Advanced Nonlinear Studies **13** (2013), 97–114

A Resonant-Superlinear Elliptic Problem Revisited

(Dedicated to Klaus Schmitt on the occasion of his retirement)

Mabel Cuesta

Université du Littoral Côte d'Opale ULCO
LMPA, 50, rue F. Buisson, B.P. 699
F-62228 Calais, France
e-mail: cuesta@lmpa.univ-littoral.fr

Colette De Coster

Université de Valenciennes et du Hainaut Cambrésis
LAMAV FR CNRS 2956
Institut des Sciences et Techniques de Valenciennes
F-59313 Valenciennes Cedex 9, France
e-mail: Colette.DeCoster@univ-valenciennes.fr

Communicated by Stephan Cantrell

Abstract

We consider the resonant-superlinear elliptic problem $-\Delta u = \lambda_1 u + (u^+)^q + f(x)$, with Dirichlet boundary conditions on a bounded regular domain of $\mathbb{R}^N$. We assume that $1 < q < \frac{N+1}{N-1}$, $f \in L^s(\Omega)$ with $s > N$ satisfies $\int_\Omega f\varphi_1 < 0$ and (λ_1, φ_1) is the first eigenpair of $-\Delta$ on $H_0^1(\Omega)$. We apply a non-well ordered lower and upper solution result on a family of modified problems and obtain a sequence of localized solutions of these modified problems. Thanks to this localization and a precise bootstrap argument we are abble to prove that for large modification, these solutions are, in fact, solutions of our initial problem.

The problem was already considered in [2] by a totally different approach.

2010 Mathematics Subject Classification. 35J25, 35J61, 35B45, 35A15, 35A16 .

Key words. Semilinear elliptic problems, superlinear, resonance, lower and upper solutions

1 Introduction

In this paper we consider the problem

$$\begin{aligned} -\Delta u &= \lambda_1 u + (u^+)^q + f(x) \quad &\text{in} \quad &\Omega, \\ u &= 0 &\text{on} \quad &\partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain ($N \geq 3$), $u^+ := \max\{u, 0\}$ (we will use also the notation $u^- := \max\{-u, 0\}$), λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, $f \in L^s(\Omega)$ with $s > N$ and $q > 1$. We will denote by φ_1 the positive eigenfunction corresponding to λ_1 with $\|\varphi_1\|_{H_0^1} = \int_\Omega |\nabla\varphi_1|^2 = 1$.

In a recent paper [2], M. Cuesta, D.G. de Figueiredo and P.N. Srikanth prove the following result.

Theorem 1.1 *Assume that $N \geq 3$, $f \in L^s(\Omega)$ with $s > N$ is such that*

$$\int_\Omega f\varphi_1 \leq 0,$$

and q satisfies

$$1 < q < \frac{N+1}{N-1}.$$

Then the problem (1.1) *possesses at least one solution.*

It is easy to observe that the condition $\displaystyle\int_\Omega f\varphi_1 \leq 0$ is necessary to have a solution. For what concerns the restriction on the growth q of the superlinear term, to our knowledge, there had not been any improvement of this condition in this situation. Observe that we cannot reduce ourself to the search of positive solutions or at least to solutions above a fixed function.

The proof of [2] uses Hardy-Sobolev type inequalities and argument as in Brezis-Turner famous paper [1]. As this approach uses deeply the selfadjointness of $-\Delta$ and the orthogonal decomposition of a function $u \in H_0^1(\Omega)$ in the form $u = a\varphi_1 + w$, it is not clear how to extend this approach for example to the p-Laplacian problem. On the other hand, it is easy to observe that the problem (1.1) possesses non well ordered lower and upper solutions. In this work, we apply then the result of C. De Coster [3] and prove the same result by a completely different approach which is more promising, for what concerns a possible extension to the p-Laplacian problem, as it does not use the orthogonal decomposition of the solution.

In [3], in case we have α and β lower and upper solutions of (1.1) with $\alpha \not\leq \beta$, the second author defines

$$c := \inf_{\gamma\in\Gamma} \max_{s\in T_\gamma} \Phi(\gamma(s))$$

with

$$\begin{aligned} \Gamma &= \{\gamma \in C([0,1], C_0^1(\overline{\Omega})) \mid \gamma(0) \in C^\beta,\ \gamma(1) \in C_\alpha\}, \\ T_\gamma &= \{s \in [0,1] \mid \gamma(s) \in C_0^1(\overline{\Omega}) \setminus (C^\beta \cup C_\alpha)\}, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} C_\alpha &= \{u \in C_0^1(\overline{\Omega}) \mid u \gg \alpha\}, \\ C^\beta &= \{u \in C_0^1(\overline{\Omega}) \mid u \ll \beta\}, \end{aligned}$$

(see Definition 2.1 for the definition of $u \ll v$) and

$$\Phi : H_0^1(\Omega) \to \mathbb{R}, u \mapsto \int_\Omega \left[\frac{|\nabla u(x)|^2}{2} - \lambda_1 \frac{|u|^2}{2} - \frac{(u^+)^{q+1}}{q+1} - fu \right] dx.$$

Then, in [3, Theorem 3.7], it is proved that, if $c \in \mathbb{R}$ and Φ satisfies the Palais-Smale condition, there exists $u \in C_0^1(\overline{\Omega}) \setminus (C^\beta \cup C_\alpha)$ solution of (1.1) such that $\Phi(u) = c$.

In our case, we are unable to prove that $c \in \mathbb{R}$. To solve this problem, as in [4], we consider a family of modified problems (2.1) for which we prove the existence of a solution $u_r \in C_0^1(\overline{\Omega}) \setminus (C^\beta \cup C_\alpha)$. Then the main difficulty is to come back to the original problem (1.1). This can be done using the localization information on the solution u_r and a precise bootstrap argument.

2 Lower, upper solutions and modified problems

In this paper, we use the following notations.

Definition 2.1 Given functions $u, v : \overline{\Omega} \to \mathbb{R}$, we write
(i) $u \leq v$ if for all $x \in \overline{\Omega}$, $u(x) \leq v(x)$;
(ii) $u < v$ if $u \leq v$ and $u \neq v$;
(iii) $u \ll v$ if, for all $x \in \Omega$, $u(x) < v(x)$ and, for all $x \in \partial\Omega$, either $u(x) < v(x)$ or $\frac{\partial u}{\partial \eta}(x) > \frac{\partial v}{\partial \eta}(x)$.
Here η denotes the exterior normal direction with respect to $\partial\Omega$.

Recall the following notions of lower and upper solutions.

Definition 2.2 A function $\alpha \in W^{2,p}(\Omega)$ (with $p > N$) is a lower solution of (1.1) if
(i) for a.e. $x \in \Omega$, $-\Delta\alpha(x) \leq \lambda_1\alpha(x) + (\alpha^+(x))^q + f(x)$;
(ii) $\alpha \leq 0$ on $\partial\Omega$.

Similarly, an upper solution of (1.1) is defined by reversing the above inequalities.

Lemma 2.1 *Under the assumptions of Theorem 1.1, the problem* (1.1) *has an upper solution* $\beta \ll 0$.

