Research Article

Yuxia Guo* and Yichen Hu

Non-degeneracy of bubble solutions for higher order prescribed curvature problem

<https://doi.org/10.1515/ans-2022-0003>

received October 27, 2021; accepted February 4, 2022

Abstract: In this article, we are concerned with the following prescribed curvature problem involving polyharmonic operator on $\mathbb{S}^N$:

$$D^m u = K(|y|)u^{m^*-1}, \quad u > 0 \text{ in } \mathbb{S}^N, \quad u \in H^m(\mathbb{S}^N),$$

where $K(|y|)$ is a positive function, $m^* = \frac{2N}{N-2m}$ is the Sobolev embedding critical exponent, $N > 2m + 2$. D^m is the $2m$ order differential operator given by

$$D^m = \prod_{l=1}^m \left(-\Delta_g + \frac{1}{4}(N-2l)(N+2l-2) \right),$$

where Δ_g is the Laplace-Beltrami operator on $\mathbb{S}^N$, $\mathbb{S}^N$ is the unit sphere with Riemann metric g . We first establish two kinds of local Pohozaev identities for polyharmonic operator, then we prove that the positive bubbling solution constructed in the study of Guo and Li is non-degenerate.

Keywords: polyharmonic operator, non-degeneracy, Pohozaev identities

MSC 2020: 35B33, 35J91

1 Introduction

We consider the following prescribed curvature problem involving polyharmonic operator on $\mathbb{S}^N$

$$D^m u = K(|y|)u^{m^*-1}, \quad u > 0 \text{ in } \mathbb{S}^N, \quad u \in H^m(\mathbb{S}^N), \quad (1.1)$$

where $K(|y|)$ is a positive function, $m^* = \frac{2N}{N-2m}$ is the critical exponent of Sobolev embedding, $N > 2m + 2$, D^m is a $2m$ order differential operator given by

$$D^m = \prod_{l=1}^m \left(-\Delta_g + \frac{1}{4}(N-2l)(N+2l-2) \right),$$

where Δ_g is the Laplace-Beltrami operator on $\mathbb{S}^N$, $\mathbb{S}^N$ is the unit sphere with Riemann metric g .

In the case of $m = 1$, problem (1.1) is reduced to the following prescribed curvature problem:

$$-\Delta_{\mathbb{S}^N} u + \frac{N(N-2)}{2}u - K(y)u^{\frac{N+2}{N-2}} = 0, \quad u > 0, \text{ on } \mathbb{S}^N. \quad (1.2)$$

* **Corresponding author: Yuxia Guo**, Department of Mathematical Science, Tsinghua University, Beijing, P. R. China, e-mail: yguo@tsinghua.edu.cn

Yichen Hu: Department of Mathematical Science, Tsinghua University, Beijing, P. R. China, e-mail: hu-yc19@mails.tsinghua.edu.cn

The so-called prescribed curvature problem is to find a conformal invariant metric g such that the curvature is $K(y)$.

By using the stereo-graphic projection, problem (1.2) is reduced to the following elliptic problem in $\mathbb{R}^N$:

$$\begin{cases} -\Delta u = K(y)u^{\frac{N+2}{N-2}}, & u > 0, \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \quad (1.3)$$

Because of its geometry background, problem (1.3) has been extensively studied in the last few decades. It is known that (see [1]) (1.3) does not always admit a solution. Hence, we are more interested in the sufficient condition on the curvature function $K(y)$, under which problem (1.3) admits a solution. Indeed, there have been a lot of existence results obtained in the literature, see, for example, [2–7] and references therein. In particular, we know that any solution of (1.3) is radially symmetric if there is an $r_0 > 0$ such that $K(|y|)$ is non-increasing in $(0, r_0]$ and non-decreasing in $[r_0, +\infty)$ (see [8]). It is natural to ask whether or not there are non-radial symmetric solutions to (1.3). This question was answered in the paper of [9].

In general case of $m \geq 1$, problem (1.1) also has attracted wide attention due to its geometry roots and various applications in physics during the last few decades. For instance, when $m = 2$, problem (1.1) is related to the Paneitz operator, which was introduced by Paneitz [10] for smooth four-dimensional Riemannian manifolds and was generalized by Branson [11] to smooth N -dimensional Riemannian manifolds. For various existence results to problems involving polyharmonic operator and other related problems, we refer the reader to the papers [12–23] and references therein. It is evident to see from these papers that compared to the problems with Laplace operator (i.e., $m = 1$), the problems involving polyharmonic operator present new and challenging features which make the problem getting more complicated and difficult.

