Research Article

Yansheng Shen*

Existence of Solutions for Choquard Type Elliptic Problems with Doubly Critical Nonlinearities

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Abstract: In this article, we first study the existence of nontrivial solutions to the nonlocal elliptic problems in $\mathbb{R}^N$ involving fractional Laplacians and the Hardy–Sobolev–Maz’ya potential. Using variational methods, we investigate the attainability of the corresponding minimization problem, and then obtain the existence of solutions. We also consider another Choquard type equation involving the p -Laplacian and critical nonlinearities in $\mathbb{R}^N$.

Keywords: Singular Elliptic Equation, Hardy–Sobolev–Maz’ya type Inequalities, Refinement of Hardy–Littlewood–Sobolev Inequality, Critical Nonlinearities, Variational Methods

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1 Introduction

In this paper, we first consider the existence of nontrivial weak solutions $u \in \dot{H}^s(\mathbb{R}^N)$ of the doubly critical equation of Choquard type with the Hardy–Sobolev–Maz’ya potential:

$$(-\Delta)^s u - \theta \frac{u}{|x'|^{2s}} = (\mathcal{K}_\alpha * |u|^{2_{s,\alpha}^\#}) |u|^{2_{s,\alpha}^\# - 2} u + \frac{|u|^{2_{s,\beta}^* - 2} u}{|x'|^\beta} \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

where $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $2 \leq k \leq N-1$, and $0 \leq \theta < c_{N,k,s} := \frac{c_{N,s}}{2} a_{k,s}$, with

$$c_{N,s} = 2^{2s} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}, \quad a_{k,s} := 2\pi^{\frac{N}{2}} \frac{\Gamma^2(\frac{k+2s}{4})}{\Gamma^2(\frac{k-2s}{4})} \frac{|\Gamma(-s)|}{\Gamma(\frac{N+2s}{2})};$$

also $\mathcal{K}_\alpha(x) = \frac{1}{|x|^{N-\alpha}}$ is a Riesz potential, $0 < \alpha, \beta < 2s < N$, $2_{s,\beta}^* = \frac{2(N-\beta)}{N-2s}$ is the critical Hardy–Sobolev exponent and $2_{s,\alpha}^\# = \frac{N+\alpha}{N-2s}$ is the fractional critical exponent for the Sobolev type embedding. We note that $a_{k,s}$ is the best constant of the following fractional Hardy–Maz’ya inequality:

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \geq a_{k,s} \int_{\mathbb{R}^N} \frac{u(x)^2}{|x'|^{2s}} dx \quad \text{for all } u \in C_c^{0,1}(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k}).$$

The fractional Laplacian operator $(-\Delta)^s$ is a nonlocal pseudo-differential operator, taking the form

$$(-\Delta)^s u(x) := c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad s \in (0, 1),$$

*Corresponding author: Yansheng Shen, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P. R. China, e-mail: ysshenn@mail.bnu.edu.cn

where $u \in C_c^\infty(\mathbb{R}^N)$, P.V. stands for the Cauchy principle value and $c_{N,s} = 2^{2s} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{|\Gamma(-s)|}$, see [11] and the references therein for the basics on the fractional Laplacian.

The weak solutions of (1.1) will be found in the space $\dot{H}^s(\mathbb{R}^N)$, which is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(\xi)|^2 d\xi.$$

By a weak solution u of the problem $(-\Delta)^s u = f$ in $\mathbb{R}^N$ for $f \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$, we mean that $u \in \dot{H}^s(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx = \int_{\mathbb{R}^N} f \varphi dx \quad \text{for all } \varphi \in \dot{H}^s(\mathbb{R}^N).$$

Problems with two nonlinearities recently have been studied by several authors. For the case of local operators, such problems were considered in [2, 13, 18] for the Laplacian $-\Delta$, the p -Laplacian $-\Delta_p$ and the bi-harmonic operator Δ^2 . Corresponding to the local case of such problems, the nonlocal fractional elliptic equations have been investigated widely in recent years. In [17], by using the harmonic extension [5] and concentration compactness principle [24, 25], Ghoussoub and Shakerian proved the existence of nontrivial weak solutions for a fractional Laplacian problem with critical Sobolev and Hardy–Sobolev terms. Then, by using the same methods, Chen [9] investigated such a problem with two critical Hardy–Sobolev terms. It is worth noting that the problems above are related to the Sobolev and Hardy–Sobolev type inequalities, see [10, 15, 32] and the references therein. In 2017, Yang and Wu [33] extended such problems to a broader scope. They studied the following fractional Choquard equation with critical nonlocal Hartree term and Hardy–Sobolev term:

$$(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = (\mathcal{K}_\alpha * |u|^{2_{h,a}^*}) |u|^{2_{h,a}^*-2} u + \frac{|u|^{2_{s,\beta}^*-2} u}{|x|^\beta} \quad \text{in } \mathbb{R}^N,$$

where $\mathcal{K}_\alpha = |x|^{a-N}$ and $0 < \alpha < 2s < N$, $0 \leq \gamma < 4^s \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$. The method they used is elementary, and they did not use the tools mentioned above, such as the extension methods or the concentration compactness principle, etc. Instead, they use an improved Sobolev inequality, which was proved in [29]. For more about nonlinear Choquard equations involving the singular Hardy potential and critical nonlinearities, see [14, 16, 28] and the references therein.

The problems mentioned above have a common characteristic, that is, they involve the singular Hardy potential and critical nonlinearities. Recently, as an application of the Hardy–Sobolev–Maz'ya inequality, a great attention has been focused on the study of fractional elliptic problems involving the Hardy–Sobolev–Maz'ya potential. In 2017, Cai, Chu and Lei [6] studied the existence of solutions for a fractional elliptic problem with the Hardy–Sobolev–Maz'ya potential and critical nonlinearities in a bounded domain. In 2019, Mallick [27] proved the following Hardy–Sobolev–Maz'ya type constant:

$$S(N, s, \gamma, \theta) := \inf_{u \in C_c^{0,1}(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx dy - \gamma \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{2s}} dx}{\left(\int_{\mathbb{R}^N} \frac{|u|^{2_{s,\theta}^*}}{|x'|^\theta} dx \right)^{\frac{2}{2_{s,\theta}^*}}},$$

achieved by a nontrivial and nonnegative function in $\dot{H}_\gamma^s(\mathbb{R}^N)$, where $s \in (0, 1)$, $x' \in \mathbb{R}^k$, $2 \leq k \leq N-1$, $0 \leq \theta < 2s < N$, $0 \leq \gamma < a_{k,s}$. (We refer to [7, 12], and many more references therein, for classical Hardy–Sobolev–Maz'ya inequalities.) As an application of this result, Mallick derived the existence of positive solutions for the Euler–Lagrange equation that related to the fractional Hardy–Sobolev–Maz'ya inequality in the entire space:

$$\frac{2}{c_{N,s}} (-\Delta)^s u - \gamma \frac{u}{|x'|^{2s}} = \frac{|u|^{2_{s,\theta}^*-2} u}{|x'|^\theta}.$$

Furthermore, Mallick investigated the symmetry properties and a precise asymptotic behavior of the solutions (see also [26]).

Now, it is fairly natural to ask whether the nontrivial weak solutions exists for the fractional elliptic equations with Hardy–Sobolev–Maz'ya potential and doubly critical nonlinearities. To the best of our knowledge,

there is not any such result in the literature on such problems. Hence, based on the results obtained by Mallick and inspired by the work of Yang and Wu, our first aim is to study the existence of nontrivial solutions for problem (1.1).

Solutions of problem (1.1) were found as critical points of a suitable functional, by the Mountain Pass Lemma without the $(PS)_c$ condition. In this case, the $(PS)_c$ condition only holds true for c in certain intervals related to the best Hardy–Sobolev–Maz’ya type constant. In the control of the Mountain–Pass level, the extremal function of the best Hardy–Sobolev–Maz’ya type constant plays an important role. The explicit form of this function is used in the process.

