Research Article

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Multiple Periodic Solutions of a Class of Fractional Laplacian Equations

<https://doi.org/10.1515/ans-2020-2113>

Received May 28, 2020; revised October 26, 2020; accepted October 31, 2020

Abstract: In this paper, we study the existence of multiple periodic solutions for the following fractional equation:

$$(-\Delta)^s u + F'(u) = 0, \quad u(x) = u(x + T) \quad x \in \mathbb{R}.$$

For an even double-well potential, we establish more and more periodic solutions for a large period T . Without the evenness of F we give the existence of two periodic solutions of the problem. We make use of variational arguments, in particular Clark's theorem and Morse theory.

Keywords: Periodic Solution, Fractional Laplacian, Minimax Method, Morse Index

MSC 2010: 35A15, 35B10, 35J60

Communicated by: Changfeng Gui

1 Introduction

In recent years, a great deal of attention has been devoted to the study of nonlinear elliptic equations involving the nonlocal operators. Nonlocal operators such as the fractional Laplacian $(-\Delta)^s$ appear in continuum mechanics, phase transition phenomena, population dynamics and game theory (e.g., [18, 20]). The fractional Laplacian $(-\Delta)^s$ is a pseudo-differential operator defined by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where $s \in (0, 1)$ and $C_{N,s} > 0$ is a constant given by

$$C_{N,s} = \left[\int_{\mathbb{R}^N} \frac{1 - \cos \zeta_1}{|\zeta|^{N+2s}} d\zeta \right]^{-1}.$$

For any $u \in \mathcal{S}$, the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$, the fractional Laplacian $(-\Delta)^s$ can also be defined as follows:

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u))(x) \quad \text{for all } \xi \in \mathbb{R}^N,$$

where

$$\mathcal{F}u = \hat{u} = (2\pi)^{-N/2} \int_{\mathbb{R}^N} u(x) e^{-ix \cdot \xi} dx$$

denotes the Fourier transform of u .

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In this paper, we consider the following problem:

$$(-\Delta)^s u(x) + F'(u(x)) = 0, \quad x \in \mathbb{R}. \quad (1.1)$$

This nonlocal equation has been studied in many works for different types of potentials F . Cabré and Sire studied layer solutions of this equation with some special potential F (see [5, 6]). Under the general Berestycki–Lions-type assumptions on $-F'(u)$, Chang and Wang [10] obtained a ground state positive solution of (1.1). Frank and Lenzmann [16] gave the uniqueness of the ground state solution of (1.1) with $F'(u) = u - |u|^{p-2}u$. And the regularity of solutions was examined in [7, 8, 22, 23].

In this paper, we focus on the existence of periodic solutions of (1.1). As far as we know, not much study of the periodic solution to elliptic equations involving $(-\Delta)^s$ was yet conducted. Only [12, 15, 17] considered the periodic solution of (1.1), and [13] considered the periodic solution of an elliptic system involving $(-\Delta)^s$. We refer the readers to [2–4] and the references therein, where periodic solutions have been studied for nonlocal problems involving a spectral fractional Laplacian. In [12, 17], the existence of periodic solutions was studied by using the variational method. Our study is motivated by these works, in particular by the works [12, 17]. Our goal here is to study the multiplicity of periodic solutions, which was not considered in these works. We make use of Clark's theorem and Morse theory to establish multiple solutions. These are classical tools in the study of periodic solutions for local operators (e.g., [9, 19, 21]). We develop a framework so these machinery can be applied to fractional Laplacian equations. This may be of independent interest.

To study our multiplicity results of periodic solutions of (1.1), we make the following assumptions:

(F1) F is a smooth enough double-well potential with the well bottom at ± 1 , i.e. it satisfies

$$F(\pm 1) = 0 < F(t), \quad F'(\pm 1) = 0, \quad t \in (-1, 1),$$

(F2) F is non-decreasing in $(-1, 0)$ and is non-increasing in $(0, 1)$.

Now our results are stated as follows.

Theorem 1.1. Assume F satisfies (F1), (F2) and is even. Then for any $k \geq 1$ there exists a T_k such that for any $T > T_k$ problem (1.1) has k pairs of odd periodic solutions $\pm u_{i,T}$ ($i = 1, \dots, k$) of period T and $|u_{i,T}|_\infty \in (0, 1)$.

Theorem 1.2. Assume F is even and satisfies (F1). If $F''(0) < 0$, then for any $k \geq 1$ there exists a $\tilde{T}_k$ such that for any $T > \tilde{T}_k$, problem (1.1) has k pairs of odd periodic solutions $\pm \tilde{u}_{i,T}$ of period T and $|\tilde{u}_{i,T}|_\infty \in (0, 1)$, $i = 1, \dots, k$.

Theorem 1.3. Assume F satisfies (F1) and $F''(\pm 1) > 0$. Suppose F also satisfies the following conditions:
(F2') F is increasing in $(-1, 0)$ and is decreasing in $(0, 1)$.

(F3) If

$$\frac{T(-F''(0))^{\frac{1}{2s}}}{2\pi}$$

is a positive integer, then either $\frac{F'(t)}{t} \geq F''(0)$ or $\frac{F'(t)}{t} \leq F''(0)$ holds for $t \neq 0$ in a neighborhood of 0.

Then there exists a $\mathcal{T}$ such that for any $T > \mathcal{T}$ problem (1.1) has two periodic solutions $v_{1,T}$ and $v_{2,T}$ of period T , and $|v_{1,T}|_\infty, |v_{2,T}|_\infty \in (0, 1)$.

Remark 1.4. We note that Theorem 1.1 and Theorem 1.2 require F to be even and we establish multiple solutions, while Theorem 1.3 does not require the evenness of F and we give two solutions. Our work is motivated by [17] and is the further study of [17]. Gui, Zhang and Du [17] obtained one nonconstant periodic solution of (1.1) under the assumptions of Theorem 1.1 and Theorem 1.3 without (F3). In Theorem 1.2, we do not assume (F2) and there may exist some other constant solutions than 0 and ± 1 .

We shall use a localization argument developed by Caffarelli–Silvestre [8] which transforms the nonlocal problem to a local one in a domain in $\mathbb{R}^2$ for which we will develop necessary estimates so that some classical minimax arguments and Morse theory may be applied. These will enable us to establish the multiplicity of periodic solutions for the original nonlocal problem.

This paper is organized as follows. In Section 2, we will introduce some notation and give some preliminary results. In Section 3, we consider a truncated problem proving the k pairs solutions exploiting Clark's theorem, and use the maximum principle proving that these k pairs solutions are actually solutions of (1.1). Section 4 is devoted to the proof of Theorem 1.3, mainly using a truncated argument, the mountain pass theorem and the Morse index.

2 Preliminaries

In this section, for the convenience of the readers, we collect some basic results that will be used in the forthcoming sections. As we mention in Section 1, we will use a localization argument developed by Caffarelli–Silvestre [8]. For a given function u , the solution of the boundary value problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0, & (x, y) \in \mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}, \\ w(x, 0) = u(x), & x \in \partial\mathbb{R}_+^2 = \{(x, 0) \in \mathbb{R}^2\}. \end{cases}$$

is called s -harmonic extension of u , denoted by $w = h_s(u)$. In particular, $w = P_s(\cdot, y) * u$, where

$$P_s(x, y) = p_s \frac{y^{2s}}{(x^2 + y^2)^{\frac{1+2s}{2}}},$$

is the s -Poisson kernel and p_s is a constant so that $\int_{\mathbb{R}} P_s(x, y) dx = 1$. From [8] we know that

$$\lim_{y \rightarrow 0} y^{1-2s} \frac{\partial w}{\partial y}(x, y) = -k_s(-\Delta)^s u(x),$$

where

$$k_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}.$$

By $w = P_s(\cdot, y) * u$, we can conclude that $w(x, y)$ is odd with respect to x if $u(x)$ is odd, and $w(x, y)$ is periodic with respect to x if $u(x)$ is periodic. Hence, we can reformulate problem (1.1) in a relatively local way as follows:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}_+^2, \\ -\frac{1}{k_s} \frac{\partial w}{\partial y} = -F'(w(x, 0)) & \text{in } \mathbb{R}, \end{cases} \quad (2.1)$$

where

$$\frac{\partial w}{\partial y} = \lim_{y \rightarrow 0} y^{1-2s} \frac{\partial w}{\partial y}(x, y) = -k_s(-\Delta)^s u(x).$$

We will omit the constant k_s in the following.