This article is concerning on the non-degeneracy of bubbling solutions for problem (1.1). It is also known that by using the stereographic projection, the prescribed scalar curvature problem for polyharmonic operator on $\mathbb{S}^N$ can be reduced to the following problem in $\mathbb{R}^N$:

$$(-\Delta)^m u = K(|y|)u^{m^*-1}, \quad u > 0, \text{ in } \mathbb{R}^N, \quad u \in D^{m,2}(\mathbb{R}^N), \quad (1.4)$$

where $m^* = \frac{2N}{N-2m}$ is the Sobolev critical exponent of the embedding from $D^{m,2}(\mathbb{R}^N)$ onto $L^p(\mathbb{R}^N)$, and $D^{m,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm induced by the scalar product:

$$(u, v) = \begin{cases} \int_{\mathbb{R}^N} \Delta^{\frac{m}{2}} u \cdot \Delta^{\frac{m}{2}} v, & \text{if } m \text{ is even,} \\ \int_{\mathbb{R}^N} \nabla \Delta^{\frac{m-1}{2}} u \cdot \nabla \Delta^{\frac{m-1}{2}} v, & \text{if } m \text{ is odd.} \end{cases}$$

In particular, we are interested in the existence of non-radial solutions for (1.4). This was done in [24], where infinitely many non-radial bubble solutions were constructed under the following assumption on $K(|y|)$:

(K) There are $r_0 > 0$ and $c_0 > 0$, such that

$$K(r) = K(r_0) - c_0(r - r_0)^2 + O(|r - r_0|^3), \quad r \in (r_0 - \delta, r_0 + \delta). \quad (1.5)$$

Without loss of generality, we may assume that $K(r_0) = 1$.

The main purpose of this article is to discuss the non-degeneracy of positive bubble solution constructed in [24]. We would like to mention that the non-degeneracy of the solution is very important for the further construction of new solutions for problem (1.1). For example, we may ask whether we can obtain the similar result as in [24] without the symmetry assumptions on the curvature function $k(y)$. To answer this question, among any other things, one main task is to get a better understanding of the corresponding linearized problem around the approximation solution. That is, the non-degeneracy of the linearized operator. Another application in our mind is to construct new-type bubble solutions for (1.1). We will consider these problems in our later work.

Before the statement of the main results, let us briefly introduce the bubble solutions constructed in [25].

It is known (see [9,26]) that a family of positive solutions to the following limit problem:

$$(-\Delta)^m u = u^{m^*-1}, \quad u > 0 \text{ in } \mathbb{R}^N, \quad u \in D^{m,2}(\mathbb{R}^N) \quad (1.6)$$

are given by

$$U_{x,\mu}(y) = \frac{C_{N,m} \mu^{\frac{N-2m}{2}}}{(1 + \mu^2|y-x|^2)^{\frac{N-2m}{2}}}, \quad x \in \mathbb{R}^N, \mu > 0, \quad (1.7)$$

where $C_{N,m}$ is a constant depending on N, m .

Let k be an integer number and we consider the vertices of a regular polygon with k edges in the (y_1, y_2) -plane given by

$$x_j = \left(r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \dots, k,$$

where 0 denotes the zero vector in $\mathbb{R}^{N-2}$ and $r \in (r_0 - \delta, r_0 + \delta)$.

For any point $y \in \mathbb{R}^N$, we set $y = (y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$. Define

$$H_s = \left\{ u : u \text{ is even in } y_h, \quad h = 2, \dots, N, \quad u(r \cos \theta, r \sin \theta, y'') = u \left(r \cos \left(\theta + \frac{2\pi j}{k} \right), r \sin \left(\theta + \frac{2\pi j}{k} \right), y'' \right) \right\},$$

and set

$$W_{r,\mu}(y) = \sum_{j=1}^k U_{x_j,\mu}(y).$$

Furthermore, for a function $u \in H_s \cap D^{m,2}(\mathbb{R}^N)$, we introduce the norm $\|u\|_*$ as follows:

$$\|u\|_* = \sup_{y \in \mathbb{R}^N} |u(y)| \left(\sum_{j=1}^k \frac{\mu^{\frac{N-2m}{2}}}{(1 + \mu|y-x_j|)^{\frac{N-2m}{2}} + \tau} \right)^{-1},$$

where τ is any fixed number in $\left(\frac{N-2m-2}{N-2m}, 1 + \theta \right)$, $\theta > 0$ is a small constant. The result obtained in [24] states the following.

