2), from [20], we see that (4.14) holds with some } C_\alpha \text{ which may depend on } \alpha. \text{ Therefore, we only have to show that we can take } C_\alpha \text{ independent of } \alpha.
$$

Now, (4.14) is equivalent to

$$
\|v\|_{W^{2,q}} \leq C\|(-\Delta + \mu_\alpha^{-2} V(\cdot/\mu_\alpha) - \mu_\alpha^{-2} \nu_\alpha)v\|_{L^q}.
$$

Since, $-2\tilde{\nu}_1 > \mu_\alpha^{-2} \nu_\alpha > -\tilde{\nu}_1/2 > 0$, we see by the elliptic regularity theorem that

$$
\|v\|_{W^{2,q}} \leq C\|(-\Delta - \mu_\alpha^{-2} \nu_\alpha)v\|_{L^q},
$$

where $C$ is independent of $\alpha$. Therefore, we only have to show

$$
\|(-\Delta - \mu_\alpha^{-2} \nu_\alpha)v\|_{L^q} \leq C\|(-\Delta + \mu_\alpha^{-2} V(\cdot/\mu_\alpha) - \mu_\alpha^{-2} \nu_\alpha)v\|_{L^q}.
$$
for a constant $C$ independent of $\alpha$.

Further, since

$$-\Delta - \mu_\alpha^{-2} \nu_\alpha = \left(1 - \mu_\alpha^{-2} V(\cdot/\mu_\alpha) \left(-\Delta + \mu_\alpha^{-2} V(\cdot/\mu_\alpha) - \mu_\alpha^{-2} \nu_\alpha\right)^{-1}\right) \times \left((-\Delta + \mu_\alpha^{-2} V(\cdot/\mu_\alpha) - \mu_\alpha^{-2} \nu_\alpha\right),$$

we only have to show

$$\sup_{\alpha \geq \alpha_0} \|\mu_\alpha^{-2} V(\cdot/\mu_\alpha) \left(-\Delta + \mu_\alpha^{-2} V(\cdot/\mu_\alpha) - \mu_\alpha^{-2} \nu_\alpha\right)^{-1}\| < \infty,$$

where $\|\cdot\|$ is an operator norm for an operator on $L^q$. By the uniform boundedness principle, it suffices to show that for any $u \in L^q$,

$$\sup_{\alpha \geq \alpha_0} \|\mu_\alpha^{-2} V(\cdot/\mu_\alpha) \left(-\Delta + \mu_\alpha^{-2} V(\cdot/\mu_\alpha) - \mu_\alpha^{-2} \nu_\alpha\right)^{-1} u\|_{L^q} < \infty. \quad (4.15)$$

Dividing $u$ to $u^+ := \max(u,0)$ and $u^- := u - u^+$, it suffices to show (4.15) for $u \leq 0$.

Now, by the Laplace transform we have

$$\|\mu_\alpha^{-2} V(\cdot/\mu_\alpha) \left(-\Delta + \mu_\alpha^{-2} V(\cdot/\mu_\alpha) - \mu_\alpha^{-2} \nu_\alpha\right)^{-1} u\|_{L^q}^q = \int_{\mathbb{R}^N} \left|\mu_\alpha^{-2} V(x/\mu_\alpha) \int_0^\infty e^{\mu_\alpha^{-2} \nu_\alpha t} e^{-\left(-\Delta + \mu_\alpha^{-2} V(\cdot/\mu_\alpha)\right)t} u dt\right|^q \, dx$$

$$= \mu_\alpha^q \int_{\mathbb{R}^N} \left|V(y) \int_0^\infty e^{\mu_\alpha^{-2} \nu_\alpha t} e^{-\left(-\Delta + V\right)t} u(\mu_\alpha y) dt\right|^q \, dy$$

$$= \mu_\alpha^q \left\|V(-\Delta + V - \nu_\alpha)^{-1} u(\mu_\alpha)\right\|^q_{L^q}.$$ 

Next, take $\alpha$ sufficiently large so that $-\nu_\alpha > 1$. Let $f_\alpha(x) := (-\Delta + V - \nu_\alpha)^{-1} u(\mu_\alpha x)$ and $f_1 := (-\Delta + V + 1)^{-1} u(\mu_\alpha x)$. Then, by the Laplace transform,

$$f_\alpha(x) = \int_0^\infty e^{t(\nu_\alpha + \Delta - V)} u(\mu_\alpha x) \, dx$$

$$f_1(x) = \int_0^\infty e^{t(-1+\Delta - V)} u(\mu_\alpha x) \, dx.$$ 

One can show that $e^{t(\Delta - V)}$ is positivity preserving by the Trotter product formula and the fact that $e^{t\Delta}$ is positivity preserving (See [22]). Therefore, we see that by the positivity of $u(\mu_\alpha x)$, $f_\alpha$ and $f_1$ are nonnegative. Further, subtracting the equations which $f_\alpha$ and $f_1$ satisfies we have

$$(-\Delta + V + 1)(f_1 - f_\alpha) = (-\nu_\alpha - 1)f_\alpha.$$ 

Since the right hand side of the equation is nonnegative, we have $f_\alpha \leq f_1$. Therefore, we have

$$\mu_\alpha \left\|V(-\Delta + V - \nu_\alpha)^{-1} u(\mu_\alpha)\right\|_{L^q}^q \leq \mu_\alpha \left\|V(-\Delta + V + 1)^{-1} u(\mu_\alpha)\right\|_{L^q}^q.$$
On nonlinear Schrödinger equation with potential

Since $V(−\Delta + V + 1)^{-1}$ is a bounded operator in $L^q$, we have

$$
\|\mu_α^{-2}V(·/μ_α)(−\Delta + \mu_α^{-2}V(·/μ_α) − \mu_α^{-2}ν_α)^{-1}u\|_{L^q} ≤ C\mu_α^N\|u(μ_α)\|_{L^q},
$$

which shows (4.15). Therefore we have the conclusion.