2.1 Variational Framework

We first introduce some function spaces:

$$\begin{aligned} \mathcal{H}_T &= \left\{ w(x, y) : w(x+T, y) = w(x, y) \text{ for all } y \geq 0, \right. \\ &\quad \left. \|w\|^2 = \int_0^\infty \int_{-T/2}^{T/2} y^{1-2s} |\nabla w(x, y)|^2 dx dy + \int_{-T/2}^{T/2} |w(x, 0)|^2 dx < \infty \right\}, \\ H_T^s &= \left\{ u(x) : u(x+T) = u(x), \|u\|_s^2 = \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \frac{|u(x) - u(z)|^2}{|x - z|^{1+2s}} dx dz + \int_{-T/2}^{T/2} |u(x)|^2 dx < \infty \right\}, \\ \hat{H}_T^s &= \left\{ u \in H_T^s : \|u\|_{\hat{H}_T^s}^2 = \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \sum_{j=-\infty, j \in \mathbb{Z}}^{+\infty} \frac{|u(x) - u(z)|^2}{|x - z + jT|^{1+2s}} dx dz + \int_{-T/2}^{T/2} |u(x)|^2 dx < \infty \right\}, \\ L_T^p &= \left\{ u(x) : u(x+T) = u(x), \|u\|_p^p = \int_{-T/2}^{T/2} |u(x)|^p dx < \infty \right\}. \end{aligned}$$

Clearly, $\hat{H}_T^s$ continuously imbeds into H_T^s . The following results are well-known.

Lemma 2.1 (see [11]). *Let $s \in (0, 1)$. The following imbeddings are continuous:*

$$\begin{aligned} H_\tau^s &\hookrightarrow L_\tau^p, \quad p \in [2, 2_s^*], s > \frac{1}{2}, \\ H_\tau^s &\hookrightarrow L_\tau^p, \quad p \in [2, 2_s^*), s = \frac{1}{2}, \\ H_\tau^s &\hookrightarrow L_\tau^p, \quad p \in [2, 2_s^*], s < \frac{1}{2}, \end{aligned}$$

where $2_s^* = \frac{2}{1-2s}$ if $s < \frac{1}{2}$, and $2_s^* = \infty$ if $s \geq \frac{1}{2}$. Moreover, H_τ^s is compactly imbedded in L_τ^p for all $2 \leq p < 2_s^*$.

Lemma 2.2 (see [12]). *Let $s \in (0, 1)$. There exists a constant $C = C(s) > 0$ such that*

$$\|\text{Tr}(w)\|_{\dot{H}_\tau^s} \leq C\|w\|,$$

where $\text{Tr}(w)$ is the trace of w .

We shall say that $w \in \mathcal{H}_\tau$ is a weak periodic solution of (2.1) if

$$\int_{\Omega_T} y^{1-2s} \nabla w(x, y) \nabla v(x, y) dx dy + \int_{-T/2}^{T/2} F'(w(x, 0))v(x, 0) dx = 0$$

for all $v \in \mathcal{H}_\tau$. For all $w \in \mathcal{H}_\tau$ we set

$$J(w) = \frac{1}{2} \int_{\Omega_T} y^{1-2s} |\nabla w(x, y)|^2 dx dy + \int_{-T/2}^{T/2} F(w(x, 0)) dx, \quad (2.2)$$

where $\Omega_T = [-\frac{T}{2}, \frac{T}{2}] \times (0, \infty)$, and for any $w, v \in \mathcal{H}_\tau$ it holds that

$$\langle J'(w), v \rangle = \int_{\Omega_T} y^{1-2s} \nabla w(x, y) \nabla v(x, y) dx dy + \int_{-\frac{T}{2}}^{\frac{T}{2}} F'(w(x, 0))v(x, 0) dx.$$

Therefore, weak solutions to (2.1) correspond to critical points of the functional (2.2).

For $c \in \mathbb{R}$, we say that J satisfies the $(PS)_c$ condition if for any sequence $\{w_n\}_{n \geq 1} \subset \mathcal{H}_\tau$ with $J(w_n) \rightarrow c$, the sequence $J'(w_n) \rightarrow 0$ in $\mathcal{H}_\tau^*$ has a convergent subsequence in $\mathcal{H}_\tau$.

The standard version of the mountain pass theorem is due to Ambrosetti–Rabinowitz [1] and goes as follows.

Lemma 2.3. *Let X be a real Banach space and let $I \in C^1(X, \mathbb{R})$ satisfy the $(PS)_c$ condition for any $c \in \mathbb{R}$. Suppose there exist $u_0, u_1 \in X$ and $r > 0$ such that $u_1 \notin B_r(u_0)$ and $I|_{\partial B_r(u_0)} > \max\{I(u_0), I(u_1)\}$. Then*

$$c = \inf_{g \in \Gamma} \max_{t \in [0, 1]} I(g(t))$$

is a critical value of I , where

$$\Gamma = \{g \in C([0, 1]) : g(0) = u_0, g(1) = u_1\}.$$

The following version of Clark's theorem can be found in [21].

Lemma 2.4. *Let X be a real Banach space and let $I \in C^1(X, \mathbb{R})$ with I even, bounded from below, and satisfying the $(PS)_c$ condition for any $c \in \mathbb{R}$. Suppose $I(0) = 0$. There exists a set $K \subset X$ such that K is homeomorphic to ∂B^k by an odd map where B^k is the k -dimensional unit ball, and $\sup_K I < 0$. Then I possesses at least k distinct pairs of critical points.*

2.2 Critical Groups

We now recall some results about critical group and Morse theory, for which we refer the readers to [9, 19] for further information. Let X be a Hilbert space with $Y \subset X$ and let $H_q(X, Y, G)$ be the q -th singular relative

homology group with the abelian coefficient group G . Assume u_0 is an isolated critical point of $f \in C^2(X, \mathbb{R})$ with $f(u_0) = c$. Let U be a neighborhood of u_0 such that $U \cap K \cap f^c = \{u_0\}$, where K is the critical set of f and $f^c = \{u \in X : f(u) \leq c\}$. The group

$$C_q(f, u_0) = H_q(f^c \cap U, f^c \cap U \setminus \{u_0\}, G), \quad q \in \mathbb{N},$$

is said to be the q -th critical group of f at u_0 . If $f''(u_0)$ has a bounded inverse, then u_0 is called a nondegenerate critical point. And we call the dimension of the negative subspace of $f''(u_0)$ the Morse index of u_0 , denoted by $i(f, u_0)$.

We recall some special cases in which the computation of critical groups is immediate. Let $\delta_{h,k}$ be the Kronecker symbol.