Theorem A. Suppose that $K(r)$ satisfies (K) and $N > 2m + 2$. Then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, (1.4) has a solution u_k of the form

$$u_k = W_{r_k,\mu_k}(y) + \omega_k,$$

where $\omega_k \in H_s \cap D^{m,2}(\mathbb{R}^N)$, and as $k \rightarrow +\infty$, $|r_k - r_0| = O\left(\frac{1}{\mu_k^{1+\sigma}}\right)$, $\mu_k \sim k^{\frac{N-2m}{N-2m-2}}$,

$$\|\omega_k\|_* = O\left(\frac{1}{\mu_k^{1+\sigma}}\right)$$

for some $\sigma > 0$.

Let L_k be the linear operator defined by

$$L_k \xi = (-\Delta)^m \xi - (m^* - 1)K(y)u_k^{m^*-2}\xi.$$

The main result of the present article is the following.

Theorem 1.1. Suppose that $K(y)$ satisfies (K), $K(y) \in C^3(B_\vartheta(r_0))$ for some $\vartheta > 0$, and $K'(r) \in L^\infty(\mathbb{R}^N)$. Then If $N > 2m + 2$, there exists $K_0 > 0$, such that for any integer $k > K_0$, the positive bubble solution u_k obtained in Theorem A is non-degenerate in the sense that if $\xi \in H_s \cap D^{m,2}(\mathbb{R}^N)$ is a solution of $L_k \xi = 0$, then $\xi = 0$.

As a direct consequence of Theorem 1.1, we have

Theorem 1.2. *Under the assumption in Theorem 1.1. There exists $K_0 > 0$, such that for any integer $k > K_0$, the bubble solution of the prescribed curvature problem (1.1) is non-degenerate in the sense that if $\xi \in H_s \cap D^{m,2}(\mathbb{R}^N)$ is a solution of $L_k \xi = 0$, then $\xi = 0$.*

For the proof of Theorem 1.1, we shall proceed contrary arguments by using the local Pohozaev identities for polyharmonic equations. However, different from the case of $m = 1$, the local Pohozaev identities will get more complex and there are even infinite many items (as m is large) to be dealt with. As a better understanding of different Pohozaev identities for polyharmonic equations is very important for us, some extra techniques and more detailed calculation are needed. We believe that these Pohozaev identities for polyharmonic operator can be applied for more elliptic problems involving polyharmonic operator.

The article is organized as follows. In Section 2, by delicate computations, we establish two types of local Pohozaev identities for polyharmonic. Section 3 is devoted to the proof of Theorem 1.1. For this purpose, we will first establish a finite estimate for the bubble solution u_k by using the local Pohozaev identities, then we proceed a contrary argument to prove its non-degeneracy. Some essential estimates are attached in Appendices.

2 Local Pohozaev identities

In this section, we first establish the local Pohozaev identities for polyharmonic operator. Let

$$(-\Delta)^m u = K(|y|)u^{m^*-1} \quad (2.1)$$

and

$$(-\Delta)^m \xi = (m^* - 1)K(|y|)u^{m^*-2}\xi. \quad (2.2)$$

Assume that Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We have the following identities.

Lemma 2.1. *If $m = 2l$, $l \in \mathbb{N}^+$, then*

$$\begin{aligned} \int_{\Omega} u^{m^*-1} \xi \frac{\partial K(|y|)}{\partial y_s} &= \int_{\partial\Omega} K(|y|)u^{m^*-1} \xi \nu_s - \int_{\partial\Omega} \Delta^l u \Delta^l \xi \nu_s + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial^2 \Delta^{i-1} \xi}{\partial y_s \partial \nu} - \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \frac{\partial \Delta^{i-1} \xi}{\partial y_s} \\ &\quad + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial^2 \Delta^{i-1} u}{\partial y_s \partial \nu} - \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \frac{\partial \Delta^{i-1} u}{\partial y_s} \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} \int_{\Omega} u^{m^*-1} \xi \langle \nabla K(|y|), y - x_0 \rangle &= \int_{\partial\Omega} K(|y|)u^{m^*-1} \xi \langle \nu, y - x_0 \rangle - \int_{\partial\Omega} \Delta^l u \Delta^l \xi \langle \nu, y - x_0 \rangle \\ &\quad - \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle}{\partial \nu} \\ &\quad - \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} \langle \nabla u, y - x_0 \rangle + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} \langle \nabla u, y - x_0 \rangle}{\partial \nu} \\ &\quad + \frac{N-2m}{2} \sum_{i=1}^l \left(\int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \xi}{\partial \nu} + \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} u}{\partial \nu} - \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \xi - \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} u \right). \end{aligned} \quad (2.4)$$