Next, we show that $ψ_α$ converges to $φ_1$ in $L^∞$-norm.

**Lemma 4.9** Let $1 < p < 1 + 4/N$. Assume (V1), (V2) and (b1). Let $ψ_α$ and $z_α$ as in the proof of Lemma 4.7. Then,

$$
ψ_α(· + z_α) → φ_1, \ (α → ∞) \ \text{in} \ L^∞.
$$

**Proof.** Since $L^∞(R) \hookrightarrow H^1(R)$, for the case $N = 1$, Lemma 4.9 is obvious. We consider for $N ≥ 2$.

First, rewriting (4.13), we have

$$(−\Delta + \mu_α^{-2}V(·/μ_α + y_α,u_α) − \mu_α^{-2}ν_α)ψ_α = b(x/μ_α + y_α,u_α)ψ_α^p,$$

where $y_α,u_α = μ_α^{-1}z_α$. So, by (4.14) and by a standard bootstrap argument, there exists a constant $M < ∞$ such that

$$
\sup_{α > 1} \|ψ_α\|_{L^p} \leq M.
$$

Next, by Theorem 9.20 of [13], we have

$$
\sup_{|x−y|<r} \psi_α \leq C \left( \frac{1}{r^N} \int_{|x−y|<2r} \psi_α^2 \, dx \right)^{1/2} + r \|\psi_α^p\|_{L^N},
$$

(4.16)

where $C$ does not depend on $y ∈ R^N$, $r ≤ 1$ and $α ≥ 1$.

Now, fix $ε > 0$ arbitrary. Take $r > 0$ so that $CrM^p < ε/2$. Next, since $ψ_α → φ_1$ in $H^1$, and $φ_1$ decays exponentially, we have that for sufficiently large $R_0, α_0$ that

$$
Cr^{-N} \int_{|x−y|<2r} \psi_α^2 \, dx < \frac{1}{2} ε
$$

for $y ≥ R_0$ and $α ≥ α_0$.

On the other hand, using the bootstrap argument, we can show that $ψ_α$ converges to $φ_1$ uniformly on compacts. Therefore we have the conclusion.

Now, we prove Theorem 1.2.

**Proof.** [Proof of Theorem 1.2 (i)] First, by Lemma 4.9 and the exponential decay of $φ_1$, we have, for any $ε > 0$ and sufficiently large $α$,

$$
ψ_α(x) ≤ C \exp(−C|x|) + ε.
$$
Therefore, since \( y_{\alpha,u,a} = \mu_{\alpha}^{-1} z_{\alpha} \), we have
\[
\phi_{\alpha}(x) \leq C\mu_{\alpha}^{N/2} \exp(-C\mu_{\alpha}|x - y_{\alpha,u,a}|) + \mu_{\alpha}^{N/2}\varepsilon.
\]

Now, by Lemma 4.7, \( \nu_{\alpha} \sim \mu_{\alpha}^{2} = \alpha^{\frac{4(p-1)}{p^2 - 1}} \). So,
\[
\left( \frac{\nu_{\alpha}}{2\alpha p - 1} \right)^{\frac{1}{2-p}} \sim \alpha^{\frac{4(p-1)}{p^2 - 1} - 1} = \alpha^{\frac{N(p-1)}{p^2 - 1}} = \mu_{\alpha}^{N/2}.
\]

Therefore, taking \( \varepsilon > 0 \) sufficiently small and taking \( \alpha \) sufficiently large, we see that
\[
u_{\alpha}(x) \leq \left( \frac{\nu_{\alpha}}{2\alpha p - 1} \right)^{\frac{1}{2-p}} , \quad \text{for } |x - y_{\alpha,u,a}| > L\mu_{\alpha}^{-1}.
\]

(4.17)

where \( L > 0 \) is a constant sufficiently large and independent of \( \alpha \).

Next, set \( P = -\Delta + \frac{1}{2}\nu_{\alpha} \). Then, by (4.17) and since \( u_{\alpha} \) satisfies
\[
-\Delta \phi_{\alpha} + V \phi_{\alpha} - \alpha^{p-1} \phi_{\alpha} + \nu_{\alpha} \phi_{\alpha} = 0,
\]
we obtain \( P u_{\alpha} \leq 0 \) for \( |x - y_{\alpha,u,a}| > L\mu_{\alpha}^{-1} \). Further, set
\[
v_{L,\tilde{L}} := ||\psi||_{L^{\infty}} \left( \exp(-\delta(|x - y_{\alpha,u,a}| - L\mu_{\alpha}^{-1})) + \exp(-\delta(\tilde{L} - |x - y_{\alpha,u,a}|)) \right).
\]