Proof. Let w be the solution of

$$\begin{aligned} -\Delta w &= \lambda_1 w + f(x) - \lambda_1 \Big(\int_\Omega f\varphi_1\Big)\varphi_1(x), && \text{in} \quad \Omega, \\ w &= 0, && \text{on} \quad \partial\Omega. \end{aligned}$$

We can choose $k > 0$ such that $\beta = -k\varphi_1 + w \ll 0$. Then we have

$$\begin{aligned} -\Delta\beta &= \lambda_1\beta + f(x) - \Big(\int_\Omega f\varphi_1\Big)\varphi_1(x) \geq \lambda_1\beta + (\beta^+)^q + f(x), && \text{in} \quad \Omega, \\ \beta &= 0, && \text{on} \quad \partial\Omega, \end{aligned}$$

which means that β is an upper solution of (1.1).

Lemma 2.2 *Under the assumptions of Theorem 1.1, the problem* (1.1) *has a lower solution* $\alpha \gg 0$.

Proof. Let $\epsilon > 0$ such that $\lambda_1 + \epsilon \notin \sigma(-\Delta)$. Observe that, for all $u \geq 0$,

$$u^q \geq \epsilon u - \epsilon(\frac{\epsilon}{q})^{\frac{1}{q-1}}(\frac{q-1}{q}).$$

Define w as the solution of

$$\begin{array}{rlll} -\Delta w = (\lambda_1 + \epsilon)w + f(x) - \epsilon(\frac{\epsilon}{q})^{\frac{1}{q-1}}(\frac{q-1}{q}), & \text{in} & \Omega, \\ w = 0, & \text{on} & \partial\Omega. \end{array}$$

Choose $k > 0$ such that $\alpha = k\varphi_1 + w \gg 0$. Then we have

$$-\Delta\alpha = \lambda_1\alpha + f(x) - \epsilon k\varphi_1 + \epsilon\alpha - \epsilon(\frac{\epsilon}{q})^{\frac{1}{q-1}}(\frac{q-1}{q}) \leq \lambda_1\alpha + f(x) + (\alpha^+)^q,$$

in Ω and $\alpha = 0$ on $\partial\Omega$, which means that α is a lower solution of (1.1).

We have proved so far that problem (1.1) possesses a pair of non ordered lower and upper solutions. In order to be able to apply [3, Theorem 3.7], we are going to modify our problem in the following way.

For $r > 1 - \min\beta$, consider the modified problem

$$\begin{array}{rlll} -\Delta u = (\lambda_1 - \frac{1}{r}h_r(u))u + (u^+)^q + f(x) =: g_r(x,u), & \text{in} & \Omega, \\ u = 0, & \text{on} & \partial\Omega, \end{array} \tag{2.1}$$

where

$$\begin{array}{rcll} h_r(u) & = & 0, & \text{if } u > -r, \\ & = & -(u+r), & \text{if } u \in [-r-1, -r], \\ & = & 1, & \text{if } u < -r-1. \end{array}$$

As $r > 1 - \min\beta$, the lower and upper solutions α and β given by Lemmas 2.1 and 2.2 are still lower and upper solutions of (2.1).

Lemma 2.3 *The modified problem* (2.1) *has a lower solution $\alpha_r \ll \beta$.*

Proof. Observe first that, for all $u \leq 0$,

$$\begin{array}{rcl} (\lambda_1 - \frac{1}{r}h_r(u))u & \geq & (\lambda_1 - \frac{1}{r})u - \frac{1}{r}(h_r(u) - 1)u \\ & \geq & (\lambda_1 - \frac{1}{r})u + \frac{1}{r}(h_r(u) - 1)(r+1) \\ & \geq & (\lambda_1 - \frac{1}{r})u - \frac{1}{r}(r+1). \end{array}$$

Let w_r be the solution of

$$\begin{array}{rlll} -\Delta w = (\lambda_1 - \frac{1}{r})w + f(x) - \frac{r+1}{r}, & \text{in} & \Omega, \\ w = 0, & \text{on} & \partial\Omega, \end{array}$$

and $k_r > 0$ be large enough so that $\alpha_r = w_r - k_r\varphi_1 \ll \beta$, for example $k_r = \sup\{\frac{w_r(x)-\beta(x)}{\varphi_1(x)} \mid x \in \Omega\} + 1$. It is then easy to observe that

$$\begin{array}{rcll} -\Delta\alpha_r & = & (\lambda_1 - \frac{1}{r})\alpha_r - \frac{r+1}{r} - \frac{1}{r}k_r\varphi_1 + f(x) & \\ & \leq & (\lambda_1 - \frac{1}{r}h_r(\alpha_r))\alpha_r + f(x) & \\ & = & (\lambda_1 - \frac{1}{r}h_r(\alpha_r))\alpha_r + (\alpha_r^+)^q + f(x), & \text{in } \Omega, \\ & & \alpha_r = 0, & \text{on } \partial\Omega, \end{array}$$

which proves the result.

We use the lower solution α_r to truncate the previous nonlinearity. Let us consider, for every $r > 1 - \min\beta$, the (second) modified problem

$$\begin{aligned} -\Delta u &= g_r(x, \gamma_r(x, u)), && \text{in} \quad \Omega, \\ u &= 0, && \text{on} \quad \partial\Omega, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \gamma_r(x, u) &= u, && \text{if } u \geq \alpha_r(x), \\ &= \alpha_r(x), && \text{if } u < \alpha_r(x). \end{aligned}$$

Applying the maximum principle, it is easy to verify that every solution u of (2.2) satisfies $u \geq \alpha_r$.

3 Proof of theorem 1.1

In this section we want to apply [3, Theorem 3.7] to problem (2.2). Let us define the C^1-functional $\overline{\Phi}_r : H_0^1(\Omega) \to \mathbb{R}$ corresponding to (2.2):

$$\overline{\Phi}_r(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \overline{G}_r(x, u)\,dx$$

where $\overline{G}_r(x, s) := \displaystyle\int_0^s g_r(x, \gamma_r(x, \rho))\,d\rho$. Trivially, if $r > 2/\lambda_1$, the function

$$u \mapsto g_r(x, \gamma_r(x, u)) + u$$

is increasing in u and locally Lipschitz in u uniformly in x. This means that, for all $r > 0$, the assumption (H) of [3] is satisfied with $m = 1$.

Lemma 3.1 *Let $u_n \in H_0^1(\Omega)$ be a sequence of functions satisfying*

$$\int_\Omega (|\nabla u_n|^2 - \lambda_1|u_n|^2) = o_\infty(\|u_n\|^2_{H_0^1}). \tag{3.1}$$

Then either $(\|u_n\|_{H_0^1})_n$ is bounded or, up to a subsequence, $(\frac{u_n}{\|u_n\|_{H_0^1}})_n$ converges strongly in $H_0^1(\Omega)$ to $\pm\varphi_1$.