Proof. Proof of (2.3). Multiplying the two equations (2.1) and (2.2) by $\frac{\partial \xi}{\partial y_s}$ and $\frac{\partial u}{\partial y_s}$, respectively, and subtracting the obtained two equations, we have

$$\int_{\Omega} \left(\Delta^m u \frac{\partial \xi}{\partial y_s} + \Delta^m \xi \frac{\partial u}{\partial y_s} \right) = \int_{\Omega} K(|y|) \left(u^{m^*-1} \frac{\partial \xi}{\partial y_s} + (m^* - 1) u^{m^*-2} \xi \frac{\partial u}{\partial y_s} \right). \quad (2.5)$$

Similarly, we have

$$\int_{\Omega} K(|y|) \left(u^{m^*-1} \frac{\partial \xi}{\partial y_s} + (m^* - 1) u^{m^*-2} \xi \frac{\partial u}{\partial y_s} \right) = \int_{\Omega} K(|y|) \frac{\partial (u^{m^*-1} \xi)}{\partial y_s} = - \int_{\Omega} u^{m^*-1} \xi \frac{\partial K(|y|)}{\partial y_s} + \int_{\partial \Omega} K(|y|) u^{m^*-1} \xi v_s. \quad (2.6)$$

A direct computation shows that

$$\begin{aligned} & \int_{\Omega} \left(\Delta^m u \frac{\partial \xi}{\partial y_s} + \Delta^m \xi \frac{\partial u}{\partial y_s} \right) \\ &= \int_{\Omega} \left(\Delta^{m-1} u \frac{\partial \Delta \xi}{\partial y_s} + \Delta^{m-1} \xi \frac{\partial \Delta u}{\partial y_s} \right) - \int_{\partial \Omega} \Delta^{m-1} u \frac{\partial^2 \xi}{\partial \nu \partial y_s} + \int_{\partial \Omega} \frac{\partial \Delta^{m-1} u}{\partial \nu} \frac{\partial \xi}{\partial y_s} \\ &\quad - \int_{\partial \Omega} \Delta^{m-1} \xi \frac{\partial^2 u}{\partial \nu \partial y_s} + \int_{\partial \Omega} \frac{\partial \Delta^{m-1} \xi}{\partial \nu} \frac{\partial u}{\partial y_s} \\ &= \int_{\Omega} \left(\Delta^l u \frac{\partial \Delta^l \xi}{\partial y_s} + \Delta^l \xi \frac{\partial \Delta^l u}{\partial y_s} \right) - \sum_{i=1}^l \int_{\partial \Omega} \Delta^{m-i} u \frac{\partial^2 \Delta^{i-1} \xi}{\partial y_s \partial \nu} + \sum_{i=1}^l \int_{\partial \Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \frac{\partial \Delta^{i-1} \xi}{\partial y_s} \\ &\quad - \sum_{i=1}^l \int_{\partial \Omega} \Delta^{m-i} \xi \frac{\partial^2 \Delta^{i-1} u}{\partial y_s \partial \nu} + \sum_{i=1}^l \int_{\partial \Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \frac{\partial \Delta^{i-1} u}{\partial y_s} \\ &= \int_{\partial \Omega} \Delta^l u \Delta^l \xi v_s - \sum_{i=1}^l \int_{\partial \Omega} \Delta^{m-i} u \frac{\partial^2 \Delta^{i-1} \xi}{\partial y_s \partial \nu} + \sum_{i=1}^l \int_{\partial \Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \frac{\partial \Delta^{i-1} \xi}{\partial y_s} \\ &\quad - \sum_{i=1}^l \int_{\partial \Omega} \Delta^{m-i} \xi \frac{\partial^2 \Delta^{i-1} u}{\partial y_s \partial \nu} + \sum_{i=1}^l \int_{\partial \Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \frac{\partial \Delta^{i-1} u}{\partial y_s}. \end{aligned} \quad (2.7)$$

This completes the proof of (2.3).

Proof of (2.4). For this formula, we multiply the two equations (2.1) and (2.2) by $\langle \nabla \xi, y - x_0 \rangle$ and $\langle \nabla u, y - x_0 \rangle$, respectively, and again subtracting the two resulted equations, we have

$$\int_{\Omega} (\Delta^m u \langle \nabla \xi, y - x_0 \rangle + \Delta^m \xi \langle \nabla u, y - x_0 \rangle) = \int_{\Omega} K(|y|) (u^{m^*-1} \langle \nabla \xi, y - x_0 \rangle + (m^* - 1) u^{m^*-2} \xi \langle \nabla \xi, y - x_0 \rangle). \quad (2.8)$$