Then, we have
\[
P v_{L,\tilde{L}} = ||\psi||_{L^{\infty}} \left( -\delta^2 + \frac{N-1}{r} \delta + \frac{1}{2}\nu_{\alpha} \right) e^{-\delta(\tilde{L} - r\mu_{\alpha}^{-1})}
\]
\[+ ||\psi||_{L^{\infty}} \left( -\delta^2 + \frac{N-1}{r} \delta + \frac{1}{2}\nu_{\alpha} \right) e^{-\delta(\tilde{L} - r)},
\]

where \( r = |x - y_{\alpha}| \). So, taking \( \delta = \nu_{\alpha}^{1/2}/2 \) and taking \( \alpha \) sufficiently large, we have \( P v_{L,\tilde{L}} \geq 0 \). Since
\[v_{L,\tilde{L}}(x) \geq u_{\alpha}(x) \quad \text{on } \{|x - y_{\alpha,u,a}| = L\mu_{\alpha}^{-1}\} \cup \{|x - y_{\alpha,u,a}| = \tilde{L}\}
\]
and
\[P(v_{L,\tilde{L}} - u_{\alpha}) \geq 0 \quad \text{in } \{|L\mu_{\alpha}^{-1} < |x - y_{\alpha,u,a}| < \tilde{L}\},
\]
we have
\[u_{\alpha}(x) \leq v_{L,\tilde{L}} \quad \text{in } \{|L\mu_{\alpha}^{-1} < |x - y_{\alpha,u,a}| < \tilde{L}\}.
\]

Taking \( \tilde{L} \to \infty \), we have
\[
\phi_{\alpha} \leq C\mu_{\alpha}^{N/2} \exp(-C\mu_{\alpha}(|x - y_{\alpha,u,a}| - L\mu_{\alpha}^{-1})).
\]

Therefore we have the exponential decay result.
We next prove the uniqueness of local maximum point. It suffices to show that \( \psi_\alpha \) has only one local maximum point. Suppose, \( z_0 \in \mathbb{R}^N \) is a local maximum point of \( \psi_\alpha \). Then, since \( \psi_\alpha \) satisfies
\[
-\Delta \psi_\alpha + \tilde{\nu}_1 \psi_\alpha - (\tilde{\nu}_1 - \mu_\alpha^{-2} \nu_\alpha) \psi_\alpha \leq b(x/\mu_\alpha + y_{\alpha,u_n}) \psi_\alpha^p
\]
and \( -\Delta \psi_\alpha(z_0) \geq 0 \), we have
\[
\tilde{\nu}_1/2 \leq \psi_\alpha(z_0)^{p-1}, \tag{4.18}
\]
for sufficiently large \( \alpha \), because \( b \leq 1 \). So, by the exponential decay result, the local maximum point has to be near 0. However, \( \psi_\alpha \) converges to \( \phi_1 \) in \( C^2 \) on compacts, so there can be only one local maximum point for sufficiently large \( \alpha > 0 \).

5 Proof of Theorem 1.3

As the previous section, we set \( \phi_1 \) to be the positive radial ground state of \( \tilde{J}_1 \) under the constraint \( \|u\|_{L^2} = 1 \).

Lemma 5.1 Let \( 1 < p < 1 + 4/N \). Assume \( V \) and \( b \) satisfies the assumptions of Theorem 1.3. Take \( \{\alpha_n\} \) to be a positive sequence which goes to infinity and \( u_n \in J_{\alpha_n} \) to be a ground state of \( J_{\alpha_n} \). Let \( y_n := y_{\alpha_n,u_n} \) be the unique local maximum point of \( u_n \). Suppose \( y_n \) converges to \( y_0 \in b^{-1}(\{1\}) \).

(i) If \( 2 - \beta(y_0) > 0 \), then
\[
\lim_{n \to \infty} \mu_\alpha^{-\beta(y_0)} (\lambda_{\alpha_n} - \tilde{\lambda}_{\alpha_n}) = \frac{1}{p+1} \int_{\mathbb{R}^N} Q_{y_0}(x) \phi_1^{p+1} dx.
\]

(ii) If \( 2 - \beta(y_0) \leq 0 \), then
\[
\lim_{n \to \infty} (\lambda_{\alpha_n} - \tilde{\lambda}_{\alpha_n}) = \sigma(\beta(y_0)) \frac{1}{p+1} \int_{\mathbb{R}^N} Q_{y_0}(x) \phi_1^{p+1} dx + \frac{1}{2} V(y_0).
\]

Here, \( Q_{y_0} \) and \( \sigma \) are defined in the claim of Theorem 1.3.

Remark 5.1 By lemma 5.1, Theorem 1.2 immediately follows.

Proof. We calculate \( J_{\alpha_n}(u_n) \). Set \( \psi_n(x) = \mu_\alpha^{-N/2} u_n(\mu_\alpha^{-1} x + y_0) \). Then, since we have assumed \( y_{\alpha_n,u_n} \to y_0 \) and by Lemmas 4.6 and 4.9, we have \( \psi_n \to \phi_1 \) in \( L^\infty \).
and $H^1$.

\[
J_{\alpha_n}(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u_n^2 \, dx - \frac{\alpha_{\alpha_n}^{p-1}}{p+1} \int_{\mathbb{R}^N} b(x) u_n^{p+1} \, dx \\
= \mu_{\alpha_n}^2 \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi_n|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} \psi_n^{p+1} \, dx \right) \\
+ \frac{1}{p+1} \mu_{\alpha_n}^{2-\beta(y_0)} \int_{B(y_0, \mu_{\alpha_n} r)} Q_{\gamma_0}(x) \psi_n^{p+1} \, dx \\
- \frac{1}{p+1} \mu_{\alpha_n}^{2+\gamma(y_0)-\gamma(\theta)} \int_{B(y_0, \mu_{\alpha_n} r)} R_{\gamma_0}(x) \psi_n^{p+1} \, dx \\
- \frac{\alpha_{\alpha_n}^{p-1}}{p+1} \int_{B^{\varepsilon}(y_0,r)} (b(x) - 1) u_n \, dx.
\]