Then, by applying the same strategy of (1.1), and as an application of the results obtained by Su and Chen [30], we continue to study another Choquard equation involving critical nonlinearities and the local p -Laplace operator:

$$-\Delta_p u - \mu \frac{|u|^{p^*-2} u}{|x|^p} = (\mathcal{K}_\lambda * |u|^{p_\lambda^*}) |u|^{p_\lambda^*-2} u + |u|^{p^*-2} u \quad \text{in } \mathbb{R}^N, \quad (1.2)$$

where $N \geq 3$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplace operator, $\mu \in [0, (\frac{N-p}{p})^p]$, $p \in (1, N)$, $\lambda \in (0, N)$, $p^* = \frac{Np}{N-p}$ and $p_\lambda^* = \frac{p}{2} \cdot \frac{2N-\lambda}{N-p}$ is the Hardy–Littlewood–Sobolev upper critical exponent; the Riesz potential $\mathcal{K}_\lambda(x)$ here is taking the form $\mathcal{K}_\lambda(x) = |x|^{-\lambda}$. The Sobolev space $W^{1,p}(\mathbb{R}^N)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{W^{1,p}(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

In [30], Su and Chen investigated the following minimizing problem:

$$\Lambda_{\mu,\lambda} := \inf_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx}{\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\lambda^*} |u(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy \right)^{\frac{p}{2p_\lambda^*}}},$$

where $N \geq 3$, $\lambda \in (0, N)$, $p \in (1, N)$ and $p_\lambda^* = \frac{p}{2} \cdot \frac{2N-\lambda}{N-p}$. By using a refinement of the Hardy–Littlewood–Sobolev inequality, they proved that $\Lambda_{\mu,\lambda}$ is achieved in $\mathbb{R}^N$ by a radially symmetric decreasing and nonnegative function. Moreover, they gave an estimation of the extremal functions; we refer the reader to [30] for details.

This paper is organized as follows. In Section 2, we establish the appropriate space which is applicable to the study of problems (1.1) and (1.2), and we give some preliminary results and the main results of this paper. In Section 3, we investigate the attainability of the minimization problem related to problem (1.1), and then we give the proof of the existence of a solution to problem (1.1) in Section 4. Finally, we give the proof of the existence result for problem (1.2).

2 Preliminaries and Statements of the Main Results

Equations (1.1) and (1.2) are related to some specific functional embedding and geometric inequalities.

Towards problem (1.1), for fixed $2 \leq k \leq N-1$, $0 \leq \theta < c_{N,k,s}$, we define $\mathcal{H}_\theta^s(\mathbb{R}^N)$ as the completion of $C_c^{0,1}(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$ under the following norm:

$$[u]_{s,\theta,\mathbb{R}^N}^2 := \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy - \theta \int_{\mathbb{R}^N} \frac{u(x)^2}{|x'|^{2s}} dx = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \theta \int_{\mathbb{R}^N} \frac{u(x)^2}{|x'|^{2s}} dx,$$

and it is evident that $\mathcal{H}_\theta^s(\mathbb{R}^N) = \dot{H}^s(\mathbb{R}^N)$ for $0 \leq \theta < c_{N,k,s}$.

Thus, in the present paper, we work in the space $\dot{H}^s(\mathbb{R}^N)$ for problem (1.1). Note that if $0 \leq \theta < c_{N,k,s}$, it follows from the Hardy–Maz’ya inequality that

$$\|u\|_H^2 = \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u|^2 - \theta \frac{|u|^2}{|x'|^{2s}} \right) dx \quad (2.1)$$

is comparable to the norm $\|u\|_{\dot{H}^s(\mathbb{R}^N)}^2$, since the following inequalities hold:

$$\left(1 - \frac{\theta_+}{c_{N,k,s}}\right) \|u\|_{\dot{H}^s(\mathbb{R}^N)} \leq \|u\|_H \leq \left(1 + \frac{\theta_-}{c_{N,k,s}}\right) \|u\|_{\dot{H}^s(\mathbb{R}^N)},$$

for all $u \in \dot{H}^s(\mathbb{R}^N)$, where $\theta_+ = \max\{\theta, 0\}$ and $\theta_- = \max\{-\theta, 0\}$. Next, we work in the $\dot{H}^s(\mathbb{R}^N)$ for problem (1.1) endowed with the norm (2.1).

For problem (1.2), the standard Hardy inequality asserts that $W^{1,p}(\mathbb{R}^N)$ is embedded in the weighted space $L^p(\mathbb{R}^N, |x|^{-p})$ and this embedding is continuous. More precisely,

$$\mu_1 \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \int_{\mathbb{R}^N} |\nabla u|^p dx, \quad \mu_1 := \left(\frac{N-p}{p}\right)^p, \quad (2.2)$$

for all $u \in W^{1,p}(\mathbb{R}^N)$. Moreover, the constant μ_1 is optimal. If $\mu < \mu_1$, it follows from the Hardy inequality (2.2) that

$$\|u\|_W := \left(\int_{\mathbb{R}^N} |\nabla u|^p dx - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \right)^{\frac{1}{p}} \quad (2.3)$$

is well defined on $W^{1,p}(\mathbb{R}^N)$. By using the same arguments mentioned above, we know that $\|\cdot\|_W$ is an equivalent norm in $W^{1,p}(\mathbb{R}^N)$. Thus, we work in $W^{1,p}(\mathbb{R}^N)$ for problem (1.2) and we take the equivalent norm (2.3).

Solutions of problems (1.1) and (1.2) are equivalent to a nontrivial critical point of the following functionals, respectively:

$$I_0(u) = \frac{1}{2} \|u\|_H^2 - \frac{1}{2 \cdot 2_{s,\alpha}^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_{s,\alpha}^*} |u(y)|^{2_{s,\alpha}^*}}{|x-y|^{N-\alpha}} dx dy - \frac{1}{2_{s,\beta}^*} \int_{\mathbb{R}^N} \frac{|u|^{2_{s,\beta}^*}}{|x'|^\beta} dx,$$

where $u \in \dot{H}^s(\mathbb{R}^N)$, and

$$I_1(u) = \frac{1}{p} \|u\|_W^p - \frac{1}{2 \cdot p_\lambda^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\lambda^*} |u(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx,$$

where $u \in W^{1,p}(\mathbb{R}^N)$.

Now, we introduce the following improved version of Sobolev inequality, which can be founded in [27, 29]. We say that a measurable function $u: \mathbb{R}^N \rightarrow \mathbb{R}$ belongs to the Morrey space $\|u\|_{\mathcal{L}^{r,\varpi}(\mathbb{R}^N)}$, with $r \in [1, \infty)$ and $\varpi \in (0, N]$, if only and if

$$\|u\|_{\mathcal{L}^{r,\varpi}(\mathbb{R}^N)}^r = \sup_{R>0, x \in \mathbb{R}^N} R^{\varpi-N} \int_{B(x,R)} |u(y)|^r dy < \infty.$$

Mallick [27] extended the improved version of fractional Sobolev inequality that originated in [29], we state it below and give a proof for the convenience of the readers. This can be viewed as a generalization of the classical Caffarelli–Kohn–Nirenberg inequalities (see [4] and also [21]).

Lemma 2.1 ([27]). *Let $0 \leq b < 2s$ and $u \in \dot{H}^s(\mathbb{R}^N)$. Then there exist constants $C > 0$, $\frac{2}{2_s^*} \leq \theta < 1$, independent of u , such that*

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2_{s,b}^*}}{|x'|^b} dx \right)^{\frac{1}{2_{s,b}^*}} \leq C \|u\|_{\dot{H}^s(\mathbb{R}^N)}^{\theta_1} \|u\|_{L^{2,N-2s}}^{\theta_2}, \quad (2.4)$$

where $2_s^* = \frac{2N}{N-2s}$ is the critical Sobolev exponent,

$$\|u\|_{L^{2,N-2s}}^2 := \sup_{R>0, x \in \mathbb{R}^N} \frac{R^{N-2s}}{|B_R^N(x)|} \int_{B_R^N(x)} |u|^2 dy,$$

$\theta_1 + \theta_2 = 1$ and θ_1 is the constant that satisfies

$$\theta_1 = \begin{cases} \theta, & b = 0, \\ [(2_{s,b}^* - 2)\theta + 2]/2_{s,b}^*, & 0 < b < 2s. \end{cases}$$

Proof. Case I: $b = 0$. In the case, we obtain the conclusion from [29] at once. That is, there exists constants $C > 0$, $\frac{2}{2_s^*} \leq \theta < 1$, such that

$$\|u\|_{L^{2_s^*}} \leq C \|u\|_{\dot{H}^s(\mathbb{R}^N)}^\theta \|u\|_{L^{2, N-2s}}^{1-\theta}. \quad (2.5)$$

Case II: $0 < b < 2s$. By the Hölder inequality, we have

$$\int_{\mathbb{R}^N} \frac{|u|^{2_{s,b}^*}}{|x'|^b} dx \leq \left(\int_{\mathbb{R}^N} |u|^{2_s^*} dx \right)^{\frac{2_{s,b}^*-2}{2_s^*}} \left(\int_{\mathbb{R}^N} \frac{|u|^{\frac{2 \cdot 2_s^*}{2_s^*-2_{s,b}^*+2}}}{|x'|^{\frac{b \cdot 2_s^*}{2_s^*-2_{s,b}^*+2}}} dx \right)^{\frac{2_s^*-2_{s,b}^*+2}{2_s^*}}. \quad (2.6)$$