Lemma 2.5. Assume $f \in C^2(X, \mathbb{R})$ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. Let u be an isolated critical point of f . Then the following assertions hold:

(i) If u is a local minimum of f , then

$$C_q(f, u) = \delta_{q,0}G \quad \text{for all } q \in \mathbb{N}.$$

(ii) If u is a nondegenerate critical point of f with Morse index j , then

$$C_q(f, u) = \delta_{q,j}G \quad \text{for all } q \in \mathbb{N}.$$

(iii) If u is a mountain-pass critical point of f with Morse index 1, then

$$C_q(f, u) = \delta_{q,1}G \quad \text{for all } q \in \mathbb{N}.$$

(iv) If $\dim X = n < \infty$ and u is a local maximum of f , then

$$C_q(f, u) = \delta_{q,n}G \quad \text{for all } q \in \mathbb{N}.$$

Lemma 2.6 (Shifting Theorem). Assume $f \in C^2(X, \mathbb{R})$ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. Let u be an isolated critical point of f with Morse index j . Then

$$C_q(f, u) = C_{q-j}(\tilde{f}, u) \quad \text{for all } q \in \mathbb{N},$$

where $\tilde{f} = f|_N$ and $N = \ker f''(u)$.

We now recall the following Morse identity.

Lemma 2.7. Assume that $f \in C^2(X, \mathbb{R})$ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$ and that f has only m critical points $\{u_i\}_{1 \leq i \leq m}$. Set

$$\beta_q = \text{rank } H_q(X, f^\eta, G) \quad \text{for all } q \in \mathbb{N},$$

where $\eta < \inf_{i \in [1, m]} f(u_i)$, and

$$M_q = \sum_{i=1}^m \text{rank } C_q(f, u_i) \quad \text{for all } q \in \mathbb{N}.$$

Then there exists a formal power series $Q(t) = \sum_{k=0}^{\infty} q_k t^k$ ($q_k \geq 0$ for all $k \in \mathbb{N}$) such that for all $t \in \mathbb{R}$,

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1+t)Q(t).$$

2.3 Some Regularity Results

This section is devoted to give some regularity results about the fractional equation. We first introduce

$$\begin{aligned} B_R &= \{(x, y) \in \mathbb{R}^2 : |(x, y)| < R\}, \\ B_R^+ &= \{(x, y) \in \mathbb{R}^2 : y > 0, |(x, y)| < R\}, \\ \Gamma_R^0 &= \{(x, 0) \in \mathbb{R}^2 : |(x, 0)| < R\}, \\ \Gamma_R^+ &= \{(x, y) \in \mathbb{R}^2 : y \geq 0, |(x, y)| = R\}, \end{aligned}$$

and for a smooth domain $\Omega \subset \mathbb{R}_+^2$ and $a \in (-1, 1)$ we set

$$H^1(\Omega, y^a dx dy) = \{w \in L^2(\Omega, y^a dx dy) : |\nabla w| \in L^2(\Omega, y^a dx dy)\}.$$

We consider the following equation:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0, & (x, y) \in B_R^+, \\ \frac{\partial w}{\partial \nu} = -d(x)w(x, 0), & (x, 0) \in \Gamma_R^0, \end{cases} \quad (2.3)$$

where $d(x) \in L^\infty(\Gamma_R^0)$.

We say w is a weak solution of (2.3) if

$$\int_{B_R^+} y^{1-2s} |\nabla w|^2 dx dy + \int_{\Gamma_R^0} w^2(x, 0) dx < +\infty$$

and

$$\int_{B_R^+} y^{1-2s} \nabla w(x, y) \nabla v(x, y) dx dy + \int_{\Gamma_R^0} d(x)w(x, 0)v(x, 0) dx = 0$$

for all $v \in C^1(\overline{B_R^+})$ such that $v \equiv 0$ on Γ_R^+ .

We now recall a maximum principle.

Lemma 2.8 ([5, Corollary 4.12]). *Let $R > 0$, let d be a Hölder continuous function in Γ_R^0 and let*

$$w \in L^\infty(B_R^+) \cap H^1(B_R^+, y^{1-2s})$$

with $w \geq 0$ in B_R^+ be a weak solution of (2.3). Then $w > 0$ in $B_R^+ \cup \Gamma_R^0$ unless $w \equiv 0$ in B_R^+ .

The following estimate is from [7, Corollary 2.5] and [14, Theorem 2.3.12].

Lemma 2.9. *Let $w \in H^1(B_{2R}, y^{1-2s} dx dy)$ be a weak solution of*

$$-\operatorname{div}(|y|^{1-2s}\nabla w) = 0, \quad (x, y) \in B_{2R}.$$

Then there exists small $\alpha > 0$ depending on s such that

$$\begin{aligned} \sup_{B_R} |D_x^k w(x, y)| &\leq \frac{C}{R^k} (\sup_{B_{2R}} w - \inf_{B_{2R}} w), \\ [D_x^k w(x, y)]_{C^\alpha(B_R)} &\leq \frac{C}{R^{k+\alpha}} (\sup_{B_{2R}} w - \inf_{B_{2R}} w), \end{aligned}$$

Adopting some ideas of [5], we have the following lemma.

Lemma 2.10. *Let $f \in C^{1,\gamma}(\mathbb{R})$ with $\gamma > \max\{0, 1 - 2s\}$ and let*

$$w \in L^\infty(B_{4R}^+) \cap H^1(B_{4R}^+, y^{1-2s} dx dy)$$

with $w(x, 0) \in L^\infty(\Gamma_{4R}^0)$ be a weak solution of

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0, & (x, y) \in B_{4R}^+, \\ \frac{\partial w}{\partial \nu} = f(w(x, 0)), & (x, 0) \in \Gamma_{4R}^0. \end{cases}$$

Then $w(x, 0) \in C^{2,\alpha}(\overline{\Gamma_R^0})$ for some $0 < \alpha < 1$ depending only on s and γ .

Proof. Fix $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta = 1$ on Γ_{3R}^0 , $0 \leq \eta \leq 1$ on $\mathbb{R}$ and $\operatorname{supp} \eta \subset \Gamma_{4R}^0$. Multiply $f(w(x, 0))$ by η and call it $\tilde{f}(x, 0)$. Let $v(x, y) = \Gamma_s(\cdot, y) * \tilde{f}(x, 0)$. Then $v(x, y)$ is a bounded weak solution of

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) = 0, & (x, y) \in \mathbb{R}_+^2, \\ \frac{\partial v}{\partial \nu} = \tilde{f}(x, 0), & (x, 0) \in \partial\mathbb{R}_+^2, \end{cases}$$

where we used that $\Gamma_s(x, y) = c_s |(x, y)|^{2s-1}$ ($c_s > 0$ is a constant) is the solution of

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla \Gamma_s) = 0, & (x, y) \in \mathbb{R}_+^2, \\ \frac{\partial \Gamma_s}{\partial \nu} = \delta_0, & (x, 0) \in \partial \mathbb{R}_+^2, \end{cases}$$

where δ_0 is the Dirac delta distribution at the origin. And the function $v(x, 0)$ solves

$$(-\Delta)^s v = \tilde{f}(x, 0) \quad \text{for all } (x, 0) \in \mathbb{R}. \quad (2.4)$$

Applying [23, Proposition 2.9], we have the following assertions:

- If $s \leq \frac{1}{2}$, then $v(x, 0) \in C^{0,\beta}(\mathbb{R})$ for any $\beta < 2s$.
- If $s > \frac{1}{2}$, then $v(x, 0) \in C^{1,\beta}(\mathbb{R})$ for any $\beta < 2s - 1$.