Now, by Theorem 1.2 $u_n$ decays exponentially, so we have

\[
\frac{\alpha_{\alpha_n}^{p-1}}{p+1} \int_{B^{\varepsilon}(y_0,r)} (1 - b(x)) u_n^{p+1} \, dx \leq C_1 e^{-C_2 r \alpha_{\alpha_n}^3},
\]

for some $C_j > 0$, $j = 1, 2, 3$. Further, since $\psi_n \rightarrow \phi_1$ in $H^1$, by Theorem 1.2 and $V(x) \leq C(1 + |x|)^p$, we obtain

\[
\mu_{\alpha_n}^2 \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi_n|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} \psi_n^{p+1} \, dx \right) \geq \tilde{\lambda}_{\alpha_n},
\]

\[
\frac{1}{p+1} \mu_{\alpha_n}^{2-\beta(y_0)} \int_{B(y_0, \mu_{\alpha_n} r)} Q_{\gamma_0}(x) \psi_n^{p+1} \, dx = \frac{1}{p+1} \mu_{\alpha_n}^{2-\beta(y_0)} \int_{\mathbb{R}^N} Q_{\gamma_0} \phi_1^{p+1} \, dx + o(\mu_{\alpha_n}^{2-\beta(y_0)}),
\]

as $\alpha_n \rightarrow \infty$. Thus, we obtain

\[
J_{\alpha_n}(u_n) \geq \tilde{\lambda}_{\alpha_n} + \frac{1}{2} V(y_0) + o(1) + \mu_{\alpha_n}^{2-\beta(y_0)} \frac{1}{p+1} \int_{\mathbb{R}^N} Q_{\gamma_0}(x) \phi_1^{p+1} \, dx + o(\mu_{\alpha_n}^{2-\beta(y_0)}).
\]

Next, take $v_n(x) = \mu_{\alpha_n}^{N/2} \phi_1(\mu_{\alpha_n}(x + y_0))$. Then, we have

\[
J_{\alpha_n}(v_n) = \tilde{\lambda}_{\alpha_n} + \frac{1}{2} V(y_0) + o(1) + \mu_{\alpha_n}^{2-\beta(y_0)} \frac{1}{p+1} \int_{\mathbb{R}^N} Q_{\gamma_0}(x) \phi_1^{p+1} \, dx + o(\mu_{\alpha_n}^{2-\beta(y_0)}).
\]

Since $J_{\alpha_n}(v_n) \geq J_{\alpha_n}(u_n)$, we have the conclusion. $\blacksquare$
6 Proof of Theorems 1.4 and 1.5

To show Theorem 1.4, we follow the way of [14] and use the results of Theorem 1.2. We first prepare a useful identity following Wang [24].

**Lemma 6.1** Let \( b, V \in C^1 \). Assume (V1), (V2), (b1) and that \(|\nabla V|, |\nabla b| \leq C(1 + |x|)^r \) for some \( r \). Let \( u_\alpha \in J_\alpha \) and let \( \psi_\alpha := \mu_\alpha^{-N/2}u_\alpha(\cdot/\mu_\alpha) \). Then

\[
\int_{\mathbb{R}^N} \frac{\mu_\alpha^{-2}}{2} \partial_{x_j} V(\mu_\alpha^{-1}x)\psi_\alpha^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} \partial_{x_j} b(\mu_\alpha^{-1}x)\psi_\alpha^{p+1} \, dx = 0. \tag{6.19}
\]

**Proof.** \( \psi_\alpha \) satisfies

\[
-\Delta \psi_\alpha + (\mu_\alpha^{-2}V(\mu_\alpha^{-1}x) - \mu_\alpha^{-2}\nu_\alpha) \psi_\alpha - b(\mu_\alpha^{-1}x)\psi_\alpha^{p-1} = 0, \tag{6.20}
\]

Multiplying (6.20) by \( \nabla \psi_\alpha \) and integrating on \( B_R(0) \), we have

\[
0 = \int_{B_R(0)} \Delta \psi_\alpha \nabla \psi_\alpha \, dx - \int_{B_R(0)} (\mu_\alpha^{-2}V(\mu_\alpha^{-1}x) - \mu_\alpha^{-2}\nu_\alpha) \psi_\alpha \nabla \psi_\alpha \, dx
+ \int_{B_R(0)} b(\mu_\alpha^{-1}x)\psi_\alpha^p \nabla \psi_\alpha \, dx
= \int_{B_R(0)} \Delta \psi_\alpha \nabla \psi_\alpha \, dx - \frac{1}{2} \int_{B_R(0)} \nabla \left( (\mu_\alpha^{-2}V(\mu_\alpha^{-1}x) - \mu_\alpha^{-2}\nu_\alpha) \psi_\alpha^2 \right) \, dx
+ \frac{1}{2} \int_{B_R} \mu_\alpha^{-3}\nabla V(\mu_\alpha^{-1}x)\psi_\alpha^2 \, dx
+ \frac{1}{p+1} \int_{B_R} \nabla (b(\mu_\alpha^{-1}x)\psi_\alpha^{p+1}) \, dx
- \frac{1}{p+1} \int_{B_R} \mu_\alpha^{-1}\nabla b(\mu_\alpha^{-1}x)\psi_\alpha^{p+1}. \]