Proof. If $\|u_n\|_{H_0^1} \to +\infty$, let us consider $w_n := \frac{u_n}{\|u_n\|_{H_0^1}}$. As the sequence $(w_n)_n$ is bounded, we have $(w_{n_k})_k$, a subsequence of $(w_n)_n$ and $w \in H_0^1(\Omega)$ such that $w_{n_k} \to w$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and a.e. Using (3.1), we find

$$0 \leq \int_\Omega (|\nabla w|^2 - \lambda_1|w|^2) \leq \lim_{k\to\infty}\int_\Omega (|\nabla w_{n_k}|^2 - \lambda_1|w_{n_k}|^2) = 0,$$

from which we deduce $w = c\varphi_1$ for some constant $c \in \mathbb{R}$. Moreover we have $\displaystyle\int_\Omega |\nabla w_{n_k}|^2 = 1$, which implies

$$1 = \lim_{k\to\infty}\int_\Omega |\nabla w_{n_k}|^2 = \lambda_1 \lim_{k\to\infty}\int_\Omega |w_{n_k}|^2 = \lambda_1 \int_\Omega |w|^2$$

and $\int_\Omega |\nabla w|^2 = \lambda_1 \int_\Omega |w|^2 = 1$. As $w_{n_k} \to w$ weakly in $H_0^1(\Omega)$ and $\|w_{n_k}\|_{H_0^1} \to \|w\|_{H_0^1}$, we then deduce that $w_{n_k} \to w$ strongly in $H_0^1(\Omega)$. By the normalization of φ_1, we have $c = \pm 1$.

Proposition 3.1 *Under the assumptions of Theorem 1.1, for all $r > r_0 := \max\{1 - \min\beta, 2/\lambda_1\}$, we have a solution u_r of* (2.1) *with $u_r \not\leq \beta$, $u_r \not\geq \alpha$, $u_r \geq \alpha_r$ and $\overline{\Phi}_r(u_r) = c_r$ where*

$$c_r = \inf_{\gamma\in\Gamma} \max_{s\in T_\gamma} \overline{\Phi}_r(\gamma(s)),$$

and Γ and T_γ are given by (1.2).

Moreover, there exists $d > 0$ such that, for all $r > r_0$, we have $c_r \leq d$.

Proof. Step 1.- For all $r > r_0$, we have $c_r \in \mathbb{R}$ and there exists $d > 0$ such that, for all $r > r_0$, we have $c_r \leq d$. For all $\gamma \in \Gamma$, let $s_0 \in T_\gamma$ such that $u_0 = \gamma(s_0) \in \partial C^\beta$. Observe that $u_0 \ll 0$ and

$$\overline{\Phi}_r(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx - \int_\Omega \overline{G}_r(x,u_0)\,dx \geq \frac{1}{2}\|u_0\|^2_{H_0^1} - c_r\|u_0\|_{H_0^1} - d_r,$$

with c_r, d_r depending on r and α_r but not on u_0. This implies the existence of $m_r \in \mathbb{R}$ such that, for all $\gamma \in \Gamma$,

$$\max_{s\in T_\gamma} \overline{\Phi}_r(\gamma(s)) \geq m_r,$$

and $c_r \in \mathbb{R}$.

Moreover, there exists a path $\gamma_0 \in \Gamma$ such that, for all $t \in [0,1]$ and all $x \in \Omega$, $\gamma_0(t)(x) \geq \max\{-r, \alpha_r(x)\}$. Hence, for all $r > r_0$, we have

$$c_r \leq d := \max_{\gamma_0(t)\notin(C^\beta\cup C_\alpha)} \Phi(\gamma_0(t)),$$

as, for $u \geq \max\{-r, \alpha_r(x)\}$, by definition, $\overline{\Phi}_r(u) = \Phi(u)$.

Step 2: $\overline{\Phi}_r$ satisfies the Palais-Smale condition.

Let $(u_n)_n$ be a (PS) sequence for $\overline{\Phi}_r$ i.e. there exists $M > 0$ and $\epsilon_n \to 0$ such that $|\overline{\Phi}_r(u_n)| \leq M$ and, for all $v \in H_0^1(\Omega)$, $|\langle \overline{\Phi}_r'(u_n), v\rangle| \leq \epsilon_n \|v\|_{H_0^1}$. Using this last condition with $v = u_n^-$ we obtain

$$\begin{aligned}
\epsilon_n\|u_n^-\|_{H_0^1} &\geq |\langle \overline{\Phi}_r'(u_n), u_n^-\rangle| \\
&\geq \int_\Omega |\nabla u_n^-|^2 - (\lambda_1 - \frac{1}{r})\int_\Omega (u_n^-)^2 - \frac{1}{r}\int_\Omega (1 - h_r(\gamma_r(x,u_n)))(u_n^-)^2 + \int_\Omega f u_n^- \\
&\geq \int_\Omega |\nabla u_n^-|^2 - (\lambda_1 - \frac{1}{r})\int_\Omega (u_n^-)^2 - \frac{(r+1)^2}{r}|\Omega| + \int_\Omega f u_n^-,
\end{aligned}$$

from which we deduce that $\|u_n^-\|_{H_0^1}$ is bounded. Moreover we have

$$\begin{aligned}
M + \epsilon_n\|u_n^+\|_{H_0^1} &\geq \overline{\Phi}_r(u_n) - \frac{1}{q+1}\langle \overline{\Phi}_r'(u_n), u_n^+\rangle \\
&\geq (\frac{1}{2} - \frac{1}{q+1})(\int_\Omega |\nabla u_n^+|^2 - \lambda_1 \int_\Omega (u_n^+)^2) \\
&\quad + \frac{1}{2}\int_\Omega |\nabla u_n^-|^2 - (1 - \frac{1}{q+1})\int_\Omega f u_n^+ + \int_\Omega f u_n^- + \frac{\lambda_1}{2}\int_\Omega \alpha_r u_n^-.
\end{aligned}$$

As $\|u_n^-\|_{H_0^1}$ is bounded, we can write

$$\int_\Omega |\nabla u_n^+|^2 - \lambda_1 \int_\Omega (u_n^+)^2 = o_\infty(\|u_n^+\|_{H_0^1}^2),$$

and, using Lemma 3.1, either $\|u_n^+\|_{H_0^1}$ is bounded or $\frac{u_n^+}{\|u_n^+\|_{H_0^1}} \to \varphi_1$ strongly in $H_0^1(\Omega)$.

Let us prove by contradiction that the second situation cannot occur. By the (PS) condition with $v = \varphi_1$, we have

$$\begin{aligned} \epsilon_n \|\varphi_1\|_{H_0^1} &\geq -\langle \overline{\Phi}_r'(u_n), \varphi_1 \rangle \\ &\geq \int_\Omega (u_n^+)^q \varphi_1 + \int_\Omega f\varphi_1 - \frac{1}{r}\int_\Omega h_r(\gamma_r(x,u_n))\gamma_r(x,u_n)\varphi_1. \end{aligned}$$

In case $\frac{u_n^+}{\|u_n^+\|_{H_0^1}} \to \varphi_1$ strongly in $H_0^1(\Omega)$, dividing by $\|u_n^+\|_{H_0^1}^q$ and passing to the limit we obtain the contradiction

$$\int_\Omega \varphi_1^{q+1} = 0.$$

Hence $\|u_n\|_{H_0^1}$ is bounded which proves the (PS) condition.

Conclusion. By the above claims, we can apply [3, Theorem 3.7] and, for all $r > r_0$, there exists a solution u_r of (2.2) with $u_r \not\leq \beta$, $u_r \not\geq \alpha$ and $\overline{\Phi}_r(u_r) = c_r$. Hence by construction, for all $r > r_0$, we have a solution u_r of (2.1) with $u_r \not\leq \beta$, $u_r \not\geq \alpha$, $u_r \geq \alpha_r$ and $\overline{\Phi}_r(u_r) = c_r$.

In order to conclude the proof of the result, we need some estimates. Let us denote by Φ_r the functional corresponding to (2.1) and u_r the solution given by Proposition 3.1.