Let $U = w - v$. Then U is a bounded weak solution of

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla U) = 0, & (x, y) \in B_{3R}^+, \\ \frac{\partial U}{\partial \nu} = 0, & (x, 0) \in \Gamma_{3R}^0. \end{cases}$$

By [8, Lemma 4.1], the extension function

$$\tilde{U}(x, y) = \begin{cases} U(x, y), & y \geq 0, \\ U(x, -y), & y < 0, \end{cases}$$

is the solution of

$$-\operatorname{div}(|y|^{1-2s} \nabla \tilde{U}) = 0, \quad (x, y) \in B_{3R}.$$

By Lemma 2.9, there exist two constants $\mu = \mu(s) > 0$ and $C = C(R, s, \|f\|_\infty, \|w\|_\infty) > 0$ such that

$$\|\tilde{U}\|_{C^\mu(B_{5R/2})} + \|D_x \tilde{U}\|_{C^\mu(B_{5R/2})} + \|D_x^2 \tilde{U}\|_{C^\mu(B_{5R/2})} \leq C.$$

Hence $w(x, 0) \in C^{1,\nu}(\overline{\Gamma_{2R}^0})$ with $\nu < 2s - 1$ for $s > \frac{1}{2}$. If $\frac{1}{4} < s \leq \frac{1}{2}$, fix $\eta_1 \in C_0^\infty(\mathbb{R})$ such that $\eta_1 = 1$ on $\Gamma_{9R/4}^0$, $0 \leq \eta_1 \leq 1$, on $\mathbb{R}$ and $\operatorname{supp} \eta_1 \subset \Gamma_{5R/2}^0$. Multiply $f(w(x, 0))$ by η_1 and still call it $\tilde{f}(x, 0)$. Moreover, still set

$$v(x, y) = \Gamma_s(\cdot, y) * \tilde{f}(x, 0).$$

Then, using the above procedure again, we can see that $w \in C^{1,\nu}(\overline{\Gamma_{2R}^0})$ with $\nu < 4s - 1$. The only difference is that in equation (2.4), for $\tilde{f}(x, 0) \in C^\beta$, we give the following estimates by applying [23, Proposition 2.8]:

- If $\beta + 2s \leq 1$, then $v(x, 0) \in C^{0,\beta+2s}(\mathbb{R})$.
- If $\beta + 2s > 1$, then $v(x, 0) \in C^{1,\beta+2s-1}(\mathbb{R})$.

And for any $s \in (0, \frac{1}{4}]$, we can iterate the above procedure a finite number of times, until $\beta + 2ks > 1$ for some integer k . Then we have

$$w(x, 0) \in C^{1,\nu}(\overline{\Gamma_{2R}^0})$$

for some $\nu > 0$.

For the $C^{2,\alpha}$ regularity, we can differentiate equation (2.4) to obtain that

$$(-\Delta)^s D_x v = D_x \tilde{f}(x, 0) \quad \text{for all } (x, 0) \in \mathbb{R},$$

with $D_x \tilde{f}(x, 0)$ belonging to C^σ ($\sigma < \min\{\nu, \nu\}$). Therefore, by [23, Proposition 2.8], we can see that $D_x v(x, 0)$ belongs to $C^{0,\sigma+2s}(\mathbb{R})$ if $\sigma + 2s < 1$, and it belongs to $C^{1,\sigma+2s-1}(\mathbb{R})$ if $\sigma + 2s > 1$. Adding this estimate to the procedure of obtaining the $C^{1,\nu}$ regularity, we iterate this new procedure a finite number of times. Since $\nu + 2s > 1$, we finally arrive at $D_x v(x, 0) \in C^{1,\alpha}(\mathbb{R})$. Then

$$w(x, 0) \in C^{2,\alpha}(\overline{\Gamma_R^0})$$

for some α depending only on s, γ . □

2.4 The Eigenvalue Problem

In this subsection, we consider the following equation:

$$(-\Delta)^s v = \mu v, \quad v(x) = v(x + 2\pi). \quad (2.5)$$

Lemma 2.11 ([12, Lemma 3.1]). *The only eigenvalues of problem (2.5) are $\mu_0 = 0$, with a constant eigenfunction, and $\mu_k = k^{2s}$, $k = 1, 2, \dots$, with an eigenspace spanned by $\cos(kx)$ and $\sin(kx)$.*

Corollary 2.12. *The only eigenvalues of*

$$(-\Delta)^s v = \mu v, \quad v(x) = v(x + 2T), \quad (2.6)$$

are $\mu_0 = 0$, with a constant eigenfunction, and $\mu_k = (\frac{k\pi}{T})^{2s}$, $k = 1, 2, \dots$, with an eigenspace spanned by $\cos(\frac{k\pi x}{T})$ and $\sin(\frac{k\pi x}{T})$.

Corollary 2.13. *The only eigenvalues of*

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0, & (x, y) \in [-T, T] \times \mathbb{R}_+, \\ \frac{\partial w}{\partial \nu} = -\mu w(x, 0), & (x, 0) \in [-T, T], \\ w(x, y) = w(x + 2T, y), & (x, y) \in [-T, T] \times \overline{\mathbb{R}_+}, \end{cases} \quad (2.7)$$

are $\mu_0 = 0$, with a constant eigenfunction, and $\mu_k = (\frac{k\pi}{T})^{2s}$, $k = 1, 2, \dots$, with an eigenspace spanned by $P_s(\cdot, y) * \cos(\frac{k\pi x}{T})$ and $P_s(\cdot, y) * \sin(\frac{k\pi x}{T})$.

Proof. The conclusion can be easily verified by the one-to-one correspondence between the eigenvalue of (2.7) and the eigenvalue of (2.6). $\square$

3 Existence of k Solutions

In this section, we will work in the following Hilbert space:

$$X_T = \left\{ w(x, y) : w\left(\frac{T}{2}, y\right) = w(0, y) = 0 \text{ for all } y \geq 0, w \in \mathcal{H}_T \right\}.$$

We choose $\hat{F}$ such that

$$\begin{cases} \hat{F}(t) = \frac{(t-1)^2}{2}, & t > 1, \\ \hat{F}(t) = F(t), & |t| \leq 1, \\ \hat{F}(t) = \frac{(t+1)^2}{2}, & t < -1. \end{cases}$$

Let $D_T = [0, \frac{T}{2}] \times (0, \infty)$ and consider the following functional:

$$\hat{J}(w) = \frac{1}{2} \int_{D_T} y^{1-2s} |\nabla w|^2 dx dy + \int_0^{T/2} \hat{F}(w(x, 0)) dx.$$

3.1 Proof of Theorem 1.1

Now we give the proof of Theorem 1.1. It is standard to check $\hat{J} \in C^1(X_T, \mathbb{R})$ by Lemma 2.1, Lemma 2.2 and the assumption of $\hat{F}$. Clearly, $\hat{J}$ is even and $\hat{J}(w) \geq 0$ for all $w \in X_T$. It is easily to see that there exists a constant $C > 0$ independent of $w \in X_T$ such that

$$\hat{J}(w) \geq \frac{1}{2} \int_{D_T} y^{1-2s} |\nabla w|^2 dx dy + \frac{1}{4} \int_0^{T/2} w^2(x, 0) dx - C. \quad (3.1)$$

Then $\hat{J}$ is bounded below and coercive. Now let $\{w_n\}_{n \geq 1}$ be a $(PS)_c$ sequence, that is, $\hat{J}(w_n) \rightarrow c$ and $\hat{J}'(w_n) \rightarrow 0$. Then, by (3.1), we can easily see that $\{w_n\}_{n \geq 1}$ is bounded. Assume without loss of generality that w_n converges to w weakly in X_T and strongly in $L^2([0, \frac{T}{2}])$ by Lemma 2.1 and Lemma 2.2. Then $\hat{J}'(w) = 0$. Note that

$$\begin{aligned} & \int_{D_T} y^{1-2s} |\nabla(w_n(x, y) - w(x, y))|^2 dx dy \\ &= \langle \hat{J}'(w_n) - \hat{J}'(w), w_n - w \rangle - \int_0^{T/2} [\hat{F}'(w_n(x, 0)) - \hat{F}'(w(x, 0))](w_n(x, 0) - w(x, 0)) dx. \end{aligned}$$

Clearly,

$$\langle \hat{J}'(w_n) - \hat{J}'(w), w_n - w \rangle \rightarrow 0,$$

and by the choice of $\hat{F}$ we have

$$\begin{aligned} & \int_0^{T/2} [\hat{F}'(w_n(x, 0)) - \hat{F}'(w(x, 0))](w_n(x, 0) - w(x, 0)) dx \\ & \leq C(1 + \|w_n(x, 0)\|_{L^2([0, \frac{T}{2}])} + \|w(x, 0)\|_{L^2([0, \frac{T}{2}])}) \|w_n(x, 0) - w(x, 0)\|_{L^2([0, \frac{T}{2}])}. \end{aligned}$$

Therefore, w_n converges to w strongly in X_T and the $(PS)_c$ condition holds for $\hat{J}$.