Let $t = \frac{b \cdot 2_s^*}{2_s^*-2_{s,b}^*+2}$. Then $0 < t < 2s$ and $2_{s,t}^* = \frac{2 \cdot 2_s^*}{2_s^*-2_{s,b}^*+2}$. Hence, by the fractional Hardy–Sobolev–Maz’ya inequality, and by combining (2.5) and (2.6), we get the desired inequality (2.4) with $\theta_1 = ((2_{s,b}^* - 2)\theta + 2)/2_{s,b}^*$. The proof of Lemma 2.1 is complete. $\square$

Lemma 2.2 ([29]). *For any $1 < p < N$, there exists a constant $C > 0$, depending only on N and p such that for any θ and ϑ satisfying $\frac{p}{p^*} \leq \theta < 1$, $1 \leq \vartheta < p^*$, we have*

$$\left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \leq C \|u\|_{W^{1,p}(\mathbb{R}^N)}^\theta \|u\|_{\mathcal{L}^{\vartheta, \frac{\vartheta(N-p)}{p}}(\mathbb{R}^N)}^{1-\theta} \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N).$$

Lemma 2.3 (Hardy–Littlewood–Sobolev Inequality [22, 23]). *Let $t, r > 1$ and $0 < \lambda < N$, with $\frac{1}{t} + \frac{1}{r} + \frac{\lambda}{N} = 2$, and let $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C > 0$, independent of f, g , such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)||h(y)|}{|x-y|^\lambda} dx dy \leq C \|f\|_t \|h\|_r. \quad (2.7)$$

Moreover, when $t = r = \frac{2N}{2N-\lambda}$, the equality in (2.7) holds if and only if the extremal u is given by

$$u(x) = \frac{C}{(\lambda^2 + |x - x_0|^2)^{\frac{N-2s}{2}}}$$

for some $C, \lambda > 0$ and $x_0 \in \mathbb{R}^N$.

By the Hardy–Littlewood–Sobolev inequality, we have

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_{s,\alpha}^*} |u(y)|^{2_{s,\alpha}^*}}{|x-y|^{N-\alpha}} dx dy \right)^{\frac{1}{2_{s,\alpha}^*}} \leq C \|u\|_{L^{2_s^*}}^2. \quad (2.8)$$

The exponent $2_{s,\alpha}^*$ is critical in the sense that it is the limit exponent for the Sobolev type inequality. Combining (2.4) and (2.8), we have the following inequality:

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_{s,\alpha}^*} |u(y)|^{2_{s,\alpha}^*}}{|x-y|^{N-\alpha}} dx dy \right)^{\frac{1}{2_{s,\alpha}^*}} \leq C \|u\|_{\dot{H}^s(\mathbb{R}^N)}^{2\theta_1} \|u\|_{L^{2, N-2s}}^{2\theta_2} \quad \text{for all } u \in \dot{H}^s(\mathbb{R}^N). \quad (2.9)$$

Similarly, combining Lemma 2.2 and Lemma 2.3, Su and Chen [30] derived the following result:

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\lambda^*} |u(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy \right)^{\frac{1}{p_\lambda^*}} \leq C \|u\|_{W^{1,p}(\mathbb{R}^N)}^{2\theta} \|u\|_{\mathcal{L}^{p, N-p}(\mathbb{R}^N)}^{2(1-\theta)}, \quad (2.10)$$

where $u \in W^{1,p}(\mathbb{R}^N)$.

For $0 \leq \theta < c_{N,k,s}$, we define the following best fractional Hardy–Sobolev–Maz’ya constant:

$$S_{\theta,\beta} = \inf_{u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 - \theta \frac{|u|^2}{|x'|^{2s}}) dx}{\left(\int_{\mathbb{R}^N} \frac{|u|^{2_{s,\beta}^*}}{|x'|^\beta} dx \right)^{\frac{2}{2_{s,\beta}^*}}}. \quad (2.11)$$

We may easily see, as in [27] with minor changes, that the infimum is achieved in $\dot{H}^s(\mathbb{R}^N)$.

We first consider problem (1.1). Since solutions of (1.1) can be founded as a nontrivial critical point of the functional I_0 , the Nehari manifold related to I_0 is given by

$$\mathcal{N}_0 = \{u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\} : \langle I'_0(u), u \rangle = 0\}.$$

And a minimizer of the minimization problem

$$c_0 = \inf_{u \in \mathcal{N}_0} I_0(u)$$

is a solution of problem (1.1). In order to establish the existence of solutions for problem (1.1) for $0 \leq \theta < c_{N,k,s}$, we investigate the minimization problem

$$S_{\theta,\alpha} = \inf_{u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 - \theta \frac{|u|^2}{|x'|^{2s}}) dx}{\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_{s,\alpha}^*} |u(y)|^{2_{s,\alpha}^*}}{|x-y|^{N-\alpha}} dx dy \right)^{\frac{1}{2_{s,\alpha}^*}}}. \quad (2.12)$$

We will show that $S_{\theta,\alpha}$ is achieved by a nontrivial and nonnegative function in $\dot{H}^s(\mathbb{R}^N)$ and this fact allow us to establish the existence of solution for problem (1.1).

Theorem 2.4. Suppose that $0 \leq \theta < c_{N,k,s}$. Then $S_{\theta,\alpha}$ can be achieved by a positive function in $\dot{H}^s(\mathbb{R}^N)$.

Theorem 2.5. Suppose that $0 \leq \theta < c_{N,k,s}$. Then problem (1.1) has at least one nontrivial solution in $\dot{H}^s(\mathbb{R}^N)$.

Based on the results in [30] that the sharp constants $\Lambda_{\mu,\lambda}$ and

$$\Lambda_{\mu} := \inf_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p dx - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{\frac{p}{p^*}}}$$

are achieved, by using similar arguments as in the proof of Theorem 2.5, we obtain the following result.

Theorem 2.6. Suppose $\mu \in [0, (\frac{N-p}{p})^p]$, $\lambda \in (0, N)$. Then problem (1.2) has at least one nontrivial solution in $W^{1,p}(\mathbb{R}^N)$.

Throughout this paper, we assume that the projection of a point $x \in \mathbb{R}^N$ to $\mathbb{R}^k$ ($2 \leq k \leq N-1$) and $\mathbb{R}^{N-k}$ is denoted by x' and x'' , respectively; $B_R^k(z)$ denotes the k -dimensional ball of radius $R > 0$ centered at $z \in \mathbb{R}^k$, $o(1)$ is a generic infinitesimal value, and we always denote positive constants with C for convenience.

3 The Proof of Theorem 2.4

Proof. Let $\{u_n\}_n$ be a minimizing sequence of $S_{\theta,\alpha}$. Thanks to the homogeneity of (2.12), we do assume that

$$\int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_n|^2 - \theta \frac{|u_n|^2}{|x'|^{2s}} \right] dx \rightarrow S_{\theta,\alpha}, \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_{s,\alpha}^*} |u_n(y)|^{2_{s,\alpha}^*}}{|x-y|^{N-\alpha}} dx dy = 1.$$

Using inequality (2.9), we can find $C > 0$ such that

$$\|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)} \geq C.$$

Also, the Sobolev imbedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2,N-2s}(\mathbb{R}^N)$ implies that

$$\|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)} \leq C.$$

So we can find $\lambda_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\lambda_n^{-2s} \int_{B_{\lambda_n}^N(x_n)} |u_n(y)|^2 dy \geq \|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)}^2 - \frac{C}{2n} \geq C_1 > 0.$$

Let $\hat{u}_n(x) = \lambda_n^{\frac{N-2s}{2}} u_n(\lambda_n x', \lambda_n x'' + x_n'')$. Then we have

$$\int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} |\hat{u}_n(y)|^2 dy \geq C_1 > 0.$$

Note that

$$\begin{cases} \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} \hat{u}_n|^2 - \theta \frac{|\hat{u}_n|^2}{|x'|^{2s}} \right] dx = \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_n|^2 - \theta \frac{|u_n|^2}{|x'|^{2s}} \right] dx \rightarrow S_{\theta, \alpha}, \\ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\hat{u}_n(x)|^{2_{s, \alpha}^*} |\hat{u}_n(y)|^{2_{s, \alpha}^*}}{|x-y|^{N-\alpha}} dx dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_{s, \alpha}^*} |u_n(y)|^{2_{s, \alpha}^*}}{|x-y|^{N-\alpha}} dx dy = 1, \end{cases} \quad (3.1)$$

which implies that $\{\hat{u}_n\}_n$ is also a minimizing sequence of $S_{\theta, \alpha}$. Moreover, from (3.1), we can show that $\{\hat{u}_n\}_n$ is bounded in $\dot{H}^s(\mathbb{R}^N)$. Hence, we assume that

$$\hat{u}_n(x) \rightharpoonup \hat{u}(x) \text{ in } \dot{H}^s(\mathbb{R}^N), \quad \hat{u}_n(x) \rightarrow \hat{u}(x) \text{ a.e. in } \mathbb{R}^N, \quad \hat{u}_n(x) \rightarrow \hat{u}(x) \text{ in } L_{\text{loc}}^p(\mathbb{R}^N), \quad 1 \leq p < 2_s^*.$$