On the other hand, we have

$$\begin{aligned} & \int_{\Omega} K(|y|) (u^{m^*-1} \langle \nabla \xi, y - x_0 \rangle + (m^* - 1) u^{m^*-2} \xi \langle \nabla \xi, y - x_0 \rangle) \\ &= \int_{\Omega} K(|y|) \langle \nabla (u^{m^*-1} \xi), y - x_0 \rangle \\ &= \int_{\partial \Omega} K(|y|) u^{m^*-1} \xi \langle \nu, y - x_0 \rangle - \int_{\Omega} u^{m^*-1} \xi \langle \nabla K(|y|), y - x_0 \rangle - N \int_{\Omega} K(|y|) u^{m^*-1} \xi, \end{aligned} \quad (2.9)$$

where ν is the outward unit normal of $\partial \Omega$ at $y \in \partial \Omega$. Moreover,

$$\begin{aligned}
& \int_{\Omega} (\Delta^m u \langle \nabla \xi, y - x_0 \rangle + \Delta^m \xi \langle \nabla u, y - x_0 \rangle) \\
&= \int_{\Omega} (\Delta^l u \Delta^l \langle \nabla \xi, y - x_0 \rangle + \Delta^l \xi \Delta^l \langle \nabla u, y - x_0 \rangle) \\
&\quad - \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle}{\partial \nu} \\
&\quad - \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} \langle \nabla u, y - x_0 \rangle + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} \langle \nabla u, y - x_0 \rangle}{\partial \nu}
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
& \int_{\Omega} \Delta^l \xi \Delta^l \langle \nabla u, y - x_0 \rangle + \int_{\Omega} \Delta^l u \Delta^l \langle \nabla \xi, y - x_0 \rangle \\
&= \int_{\Omega} \Delta^l \xi (\Delta^{l-1} \langle \nabla \Delta u, y - x_0 \rangle + 2 \Delta u) + \int_{\Omega} \Delta^l u (\Delta^{l-1} \langle \nabla \Delta \xi, y - x_0 \rangle + 2 \Delta \xi) \\
&= \int_{\Omega} \Delta^l \xi \Delta^{l-1} \langle \nabla \Delta u, y - x_0 \rangle + \int_{\Omega} \Delta^l u \Delta^{l-1} \langle \nabla \Delta \xi, y - x_0 \rangle + 2 \int_{\Omega} \Delta^l \xi \Delta^l u \\
&= \int_{\Omega} \Delta^l \xi \langle \nabla \Delta u, y - x_0 \rangle + \int_{\Omega} \Delta^l u \langle \nabla \Delta \xi, y - x_0 \rangle + 4l \int_{\Omega} \Delta^l \xi \Delta^l u \\
&= \int_{\partial\Omega} \Delta^l \xi \Delta^l u \langle \nu, y - x_0 \rangle + (4l - N) \int_{\Omega} \Delta^l \xi \Delta^l u.
\end{aligned} \tag{2.11}$$

We also have

$$\begin{aligned}
& m^* \int_{\Omega} K(|y|) u^{m^*-1} \xi \\
&= \int_{\Omega} (\Delta^m u \xi + \Delta^m \xi u) \\
&= 2 \int_{\Omega} \Delta^l \xi \Delta^l u + \sum_{i=1}^l \left(\int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \xi}{\partial \nu} + \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} u}{\partial \nu} - \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \xi - \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} u \right),
\end{aligned} \tag{2.12}$$

which gives

$$\begin{aligned}
\int_{\Omega} \Delta^l u \Delta^l \xi &= \frac{m^*}{2} \int_{\Omega} K(|y|) u^{m^*-1} \xi - \frac{1}{2} \sum_{i=1}^l \left(\int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \xi}{\partial \nu} + \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} u}{\partial \nu} \right. \\
&\quad \left. - \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \xi - \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} u \right).
\end{aligned} \tag{2.13}$$