For $\sigma \in (0, \frac{1}{2})$, we define k functions as follows:

$$h_i(x) = (-1)^j g(x), \quad x \in \left[0, \frac{T}{2^i}\right] + \frac{jT}{2^i}, \quad j \in \{0, \dots, 2^{i-1} - 1\}, \quad i = 1, \dots, k,$$

where g satisfies $g(x) = g(x + \frac{T}{2^k})$ and

$$g(x) = \begin{cases} \frac{2^{k+1}}{\sigma T} x, & x \in \left[0, \frac{\sigma T}{2^{k+1}}\right], \\ 1, & x \in \left[\frac{\sigma T}{2^{k+1}}, \frac{T}{2^k} - \frac{\sigma T}{2^{k+1}}\right], \\ \frac{2}{\sigma} - \frac{2^{k+1}}{\sigma T} x, & x \in \left[\frac{T}{2^k} - \frac{\sigma T}{2^{k+1}}, \frac{T}{2^k}\right]. \end{cases}$$

Then we can construct k functions $\psi_i(x, y) = \exp\{-\frac{y}{2^{b+1}}\} h_i(x)$, $i = 1, \dots, k$, for some $b > 1$. Set

$$K = \left\{ \sum_{i=1}^k \alpha_i \psi_i(x, y) : \sum_{i=1}^k |\alpha_i| = 1 \right\}.$$

It is clear that K is homeomorphic to ∂B^k by an odd map for any b . We now claim $\sup_K \hat{J} < \hat{J}(0)$ if b is sufficiently large. Let

$$\psi(x, y) = \sum_{i=1}^k \alpha_i \psi_i(x, y) \in K.$$

After a direct checking, we can see that

$$\begin{aligned} & \left| \sum_{i=1}^k \alpha_i h_i(x) \right| \leq 1, \\ & \left| \sum_{i=1}^k \alpha_i D_x h_i(x) \right| \leq \frac{2^{k+1}}{\sigma T} \quad \text{for any } x \in \left[0, \frac{T}{2}\right], \\ & \left| \sum_{i=1}^k \alpha_i D_x h_i(x) \right| = 0, \quad x \in \left[\frac{\sigma T}{2^{k+1}}, \frac{T}{2^k} - \frac{\sigma T}{2^{k+1}}\right] + \frac{mT}{2^k}, \quad m = 0, \dots, 2^{k-1} - 1, \\ & \left| \sum_{i=1}^k \alpha_i h_i(x) \right| = 1, \quad x \in \left[\frac{\sigma T}{2^{k+1}}, \frac{T}{2^k} - \frac{\sigma T}{2^{k+1}}\right] + \frac{m_1 T}{2^k}, \quad \text{for a suitable } m_1 \in \{0, \dots, 2^{k-1} - 1\}. \end{aligned}$$

By the assumptions (F1) and (F2), for any $\psi \in K$,

$$\int_0^{T/2} \hat{F}(\psi(x, 0)) dx = \int_0^{T/2} F\left(\sum_{i=1}^k \alpha_i \psi_i(x, 0)\right) dx < \frac{F(0)T}{2^k} (2^{k-1} - 1 + \sigma). \quad (3.2)$$

We also have

$$\begin{aligned} \int_{D_T} y^{1-2s} |\nabla \psi(x, y)|^2 dx dy &= \int_0^\infty \int_0^{T/2} y^{1-2s} \left| \nabla \left(\sum_{i=1}^k \alpha_i \psi_i(x, y) \right) \right|^2 dx dy \\ &= \int_0^\infty y^{1-2s} \exp\left\{-\frac{y}{2b}\right\} dy \int_0^{T/2} \frac{|\sum_{i=1}^k \alpha_i h_i(x)|^2}{2^{2b+2}} + \left| \sum_{i=1}^k \alpha_i D_x h_i(x) \right|^2 dx \\ &\leq \left(\frac{1}{2^{2b+2}} \cdot \frac{T}{2} + \left(\frac{2^{k+1}}{\sigma T} \right)^2 \cdot \frac{\sigma T}{2} \right) \int_0^\infty y^{1-2s} \exp\left\{-\frac{y}{2b}\right\} dy \\ &= 2^{b(2-2s)} \left(\frac{1}{2^{2b}} \cdot \frac{T}{8} + \frac{2^{2k+1}}{\sigma T} \right) \int_0^\infty z^{1-2s} \exp\{-z\} dz \\ &= \Gamma(2-2s) 2^{-2sb} \left(\frac{T}{8} + 2^{2b} \cdot \frac{2^{2k+1}}{\sigma T} \right). \end{aligned}$$

For sufficiently large b , we have that $\Gamma(2-2s)2^{-2sb}$ is small, and $2^{2b} \cdot \frac{2^{2k+1}}{\sigma T}$ is also small provided that T is large enough. Therefore, there exists $T_k > 0$ such that for any $T > T_k$ the following estimate holds true:

$$\int_{D_T} y^{1-2s} |\nabla \psi(x, y)|^2 dx dy < F(0)(1-\sigma) \frac{T}{2^k}.$$

Hence, by (3.2), we can get that $\hat{J}(\psi) < F(0) \frac{T}{2} = \hat{J}(0)$ for any $\psi \in K$. Then all the conditions of Clark's theorem are satisfied, and hence we can get k pairs solutions $\pm W_{i,T} \in X_T$ ($i = 1, \dots, k$) of

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla w) = 0 & \text{in } D_T, \\ \frac{\partial w}{\partial \nu} = \hat{F}'(w(x, 0)) & \text{in } \left[0, \frac{T}{2}\right]. \end{cases}$$

We now show that $W_{i,T} \in [-1, 1]$ for any $i \in \{1, \dots, k\}$. Let $W_\pm = \max\{\pm W, 0\}$. Then we have

$$\begin{aligned} 0 &= - \int_{D_T} y^{1-2s} \nabla W_{i,T} \nabla (W_{i,T} + 1)_- dx dy - \int_0^{T/2} \hat{F}'(W_{i,T}(x, 0))(W_{i,T}(x, 0) + 1)_- dx \\ &= \int_{D_T} y^{1-2s} |\nabla (W_{i,T} + 1)_-|^2 dx dy - \int_{[0, T/2] \cap \{W_{i,T} \leq -1\}} \hat{F}'(W_{i,T}(x, 0))(W_{i,T}(x, 0) + 1)_- dx \\ &= \int_{D_T} y^{1-2s} |\nabla (W_{i,T} + 1)_-|^2 dx dy + \int_0^{T/2} |(W_{i,T}(x, 0) + 1)_-|^2 dx, \end{aligned}$$

where we have used that $\hat{F}'(t) = t + 1$ for $t \leq -1$. We can conclude that $(W_{i,T} + 1)_- = 0$, and therefore $W_{i,T} \geq -1$. Similarly, we can prove that $W_{i,T} \leq 1$ by using that $\hat{F}'(t) = t - 1$ for $t \geq 1$. Therefore, $W_{i,T}$ ($i = 1, \dots, k$) are the solutions of