Now, we claim that $\{\frac{x_n'}{\lambda_n}\}_n$ is uniformly bounded in n . For $0 < \kappa < 2s$, by the Hölder inequality, we have

$$\begin{aligned} 0 < C_1 &\leq \int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} |\hat{u}_n|^2 dx = \int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} |x'|^{\frac{2\kappa}{2s, \kappa}} \frac{|\hat{u}_n|^2}{|x'|^{\frac{2\kappa}{2s, \kappa}}} dx \\ &\leq \left(\int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} |x'|^{\frac{2\kappa}{2s, \kappa} \cdot \frac{2s^*, \kappa}{2s^*, \kappa - 2}} dx \right)^{\frac{2s^*, \kappa - 2}{2s^*, \kappa}} \left(\int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} \frac{|\hat{u}_n|^{2 \cdot \frac{2s^*, \kappa}{2s^*, \kappa - 2}}}{|x'|^{\frac{2\kappa}{2s, \kappa} \cdot \frac{2s^*, \kappa}{2s^*, \kappa - 2}}} dx \right)^{\frac{2}{2s^*, \kappa}} \\ &= \left(\int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} |x'|^{\frac{\kappa(N-2s)}{2s-\kappa}} dx \right)^{\frac{2s-\kappa}{2s^*, \kappa}} \left(\int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} \frac{|\hat{u}_n|^{2s^*, \kappa}}{|x'|^\kappa} dx \right)^{\frac{2}{2s^*, \kappa}}. \end{aligned}$$

By the rearrangement inequality [23], we have

$$\int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} |x'|^{\frac{\kappa(N-2s)}{2s-\kappa}} dx \leq C \int_{B_1^k(0)} |x'|^{\frac{\kappa(N-2s)}{2s-\kappa}} dx' \leq C,$$

which implies that

$$\int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} \frac{|\hat{u}_n|^{2s^*, \kappa}}{|x'|^\kappa} dx \geq C > 0. \quad (3.2)$$

Now, suppose on the contrary that $|\frac{x_n'}{\lambda_n}| \rightarrow \infty$. Then for any $y \in B_1^k(\frac{x_n'}{\lambda_n})$, we have

$$|y| \geq \left| \frac{x_n'}{\lambda_n} \right| - 1$$

for n large enough. By the Hölder inequality and the fractional Sobolev inequality, it follows that

$$\begin{aligned} \int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} \frac{|\hat{u}_n|^{2s^*, \kappa}}{|x'|^\kappa} dx &\leq \frac{1}{(|\frac{x_n'}{\lambda_n}| - 1)^\kappa} \int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} |\hat{u}_n|^{2s^*, \kappa} dx \\ &\leq \frac{1}{(|\frac{x_n'}{\lambda_n}| - 1)^\kappa} \left(\int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x_n'}{\lambda_n})} |\hat{u}_n|^{2s^*} dx \right)^{\frac{N-\kappa}{N}} \\ &\leq \frac{1}{(|\frac{x_n'}{\lambda_n}| - 1)^\kappa} \|\hat{u}_n\|_{\dot{H}^s(\mathbb{R}^N)}^{\frac{2(N-\kappa)}{N-2s}} \rightarrow 0, \end{aligned}$$

which contradicts (3.2). Thus, $\{\frac{x'_n}{\lambda_n}\}_n$ is uniformly bounded, and there exists $R > 0$ such that

$$\int_{B_1^{N-k}(0)} \int_{B_R^k(0)} |\hat{u}_n|^2 dx \geq \int_{B_1^{N-k}(0)} \int_{B_1^k(\frac{x'_n}{\lambda_n})} |\hat{u}_n|^2 dx \geq C > 0.$$

The Sobolev compact embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^N)$ implies that

$$\int_{B_1^{N-k}(0)} \int_{B_R^k(0)} |\hat{u}|^2 dx \geq C > 0,$$

which means $\hat{u} \neq 0$. By the Brezis–Lieb lemma [3], we obtain

$$\int_{\mathbb{R}^N} (\mathcal{K}_\alpha * |\hat{u}_n - \hat{u}|^{2_{s,a}^*}) |\hat{u}_n - \hat{u}|^{2_{s,a}^*} dx + \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * |\hat{u}|^{2_{s,a}^*}) |\hat{u}|^{2_{s,a}^*} dx = \int_{\mathbb{R}^N} (\mathcal{K}_\alpha * |\hat{u}_n|^{2_{s,a}^*}) |\hat{u}_n|^{2_{s,a}^*} dx + o(1).$$

Hence,

$$\begin{aligned} S_{\theta,\alpha} &= \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} \hat{u}_n|^2 - \theta \frac{|\hat{u}_n|^2}{|x'|^{2s}} \right] dx + o(1) \\ &= \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} (\hat{u}_n - \hat{u})|^2 - \theta \frac{|\hat{u}_n - \hat{u}|^2}{|x'|^{2s}} \right] dx + \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} \hat{u}|^2 - \theta \frac{|\hat{u}|^2}{|x'|^{2s}} \right] dx + o(1) \\ &\geq S_{\theta,\alpha} \left(\int_{\mathbb{R}^N} [(\mathcal{K}_\alpha * |\hat{u}_n - \hat{u}|^{2_{s,a}^*}) |\hat{u}_n - \hat{u}|^{2_{s,a}^*}] dx \right)^{\frac{1}{2_{s,a}^*}} + S_{\theta,\alpha} \left(\int_{\mathbb{R}^N} [(\mathcal{K}_\alpha * |\hat{u}|^{2_{s,a}^*}) |\hat{u}|^{2_{s,a}^*}] dx \right)^{\frac{1}{2_{s,a}^*}} + o(1) \\ &\geq S_{\theta,\alpha} \left(\int_{\mathbb{R}^N} [(\mathcal{K}_\alpha * |\hat{u}_n - \hat{u}|^{2_{s,a}^*}) |\hat{u}_n - \hat{u}|^{2_{s,a}^*}] dx + \int_{\mathbb{R}^N} [(\mathcal{K}_\alpha * |\hat{u}|^{2_{s,a}^*}) |\hat{u}|^{2_{s,a}^*}] dx \right)^{\frac{1}{2_{s,a}^*}} + o(1) \\ &= S_{\theta,\alpha}. \end{aligned}$$

Since $\hat{u} \neq 0$, we have

$$S_{\theta,\alpha} = \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} \hat{u}|^2 - \theta \frac{|\hat{u}|^2}{|x'|^{2s}} \right] dx$$

and

$$\int_{\mathbb{R}^N} [(\mathcal{K}_\alpha * |\hat{u}|^{2_{s,a}^*}) |\hat{u}|^{2_{s,a}^*}] dx = 1,$$

which implies that $S_{\theta,\alpha}$ is achieved.

Let $\hat{u}$ be a minimizer, since

$$\int_{\mathbb{R}^N} [|(-\Delta)^{\frac{s}{2}} \hat{u}|^2] dx \leq \int_{\mathbb{R}^N} [|(-\Delta)^{\frac{s}{2}} \hat{u}|^2] dx.$$

Thus, we obtain that $\hat{u} \geq 0$ and the proof is complete. $\square$

4 The Proof of Theorem 2.5

In the section, we prove the existence of solutions for problem (1.1). We first introduce the Mountain Pass Lemma of Ambrosetti and Rabinowitz [1] and give some technical lemmas so that we can use the Mountain Pass Lemma to seek critical points of the well-defined functional I_0 . Establishing the existence of solutions for nonlinear PDEs with critical growth using the critical point theory and the Mountain Pass Lemma has been widely used in the literature (see, e.g., [8, 19, 20, 34] and the references therein).