Thus, the result follows. $\square$

Lemma 2.2. *If $m = 2l + 1$, $l \in \mathbb{N}^+$, then*

$$\begin{aligned}
\int_{\Omega} u^{m^*-1} \xi \frac{\partial K(|y|)}{\partial y_s} &= \int_{\partial\Omega} K(|y|) u^{m^*-1} \xi \nu_s - \int_{\partial\Omega} \langle \nabla \Delta^l u, \nabla \Delta^l \xi \rangle \nu_s - \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial^2 \Delta^{i-1} \xi}{\partial y_s \partial \nu} \\
&\quad + \sum_{i=1}^{l+1} \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \frac{\partial \Delta^{i-1} \xi}{\partial y_s} - \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial^2 \Delta^{i-1} u}{\partial y_s \partial \nu} + \sum_{i=1}^{l+1} \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \frac{\partial \Delta^{i-1} u}{\partial y_s}
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
 & \int_{\Omega} u^{m^*-1} \xi \langle \nabla K(|y|), y - x_0 \rangle \\
 &= \int_{\partial\Omega} K(|y|) u^{m^*-1} \xi \langle \nu, y - x_0 \rangle - \int_{\partial\Omega} \langle \nabla \Delta^l u, \nabla \Delta^l \xi \rangle \langle \nu, y - x_0 \rangle + \sum_{i=1}^{l+1} \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle \\
 &\quad - \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle}{\partial \nu} + \sum_{i=1}^{l+1} \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} \langle \nabla u, y - x_0 \rangle - \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} \langle \nabla u, y - x_0 \rangle}{\partial \nu} \\
 &\quad + \frac{N-2m}{2} \sum_{i=1}^l \left(- \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \xi}{\partial \nu} - \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} u}{\partial \nu} \right) + \frac{N-2m}{2} \sum_{i=1}^{l+1} \left(\int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \xi + \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} u \right). \quad (2.15)
 \end{aligned}$$

Proof. Proof of (2.14). Using the similar arguments as in the proof of (2.3), we have

$$\int_{\Omega} \left(-\Delta^m u \frac{\partial \xi}{\partial y_s} - \Delta^m \xi \frac{\partial u}{\partial y_s} \right) = \int_{\Omega} K(|y|) \left(u^{m^*-1} \frac{\partial \xi}{\partial y_s} + (m^* - 1) u^{m^*-2} \xi \frac{\partial u}{\partial y_s} \right). \quad (2.16)$$

And

$$\int_{\Omega} K(|y|) \left(u^{m^*-1} \frac{\partial \xi}{\partial y_s} + (m^* - 1) u^{m^*-2} \xi \frac{\partial u}{\partial y_s} \right) = \int_{\Omega} K(|y|) \frac{\partial (u^{m^*-1} \xi)}{\partial y_s} = - \int_{\Omega} u^{m^*-1} \xi \frac{\partial K(|y|)}{\partial y_s} + \int_{\partial\Omega} K(|y|) u^{m^*-1} \xi \nu_s. \quad (2.17)$$

Further computation leads to

$$\begin{aligned}
 & \int_{\Omega} \left(-\Delta^m u \frac{\partial \xi}{\partial y_s} - \Delta^m \xi \frac{\partial u}{\partial y_s} \right) \\
 &= \int_{\Omega} \left(-\Delta^{l+1} u \frac{\partial \Delta^l \xi}{\partial y_s} - \Delta^{l+1} \xi \frac{\partial \Delta^l u}{\partial y_s} \right) + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial^2 \Delta^{i-1} \xi}{\partial y_s \partial \nu} \\
 &\quad - \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \frac{\partial \Delta^{i-1} \xi}{\partial y_s} + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial^2 \Delta^{i-1} u}{\partial y_s \partial \nu} - \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \frac{\partial \Delta^{i-1} u}{\partial y_s} \\
 &= \int_{\Omega} \left\langle \nabla \Delta^l u, \nabla \Delta^l \frac{\partial \xi}{\partial y_s} \right\rangle + \int_{\Omega} \left\langle \nabla \Delta^l \xi, \nabla \Delta^l \frac{\partial u}{\partial y_s} \right\rangle + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial^2 \Delta^{i-1} \xi}{\partial y_s \partial \nu} \\
 &\quad - \sum_{i=1}^{l+1} \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \frac{\partial \Delta^{i-1} \xi}{\partial y_s} + \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial^2 \Delta^{i-1} u}{\partial y_s \partial \nu} - \sum_{i=1}^{l+1} \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \frac{\partial \Delta^{i-1} u}{\partial y_s} \\
 &= \int_{\partial\Omega} \langle \nabla \Delta^l u, \nabla \Delta^l \xi \rangle \nu_s - \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial^2 \Delta^{i-1} \xi}{\partial y_s \partial \nu} \\
 &\quad + \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \frac{\partial \Delta^{i-1} \xi}{\partial y_s} - \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial^2 \Delta^{i-1} u}{\partial y_s \partial \nu} + \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \frac{\partial \Delta^{i-1} u}{\partial y_s}. \quad (2.18)
 \end{aligned}$$

And (2.14) is proved.