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla w) = 0 & \text{in } D_T, \\ \frac{\partial w}{\partial \nu} = F'(w(x, 0)) & \text{in } \left[0, \frac{T}{2}\right]. \end{cases}$$

Now, we make odd extensions of $W_{i,T}$ ($i = 1, \dots, k$) (with respect to x) from $\overline{D}_T$ to $\overline{\Omega}_T = [-\frac{T}{2}, \frac{T}{2}] \times [0, \infty)$. Furthermore, we extend it periodically (with respect to x again) from $\overline{\Omega}_T$ to $\mathbb{R}_+^2$, and we still denote it by $W_{i,T}$. Then $W_{i,T}$ ($i = 1, \dots, k$) are the weak solutions of (2.1). And $u_{i,T}(x) = W_{i,T}(x, 0)$ ($i = 1, \dots, k$) are the odd periodic solutions of (1.1). By Lemma 2.8, we can conclude that $u_{i,T}(x) \in (-1, 1)$, where we used that $F'(\pm 1) = 0$.

3.2 Proof of Theorem 1.2

We now prove Theorem 1.2. Recall from the proof of Theorem 1.1 that we only need to find a set K homeomorphic to ∂B^k such that $\sup_K \hat{J} < \hat{J}(0)$. By $F'(0) = 0$ and Taylor's formula, we can see that

$$\hat{F}(u) = \hat{F}(0) + \frac{\hat{F}''(\theta u)}{2} u^2 \leq \hat{F}(0) + \frac{(\hat{F}''(0) + \epsilon)}{2} u^2 \quad \text{for all } u \in B_\varrho(0),$$

where $\theta \in (0, 1)$, $\varrho > 0$ and $\epsilon = \epsilon(\varrho) > 0$ are small. From Corollary 2.13,

$$\mu_k = \left(\frac{2k\pi}{T} \right)^{2s} \quad \text{and} \quad \varphi_k = P_s(\cdot, y) * \sin\left(\frac{2k\pi x}{T}\right)$$

are the k -th eigenvalue and eigenfunction, respectively, of equation (2.7) with $\varphi(0, y) = \varphi(\frac{T}{2}, y) = 0$. Define

$$K = \left\{ \varphi \in \sum_{i=1}^k \alpha_i \varphi_i : \sum_{i=1}^k \alpha_i^2 = r^2 \right\}, \quad r < \sqrt{k}\varrho.$$

Then K is homeomorphic to ∂B^k . It is clear that

$$\int_{D_T} y^{1-2s} \nabla \varphi_i \nabla \varphi_j \, dx \, dy = \mu_i \int_0^{T/2} \varphi_i(x, 0) \varphi_j(x, 0) \, dx \quad \text{for all } i, j \in \{1, \dots, k\}.$$

Then for any $\varphi \in K$ we have $|\varphi| \leq \varrho$ and

$$\begin{aligned} \hat{J}(\varphi) &= \frac{1}{2} \int_{D_T} y^{1-2s} |\nabla \varphi|^2 \, dx \, dy + \int_0^{T/2} \hat{F}(\varphi(x, 0)) \, dx \\ &\leq \frac{1}{2} \int_{D_T} y^{1-2s} |\nabla \varphi|^2 \, dx \, dy + \frac{(\hat{F}''(0) + \epsilon)}{2} \int_0^{T/2} \varphi^2(x, 0) \, dx + \frac{\hat{F}(0)T}{2} \\ &\leq \frac{1}{2} \left(\left(\frac{2k\pi}{T} \right)^{2s} + \hat{F}''(0) + \epsilon \right) \int_0^{T/2} \varphi^2(x, 0) \, dx + \frac{\hat{F}(0)T}{2}. \end{aligned}$$

For T large, we have $\hat{J}(\varphi) < \hat{J}(0)$ for any $\varphi \in K$ since $\hat{F}''(0) = F''(0) < 0$. Then, as in the proof of Theorem 1.1, we can complete the proof of Theorem 1.2.

4 Proof of Theorem 1.3

First, we define the extension function $\tilde{F}$ as follows:

$$\begin{cases} \tilde{F}(t) = \frac{F''(1)}{2}(t-1)^2, & t \geq 1, \\ \tilde{F}(t) = F(t), & |t| < 1, \\ \tilde{F}(t) = \frac{F''(-1)}{2}(t+1)^2, & t \leq -1. \end{cases}$$

Clearly,

$$\begin{cases} \tilde{F}'(t) = F''(1)t - F''(1), & t \geq 1, \\ \tilde{F}'(t) = F''(-1)t + F''(-1), & t \leq -1, \\ \tilde{F}''(t) = F''(1), & t \geq 1, \\ \tilde{F}''(t) = F''(-1), & t \leq -1. \end{cases}$$

Recall

$$\Omega_T = \left[-\frac{T}{2}, \frac{T}{2}\right] \times (0, \infty).$$

We now consider the following functional:

$$\tilde{J}(w) = \frac{1}{2} \int_{\Omega_T} y^{1-2s} |\nabla w|^2 dx dy + \int_{-T/2}^{T/2} \tilde{F}(w(x, 0)) dx.$$

It is easily to see that there exists a constant $C > 0$ such that

$$\tilde{J}(w) \geq \frac{1}{2} \int_{\Omega_T} y^{1-2s} |\nabla w|^2 dx dy + \frac{\min\{F''(1), F''(-1)\}}{4} \int_{-T/2}^{T/2} w^2(x, 0) dx - C,$$

that is, $\tilde{J}$ is coercive and bounded below. And as in the proof of Theorem 1.1, we can see that $\tilde{J}$ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. By the assumptions (F1) and (F2'), we also have

$$\tilde{J}(w) \geq 0 = \tilde{J}(\pm 1) \quad \text{for all } w \in \mathcal{H}_T.$$

Note that $F''(\pm 1) > 0$. Then

$$\tilde{J}''(\pm 1)\varphi^2 = \int_{\Omega_T} y^{1-2s} |\nabla \varphi|^2 dx dy + \int_{-T/2}^{T/2} \tilde{F}''(\pm 1)\varphi^2(x, 0) dx > 0 \quad \text{for all } \varphi \in \mathcal{H}_T \setminus \{0\},$$

and one can deduce that ± 1 are both local minimizers for $\tilde{J}$. Then, by the mountain pass theorem (Lemma 2.3), we can get a $V_{1,T} \in \mathcal{H}_T$ such that

$$\tilde{J}(V_{1,T}) = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \tilde{J}(\gamma(t)) > 0,$$

where the class of a path Γ is defined by

$$\Gamma = \{\gamma \in C([0, 1], \mathcal{H}_T) : \gamma(0) = -1, \gamma(1) = 1\}.$$

It is known that there exists $\mathcal{T} > 0$ such that for $T > \mathcal{T}$,

$$\tilde{J}(V_{1,T}) < \tilde{J}(0),$$

holds by a construction from [17].