Lemma 3.2 *For all $r > r_0$, let u_r be the solution of* (2.1) *given by Proposition 3.1. Then either $\|u_r^-\|_{H_0^1} \to \infty$ or $\|u_r\|_C$ is bounded if $r \to +\infty$.*

In case $\lim_{r\to+\infty} \|u_r^-\|_{H_0^1} = \infty$, the sequence $\frac{u_r^-}{\|u_r^-\|_{H_0^1}}$ converges strongly in $H_0^1(\Omega)$ to φ_1 when $r \to +\infty$.

Proof. As u_r is a solution of (2.1), we have

$$0 = |\langle \Phi_r'(u_r), u_r^- \rangle| \geq \int_\Omega |\nabla u_r^-|^2 - \int_\Omega (\lambda_1 - \frac{1}{r}h_r(u_r))(u_r^-)^2 + \int_\Omega f u_r^-,$$

from which we deduce that

$$\int_\Omega |\nabla u_r^-|^2 - \lambda_1 \int_\Omega (u_r^-)^2 \leq -\int_\Omega f u_r^-. \tag{3.2}$$

By (3.2) and Lemma 3.1, either $\|u_r^-\|_{H_0^1}$ is bounded or $\frac{u_r^-}{\|u_r^-\|_{H_0^1}} \to \varphi_1$ strongly in $H_0^1(\Omega)$. It remains then to prove that, in case $\|u_r^-\|_{H_0^1}$ is bounded, $\|u_r\|_C$ is also bounded.

Recall the existence of $d > 0$ such that, for all $r > r_0$, $c_r \leq d$. As u_r is solution of (2.1), we have in particular

$$\int_\Omega |\nabla u_r^+|^2 - \lambda_1 \int_\Omega (u_r^+)^2 - \int_\Omega (u_r^+)^{q+1} - \int_\Omega f u_r^+ = \langle \Phi_r'(u_r), u_r^+ \rangle = 0. \tag{3.3}$$

Hence, we obtain

$$\begin{aligned} d &\geq \Phi_r(u_r) - \frac{1}{q+1}\langle \Phi_r'(u_r), u_r^+\rangle \\ &\geq (\frac{1}{2} - \frac{1}{q+1})(\int_\Omega |\nabla u_r^+|^2 - \lambda_1 \int_\Omega (u_r^+)^2) - (1 - \frac{1}{q+1})\int_\Omega f u_r^+ \\ &\qquad + \frac{1}{2}(\int_\Omega |\nabla u_r^-|^2 - \lambda_1 \int_\Omega (u_r^-)^2) + \int_\Omega f u_r^-. \end{aligned}$$

This implies that

$$(\frac{1}{2} - \frac{1}{q+1})(\int_\Omega |\nabla u_r^+|^2 - \lambda_1 \int_\Omega |u_r^+|^2) \leq d + (1 - \frac{1}{q+1})\int_\Omega f u_r^+ - \int_\Omega f u_r^-. \tag{3.4}$$

In case $\|u_r^-\|_{H_0^1}$ is bounded, we deduce then that

$$\int_\Omega |\nabla u_r^+|^2 - \lambda_1 \int_\Omega (u_r^+)^2 = o_\infty(\|u_r^+\|_{H_0^1}^2).$$

Applying Lemma 3.1 we obtain that, either $\|u_r^+\|_{H_0^1} \leq C$ or $\frac{u_r^+}{\|u_r^+\|_{H_0^1}} \to \varphi_1$ strongly in $H_0^1(\Omega)$.

In the second case we have

$$\begin{aligned} 0 &= \langle \Phi_r'(u_r), \frac{u_r^+}{\|u_r^+\|_{H_0^1}^{q+1}}\rangle \\ &= \int_\Omega \frac{|\nabla u_r^+|^2}{\|u_r^+\|_{H_0^1}^{q+1}} - \lambda_1 \int_\Omega \frac{(u_r^+)^2}{\|u_r^+\|_{H_0^1}^{q+1}} - \int_\Omega \left(\frac{u_r^+}{\|u_r^+\|_{H_0^1}}\right)^{q+1} - \int_\Omega \frac{f u_r^+}{\|u_r^+\|_{H_0^1}^{q+1}}. \end{aligned}$$

Passing to the limit, we obtain the contradiction

$$0 = \int_\Omega \varphi_1^{q+1}.$$

This means that we are in the first situation i.e. $\|u_r^+\|_{H_0^1}$ is bounded and, by assumption, the same is true for $\|u_r\|_{H_0^1}$.

By a bootstrap argument, we conclude that $\|u_r\|_C$ is bounded.

Lemma 3.3 *For all $r > r_0$, let u_r be the solution of* (2.1) *given by Proposition 3.1. If* $\lim_{r\to\infty} \|u_r^-\|_{H_0^1} = \infty$ *then there exists $C > 0$ such that, for $r > r_0$, we have*

$$\|u_r^+\|_{H_0^1} \leq C\|u_r\|_{H_0^1}^{1/2}.$$

Proof. As in the previous lemma, we deduce (3.4) and (3.3). Consider the function $w_r = \frac{u_r^+}{\|u_r^-\|_{H_0^1}^{1/2}}$. By Lemma 3.2, we have $\frac{u_r^-}{\|u_r^-\|_{H_0^1}} \to \varphi_1$ and hence, from (3.4),

$$\int_\Omega |\nabla w_r|^2 - \lambda_1 \int_\Omega |w_r|^2 = o_\infty(\|w_r\|_{H_0^1}^2).$$

By Lemma 3.1, either $\frac{w_r}{\|w_r\|_{H_0^1}} \to \varphi_1$ in $H_0^1(\Omega)$ or $\|w_r\|_{H_0^1} \leq C$. If $\frac{w_r}{\|w_r\|_{H_0^1}} = \frac{u_r^+}{\|u_r^+\|_{H_0^1}} \to \varphi_1$, dividing (3.3) by $\|u_r^+\|_{H_0^1}^{q+1}$, we obtain the contradiction

$$\int_\Omega \left(\frac{u_r^+}{\|u_r^+\|_{H_0^1}} \right)^{q+1} \to 0 \quad \text{and} \quad \int_\Omega \left(\frac{u_r^+}{\|u_r^+\|_{H_0^1}} \right)^{q+1} \to \int_\Omega \varphi_1^{q+1}.$$

This implies that the first situation cannot occur. Hence, there exists $C > 0$ such that, for all $r > r_0$, we have $\|w_r\|_{H_0^1} \leq C$.