Proof of (2.15). We have

$$\int_{\Omega} (-\Delta^m u \langle \nabla \xi, y - x_0 \rangle - \Delta^m \xi \langle \nabla u, y - x_0 \rangle) = \int_{\Omega} K(|y|) (u^{m^*-1} \langle \nabla \xi, y - x_0 \rangle + (m^* - 1) u^{m^*-2} \xi \langle \nabla \xi, y - x_0 \rangle). \quad (2.19)$$

It is also easy to check that

$$\begin{aligned}
 & \int_{\Omega} K(|y|)(u^{m^*-1}\langle \nabla \xi, y - x_0 \rangle + (m^* - 1)u^{m^*-2}\xi \langle \nabla \xi, y - x_0 \rangle) \\
 &= \int_{\Omega} K(|y|)\langle \nabla(u^{m^*-1}\xi), y - x_0 \rangle \\
 &= \int_{\partial\Omega} K(|y|)u^{m^*-1}\langle \nu, y - x_0 \rangle - \int_{\Omega} u^{m^*-1}\xi \langle \nabla K(|y|), y - x_0 \rangle - N \int_{\Omega} K(|y|)u^{m^*-1}\xi,
 \end{aligned} \tag{2.20}$$

where ν is the outward unit normal of $\partial\Omega$ at $y \in \partial\Omega$. Moreover,

$$\begin{aligned}
 & \int_{\Omega} (-\Delta^m u \langle \nabla \xi, y - x_0 \rangle - \Delta^m \xi \langle \nabla u, y - x_0 \rangle) \\
 &= - \int_{\Omega} (\Delta^{l+1} u \Delta^l \langle \nabla \xi, y - x_0 \rangle + \Delta^{l+1} \xi \Delta^l \langle \nabla u, y - x_0 \rangle) \\
 &+ \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle - \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle}{\partial \nu} \\
 &+ \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} \langle \nabla u, y - x_0 \rangle - \sum_{i=1}^l \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} \langle \nabla u, y - x_0 \rangle}{\partial \nu} \\
 &= \int_{\Omega} \langle \nabla \Delta^l u, \Delta^l \langle \nabla \xi, y - x_0 \rangle \rangle + \int_{\Omega} \langle \nabla \Delta^l \xi, \Delta^l \langle \nabla u, y - x_0 \rangle \rangle \\
 &+ \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle - \sum_{i=1}^{l+1} \int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \langle \nabla \xi, y - x_0 \rangle}{\partial \nu} \\
 &+ \sum_{i=1}^l \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} \langle \nabla u, y - x_0 \rangle - \sum_{i=1}^{l+1} \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} \langle \nabla u, y - x_0 \rangle}{\partial \nu}
 \end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
 & \int_{\Omega} \langle \nabla \Delta^l u, \Delta^l \langle \nabla \xi, y - x_0 \rangle \rangle + \int_{\Omega} \langle \nabla \Delta^l \xi, \Delta^l \langle \nabla u, y - x_0 \rangle \rangle \\
 &= \int_{\Omega} \langle \nabla \Delta^l u, \langle \nabla \Delta^l \xi, y - x_0 \rangle \rangle + \int_{\Omega} \langle \nabla \Delta^l \xi, \langle \nabla \Delta^l u, y - x_0 \rangle \rangle + 4l \int_{\Omega} \langle \nabla \Delta^l u, \nabla \Delta^l \xi \rangle \\
 &= \int_{\Omega} \langle \nabla \Delta^l u, \nabla \Delta^l \xi \rangle \langle \nu, y - x_0 \rangle + (4l + 2 - N) \int_{\Omega} \langle \nabla \Delta^l u, \nabla \Delta^l \xi \rangle.
 \end{aligned} \tag{2.22}$$

On the other hand, we have

$$\begin{aligned}
 m^* \int_{\Omega} K(|y|)u^{m^*-1}\xi &= - \int_{\Omega} (\Delta^m u \xi + \Delta^m \xi u) \\
 &= 2 \int_{\Omega} \langle \nabla \Delta^l u, \nabla \Delta^l \xi \rangle - \sum_{i=1}^l \left(\int_{\partial\Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \xi}{\partial \nu} + \int_{\partial\Omega} \Delta^{m-i} \xi \frac{\partial \Delta^{i-1} u}{\partial \nu} \right) \\
 &+ \sum_{i=1}^{l+1} \left(\int_{\partial\Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \xi + \int_{\partial\Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} u \right),
 \end{aligned} \tag{2.23}$$

which gives

$$\begin{aligned} \int_{\Omega} \langle \nabla \Delta^l u, \nabla \Delta^l \xi \rangle &= \frac{m^*}{2} \int_{\Omega} K(|y|) u^{m^*-1} \xi - \frac{1}{2} \left(- \sum_{i=1}^l \left(\int_{\partial \Omega} \Delta^{m-i} u \frac{\partial \Delta^{i-1} \xi}{\partial \nu} + \int_{\partial \Omega} \Delta^{m-j} \xi \frac{\partial \Delta^{i-1} u}{\partial \nu} \right) \right. \\ &\quad \left. + \sum_{i=1}^{l+1} \left(\int_{\partial \Omega} \frac{\partial \Delta^{m-i} u}{\partial \nu} \Delta^{i-1} \xi + \int_{\partial \Omega} \frac{\partial \Delta^{m-i} \xi}{\partial \nu} \Delta^{i-1} u \right) \right). \end{aligned} \quad (2.24)$$