So, we have,

\[
\int_{B_R} \frac{\mu_\alpha^{-3}}{2} \nabla V(\mu_\alpha^{-1}x)\psi_\alpha^2 \, dx - \frac{\mu_\alpha^{-1}}{p+1} \nabla b(\mu_\alpha^{-1}x)\psi_\alpha^{p+1}
= \int_{\partial B_R} \left( \frac{1}{2} \mu_\alpha^{-2}V(\mu_\alpha^{-1}x) - \mu_\alpha^{-2}\nu_\alpha \right) \psi_\alpha^2 \eta \, dS
+ \int_{\partial B_R} \left( - \frac{1}{p+1} b(\mu_\alpha^{-1}x)\psi_\alpha^{p+1} \eta - \nabla \psi_\alpha \partial_\eta \psi_\alpha + \eta \frac{\nabla \psi_\alpha^2}{2} \right) \, dS
=: I_R,
\]

where \( \eta \) is the exterior normal field on \( \partial B_R \). Since \( \int_0^\infty |I(R)| \, dR < \infty \), there exists a subsequence \( \{R_n\} \) such that \( \mathbb{R}_n \to \infty \) and \( I(R_n) \to 0 \). By the growth condition of \(|\nabla V|\) and \(|\nabla b|\) and the dominated convergence theorem, we have (6.19) \( \blacksquare \).
Lemma 6.2 Assume that \( b, V \) satisfies the assumptions of Theorem 1.4. Set \( L_b(y) = (L_{b,j}(y))_{j=1,\ldots,N} \) and \( L_V(y) = (L_{V,j}(y))_{j=1,\ldots,N} \) as

\[
L_{b,j}(y) = \frac{1}{p+1} \int_{\mathbb{R}^N} \partial_{x_j} Q_b(x + y) \phi_1^{p+1} \, dx
\]

\[
L_{V,j}(y) = \frac{1}{2} \int_{\mathbb{R}^N} \partial_{x_j} Q_V(x + y) \phi_1^2 \, dx.
\]

Then, \( L_b(y) = 0 \) if and only if \( y = 0 \) and \( L_V(y) = 0 \) if and only if \( y = 0 \). Further, if \( 0 \) is a local minimum point of \( V \), then for any \( C > 0 \), \( L_b(y) + CL_V(y) = 0 \) if and only if \( y = 0 \).

Proof. First, since \( b, V \) are radially symmetric, we have \( Q_b \) and \( Q_V \) radially symmetric. So, \( \partial_{x_j} Q_b \) and \( \partial_{x_j} Q_V \) are odd functions respect to \( x_j \) and even respect to \( x_i, i \neq j \). This implies, \( L_b(0) = 0 \) and \( L_V(0) = 0 \).

Next, we show if \( L_b(y) = 0 \), then \( y = 0 \). Set \( \tilde{x} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1} \). Then, we have

\[
(p+1)L_{b,j}(y) = \int_{\mathbb{R}^N} \partial_{x_j} Q_b(x + y) \phi_1^{p+1} \, dx
\]

\[
= \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \partial_{x_j} Q_b(x_j, \tilde{x} + y) \phi_1^{p+1}(x_j - y_j, \tilde{x}) \, dx_j d\tilde{x}.
\]

Now, since for each fixed \( \tilde{x} \in \mathbb{R}^{N-1} \) and \( y \in \mathbb{R}^N \), \( \partial_{x_j} Q_b(x_j, \tilde{x} + y) \) is an odd monotone increasing function with respect to \( x_j > 0 \) and \( \phi_1^{p+1}(x_j, \tilde{x}) \) is a even function and monotonically decreases for \( x_j > 0 \). Suppose \( y_j > 0 \), then we have

\[
\int_{\mathbb{R}} \partial_{x_j} Q_b(x_j, \tilde{x} + y) \phi_1^{p+1}(x_j - y_j, \tilde{x}) \, dx_j
\]

\[
= \int_{x_j > 0} \partial_{x_j} Q_b(x_j, \tilde{x} + y) \phi_1^{p+1}(x_j - y_j, \tilde{x}) \, dx_j
\]

\[
+ \int_{x_j < 0} \partial_{x_j} Q_b(x_j, \tilde{x} + y) \phi_1^{p+1}(x_j - y_j, \tilde{x}) \, dx_j
\]

\[
= \int_{x_j > 0} \partial_{x_j} Q_b(x_j, \tilde{x} + y) \left( \phi_1^{p+1}(x_j - y_j, \tilde{x}) - \phi_1^{p+1}(x_j + y_j, \tilde{x}) \right) \, dx_j.
\]

Since \( |x_j - y_j| < |x_j + y_j| \) for all \( x_j > 0 \), we have \( \phi_1^{p+1}(x_j - y_j, \tilde{x}) - \phi_1^{p+1}(x_j + y_j, \tilde{x}) > 0 \). Thus, we obtain \( L_{b,j}(y) > 0 \). Similarly, for the case \( y_j < 0 \), we have \( L_{b,j}(y) < 0 \). Therefore, we see that if \( L_b(y) = 0 \), then \( y = 0 \).

We can show that if \( L_V(y) = 0 \), then \( y = 0 \) in the same way. The last claim of the lemma can be shown similarly since in this case we have \( \partial_{x_j} Q_V \) monotone increasing with respect to \( x_j \).