Remark 3.1 Observe that this implies also that, if $\lim_{r\to+\infty} \|u_n^-\|_{H_0^1} = \infty$, then

$$\lim_{r\to+\infty} \frac{\|u_n\|_{H_0^1}}{\|u_n^-\|_{H_0^1}} = 1.$$

Lemma 3.4 *For all $r > r_0$, let u_r be the solution of* (2.1) *given by Proposition 3.1. If $\lim_{r\to+\infty} \|u_r^-\|_{H_0^1} = \infty$ then, there exists $C > 0$ such that, for r large, $\int_\Omega (u_r^+)^{q+1} \leq C\|u_r\|_{H_0^1}$.*

Proof. Recall the existence of $d > 0$ such that, for all $r > r_0$, $c_r \leq d$. As u_r is a solution of (2.1), we have

$$\begin{aligned} d &\geq \Phi_r(u_r) - \frac{1}{2}\langle \Phi_r'(u_r), u_r^+ \rangle \\ &\geq \frac{1}{2}\int_\Omega |\nabla u_r^-|^2 - \frac{\lambda_1}{2}\int_\Omega |u_r^-|^2 + (\frac{1}{2} - \frac{1}{q+1})\int_\Omega (u_r^+)^{q+1} - \frac{1}{2}\int_\Omega u_r^+ f + \int_\Omega u_r^- f \\ &\geq (\frac{1}{2} - \frac{1}{q+1})\int_\Omega (u_r^+)^{q+1} - \frac{1}{2}\int_\Omega u_r^+ f + \int_\Omega u_r^- f. \end{aligned}$$

Dividing by $\|u_r\|_{H_0^1}$ and using Lemmas 3.2 and 3.3, we obtain

$$\limsup_{r\to+\infty} \int_\Omega \frac{(u_r^+)^{q+1}}{\|u_r\|_{H_0^1}} \leq -\frac{2(q+1)}{q-1}\int_\Omega f\varphi_1,$$

which proves the estimate.

Lemma 3.5 *For all $r > r_0$, let u_r be the solution of* (2.1) *given by Proposition 3.1. If $\lim_{r\to\infty} \|u_r^-\|_{H_0^1} = \infty$ then there exists $C > 0$ such that, for all $r > r_0$,*

$$\|u_r\|_C \leq C\, r. \qquad \text{and} \qquad \|u_r\|_{H_0^1} \leq C\, r.$$

Hence, there exists $C_1 > 0$ such that, for all $r > r_0$,

$$\left| \frac{1}{r}\int_\Omega h_r(u_r) u_r \right| \leq C_1.$$

Proof. Recall that, by construction $u_r \geq \alpha_r$ where $\alpha_r = w_r - k_r\varphi_1$ with w_r solution of

$$\begin{array}{rll} -\Delta w = (\lambda_1 - \frac{1}{r})w + f(x) - \frac{r+1}{r}, & \text{in} & \Omega, \\ w = 0, & \text{on} & \partial\Omega, \end{array}$$

and $k_r = \sup\{\frac{w_r(x)-\beta(x)}{\varphi_1(x)} \mid x \in \Omega\} + 1$.

Observe first that, as

$$\int_\Omega |\nabla w|^2 - (\lambda_1 - \frac{1}{r})\int_\Omega |w|^2 \geq \frac{1}{r\lambda_1}\int_\Omega |\nabla w|^2,$$

we deduce the existence of a constant C_1 such that, for all $r > r_0$,

$$\|w_r\|^2_{H^1_0} \leq r\lambda_1 \int_\Omega (f - \frac{r+1}{r})w_r \leq r\lambda_1 \||f| + 2\|_{L^2}\|w_r\|_{H^1_0},$$

and hence $\|w_r\|_{H^1_0}/r$ is bounded.

Moreover $\frac{w_r}{r}$ satisfies

$$\begin{array}{rll} -\Delta(\frac{w_r}{r}) - (\lambda_1 - 1)\frac{w_r}{r} = (1 - \frac{1}{r})\frac{w_r}{r} + \frac{f(x) - \frac{r+1}{r}}{r}, & \text{in} & \Omega, \\ \frac{w_r}{r} = 0, & \text{on} & \partial\Omega. \end{array}$$

As now the right hand side of this equation is bounded, we conclude, by a bootstrap argument, the existence of a constant C_1 such that, for all $r > r_0$,

$$\|w_r\|_{C^1_0} \leq C_1 r.$$

Hence we have also the existence of $C_2 > 0$ such that, for all $r > r_0$,

$$k_r = \sup\{\frac{w_r(x) - \beta(x)}{\varphi_1(x)} \mid x \in \Omega\} + 1 < C_2 r.$$

This allows to conclude the existence of C_3 such that, for all $r > r_0$,

$$\|u_r\|_C \leq \|\alpha_r\|_C \leq C_3 r.$$

Now multiplying (2.1) by $\frac{u_r^-}{r^2}$ and integrating we deduce that

$$\int_\Omega \frac{|\nabla u_r^-|^2}{r^2} = \int_\Omega (\lambda_1 - h_r(u_r))\frac{(u_r^-)^2}{r^2} - \int_\Omega f\frac{u_r^-}{r^2}.$$

As the right hand side of this equation is bounded, we deduce the existence of a constant $C > 0$ such that, for all $r > r_0$, $\|u_r^-\|_{H^1_0} \leq Cr$. It follows from Lemma 3.3 that

$$\|u_r^+\|_{H^1_0} \leq Cr^{1/2},$$

which concludes the proof.

Proof of Theorem 1.1. Recall that by Proposition 3.1, for all $r > r_0$, we have a solution u_r of (2.1) with $u_r \nleq \beta$, $u_r \ngeq \alpha$, $u_r \geq \alpha_r$ and $\Phi_r(u_r) = c_r$. In order to conclude the proof of the existence of a solution of problem (1.1), we wish to prove the following result:

Claim: There exists $K > r_0$ such that, for all $r > K$, every solution u_r of (2.1) *with $u_r \nleq \beta$, $u_r \ngeq \alpha$, $u_r \geq \alpha_r$ and $\Phi_r(u_r) = c_r$ satisfies $u_r > -K$.*

Assume by contradiction the existence of a sequence $(r_n)_n$ with $\lim_{n\to\infty} r_n = +\infty$ and a sequence of solutions $(u_n)_n$ of (2.1) with $u_n \nleq \beta$, $u_n \ngeq \alpha$, $u_n \geq \alpha_n$ and $\Phi_n(u_n) = c_n$ which satisfies $\min u_n \leq -n$. Here and below we simplify the notation by writing α_n instead of α_{r_n}, Φ_n instead of Φ_{r_n} and so on.

In the course of this proof we use the following notations: $a \lesssim b$ means the existence of a positive constant C, which is independent of the quantities a and b under consideration such that $a \leq Cb$.

Let us write $u_n = a_n\varphi_1 + w_n$ with $\int_\Omega w_n\varphi_1 = 0$ and $a_n = \frac{\int_\Omega u_n\varphi_1}{\int_\Omega \varphi_1^2}$. Recall that by Lemmas 3.2 and 3.3, as $\lim_{n\to\infty} \|u_n^-\|_C = \infty$, we have also $\lim_{n\to\infty} \|u_n^-\|_{H_0^1} = \infty$ and $\lim_{n\to\infty} \frac{u_n}{\|u_n\|_{H_0^1}} = -\varphi_1$ strongly in $H_0^1(\Omega)$. Hence dividing a_n by $\|u_n\|_{H_0^1}$ we obtain

$$\lim_{n\to\infty} \frac{a_n}{\|u_n\|_{H_0^1}} = \lim_{n\to\infty} \frac{1}{\int_\Omega \varphi_1^2} \int_\Omega \frac{u_n}{\|u_n\|_{H_0^1}}\varphi_1 = -1,$$

and also

$$\lim_{n\to\infty} \frac{w_n}{|a_n|} = \lim_{n\to\infty} \frac{u_n}{|a_n|} + \varphi_1 = 0 \quad \text{in } H_0^1(\Omega).$$

We would have the claim if we prove $\frac{w_n}{|a_n|} \to 0$ in $C_0^1(\overline{\Omega})$, as in that case, for n large enough, we obtain

$$u_n = |a_n|(-\varphi_1 + \frac{w_n}{|a_n|}) \leq -\frac{|a_n|}{2}\varphi_1 \ll \beta,$$

which contradicts the localization $u_n \nleq \beta$.