Lemma 4.1 ([1]). *Let $(X, \|\cdot\|)$ be a Banach space and let $J: X \rightarrow \mathbb{R}$ be a C^1 -functional satisfying the following conditions:*

- (a) $J(0) = 0$.
 - (b) *There exist $\rho, R > 0$ such that $J(u) \geq \rho$ for all $u \in X$, with $\|u\| = R$.*
 - (c) *There exist $v_0 \in X$ such that $\limsup_{t \rightarrow \infty} J(tv_0) < 0$.*
- Let $t_0 > 0$, $e = t_0 v_0$ be such that $\|e\| > R$ and $J(e) < 0$. Define*

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Then $c \geq \rho > 0$ and there exists a $(PS)_c$ sequence, that is, there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset X$, such that

$$\lim_{n \rightarrow \infty} J(u_n) = c \quad \text{and} \quad \lim_{n \rightarrow \infty} J'(u_n) = 0 \quad \text{strongly in } X'.$$

In addition, we say that J satisfies the $(PS)_c$ compactness condition if any $(PS)_c$ sequence $(u_n)_{n \in \mathbb{N}}$ for J in X has a convergent subsequence.

From the arguments in the preliminary, we know that the minimizer of the minimization problem $c_0 = \inf_{u \in \mathcal{N}_0} I_0(u)$ is a solution of problem (1.1). In order to apply Lemma 4.1, we set

$$c_\gamma^0 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_0(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], \dot{H}^s(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = e\}$ and

$$c_s^0 = \inf_{u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} I_0(tu).$$

From the definition above, under the condition $0 < \alpha, \beta < 2s < N$, we have following conclusion:

$$c_0 = c_\gamma^0 = c_s^0. \quad (4.1)$$

The proof for (4.1) is standard, see [31]. Moreover, we define

$$c^* := \min \left\{ \frac{\alpha + 2s}{2(N + \alpha)} S_{\theta, \alpha}^{\frac{N+\alpha}{\alpha+2s}}, \frac{2s - \beta}{2(N - \beta)} S_{\theta, \beta}^{\frac{N-\beta}{2s-\beta}} \right\},$$

which is a suitable energy threshold that ensures problem (1.1) admits a nontrivial solution.

Lemma 4.2. *If $0 < \alpha, \beta < 2s < N$, then $c_s^0 \leq c_0^*$.*

Proof. By Theorem 2.4, there exist a minimizer $u_1 \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}$ of $S_{\theta, \alpha}$, and, by the existence of minimizers of $S_{\theta, \beta}$, without loss of generality, we assume u_2 is a minimizer of $S_{\theta, \beta}$. For $t \geq 0$, we define

$$\begin{aligned} f_\alpha(t) &= \frac{t^2}{2} \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_1|^2 - \theta \frac{|u_1|^2}{|x'|^{2s}} \right] dx - \frac{t^{2 \cdot 2_{s, \alpha}^*}}{2 \cdot 2_{s, \alpha}^*} \int_{\mathbb{R}^{2N}} \frac{|u_1(x)|^{2_{s, \alpha}^*} |u_1(y)|^{2_{s, \alpha}^*}}{|x - y|^{N-\alpha}} dx dy, \\ f_\beta(t) &= \frac{t^2}{2} \int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_2|^2 - \theta \frac{|u_2|^2}{|x'|^{2s}} \right] dx - \frac{t^{2_{s, \beta}^*}}{2_{s, \beta}^*} \int_{\mathbb{R}^N} \frac{|u_2(x)|^{2_{s, \beta}^*}}{|x'|^\beta} dx. \end{aligned}$$

Thus, we have

$$\max_{t \geq 0} I_0(tu_1) \leq \max_{t \geq 0} f_\alpha(t), \quad \max_{t \geq 0} I_0(tu_2) \leq \max_{t \geq 0} f_\beta(t).$$

In addition, from the formulation of $f_\alpha(t)$ and $f_\beta(t)$, we obtain that the function $f_\alpha(t)$ attains its maximum at

$$t_\alpha = \left(\frac{\int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u_1|^2 - \theta \frac{|u_1|^2}{|x'|^{2s}} \right] dx}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_1(x)|^{2_{s, \alpha}^*} |u_1(y)|^{2_{s, \alpha}^*}}{|x - y|^{N-\alpha}} dx dy} \right)^{\frac{1}{2 \cdot 2_{s, \alpha}^* - 2}},$$

and the function $f_\beta(t)$ attains its maximum at

$$t_\beta = \left(\frac{\int_{\mathbb{R}^N} [|(-\Delta)^{\frac{s}{2}} u_2|^2 - \theta \frac{|u_2|^2}{|x'|^{2s}}] dx}{\int_{\mathbb{R}^N} \frac{|u_2(x)|^{2^*_{s,\beta}}}{|x'|^\beta} dx} \right)^{\frac{1}{2^*_{s,\beta}-2}}.$$

Thus, we have

$$\max_{t \geq 0} f_\alpha(t) = f_\alpha(t_\alpha) = \frac{\alpha + 2s}{2(N + \alpha)} S_{\theta, \alpha}^{\frac{N+\alpha}{\alpha+2s}}, \quad \max_{t \geq 0} f_\beta(t) = f_\beta(t_\beta) = \frac{2s - \beta}{2(N - \beta)} S_{\theta, \beta}^{\frac{N-\beta}{2s-\beta}}.$$

Now we show

$$\max_{t \geq 0} I_0(tu_1) < f_\alpha(t_\alpha), \quad \max_{t \geq 0} I_0(tu_2) < f_\beta(t_\beta).$$

Indeed, if there exist $t_1 > 0$, $t_2 > 0$ such that $I_0(t_1 u_1) = f_\alpha(t_\alpha)$, $I_0(t_2 u_2) = f_\beta(t_\beta)$, then

$$\begin{aligned} f_\alpha(t_1) - \frac{t_1^{2^*_{s,\beta}}}{2^*_{s,\beta}} \int_{\mathbb{R}^N} \frac{|u_1(x)|^{2^*_{s,\beta}}}{|x'|^\beta} dx &= f_\alpha(t_\alpha), \\ f_\beta(t_2) - \frac{t_2^{2 \cdot 2^*_{s,\alpha}}}{2 \cdot 2^*_{s,\alpha}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_2(x)|^{2^*_{s,\alpha}} |u_2(y)|^{2^*_{s,\alpha}}}{|x-y|^{N-\alpha}} dx dy &= f_\beta(t_\beta), \end{aligned}$$

which yields a contradiction, and hence the assertion follows. $\square$

Now, we show that the functional I_0 satisfies the geometry structure of the Mountain Pass Lemma without the $(PS)_{c_0^*}$ compact condition.

Lemma 4.3. Suppose $0 < \alpha, \beta < 2s < N$. Then

- (i) there exist $R, \rho > 0$ such that $I_0(u)|_{\|u\|_H=R} \geq \rho$ for all $u \in \dot{H}^s(\mathbb{R}^N)$,
- (ii) there exists $e \in \dot{H}^s(\mathbb{R}^N)$ with $\|e\|_H > R$ such that $I_0(e) < 0$.

Proof. For (i), from (2.11)–(2.12), since $2^*_{s,\beta} > 2$, $2 \cdot 2^*_{s,\alpha} > 2$, it is easy to see that

$$\begin{aligned} I_0(u) &= \frac{1}{2} \|u\|_H^2 - \frac{1}{2^*_{s,\beta}} \int_{\mathbb{R}^N} \frac{|u|^{2^*_{s,\beta}}}{|x'|^\beta} dx - \frac{1}{2 \cdot 2^*_{s,\alpha}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{s,\alpha}} |u(y)|^{2^*_{s,\alpha}}}{|x-y|^{N-\alpha}} dx dy \\ &\geq \frac{1}{2} \|u\|_H^2 - \frac{C_1}{2 \cdot 2^*_{s,\alpha}} \|u\|_H^{2 \cdot 2^*_{s,\alpha}} - \frac{C_2}{2^*_{s,\beta}} \|u\|_H^{2^*_{s,\beta}} \\ &\geq \rho > 0 \end{aligned}$$

for $\|u\|_H = R > 0$ sufficiently small. Thus, the first assertion is proved.