It is easy to see that $V_{1,T} \in [-1, 1]$. Indeed, by

$$\begin{aligned} 0 &= \int_{\Omega_T} y^{1-2s} \nabla V_{1,T} \nabla (V_{1,T} - 1)_+ dx dy + \int_{-T/2}^{T/2} \tilde{F}'(V_{1,T}(x, 0))(V_{1,T}(x, 0) - 1)_+ dx \\ &= \int_{\Omega_T} y^{1-2s} |\nabla (V_{1,T} - 1)_+|^2 dx dy + \int_{\Gamma_{T/2}^0 \cap \{V_{1,T} \geq 1\}} \tilde{F}'(V_{1,T}(x, 0))(V_{1,T}(x, 0) - 1)_+ dx \\ &= \int_{\Omega_T} y^{1-2s} |\nabla (V_{1,T} - 1)_+|^2 dx dy + F''(1) \int_{-T/2}^{T/2} |(V_{1,T}(x, 0) - 1)_+|^2 dx, \end{aligned}$$

we can conclude that $V_{1,T} \leq 1$. Also, we can prove that $V_{1,T} \geq -1$.

Now we extend $V_{1,T}$ periodically (with respect to x) from $\bar{\Omega}_T$ to the whole half space $\overline{\mathbb{R}_+^2}$, and we still denote it by $V_{1,T}$. Then we know that $V_{1,T}$ is a weak solution of (2.1).

Claim 1: $i(\tilde{J}, V_{1,T}) = 1$, that is, the Morse index of $V_{1,T}$ is equal to 1. By Lemma 2.10, we have

$$V_{1,T}(x, 0) \in C^{2,\alpha}(\overline{\Gamma_{3T/8}^0})$$

for some $\alpha \in (0, 1)$. We now show that

$$V_{1,T}(x, 0) \in C^{2,\alpha}(\pm \overline{D_1}), \quad D_1 = \left[\frac{T}{8}, \frac{7T}{8}\right]. \quad (4.1)$$

For simplicity, we assume $T = 2\pi$. By Corollary 2.13, for any $\varphi(x, y) \in \mathcal{H}_{2\pi}$, we have

$$\varphi = \sum_{k=0}^{\infty} a_k P_s(\cdot, y) * \sin(kx) + b_k P_s(\cdot, y) * \cos(kx),$$

with

$$a_k = \int_{-\pi}^{\pi} \varphi(x, 0) \sin(kx) dx \quad \text{and} \quad b_k = \int_{-\pi}^{\pi} \varphi(x, 0) \cos(kx) dx.$$

Let

$$\varphi_m = \sum_{k=0}^m a_k P_s(\cdot, y) * \sin(kx) + b_k P_s(\cdot, y) * \cos(kx).$$

Then

$$\varphi(x + \pi, y) = \lim_{m \rightarrow \infty} \varphi_m(x + \pi, y) = \sum_{k=0}^{\infty} (-1)^k a_k P_s(\cdot, y) * \sin(kx) + (-1)^k b_k P_s(\cdot, y) * \cos(kx).$$

Hence, again by Corollary 2.13, we can get that

$$\begin{aligned} \int_0^{\infty} \int_{-\pi}^{\pi} y^{1-2s} |\nabla \varphi|^2 dx dy &= \lim_{m \rightarrow \infty} \int_0^{\infty} \int_{-\pi}^{\pi} y^{1-2s} |\nabla \varphi_m|^2 dx dy \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^m k^{2s} (a_k^2 + b_k^2) \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^m k^{2s} ((-1)^k a_k)^2 + ((-1)^k b_k)^2 \\ &= \int_0^{\infty} \int_0^{2\pi} y^{1-2s} |\nabla \varphi|^2 dx dy. \end{aligned}$$

This implies that for any $U, V \in \mathcal{H}_{2\pi}$,

$$\int_0^{\infty} \int_0^{2\pi} y^{1-2s} \nabla U \nabla V dx dy = \int_0^{\infty} \int_{-\pi}^{\pi} y^{1-2s} \nabla U \nabla V dx dy.$$

Then fix $\varphi \in C^{\infty}([0, T] \times [0, \infty))$ with $\varphi(x, y) = \varphi(x + T, y)$ and

$$\int_0^{\infty} \int_0^T y^{1-2s} |\nabla \varphi|^2 dx dy + \int_0^T |\varphi(x, 0)|^2 dx < +\infty.$$

By the periodicity of $V_{1,T}$, we have

$$\begin{aligned} &\int_0^{\infty} \int_0^T y^{1-2s} \nabla V_{1,T} \nabla \varphi dx dy + \int_0^T \tilde{F}'(V_{1,T}(x, 0)) \varphi(x, 0) dx \\ &= \int_{\Omega_T} y^{1-2s} \nabla V_{1,T} \nabla \varphi dx dy + \int_0^{T/2} \tilde{F}'(V_{1,T}(x, 0)) \varphi(x, 0) dx + \int_{T/2}^T \tilde{F}'(V_{1,T}(x, 0)) \varphi(x, 0) dx \\ &= \int_{\Omega_T} y^{1-2s} \nabla V_{1,T} \nabla \varphi dx dy + \int_{-T/2}^{T/2} \tilde{F}'(V_{1,T}(x, 0)) \varphi(x, 0) dx = 0. \end{aligned}$$

Therefore, $V_{1,T}$ satisfies

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla v) = 0, & (x, y) \in [0, T] \times [0, \infty), \\ \frac{\partial v}{\partial \nu} = \tilde{F}'(v(x, 0)), & (x, 0) \in [0, T], \end{cases}$$

and (4.1) follows by Lemma 2.10.

Then, by (4.1), we have $V_{1,T}(x, 0) \in C^{2,\alpha}(\mathbb{R})$, and $\psi(x, y) = P_s(\cdot, y) * D_x V_{1,T}(x, 0)$ is a bounded weak solution of the following equation:

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) = 0, & (x, y) \in \Omega_T, \\ \frac{\partial v}{\partial y} = \tilde{F}''(V_{1,T}(x, 0))v, & (x, 0) \in \Gamma_{T/2}^0. \end{cases} \quad (4.2)$$

It follows that $V_{1,T}$ is a degenerate critical point.

One can easily verify $i(\tilde{J}, V_{1,T}) \leq 1$ because $V_{1,T}$ is a mountain-pass type critical point (e.g., [9, (1.9) in Chapter 2]). We now prove $i(\tilde{J}, V_{1,T}) = 1$, and argue by contradiction. Assume $i(\tilde{J}, V_{1,T}) = 0$, that is,

$$\tilde{J}''(V_{1,T})v^2 = \int_{\Omega_T} y^{1-2s} |\nabla v|^2 dx dy + \int_{-T/2}^{T/2} \tilde{F}''(V_{1,T}(x, 0))v^2(x, 0) dx \geq 0 \quad \text{for all } v \in \mathcal{H}_T. \quad (4.3)$$

Then we can prove that 1 is the smallest positive eigenvalue of

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla v) = 0, & (x, y) \in \Omega_T, \\ \frac{\partial v}{\partial y} - v = -\lambda[1 - \tilde{F}''(V_{1,T}(x, 0))]v, & (x, 0) \in \Gamma_{T/2}^0. \end{cases} \quad (4.4)$$

Indeed, provided by (4.3), for all positive eigenvalues λ of (4.4), we have

$$\lambda = \frac{\int_{\Omega_T} y^{1-2s} |\nabla v|^2 dx dy + \int_{-T/2}^{T/2} v^2(x, 0) dx}{\int_{-T/2}^{T/2} (1 - \tilde{F}''(V_{1,T}))v^2(x, 0) dx} \geq 1.$$

We also know that

$$\begin{aligned} \lambda_1 &= \inf_{v \in \mathcal{H}_T \setminus \{0\}} \frac{\int_{\Omega_T} y^{1-2s} |\nabla v|^2 dx dy + \int_{-T/2}^{T/2} v^2(x, 0) dx}{\int_{-T/2}^{T/2} (1 - \tilde{F}''(V_{1,T}))v^2(x, 0) dx} \\ &\leq \frac{\int_{\Omega_T} y^{1-2s} |\nabla \psi|^2 dx dy + \int_{-T/2}^{T/2} \psi^2(x, 0) dx}{\int_{-T/2}^{T/2} (1 - \tilde{F}''(V_{1,T}))\psi^2(x, 0) dx} = 1, \end{aligned}$$

where we used that $\psi(x, y) = P_s(\cdot, y) * D_x V_{1,T}(x, 0)$ is the solution of (4.2). Hence 1 is the smallest positive eigenvalue of (4.4), and $\psi(x, y)$ is the corresponding eigenfunction.