Observe that w_n satisfies

$$\begin{aligned} -\Delta w_n - \lambda_1 w_n &= h_n(u_n)u_n^- + (u_n^+)^q + f(x), && \text{in} \quad \Omega,\\ w_n &= 0, && \text{on} \quad \partial\Omega,\\ \int_\Omega w_n\varphi_1 &= 0. \end{aligned}$$

By the regularity theory for linear equation, for every $s > 0$ such that $h_n(u_n)u_n^- + (u_n^+)^q + f \in L^s(\Omega)$, we have $w_n \in W^{2,s}(\Omega)$ and, by Lemma 3.5, we obtain

$$\|w_n\|_{W^{2,s}} \lesssim \|h_n(u_n)u_n^- + (u_n^+)^q + f\|_{L^s} \lesssim \|(u_n^+)^q\|_{L^s} + 1. \tag{3.5}$$

By Lemma 3.3 and Lemma 3.5, we obtain, for $s_1 = \frac{2N}{N-2}$,

$$\|(u_n^+)^q\|_{L^{\frac{s_1}{q}}} \lesssim |a_n|^{\frac{q}{2}}, \tag{3.6}$$

and by Lemma 3.4 we have

$$\|(u_n^+)^q\|_{L^{\frac{q+1}{q}}} \lesssim |a_n|^{\frac{q}{q+1}}. \tag{3.7}$$

Observe also that as $\frac{a_n}{\|u_n\|_{H_0^1}} \to -1$, for n large enough, $u_n^+ \leq w_n^+$.

Step 1: $N = 3$**.** By (3.5) and (3.6), we have

$$\|w_n\|_{W^{2,s_1/q}} \lesssim |a_n|^{\frac{q}{2}} + 1.$$

As $\frac{s_1}{q} > N$, we know that $W^{2,\frac{s_1}{q}}(\Omega)$ is compactly embedded in $C_0^1(\overline{\Omega})$ and hence

$$\|w_n\|_{C_0^1} \lesssim |a_n|^{\frac{q}{2}} + 1.$$

Observing that $\frac{q}{2} < \frac{1}{2}\,\frac{N+1}{N-1} \leq 1$, we conclude in this case that $\frac{w_n}{|a_n|} \to 0$ in $C_0^1(\overline{\Omega})$.

Step 2: $N \geq 4$ **and** $q^2 \geq q+1$**.** By (3.7) and (3.5), we have $w_n \in W^{2,\frac{q+1}{q}}(\Omega)$ with

$$\|w_n\|_{W^{2,\frac{q+1}{q}}} \lesssim |a_n|^{\frac{q}{q+1}} + 1.$$

Moreover, observe that, as $q > 1$, we have $\frac{q}{q+1} - \frac{2}{N} > 0$. By the continuous injection properties of Sobolev spaces, we obtain $w_n \in L^{s_2}(\Omega)$ for

$$\frac{1}{s_2} = \frac{q}{q+1} - \frac{2}{N},$$

with

$$\|w_n\|_{L^{s_2}} \lesssim |a_n|^{\frac{q}{q+1}} + 1. \tag{3.8}$$

This implies that

$$\|(w_n^+)^q\|_{L^{\frac{s_2}{q}}} \lesssim |a_n|^{\frac{q^2}{q+1}} + 1. \tag{3.9}$$

Observe that

$$\frac{q}{q+1} < \frac{q^2}{q+1} \qquad \text{and} \qquad \frac{s_2}{q} > \frac{q+1}{q}.$$

This means that (3.7) is better for what concerns the exponent but (3.9) is better for what concerns the regularity. The idea now is to make a bootstrap combining these two estimates in order to gain regularity but keeping an exponent less than 1.

Recall that, until this point, we did not use the condition $q^2 \geq q+1$.

Using $u_n^+ \leq w_n^+$, we have

$$\begin{aligned}
\int_\Omega (u_n^+)^{qs} &\leq \int_\Omega (u_n^+)^{q\alpha}(w_n^+)^{q\beta} \\
&\leq \|(u_n^+)^{q\alpha}\|_{L^{\frac{q+1}{q\alpha}}} \|(w_n^+)^{q\beta}\|_{L^{\frac{s_2}{q\beta}}} \\
&\leq \|(u_n^+)^q\|^\alpha_{L^{\frac{q+1}{q}}} \|(w_n^+)^q\|^\beta_{L^{\frac{s_2}{q}}} \\
&\lesssim |a_n|^{\frac{q\alpha}{q+1}+\frac{\beta q^2}{q+1}} + |a_n|^{\frac{q\alpha}{q+1}}
\end{aligned} \tag{3.10}$$

with $\frac{1}{s}\left(\frac{q\alpha}{q+1} + \frac{\beta q^2}{q+1}\right) \leq 1$ in case α, β, s are chosen in such a way that

$$\begin{gathered}
s = \alpha + \beta, \\
0 \leq \alpha (\leq \tfrac{q+1}{q}), \qquad 0 \leq \beta (\leq \tfrac{s_2}{q}), \\
\tfrac{q\alpha}{q+1} + \tfrac{q\beta}{s_2} \leq 1, \\
\tfrac{q\alpha}{q+1} + \tfrac{\beta q^2}{q+1} \leq s.
\end{gathered}$$

The conditions in parenthesis can be deduced from the third one in case $\alpha \geq 0$ and $\beta \geq 0$.

As $q^2 > q + 1$, we see that the optimal s (i.e. the largest one) is given by

$$s = (q-1)\frac{1}{\frac{q^2}{q+1} - 1 + \frac{1}{s_2}} = \frac{N(q-1)}{Nq - N - 2}.$$

This corresponds to

$$\alpha = \frac{N(q^2 - q - 1)}{q(Nq - N - 2)} \qquad \text{and} \qquad \beta = \frac{N}{q(Nq - N - 2)}.$$

We deduce then from (3.5) that

$$\|w_n\|_{W^{2,s}} \lesssim |a_n| + 1.$$

The condition $q < \frac{N+1}{N-1}$ implies $\frac{N(q-1)}{Nq-N-2} > N$ and, by the continuous injection properties of Sobolev spaces, we have, for some $\alpha > 0$, $w_n \in C_0^{1,\alpha}(\overline{\Omega})$ with

$$\|w_n\|_{C_0^{1,\alpha}} \lesssim |a_n| + 1.$$

Hence, every subsequence $(w_{n_k})_k$ of $(w_n)_n$ has a subsubsequence $(w_{n_{k_j}})_j$ such that $\frac{w_{n_{k_j}}}{|a_{n_{k_j}}|} \to w$ strongly in $C_0^1(\overline{\Omega})$ by the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$. We deduce from $\frac{w_n}{|a_n|} \to 0$ in $H_0^1(\Omega)$, that $w = 0$. As this is true for all subsequence, we obtain $\frac{w_n}{|a_n|} \to 0$ in $C_0^1(\overline{\Omega})$ and the proof is complete in this case.