We next show that the center of the rescaled ground states goes to the origin.
Lemma 6.3 Assume that \( b,V \) satisfies the assumptions of Theorem 1.4. Let \( u_\alpha \in \mathcal{J}_\alpha \) and \( y_{\alpha,u_\alpha} \to 0 \) as \( \alpha \to \infty \). Then, \( \mu_\alpha y_{\alpha,u_\alpha} \to 0 \) as \( \alpha \to \infty \).

Proof. First, suppose \( \{\mu_\alpha y_{\alpha,u_\alpha}\} \) is unbounded. Then, there exists a subsequence with \( |\mu_\alpha y_{\alpha,u_\alpha}| \to \infty \) as \( \alpha_n \to \infty \). We denote \( y_{\alpha_n,u_{\alpha_n}} \) as \( y_n \). Setting \( \psi_n = \mu_\alpha^{-N/2}u_\alpha(\cdot/\mu_\alpha) \), by Lemma 6.1, we have

\[
\frac{\mu_\alpha^2}{2} \int_{\mathbb{R}^N} \partial_{x_j} V(\mu_\alpha^{-1} x + y_n) \psi_n^2 dx - \frac{1}{p + 1} \int_{\mathbb{R}^N} \partial_{x_j} b(\mu_\alpha^{-1} x + y_n) \psi_n^{p+1} dx = 0.
\]

Therefore, we have

\[
0 = \frac{\mu_\alpha^2}{2} \int_{|x_j + y_n| \leq r} \partial_{x_j} Q_V(\mu_\alpha^{-1} x + y_n) \psi_n^2 dx + \frac{\mu_\alpha^2}{2} \int_{|x_j + y_n| \leq r} \partial_{x_j} R_V(\mu_\alpha^{-1} x + y_n) \psi_n^2 dx + \frac{\mu_\alpha^2}{2} \int_{|x_j + y_n| > r} \partial_{x_j} \psi_n^{p+1} dx - \frac{1}{p + 1} \int_{|x_j + y_n| \leq r} \partial_{x_j} Q_b(\mu_\alpha^{-1} x + y_n) \psi_n^{p+1} dx - \frac{1}{p + 1} \int_{|x_j + y_n| > r} \partial_{x_j} R_b(\mu_\alpha^{-1} x + y_n) \psi_n^{p+1} dx - \frac{1}{p + 1} \int_{|x_j + y_n| \leq r} \partial_{x_j} b(\mu_\alpha^{-1} x + y_n) \psi_n^2 dx
\]

If necessary taking a subsequence, let \( y_n/|y_n| \to \zeta \). Then as in [14], we see that

\[
I_{V,1} = \mu_\alpha^{-2}|y_n|^{-1} \partial_{x_j} Q_V(\zeta) + o(\mu_\alpha^{-2}|y_n|^{-1}),
I_{V,2}, I_{V,3} = o(\mu_\alpha^{-2}|y_n|^{-1}),
\]

and

\[
I_{b,1} = |y_n|^{\beta-1} \partial_{x_j} Q_b(\zeta) \int_{\mathbb{R}^N} \phi_j^{p+1} dx + o(|y_n|^{\beta-1}),
I_{b,2}, I_{b,3} = o(|y_n|^{\beta-1}).
\]

Therefore, for the case \( \beta < \gamma + 2 \), we have

\[
0 = |y_n|^{\beta-1} \partial_{x_j} Q_b(\zeta) \int_{\mathbb{R}^N} \phi_j^{p+1} dx + o(|y_n|^{\beta-1}), \quad j = 1, \ldots, N,
\]

this implies \( \zeta = 0 \). However \( |\zeta| = 1 \), this is a contradiction. For the case 0 is a local minimum point of \( V \), taking subsequence, we have \( \mu_\alpha^{-2}|y_n|^{\gamma-\beta} \to c \in [0, \infty] \). For
the case \(0 \leq c < \infty\), we have

\[
0 = |y_n|^{\beta - 1} \left( \partial_{x_j} Q_b(\zeta) \int_{\mathbb{R}^N} \phi^{\beta + 1} \, dx + c \partial_{x_j} Q_V(\zeta) \right) + o(|y_n|^{\beta - 1}), \quad j = 1, \ldots, N.
\]

Since \(\partial_{x_j} Q_b(\zeta) > 0\) (resp. < 0) if \(\zeta_j > 0\) (resp. < 0) and \(\partial_{x_j} Q_V(\zeta) > 0\) (resp. < 0) if \(\zeta_j > 0\) (resp. < 0), we get a contradiction. For the case \(c = \infty\), we have

\[
0 = \mu^{-2} |y_n|^{\gamma - 1} \partial_{x_j} Q_V(\zeta) + o(\mu^{-2} |y_n|^{\gamma - 1}).
\]

As before, since this equation implies \(\zeta = 0\), this is a contradiction.

Next, assume \(\mu_n y_n \to y_0\) for some \(y_0 \in \mathbb{R}^N\). Again, above we have

\[
I_{V,1} = \mu^{-\gamma - 1}_n L_{V,j}(y_0) + o(\mu^{-\gamma - 1}_n),
\]

\[
I_{V,2,3} = o(\mu^{-\gamma - 1}_n),
\]

\[
I_{b,1} = \mu^{-\gamma - 1}_n L_{b,j}(y_0) + o(\mu^{-\beta - 1}_n),
\]

\[
I_{b,2,3} = o(\mu^{-\beta - 1}_n).
\]

Therefore, by (6.19), we have

\[
c_1 L_{b,j}(y_0) + c_2 L_{V,j}(y_0) = 0, \quad j = 1, \ldots, N,
\]

where

\[
(c_1, c_2) = \begin{cases} (1, 0) & \text{if } \beta < \gamma + 2, \\ (1, 1) & \text{if } \beta = \gamma + 2, \\ (0, 1) & \text{if } \beta > \gamma + 2. \end{cases}
\]

Therefore, by Lemma 6.2, we have \(y_0 = 0\).