For (ii), note that

$$I_0(tu_0) = \frac{t^2}{2} \|u_0\|_H^2 - \frac{t^{2^*_{s,\beta}}}{2^*_{s,\beta}} \int_{\mathbb{R}^N} \frac{|u_0|^{2^*_{s,\beta}}}{|x'|^\beta} dx - \frac{t^{2 \cdot 2^*_{s,\alpha}}}{2 \cdot 2^*_{s,\alpha}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x)|^{2^*_{s,\alpha}} |u_0(y)|^{2^*_{s,\alpha}}}{|x-y|^{N-\alpha}} dx dy.$$

Since $2^*_{s,\beta} > 2$, $2 \cdot 2^*_{s,\alpha} > 2$, we have

$$I_0(tu_0) \rightarrow -\infty \quad \text{as } t \rightarrow \infty.$$

Taking t_0 large enough so that $\|t_0 u_0\|_H > R$ and $I_0(t_0 u_0) < 0$ and letting $e = t_0 u_0$, assertion (ii) follows. $\square$

Lemma 4.4. Suppose $0 < \alpha, \beta < 2s < N$. Let $\{u_n\} \subset \dot{H}^s(\mathbb{R}^N)$ be a $(PS)_c$ sequence of I_0 with $c \in (0, c_0^*)$. Then

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*_{s,\beta}}}{|x'|^\beta} dx > 0$$

and

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2^*_{s,\alpha}} |u_n(y)|^{2^*_{s,\alpha}}}{|x-y|^{N-\alpha}} dx dy > 0. \quad (4.2)$$

Proof. Since $I_0(u_n) \rightarrow c$, $I'_0(u_n) \rightarrow 0$. Hence, we have

$$\begin{aligned} c + o(1)\|u_n\|_H &= I_0(u_n) - \frac{1}{2_{s,\beta}^*} \langle I'_0(u_n), u_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_{s,\beta}^*} \right) \|u_n\|_H^2 + \left(\frac{1}{2_{s,\beta}^*} - \frac{1}{2 \cdot 2_{s,\alpha}^\#} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_{s,\alpha}^\#} |u_n(y)|^{2_{s,\alpha}^\#}}{|x-y|^{N-\alpha}} dx dy \\ &\geq \left(\frac{1}{2} - \frac{1}{2_{s,\beta}^*} \right) \|u_n\|_H^2. \end{aligned}$$

The last inequality holds, since $2 < 2_{s,\beta}^* < 2 \cdot 2_{s,\alpha}^\#$, and we obtain that $\{u_n\}_n$ is uniformly bounded in $\dot{H}^s(\mathbb{R}^N)$. Suppose

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{2_{s,\beta}^*}}{|x'|^\beta} dx = 0.$$

Then, from $I_0(u_n) \rightarrow c$ and $I'_0(u_n) \rightarrow 0$, we have

$$c + o(1) = \frac{1}{2} \|u_n\|_H^2 - \frac{1}{2 \cdot 2_{s,\alpha}^\#} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_{s,\alpha}^\#} |u_n(y)|^{2_{s,\alpha}^\#}}{|x-y|^{N-\alpha}} dx dy$$

and

$$o(1) = \|u_n\|_H^2 - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_{s,\alpha}^\#} |u_n(y)|^{2_{s,\alpha}^\#}}{|x-y|^{N-\alpha}} dx dy. \quad (4.3)$$

Thus, we get

$$c + o(1) = \left(\frac{1}{2} - \frac{1}{2 \cdot 2_{s,\alpha}^\#} \right) \|u_n\|_H^2. \quad (4.4)$$

By the definition of $S_{\theta,\alpha}$ and (4.3), we have

$$\|u_n\|_H^2 \geq S_{\theta,\alpha} \|u_n\|_{H^{\frac{2}{2_{s,\alpha}^\#}}}^{\frac{2}{2_{s,\alpha}^\#}},$$

which implies that

$$\|u_n\|_H^2 \geq S_{\theta,\alpha}^{\frac{N+\alpha}{\alpha+2s}}.$$

Therefore, from (4.4), it follows that

$$c \geq \frac{\alpha + 2s}{2(N + \alpha)} S_{\theta,\alpha}^{\frac{N+\alpha}{\alpha+2s}},$$

which is a contradiction. Similarly, we can prove (4.2), hence the proof is complete. $\square$

Now, we give the proof of the existence of solutions for problem (1.1).

Proof. By Lemma 4.3, there exists a $(PS)_{c_s^0}$ sequence $\{u_n\}_n$, which is bounded in $\dot{H}^s(\mathbb{R}^N)$. We assume that

$$u_n \rightharpoonup u \quad \text{in } \dot{H}^s(\mathbb{R}^N), \quad u_n \rightarrow u \quad \text{a.e. in } \mathbb{R}^N, \quad u_n \rightarrow u \quad \text{in } L_{\text{loc}}^p(\mathbb{R}^N) \text{ for all } 1 \leq p < 2_s^*.$$

Thus, u is a solution of (1.1). Moreover, by Lemmas 2.1, 4.2 and 4.4, and (2.9), up to a subsequence, there exists $C > 0$ such that

$$\|u_n\|_{L^{2,N-2s}(\mathbb{R}^N)} \geq C.$$

So, we can find $\lambda_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\lambda_n^{-2s} \int_{B_{\lambda_n}(x_n)} |u_n|^2 dy \geq \|u_n\|_{L^{2,N-2s}}^2 - \frac{C}{2n} \geq C_1 > 0. \quad (4.5)$$

Now, we claim that $\{\frac{x'_n}{\lambda_n}\}_n$ is bounded. First, we let $\tilde{u}_n = \lambda_n^{\frac{N-2s}{2}} u_n(\lambda_n x + x_n)$. Then

$$\begin{aligned} \tilde{I}_0(\tilde{u}_n) &= \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \tilde{u}_n|^2 dx - \frac{\theta}{2} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^2}{|x' + \frac{x'_n}{\lambda_n}|^{2s}} dx \\ &\quad - \frac{1}{2 \cdot 2_{s,\alpha}^\#} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x)|^{2_{s,\alpha}^\#} |\tilde{u}_n(y)|^{2_{s,\alpha}^\#}}{|x-y|^{N-\alpha}} dx dy - \frac{1}{2_{s,\beta}^*} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{2_{s,\beta}^*}}{|x' + \frac{x'_n}{\lambda_n}|^\beta} dx \\ &= I_0(u_n) \rightarrow c_s^0, \end{aligned}$$

and $\langle \tilde{I}'_0(\tilde{u}_n), \varphi \rangle \rightarrow 0$ for all $\varphi \in C_c^{0,1}(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$.

By using the same argument as in Lemma 4.4, we get that $\{\tilde{u}_n\}_n$ is uniformly bounded in $\dot{H}^s(\mathbb{R}^N)$. Thus, there exists $\tilde{u}$ such that

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } \dot{H}^s(\mathbb{R}^N), \quad \tilde{u}_n \rightarrow \tilde{u} \text{ a.e. in } \mathbb{R}^N, \quad \tilde{u}_n \rightarrow \tilde{u} \text{ in } L_{\text{loc}}^p(\mathbb{R}^N) \text{ for all } 1 \leq p < 2_s^*.$$

In addition, by (4.5), we obtain

$$\int_{B_1^{N-k}(0)} \int_{B_1^k(0)} |\tilde{u}|^2 dy \geq C_1 > 0,$$

which implies that $\tilde{u} \neq 0$. If $\frac{x'_n}{\lambda_n} \rightarrow \infty$, then for any $\varphi \in C_c^{0,1}(\mathbb{R}^k \setminus \{0\} \times \mathbb{R}^{N-k})$, we have

$$\int_{\mathbb{R}^N} \frac{\tilde{u}_n \varphi}{|x' + \frac{x'_n}{\lambda_n}|^{2s}} dx \rightarrow 0, \quad \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{2_{s,\beta}^*-2} \tilde{u}_n \varphi}{|x' + \frac{x'_n}{\lambda_n}|^\beta} dx \rightarrow 0.$$

Hence, $\tilde{u}$ solves the equation

$$(-\Delta)^s \tilde{u} = (\mathcal{K}_\alpha * |\tilde{u}|^{2_{s,\alpha}^\#}) |\tilde{u}|^{2_{s,\alpha}^\#-2} \tilde{u} \quad \text{in } \mathbb{R}^N, \quad (4.6)$$

and we obtain

$$\begin{aligned} c_s^0 &= \tilde{I}_0(\tilde{u}_n) - \frac{1}{2} \langle \tilde{I}'_0(\tilde{u}_n), \tilde{u}_n \rangle + o(1) \\ &= \left(\frac{1}{2} - \frac{1}{2 \cdot 2_{s,\alpha}^\#} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x)|^{2_{s,\alpha}^\#} |\tilde{u}_n(y)|^{2_{s,\alpha}^\#}}{|x-y|^{N-\alpha}} dx dy + \left(\frac{1}{2} - \frac{1}{2_{s,\beta}^*} \right) \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{2_{s,\beta}^*}}{|x' + \frac{x'_n}{\lambda_n}|^\beta} dx + o(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{2 \cdot 2_{s,\alpha}^\#} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n(x)|^{2_{s,\alpha}^\#} |\tilde{u}_n(y)|^{2_{s,\alpha}^\#}}{|x-y|^{N-\alpha}} dx dy + o(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{2 \cdot 2_{s,\alpha}^\#} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x)|^{2_{s,\alpha}^\#} |\tilde{u}(y)|^{2_{s,\alpha}^\#}}{|x-y|^{N-\alpha}} dx dy \\ &= I_0(\tilde{u}) \geq c_0^*. \end{aligned}$$

The last inequality comes from (4.6) and (2.12), which contradicts Lemma 4.2; thus we have that $\{\frac{x'_n}{\lambda_n}\}_n$ is bounded.