Step 3: $N \geq 4$, $q^2 < q + 1$ and $s_2 > qN$. By (3.8) and (3.5), we have $\|u_n\|_{L^{\frac{s_2}{q}}} \lesssim \|u_n\|_{L^{s_2}} \lesssim |a_n| + 1$ and hence, by (3.9), we have $w_n \in W^{2,\frac{s_2}{q}}(\Omega)$ with

$$\|w_n\|_{W^{2,\frac{s_2}{q}}} \lesssim |a_n|^{\frac{q^2}{q+1}} + 1. \tag{3.11}$$

As $\frac{s_2}{q} > N$, we obtain

$$\|w_n\|_{C_0^1} \lesssim |a_n|^{\frac{q^2}{q+1}} + 1,$$

with $\frac{q^2}{q+1} < 1$ and the conclusion follows.

Step 4: $N \geq 4$, $q^2 < q + 1$ and $\frac{s_2}{q} > \frac{N}{2}$. Working as in Step 3, we have (3.11) and as $\frac{s_2}{q} > \frac{N}{2}$, we obtain

$$\|w_n\|_C \lesssim |a_n|^{\frac{q^2}{q+1}} + 1.$$

Let us try to minimize the exponent keeping the regularity i.e. we want to make the same kind of bootstrap estimate as in (3.10) with α, β and s such that

$$\begin{gathered} s = \alpha + \beta, \\ 0 \leq \alpha, \qquad 0 \leq \beta, \\ \frac{q\alpha}{q+1} + \frac{q\beta}{s_2} \leq 1, \\ \frac{q\alpha}{q+1} + \frac{\beta q^2}{q+1} = \eta s, \end{gathered}$$

with $s > \frac{N}{2}$ and η minimum. We easily see that we obtain

$$\|w_n\|_C \lesssim |a_n|^\eta + 1,$$

with

$$\eta > \frac{2}{N+2-(N-2)q} = \frac{2}{N}\left(\frac{\frac{qN}{2s_2} - 1 + q(1-\frac{qN}{2(q+1)})}{\frac{q+1}{s_2} - 1}\right).$$

Now we use this information to improve the regularity. To this aim we compute

$$\begin{aligned}\int_\Omega (u_n^+)^{qs} &\leq \int_\Omega (u_n^+)^{q\alpha}(w_n^+)^{q\beta}\\ &\lesssim \|(u_n^+)^{q\alpha}\|_{L^{\frac{q+1}{q\alpha}}}\|(w_n^+)^{q\beta}\|_C\\ &\lesssim |a_n|^{\frac{q\alpha}{q+1}+\eta q\beta} + |a_n|^{\frac{q\alpha}{q+1}},\end{aligned}$$

with $\frac{1}{s}(\frac{q\alpha}{q+1} + \eta q\beta) \leq 1$ in case α, β, s are chosen in such a way that

$$\begin{gathered} s = \alpha + \beta,\\ 0 \leq \alpha \leq \frac{q+1}{q}, \qquad 0 \leq \beta,\\ \frac{q\alpha}{q+1} + \eta q\beta \leq s.\end{gathered}$$

Recall that we are looking for the larger s such that this system has a solution.

If $q < \frac{N+2}{N}$ i.e. $\frac{2q}{N+2-(N-2)q} < 1$, we can choose η such that $\frac{2q}{N+2-(N-2)q} < \eta q < 1$. In that case we have for $s > N$, $\alpha = \frac{q+1}{q}$ and $\beta = s - \frac{q+1}{q}$, the existence of $0 < \epsilon < \eta q < 1$ such that

$$\|w_n\|_{W^{2,s}} \lesssim |a_n|^\epsilon + 1,$$

and hence

$$\|w_n\|_{C_0^1} \lesssim |a_n|^\epsilon + 1,$$

which allows to conclude that $\frac{w_n}{|a_n|} \to 0$ in $C_0^1(\overline{\Omega})$.

If $q \geq \frac{N+2}{N}$ i.e. $\eta q > \frac{2q}{N+2-(N-2)q} \geq 1$, the optimal is obtained for $\alpha = \frac{q+1}{q}$ and β and s satisfying

$$\begin{gathered} s = \frac{q+1}{q} + \beta,\\ 1 + \eta q\beta = s.\end{gathered}$$

This gives

$$\beta = \frac{1}{q(\eta q - 1)} < \frac{N+2-(N-2)q}{q(Nq-(N+2))} \qquad \text{and} \qquad s = 1 + \frac{\eta}{\eta q - 1} < \frac{N(q-1)}{Nq-(N+2)}.$$

The condition $q < \frac{N+1}{N-1}$ implies again $\frac{N(q-1)}{Nq-(N+2)} > N$. Hence, choosing η close enough to $\frac{2}{N+2-(N-2)q}$, we have $s > N$, which implies that, for some $\alpha > 0$, $w_n \in C_0^{1,\alpha}(\overline{\Omega})$ with

$$\|w_n\|_{C_0^{1,\alpha}} \lesssim |a_n| + 1.$$

This allows to conclude in that case as in Step 2.

Step 5: $N \geq 4$, $\frac{s_2}{q} \leq \frac{N}{2}$ **and** $q^2 < q+1 \leq q^3$**.** If $\frac{s_2}{q} \leq \frac{N}{2}$ then, define $s_3 > 0$ by

$$\frac{1}{s_3} = \frac{q}{s_2} - \frac{2}{N}, \quad \text{if } \frac{q}{s_2} - \frac{2}{N} > 0,$$
$$s_3 > Nq, \qquad \text{if } \frac{q}{s_2} - \frac{2}{N} = 0.$$

As $q < \frac{N+1}{N-1}$, we easily see that $s_3 > s_2$ and

$$\frac{1}{s_3} < \frac{1}{N}(q^2 - 1) - (\frac{q^3}{q+1} - 1). \tag{3.12}$$

Moreover we have by (3.11) and Sobolev inequalities

$$\|w_n\|_{L^{s_3}} \lesssim |a_n|^{\frac{q^2}{q+1}} + 1,$$

and hence

$$\|(w_n^+)^q\|_{L^{\frac{s_3}{q}}} \lesssim |a_n|^{\frac{q^3}{q+1}} + 1.$$

As in Step 2, in case $q^3 \geq q+1 > q^2$, we obtain $(u_n^+)^q \in L^s(\Omega)$ with

$$\|(u_n^+)^q\|_{L^s} \lesssim |a_n| + 1$$

where

$$s = (q^2 - 1)\frac{1}{\frac{q^3}{q+1} - 1 + \frac{1}{s_3}}.$$

This corresponds to

$$\alpha = (\frac{q^3}{q+1} - 1)\,\frac{1}{\frac{q}{q+1}(\frac{q^3}{q+1} - 1 + \frac{1}{s_3})} \quad \text{and} \quad \beta = \frac{1}{q+1}\,\frac{1}{\frac{q}{q+1}(\frac{q^3}{q+1} - 1 + \frac{1}{s_3})}.$$

By (3.12) we have $s > N$. This implies that, for some $\alpha > 0$, $w_n \in C_0^{1,\alpha}(\overline{\Omega})$ with

$$\|w_n\|_{C_0^{1,\alpha}} \lesssim |a_n| + 1.$$

We conclude in that case as in Step 2.