Using Lemmas 4.9 and 6.3, we can show \(\psi_n\) converges to \(\phi_1\). By this, we can show Theorem 1.4.

**Proof.** [Proof of Theorem 1.4] By the previous lemmas 4.9, 6.3, we see that if \(u_\alpha \in J_\alpha\) concentrates at the origin, then, \(\psi_n \to \phi_1\) in \(L^\infty\). Assume that there exists a sequence of nonradial ground states \(\{u_\alpha_n\}\) with \(\alpha_n \to \infty\). Since \(u_\alpha_n\) are nonradial, \(\psi_n\) are also nonradial. Further, there exist an orthogonal matrix \(\tau_n \in O(N)\) such that \(\psi_{\alpha_n} \neq \psi_{\alpha_n} \circ \tau_n\). Set \(\psi_{1,n} := \psi_{\alpha_n}\) and \(\psi_{2,n} := \psi_{\alpha_n} \circ \tau_n\). Since \(\psi_{1,n} \neq \psi_{2,n}\), we have \(\|\psi_{1,n} - \psi_{2,n}\|_{L^\infty} > 0\). So, set

\[
w_n := \frac{\psi_{1,n} - \psi_{2,n}}{\|\psi_{1,n} - \psi_{2,n}\|_{L^\infty}}.
\]

Then, \(w_n\) satisfies

\[
-\Delta w_n + \left(\mu_{\alpha_n}^{-2} V(\mu_{\alpha_n}^{-1} x) - \mu_{\alpha_n}^{-2} \nu_\alpha\right) w_n = c_n(x) w_n,
\]

(6.21)
with 
\[ c_n(x) = pb(\mu_{\alpha_n}^{-1} x) \int_0^1 (t\psi_{1,n}(x) + (1-t)\psi_{2,n}(x))^{p-1} dt, \]
and \( c_n \to p\phi_1^{p-1} \) in \( C^{1}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). Passing to the limit of (6.21), we get that \( w_n \to w \) in \( C^{1}_{loc} \) where \( w \) is the solution of 
\[ -\Delta w + w - p\phi_1^{p-1} w = 0. \]
It is well known that \( \text{Ker}(-\Delta + 1 - p\phi_1^{p-1}) = \langle \partial_{x_j}\phi_1, j = 1, \cdot, N \rangle \). So, we have 
\[ w(x) = \sum_{j=1}^N a_j \partial_{x_j}\phi_1. \]

From (6.19) we get 
\[ \frac{\mu_{\alpha_n}}{2} \int_{\mathbb{R}^N} \partial_{x_j}V(\mu_{\alpha_n}^{-1} x)\psi_1^{2} dx - \frac{1}{p+1} \int_{\mathbb{R}^N} \partial_{x_j}b(\mu_{\alpha_n}^{-1} x)\psi_1^{p+1} dx \]
\[ = \frac{\mu_{\alpha_n}}{2} \int_{\mathbb{R}^N} \partial_{x_j}V(\mu_{\alpha_n}^{-1} x)\psi_2^{2} dx - \frac{1}{p+1} \int_{\mathbb{R}^N} \partial_{x_j}b(\mu_{\alpha_n}^{-1} x)\psi_2^{p+1} dx = 0. \]
Then we have 
\[ \frac{\mu_{\alpha_n}}{2} \int_{\mathbb{R}^N} \partial_{x_j}V(\mu_{\alpha_n}^{-1} x)w_n(x)(\psi_1 + \psi_2) dx \]
\[ - \frac{1}{p+1} \int_{\mathbb{R}^N} \partial_{x_j}b(\mu_{\alpha_n}^{-1} x)w_n(x)\psi_1 dx = 0. \]
In the same way as the proof of Lemma 6.3, we obtain 
\[ 0 = \alpha_n^{-A(\gamma+1)} \sum_{k=1}^N a_k \int_{\mathbb{R}^N} \partial_{x_j}Q \psi_1 \partial_{x_j}\phi_1 dx + o(\alpha_n^{-A(\gamma+1)}) \]
\[ + \alpha_n^{-A(\beta-1)} \sum_{k=1}^N a_k \int_{\mathbb{R}^N} \partial_{x_j}Q \phi_1^{p-1} \partial_{x_j}\phi_1 dx + o(\alpha_n^{-A(\beta-1)}). \]
Therefore \( a_j = 0, j = 1, \cdot, N \).

Next, let \( x_n \) be such that \( w_n(x_n) = ||w_n||_{L^\infty} = 1 \). If \( \{x_n\} \) is bounded, then it is a contradiction since \( w_n \to 0 \) in \( C^1_{loc} \). On the other hand, if \( |x_n| \to \infty \) we have \( \Delta w_n(x_n) \leq 0 \) and by (6.21), we have 
\[ \mu_{\alpha_n}^{-1}V(x_n/\mu_{\alpha_n}) - \mu_{\alpha_n}V \alpha_n \leq b(x_n/\mu_{\alpha_n})c_n(x_n). \]
However, by Lemma 4.7, \( -\mu_{\alpha_n}V \alpha_n \) converges to \( -\tilde{\nu}_1 > 0 \). So, for sufficiently large \( n \), the right hand side is larger than \( -\tilde{\nu}_1/2 \). On the other hand, because the exponential decay of the ground states, the right hand side converges to 0. This is a contradiction. \[ \blacksquare \]
By Theorem 1.4 and by Corollary 4.1 of [17], we can show Theorem 1.5.