Then we define $\hat{u}_n(x) = \lambda_n^{\frac{N-2s}{2}} u_n(\lambda_n x', \lambda_n x'' + x_n'')$, and we have

$$\begin{aligned} I_0(\hat{u}_n) &= \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}_n|^2 dx - \frac{\theta}{2} \int_{\mathbb{R}^N} \frac{|\hat{u}_n|^2}{|x'|^{2s}} dx - \frac{1}{2 \cdot 2_{s,\alpha}^\#} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\hat{u}_n(x)|^{2_{s,\alpha}^\#} |\hat{u}_n(y)|^{2_{s,\alpha}^\#}}{|x-y|^{N-\alpha}} dx dy \\ &\quad - \frac{1}{2_{s,\beta}^*} \int_{\mathbb{R}^N} \frac{|\hat{u}_n|^{2_{s,\beta}^*}}{|x'|^\beta} dx \\ &= I_0(u_n), \end{aligned}$$

and

$$I'_0(\hat{u}_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Arguing as before, we have

$$\hat{u}_n \rightharpoonup \hat{u} \text{ in } \dot{H}^s(\mathbb{R}^N),$$

which yields that $\hat{u}$ is a solution of (1.1) as well. Furthermore, since $\{\frac{x'_n}{\lambda_n}\}_n$ is bounded, we conclude that there exists $R > 0$ such that

$$\int_{B_1^{N-k}(0)} \int_{B_R^k(0)} |\hat{u}_n(y)|^2 dy \geq \lambda_n^{-2s} \int_{B_{\lambda_n}(x_n)} |u_n|^2 dy \geq C_1 > 0,$$

the compact embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^N)$ implies that $\hat{u}_n \rightarrow \hat{u}$ in $L^2(B_1^{N-k}(0) \times B_R^k(0))$, from which it follows that $\hat{u} \neq 0$. Moreover,

$$c_0 = c_s^0 \geq I_0(\hat{u}) \geq c_0,$$

hence, $\hat{u}$ is a nontrivial solution of (1.1) satisfying $I_0(\hat{u}) = c_0$. This ends the proof of Theorem 2.5. $\square$

5 The Proof of Theorem 2.6

Following as before, we define the Nehari manifold:

$$\mathcal{N}_1 = \{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} : \langle I'_1(u), u \rangle = 0\}.$$

And a minimizer of the minimization problem

$$c_1 = \inf_{u \in \mathcal{N}_1} I_1(u)$$

is a solution of problem (1.2). Furthermore, we set

$$c_\gamma^1 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_1(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], W^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = e\}$, and we know that

$$c_1 = c_\gamma^1 = \inf_{u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} I_1(tu).$$

Lemma 5.1. Suppose $\mu \in [0, (\frac{N-p}{p})^p)$, $\lambda \in (0, N)$. Then

$$c_\gamma^1 < c_1^* := \min \left\{ \frac{1}{p} \cdot \frac{N-\lambda+p}{2N-\lambda} \Lambda_{\mu,\lambda}^{\frac{2N-\lambda}{N-\lambda+p}}, \frac{1}{N} \cdot \Lambda_\mu^{\frac{N}{p}} \right\}.$$

Proof. Since $\Lambda_{\mu,\lambda}$ and Λ_μ are achieved in $W^{1,p}(\mathbb{R}^N)$, respectively, without loss of generality, we assume that $u_1, u_2 \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$ are the minimizers of $\Lambda_{\mu,\lambda}$ and Λ_μ , respectively. For $t \geq 0$, we define

$$\begin{aligned} f_1(t) &= \frac{t^p}{p} \int_{\mathbb{R}^N} \left(|\nabla u_1|^p - \mu \frac{|u_1|^p}{|x|^p} \right) dx - \frac{t^{2 \cdot p_\lambda^*}}{2 \cdot p_\lambda^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_1(x)|^{p_\lambda^*} |u_1(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy, \\ f_2(t) &= \frac{t^p}{p} \int_{\mathbb{R}^N} \left(|\nabla u_2|^p - \mu \frac{|u_2|^p}{|x|^p} \right) dx - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} |u_2|^{p^*} dx. \end{aligned}$$

Obviously,

$$\max_{t \geq 0} I_1(tu_1) \leq \max_{t \geq 0} f_1(t), \quad \max_{t \geq 0} I_1(tu_2) \leq \max_{t \geq 0} f_2(t).$$

Moreover, we can verify that the functions $f_1(t)$ and $f_2(t)$ attain their maximization at

$$\begin{aligned} t_1 &= \left(\frac{\int_{\mathbb{R}^N} (|\nabla u_1|^p - \mu \frac{|u_1|^p}{|x|^p}) dx}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_1(x)|^{p_\lambda^*} |u_1(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy} \right)^{\frac{1}{2 \cdot p_\lambda^* - p}}, \quad \text{with } f_1(t_1) = \frac{1}{p} \cdot \frac{N-\lambda+p}{2N-\lambda} \Lambda_{\mu,\lambda}^{\frac{2N-\lambda}{N-\lambda+p}}, \\ t_2 &= \left(\frac{\int_{\mathbb{R}^N} (|\nabla u_2|^p - \mu \frac{|u_2|^p}{|x|^p}) dx}{\int_{\mathbb{R}^N} |u_2|^{p^*} dx} \right)^{\frac{1}{p^* - p}}, \quad \text{with } f_2(t_2) = \frac{1}{N} \cdot \Lambda_\mu^{\frac{N}{p}}. \end{aligned}$$

Now we show that

$$\max_{t \geq 0} I_1(tu_1) < f_1(t_1), \quad \max_{t \geq 0} I_1(tu_2) < f_2(t_2).$$

Indeed, if there exist $\tilde{t}_1 > 0$, $\tilde{t}_2 > 0$ such that $I_1(\tilde{t}_1 u_1) = f_1(t_1)$, $I_1(\tilde{t}_2 u_2) = f_2(t_2)$, then

$$\begin{aligned} f_1(\tilde{t}_1) - \frac{\tilde{t}_1^{p^*}}{p^*} \int_{\mathbb{R}^N} |u_1|^{p^*} dx &= f_1(t_1), \\ f_2(\tilde{t}_2) - \frac{\tilde{t}_2^{2 \cdot p_\lambda^*}}{2 \cdot p_\lambda^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_2(x)|^{p_\lambda^*} |u_2(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy &= f_2(t_2), \end{aligned}$$

which yield a contradiction and hence the assertion follows. $\square$

Lemma 5.2. Suppose $\mu \in [0, (\frac{N-p}{p})^p]$, $\lambda \in (0, N)$. Then there exists a $(PS)_{c_1^1}$ sequence of the functional I_1 .

Proof. We show that functional I_1 satisfies the geometry structure of the Mountain Pass Lemma without the $(PS)_{c_1^1}$ compact condition. Since $2 \cdot p_\lambda^* > p^* > p$, this simply follows from

$$\begin{aligned} I_1(u) &= \frac{1}{p} \|u\|_W^p - \frac{1}{2 \cdot p_\lambda^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\lambda^*} |u(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx \\ &\geq \frac{1}{p} \|u\|_W^p - \frac{\Lambda_{\mu, \lambda}^{-\frac{2 \cdot p_\lambda^*}{p}}}{2 \cdot p_\lambda^*} \|u\|_W^{2 \cdot p_\lambda^*} - \frac{\Lambda_\mu^{-\frac{p^*}{p}}}{p^*} \|u\|_W^{p^*} \\ &\geq \rho > 0 \end{aligned}$$

for $\|u\|_W = R > 0$ small, and from $I_1(tu_0) < 0$ for a fixed $u_0 \in W^{1,p}(\mathbb{R}^N)$ and $t > 0$ large. $\square$

The following result implies the non-vanishing of a $(PS)_c$ sequence.