Step 6. We now proceed by recurrence. Let us define the sequence $(s_j)_{j\geq 3}$ by

$$\frac{1}{s_j} = \frac{q}{s_{j-1}} - \frac{2}{N}. \tag{3.13}$$

As we have proved in Step 5 that $\frac{1}{s_3} < \frac{1}{s_2}$, it is easy to prove, by recurrence, that the sequence $(s_j)_{j\geq 3}$ is increasing. Moreover

$$\lim_{j\to\infty} s_j = +\infty,$$

as otherwise, if $s_j \to l$, we can easily prove that $l = \frac{N(q-1)}{2} < s_2$ which contradicts the fact that $(s_j)_j$ is increasing. This implies the existence of J such that $s_J > Nq$.

Now observe that, by bootstrap, for all $j \in \{2, \dots, J\}$,

$$\|w_n\|_{W^{2,\frac{s_j}{q}}} \lesssim |a_n|^{\frac{q^j}{q+1}} + 1. \tag{3.14}$$

Let us prove that the result is valid if $N \geq 4$, q is such that $q^j < q+1$ and, either $\frac{s_j}{q} > \frac{N}{2}$ or $q + 1 \leq q^{j+1}$.

Steps 3-4-5 concern the initial step $j = 2$. There, we have proved that the result is valid if $N \geq 4$, $q^2 < q+1$ and, either $\frac{s_2}{q} > \frac{N}{2}$ or $q+1 \leq q^3$. Moreover, we have proved that, in case $q+1 \leq q^3$, we have

$$\frac{q^2 - 1}{\frac{q^3}{q+1} - 1 + \frac{1}{s_3}} > N.$$

So assume that the result is valid if $N \geq 4$, $q^{j-1} < q+1$ and, either $\frac{s_{j-1}}{q} > \frac{N}{2}$ or $q+1 \leq q^j$, and that, in case $q+1 \leq q^j$ we have

$$\frac{q^{j-1} - 1}{\frac{q^j}{q+1} - 1 + \frac{1}{s_j}} > N.$$

Let us prove that the result is valid if $N \geq 4$, $q^j < q+1$ and, either $\frac{s_j}{q} > \frac{N}{2}$ or $q+1 \leq q^{j+1}$, and that, in case $q+1 \leq q^{j+1}$ we have

$$\frac{q^j - 1}{\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_{j+1}}} > N.$$

Case $\frac{s_j}{q} > N$. In that case, the result is proved as in Step 1 as, by (3.14), we have

$$\|w_n\|_{C^1_0} \lesssim |a_n|^{\frac{q^j}{q+1}} + 1 \qquad \text{with} \qquad \frac{q_j}{q+1} < 1.$$

Case $\frac{s_j}{q} > \frac{N}{2}$. We obtain, by (3.14),

$$\|w_n\|_C \lesssim |a_n|^{\frac{q^j}{q+1}} + 1.$$

As in Step 4, let us try now to minimize the exponent keeping the regularity. This means that we are looking for α, β and s such that

$$\begin{gathered} s = \alpha + \beta, \\ 0 \leq \alpha, \qquad 0 \leq \beta, \\ \frac{q\alpha}{q+1} + \frac{q\beta}{s_j} \leq 1, \\ \frac{q\alpha}{q+1} + \frac{\beta q^j}{q+1} = \eta s, \end{gathered}$$

with $s > \frac{N}{2}$ and η minimum.

By recurrence, we easily prove that

$$\frac{2}{N}\left(\frac{(\frac{qN}{2s_j} - 1) + q^{j-1}(1 - \frac{qN}{2(q+1)})}{\frac{q+1}{s_j} - 1}\right) = \frac{2}{N + 2 - (N-2)q}.$$

Hence, as in Step 4, we obtain

$$\|w_n\|_C \lesssim |a_n|^\eta + 1 \qquad \text{with} \qquad \eta > \frac{2}{N+2-(N-2)q}.$$

and we conclude as in Step 4.

Case $\frac{s_j}{q} < \frac{N}{2}$ *and* $q^j < q+1 \leq q^{j+1}$. In that case, we prove as in Step 2 that

$$\|(u_n^+)^q\|_{L^s} \lesssim |a_n| + 1,$$

with

$$s = \frac{q^j - 1}{\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_{j+1}}}.$$

This corresponds to

$$\alpha = (\frac{q^{j+1}}{q+1} - 1)\frac{1}{\frac{q}{q+1}(\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_{j+1}})} \qquad \text{and} \qquad \beta = \frac{1}{q+1}\frac{1}{\frac{q}{q+1}(\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_{j+1}})}.$$

It remains to prove that $s > N$ to conclude.

Assume that

$$\frac{q^{j-1} - 1}{\frac{q^j}{q+1} - 1 + \frac{1}{s_j}} > N,$$

and let us prove that

$$\frac{q^j - 1}{\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_{j+1}}} > N.$$

In fact, we have, by the recurrence assumption and as $q < \frac{N+1}{N-1}$,

$$\begin{aligned} N(\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_{j+1}}) &= N(\frac{q^{j+1}}{q+1} - 1 + \frac{q}{s_j} - \frac{2}{N}) \\ &= qN(\frac{q^j}{q+1} - 1 + \frac{1}{s_j}) + qN - N - 2 \\ &< q(q^{j-1} - 1) + qN - N - 2 \\ &= q^j - 1 + (N-1)q - (N+1) \\ &< q^j - 1. \end{aligned}$$

which proves the recurrence.

Hence in case $\frac{s_j}{q} < \frac{N}{2}$ and $q^j < q+1 \leq q^{j+1}$, we have

$$\|w_n\|_{W^{2,s}} \lesssim |a_n| + 1 \qquad \text{with} \qquad s > N,$$

and we conclude as in Step 2.

Conclusion. If $N = 3$, the Claim is obtained by Step 1. So we assume $N \geq 4$.

Let us choose $j \in \{1, \dots, J\}$ in the following way

$$\begin{array}{rl} j = 1, & \text{if } q+1 \leq q^2, \\ j = J, & \text{if } q+1 > q^J, \\ j = k \in \{2, \dots, J-1\}, & \text{if } q^k < q+1 \leq q^{k+1}. \end{array}$$

The case $j = 1$ is considered in Step 2.

In case $j = J$, we have by construction

$$\|w_n\|_{W^{2,\frac{s_J}{q}}} \lesssim |a_n|^{\frac{q^J}{q+1}} + 1,$$

with $\frac{s_J}{q} > N$ and $\frac{q^J}{q+1} < 1$. Hence we obtain

$$\|w_n\|_{C_0^1} \lesssim |a_n|^{\frac{q^J}{q+1}} + 1, \qquad \text{with} \qquad \frac{q^J}{q+1} < 1,$$

which allows to conclude that $\frac{w_n}{|a_n|} \to 0$ in $C_0^1(\overline{\Omega})$.

In case $j \in \{2, \dots, J-1\}$ the Claim is proved by Step 3-4-5 (initial case $j = 2$) or Step 6 (recurrence case). Hence the Claim is proved in every case.

By the Claim, we see that for $r > K$, the solution u_r of (2.1) with $u_r \nleq \beta$, $u_r \ngeq \alpha$, $u_r \geq \alpha_r$ and $\Phi_r(u_r) = c_r$ given by Proposition 3.1 satisfies $u_r > -K > -r$. This implies that, for this value of r, u_r is a solution of (1.1) which concludes the proof.

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