We now claim that $\psi(x, y)$ does not change sign in $x \in B_{T/2}^+ \cup \Gamma_{T/2}^0$. Otherwise, let

$$\psi_{\pm}(x, y) = \max\{\pm\psi(x, y), 0\}.$$

Since they are both not zero, we have

$$\begin{aligned} \int_{-T/2}^{T/2} (1 - \tilde{F}''(V_{1,T}))\psi^2(x, 0) dx &= \int_{-T/2}^{T/2} (1 - \tilde{F}''(V_{1,T}))\psi_+^2(x, 0) dx + \int_{-T/2}^{T/2} (1 - \tilde{F}''(V_{1,T}))\psi_-^2(x, 0) dx \\ &= a_+ + a_- \end{aligned}$$

and

$$\begin{aligned} &\int_{\Omega_T} y^{1-2s} |\nabla \psi|^2 dx dy + \int_{-T/2}^{T/2} \psi^2(x, 0) dx \\ &= \int_{\Omega_T} y^{1-2s} |\nabla \psi_+|^2 dx dy + \int_{-T/2}^{T/2} \psi_+^2(x, 0) dx + \int_{\Omega_T} y^{1-2s} |\nabla \psi_-|^2 dx dy + \int_{-T/2}^{T/2} \psi_-^2(x, 0) dx \\ &= b_+ + b_-. \end{aligned}$$

It is easily to see that $a_{\pm} > 0$. Otherwise, if $a_{-} < 0$, then

$$\frac{b_{+}}{a_{+}} < \frac{b_{+}}{a_{+} + a_{-}} < \frac{b_{+} + b_{-}}{a_{+} + a_{-}} = 1.$$

This contradicts the fact that 1 is the smallest eigenvalue.

We have either

$$\frac{b_{+} + b_{-}}{a_{+} + a_{-}} = \frac{b_{+}}{a_{+}} = \frac{b_{-}}{a_{-}} \quad \text{or} \quad \frac{b_{+} + b_{-}}{a_{+} + a_{-}} > \min\left\{\frac{b_{+}}{a_{+}}, \frac{b_{-}}{a_{-}}\right\}.$$

However, the latter case cannot occur due to the fact the $1 = \frac{b_{+} + b_{-}}{a_{+} + a_{-}}$ is the smallest positive eigenvalue. In the former case, both $\psi_{\pm}(x, y)$ are eigenfunctions with respect to 1. Then by Lemma 2.8, we have $\psi_{\pm}(x, y) > 0$ for a.e. $x \in B_{T/2}^{+} \cup \Gamma_{T/2}^0$. This is impossible.

However, by the periodicity of $V_{1,T}(x, 0)$, we have that $D_x V_{1,T}(x, 0)$ must change sign in $\Gamma_{T/2}^0$. This is a contradiction. Then we deduce that $i(\tilde{J}, V_{1,T}) = 1$.

We now come back to find the another nonconstant periodic solution.

By Lemma 2.5, we can get that

$$C_q(\tilde{J}, V_{1,T}) = \begin{cases} G, & q = 1, \\ 0, & q \neq 1, \end{cases}$$

due to $i(\tilde{J}, V_{1,T}) = 1$. By $F''(\pm 1) > 0$, we can claim that ± 1 are local minimizers of $\tilde{J}$. Again applying Lemma 2.5, we have

$$C_q(\tilde{J}, \pm 1) = \begin{cases} G, & q = 0, \\ 0, & q \neq 0. \end{cases}$$

We now consider the trivial solution θ . If $F''(0) = 0$, then $\mathcal{N} = \ker(\tilde{J}''(\theta)) = \mathbb{R}$. We recall assumption (F2'): $F(t) < F(0)$ for $t \in (-1, 1)$. We can conclude that θ is a local maximum of $\tilde{J}|_{\mathcal{N}}$. Then, by Lemmas 2.5 and 2.6,

$$C_q(\tilde{J}, \theta) = \begin{cases} G, & q = 1, \\ 0, & q \neq 1. \end{cases}$$

If there exists a positive integer k such that

$$\frac{T(-F''(0))^{\frac{1}{2s}}}{2\pi} = k,$$

then

$$\mathcal{N} = \ker(\tilde{J}''(\theta)) = \text{span}\left\{P_s(\cdot, y) * \cos\left(\frac{2\pi kx}{T}\right), P_s(\cdot, y) * \sin\left(\frac{2\pi kx}{T}\right)\right\}.$$

And if $\frac{F'(t)}{t} \leq F''(0)$ for $t \neq 0$ in a neighborhood U of 0, then θ is a local maximum of $\tilde{J}|_{\mathcal{N}}$. Let $i(\tilde{J}, \theta)$ be the Morse index of θ . By Lemma 2.5 and Lemma 2.6,

$$C_q(\tilde{J}, \theta) = \begin{cases} G, & q = i(\tilde{J}, \theta) + 2, \\ 0, & q \neq i(\tilde{J}, \theta) + 2. \end{cases}$$

Similarly, in the cases that either $\frac{T(-F''(0))^{1/2s}}{2\pi}$ is not equal to a positive integer or $\frac{T(-F''(0))^{1/2s}}{2\pi}$ is a positive integer but $\frac{F'(t)}{t} \geq F''(0)$ for $t \neq 0$ in a neighborhood U of 0, then θ is a local minimum of $\tilde{J}|_{\mathcal{N}}$. Thus

$$C_q(\tilde{J}, \theta) = \begin{cases} G, & q = i(\tilde{J}, \theta), \\ 0, & q \neq i(\tilde{J}, \theta). \end{cases}$$

If there were no other critical points, then a contradiction would occur due to the Morse inequalities as follows. We have $\beta_q = \text{rank } H_q(X, \tilde{J}^{-c}, G) = \delta_{0,q}$ ($q \in \mathbb{N}$) since $\tilde{J}$ is bounded below. Now for $k > 2 + i(\tilde{J}, \theta)$ we would have

$$M_k - M_{k-1} + \cdots + (-1)^k M_0 = \beta_k - \beta_{k-1} + \cdots + (-1)^k \beta_0.$$

The left-hand side is even, but the right-hand side is odd, a contradiction. Therefore, there must be another solution $V_{2,T}$.

Arguing as in the proof of $V_{1,T} \in [-1, 1]$, we have $V_{2,T} \in [-1, 1]$.

Now we extend $V_{2,T}$ periodically (with respect to x) from $\bar{\Omega}_T$ to the whole half space $\overline{\mathbb{R}_+^2}$, and we still denote it by $V_{2,T}$. Then we know that $V_{2,T}$ is a weak solution of (2.1). Further, $v_{i,T}(x) = V_{i,T}(x, 0)$ ($i = 1, 2$) are periodic solutions of (1.1). By Lemma 2.8, we can conclude that $v_{i,T}(x) \in (-1, 1)$, for which we use $F'(\pm 1) = 0$.

Acknowledgment: The authors thank the referees for their comments on improving the paper.

Funding: This work is supported by NSFC (11771324, 11831009).

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