Proof. [Proof of Theorem 1.5] By Corollary 4.1 of [17], one sees that for sufficiently large $L$ and there exists an $\alpha_1$, the symmetry breaking occurs at $\alpha_1$. However, by Theorem 1.4, one sees that for sufficiently large $\alpha$, the ground states are radially symmetric.

A Appendix. The Hartree case

In this appendix, we consider the ground states of the following energy functional

$$
\mathcal{H}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V|u|^2) \, dx - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x - y|} \, dx \, dy
$$

This functional was considered in [3]. However, in [3], only the concentration of $L^2$ norm was shown and the uniqueness for small constraint $\alpha$ was not. So, in this appendix we show the uniqueness and the exponential decay results. The proof is similar to the power type case.

Definition A.1 Set

$$
\mathcal{G}_\alpha := \{ u \in H^1_V \mid \|u\|_{L^2} = \alpha, \, \mathcal{H}(u) = H_\alpha, \, u > 0 \}
$$

where

$$
H_\alpha := \inf \{ \mathcal{H}(u) \mid u \in H^1_V, \|u\|_{L^2} = \alpha \}.
$$

We call the elements of $\mathcal{G}_\alpha$, the ground states, which are the minimizers of $\mathcal{H}$ under the constraint $\|u\|_{L^2} = \alpha$.

The existence of the ground states are well known.

Proposition A.1 Assume (V1). Then for all $\alpha > 0$, $\mathcal{G}_\alpha \neq \emptyset$

The following are the uniqueness and the exponential decay results.

Theorem A.1 Assume (V1), (V2).

1. Assume $\lambda_0 := \inf \sigma(-\Delta + V)$ is an eigenvalue. Let $\alpha > 0$ sufficiently small. Then the ground states are unique.

2. Let $\alpha > 0$ sufficiently large. Then all ground states $u \in \mathcal{G}_\alpha$ has only one local maximum point. Further, we have

$$
u(x) \leq C_1 \alpha^4 \exp \left( -C_2 \alpha^2 \|x - y_\alpha\| \right),
$$

where $C_1$ and $C_2$ are positive constants independent of $\alpha$ and $y_\alpha \in \mathbb{R}^N$ is the unique maximum point of $u$. $y_\alpha$ converges to the set

$$
\left\{ x \in \mathbb{R}^N \mid x \in V^{-1}(\{0\}), \, \Delta V(x) = \min_{y \in V^{-1}(\{0\})} \Delta V(y) \right\},
$$

as $\alpha \to \infty$. 

Remark A.1 For the Hartree case, we do not need the assumption (V2) for the exponential decay.

Proof. [Outline of the proof of Theorem A.1] Set
\[ K_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V|u|^2) \, dx - \frac{\alpha^2}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} \, dx \, dy. \]

Then as the power type case, we see that for \( u \in \mathcal{G}_\alpha, \) \( \alpha^{-1} u \) minimizes \( K_\alpha \) under the constraint \( ||v||_{L^2} = 1. \) We show the uniqueness. First,
\[ ||| x^{-1} * |u|^2 |||_{L^\infty} \leq C ||| u |||_{L^4}^2 + ||u||_{L^2}^2. \]

So, for \( Y_{\alpha^*} := \{ u \in H^1_V(\mathbb{R}^N) \mid ||u||_2 = 1, K_\alpha(u) = \inf_{||v||_{L^2} = 1} K_\alpha(v) \text{ for } \alpha \in [0, \alpha^*) \}, \)
we have
\[ \sup_{u \in Y_{\alpha^*}} ||| x^{-1} * |u|^2 |||_{L^\infty} < \infty. \]

Therefore, as the proof of Theorem 1.1, we set
\[ H^\psi(\alpha) = -\Delta + V - \alpha^2 |x|^{-1} * |\psi|^2 \]
and use Kato-Rellich theorem. Further, using boot strap argument, we also have the uniform bound of \( L^\infty \) norm. Since,
\[ ||| x^{-1} * (\psi^2 - \tilde{\psi}^2) |||_{L^\infty} \leq C \left( (||\psi||_{L^4}^2 + ||\tilde{\psi}||_{L^4}^2) + 2 \right) ||\psi - \tilde{\psi}||_{L^2} \]
we can do the same thing as the proof of Theorem 1.1.

Next we consider the exponential decay result. First, we have
\[ || (|x|^{-1} * \psi^2) \psi ||_{L^3} \leq C (||\psi||_{L^4}^2 + ||\psi||_{L^2}^2) ||\psi||_{L^3}. \]

For the Hartree case the local maximum principle (4.16) becomes
\[ \sup_{|x-y| < r} \psi_u \leq C \left( \left( \frac{1}{r^3} \int_{|x-y| < 2r} \psi_\alpha^2 \, dx \right)^{1/2} + r ||| x^{-1} * \psi^2 \psi |||_{L^3} \right), \]
where, \( \psi_\alpha \) is the rescaled minimizer as in Lemma 4.9. Therefore, we see \( \psi_\alpha \rightarrow \phi_1 \) in \( L^\infty \) as Lemma 4.9, where \( \phi_1 \) is the minimizer of
\[ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} \, dx \, dy. \]

The left is completely the same as the power type case. \( \blacksquare \)
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