Lemma 5.3. Suppose $\mu \in [0, (\frac{N-p}{p})^p]$, $\lambda \in (0, N)$, and let $\{u_n\} \subset W^{1,p}(\mathbb{R}^N)$ be a $(PS)_c$ sequence of I_1 , with $c \in (0, c_1^*)$. Then

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{p_\lambda^*} |u_n(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy > 0$$

and

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n(x)|^{p^*} dx > 0. \quad (5.1)$$

Proof. Since $I_1(u_n) \rightarrow c$, $I_1'(u_n) \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\begin{aligned} c + o(1) \|u_n\|_W &= I_1(u_n) - \frac{1}{p^*} \langle I_1'(u_n), u_n \rangle \\ &= \left(\frac{1}{p} - \frac{1}{p^*} \right) \|u_n\|_W^p + \left(\frac{1}{p^*} - \frac{1}{2 \cdot p_\lambda^*} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{p_\lambda^*} |u_n(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*} \right) \|u_n\|_W^p. \end{aligned}$$

The last inequality holds since $p < p^* < 2 \cdot p_\lambda^*$, and we obtain that $\{u_n\}$ is uniformly bounded in $W^{1,p}(\mathbb{R}^N)$. Suppose now, on the contrary, that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{p_\lambda^*} |u_n(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy = 0.$$

Then we have

$$c + o(1) = \frac{1}{p} \|u_n\|_W^p - \frac{1}{p^*} \int_{\mathbb{R}^N} |u_n|^{p^*} dx$$

and

$$o(1) = \|u_n\|_W^p - \int_{\mathbb{R}^N} |u_n|^{p^*} dx.$$

This yields

$$c + o(1) = \frac{1}{N} \|u_n\|_W^p.$$

Moreover,

$$\|u_n\|_W^p \geq \Lambda_\mu \left(\int_{\mathbb{R}^N} |u_n|^{p^*} dx \right)^{\frac{p}{p^*}},$$

implying

$$\|u_n\|_W^p \geq \Lambda_\mu^{\frac{N}{p}}.$$

Therefore, we obtain

$$c \geq \frac{1}{N} \Lambda_\mu^{\frac{N}{p}},$$

which is a contradiction. Similarly, we can prove (5.1), and the proof is complete. $\square$

Finally, we complete the proof of the Theorem 2.6.

Proof. By Lemma 5.2, we know that the functional I_1 satisfies the Mountain-Pass geometry structure. Thus, there exists a $(PS)_{c_Y^1}$ sequence $\{u_n\}$ of I_1 which is bounded in $W^{1,p}(\mathbb{R}^N)$. Combining Lemmas 5.1, 5.3 and 2.2, and (2.10), up to a subsequence, there exists $C > 0$ such that

$$\|u_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \geq C > 0.$$

So we may find $\lambda_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\frac{1}{\lambda_n^p} \int_{B(x_n, \lambda_n)} |u_n(y)|^p dy \geq \|u_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)}^p - \frac{C}{2n} \geq \hat{C} > 0. \quad (5.2)$$

Let $\hat{u}_n(x) = \lambda_n^{\frac{N-p}{p}} u_n(\lambda_n x)$. By using scaling invariance, we have

$$I_1(\hat{u}_n) = I_1(u_n) \rightarrow c_Y^1, \quad \langle I_1'(\hat{u}_n), \varphi \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Processing as in Lemma 5.3, we know that $\{\hat{u}_n\}$ is uniformly bounded in $W^{1,p}(\mathbb{R}^N)$. Thus, we have

$$\hat{u}_n \rightharpoonup \hat{u} \quad \text{in } W^{1,p}(\mathbb{R}^N), \quad \hat{u}_n \rightarrow \hat{u} \quad \text{a.e. in } \mathbb{R}^N, \quad \hat{u}_n \rightarrow \hat{u} \quad \text{in } L_{\text{loc}}^q(\mathbb{R}^N) \text{ for all } q \in [p, p^*),$$

and $\hat{u}$ is a weak solution of (1.2). Now we verify that $\{\frac{x_n}{\lambda_n}\}$ is bounded. From (5.2), we have

$$\int_{B(\frac{x_n}{\lambda_n}, 1)} |\hat{u}_n(y)|^p dy \geq \hat{C} > 0.$$

For any $0 < a < p$, by the Hölder inequality, we have

$$\begin{aligned} 0 < \hat{C} &\leq \int_{B(\frac{x_n}{\lambda_n}, 1)} |\hat{u}_n(y)|^p dy = \int_{B(\frac{x_n}{\lambda_n}, 1)} |y|^{\frac{pa}{p(N-a)}} \cdot \frac{|\hat{u}_n|^p}{|y|^{\frac{pa}{p(N-a)}}} dy \\ &\leq \left(\int_{B(\frac{x_n}{\lambda_n}, 1)} |y|^{\frac{a(N-p)}{p-a}} dy \right)^{1-\frac{N-p}{N-a}} \left(\int_{B(\frac{x_n}{\lambda_n}, 1)} \frac{|\hat{u}_n(y)|^p}{|y|^a} dy \right)^{\frac{N-p}{N-a}}. \end{aligned}$$

By the rearrangement inequality [23], we have

$$\int_{B(\frac{x_n}{\lambda_n}, 1)} |y|^{\frac{a(N-p)}{p-a}} dy \leq \int_{B(0,1)} |y|^{\frac{a(N-p)}{p-a}} dy \leq C.$$

Therefore,

$$0 < C \leq \int_{B(\frac{x_n}{\lambda_n}, 1)} \frac{|\hat{u}_n|^{\frac{p(N-a)}{N-p}}}{|y|^a} dy. \quad (5.3)$$

Now, suppose on the contrary that $\frac{x_n}{\lambda_n} \rightarrow \infty$ as $n \rightarrow \infty$. Then for any $y \in B(\frac{x_n}{\lambda_n}, 1)$, we have $|y| \geq |\frac{x_n}{\lambda_n}| - 1$ for n large. Thus, by the Hölder and the Sobolev inequalities, we have

$$\begin{aligned} \int_{B(\frac{x_n}{\lambda_n}, 1)} \frac{|\hat{u}_n|^{\frac{p(N-a)}{N-p}}}{|y|^a} dy &\leq \frac{1}{(|\frac{x_n}{\lambda_n}| - 1)^a} \int_{B(\frac{x_n}{\lambda_n}, 1)} |\hat{u}_n|^{\frac{p(N-a)}{N-p}} dy \\ &\leq \frac{|B(\frac{x_n}{\lambda_n}, 1)|^{\frac{a}{N}}}{(|\frac{x_n}{\lambda_n}| - 1)^a} \left(\int_{B(\frac{x_n}{\lambda_n}, 1)} |\hat{u}_n|^{\frac{Np}{N-p}} dy \right)^{\frac{N-a}{N}} \\ &\leq \frac{|B(\frac{x_n}{\lambda_n}, 1)|^{\frac{a}{N}}}{(|\frac{x_n}{\lambda_n}| - 1)^a} \|\hat{u}_n\|_W^{\frac{N-a}{N}} \\ &\leq \frac{1}{(|\frac{x_n}{\lambda_n}| - 1)^a} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which contradicts (5.3). Hence, $\{\frac{x_n}{\lambda_n}\}$ is bounded and there exists $R > 0$ such that

$$\int_{B(0, R)} |\hat{u}_n(y)|^p dy \geq \int_{B(\frac{x_n}{\lambda_n}, 1)} |\hat{u}_n(y)|^p dy \geq \hat{C} > 0.$$

Since the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L_{\text{loc}}^q(\mathbb{R}^N)$ ($q \in [p, p^*)$) is compact, we deduce that

$$\int_{B(0, R)} |\hat{u}(y)|^p dy \geq \hat{C} > 0,$$

which means $\hat{u} \neq 0$. Furthermore,

$$\begin{aligned} c_1 = c_Y^1 &= I_1(\hat{u}_n) - \frac{1}{p} \langle I_1'(\hat{u}_n), \hat{u}_n \rangle + o(1) \\ &= \left(\frac{1}{p} - \frac{1}{2 \cdot p_\lambda^*} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\hat{u}_n(x)|^{p_\lambda^*} |\hat{u}_n(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy + \left(\frac{1}{p} - \frac{1}{p^*} \right) \int_{\mathbb{R}^N} |\hat{u}_n(x)|^{p^*} dx + o(1) \\ &\geq \left(\frac{1}{p} - \frac{1}{2 \cdot p_\lambda^*} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\hat{u}(x)|^{p_\lambda^*} |\hat{u}(y)|^{p_\lambda^*}}{|x-y|^\lambda} dx dy + \left(\frac{1}{p} - \frac{1}{p^*} \right) \int_{\mathbb{R}^N} |\hat{u}(x)|^{p^*} dx + o(1) \\ &= I_1(\hat{u}) - \frac{1}{p} \langle I_1'(\hat{u}), \hat{u} \rangle \\ &= I_1(\hat{u}) \geq c_1, \end{aligned}$$

which implies that $\hat{u}$ is a nontrivial solution of (1.2) satisfying $I_1(\hat{u}) = c_1$, and the proof of Theorem 2.6 is complete. $\square$

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