Thus, the result follows. $\square$

3 Non-degeneracy of the bubble solutions

In this section, we first use Lemmas 2.1 and 2.2 to establish a fine estimate on the k -bubble solution u_k obtained in Theorem A. Then we prove a non-degeneracy result by using a contradiction argument. We introduce following norms by:

$$\|u\|_* = \sup_{y \in \mathbb{R}^N} |u(y)| \left(\sum_{j=1}^k \frac{\mu_k^{\frac{N-2m}{2}}}{(1 + \mu_k |y - x_{k,j}|)^{\frac{N-2m}{2} + \tau}} \right)^{-1}$$

and

$$\|u\|_{**} = \sup_{y \in \mathbb{R}^N} |u(y)| \left(\sum_{j=1}^k \frac{\mu_k^{\frac{N+2m}{2}}}{(1 + \mu_k |y - x_{k,j}|)^{\frac{N+2m}{2} + \tau}} \right)^{-1},$$

where $x_{k,j} = (r_k \cos \frac{2(i-1)\pi}{k}, r_k \sin \frac{2(i-1)\pi}{k}, 0)$, and τ is any fixed number in $(\frac{N-2m-2}{N-2m}, 1 + \theta)$, $\theta > 0$ is a small constant. Noting that $\mu_k \sim k^{\frac{N-2m}{N-2m-2}}$ and the choice of $x_{k,j}$ and τ , we find

$$\sum_{j=2}^k \frac{1}{(\mu_k |x_{k,1} - x_{k,j}|)^{\tau}} \leq \frac{Ck^{\tau}}{\mu_k^{\tau}} \sum_{j=1}^k \frac{1}{j^{\tau}} \leq \frac{C_1 k}{\mu_k^{\tau}} \leq C'.$$

It is known that (see [15]), $\lambda_1 = 1, \lambda_2 = \lambda_3 = \dots = \lambda_{N+2}$ are the eigenvalues of the following eigenvalue problem:

$$(-\Delta)^m v = \lambda U_{x_i, \mu}^{m^*-2} v, \quad \text{for } v \in D^{m,2}(\mathbb{R}^N),$$

and the corresponding linear independent eigenvectors $v_1, v_2, \dots, v_{N+2}$ are given by

$$\begin{aligned} v_1 &= U_{x_i, \mu}, \quad v_{j+1} = \frac{\partial U_{x_i, \mu}}{\partial y_j}, \quad j = 1, 2, \dots, N, \\ v_{N+2} &= x \cdot \nabla U_{x_i, \mu} + \frac{N-2m}{2} U_{x_i, \mu}. \end{aligned}$$

Let

$$\Omega_j = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^N : \left\langle \frac{(y', 0)}{|y'|}, \frac{x_{k,j}}{|x_{k,j}|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

We define the linear operator L_k by

$$L_k \xi = (-\Delta)^m \xi - (m^* - 1) K(|y|) u_k^{m^*-2} \xi. \quad (3.1)$$

Lemma 3.1. *There exists a constant $C > 0$ such that for all $y \in \mathbb{R}^N$,*

$$|u_k(y)| \leq C \sum_{j=1}^k \frac{\mu_k^{\frac{N-2m}{2}}}{(1 + \mu_k |y - x_{k,j}|)^{N-2m}}.$$

Proof. For convenience, we set $\hat{u}_k = \mu_k^{-\frac{N-2m}{2}} u_k(\mu_k^{-1}y)$, then

$$(-\Delta)^m \hat{u}_k = K(\mu_k^{-1}y) \hat{u}_k^{m^*-1}.$$

Using Green's formula, we know that

$$\hat{u}_k(y) = \int_{\mathbb{R}^N} \frac{C_{N,m}}{|z - y|^{N-2m}} K(\mu_k^{-1}z) \hat{u}_k^{m^*-1}(z) dz.$$

Recall that the bubble solution $u_k = \sum_{j=1}^k U_{x_j, \mu_k} + \omega_k$ with $\|\omega_k\|_* = O\left(\frac{1}{\mu_k^{1+\sigma}}\right)$ for some $\sigma > 0$, we find

$$\begin{aligned} |\hat{u}_k(y)| &\leq C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2m}} |\hat{u}_k^{m^*-1}(z)| dz \\ &\leq C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2m}} \left(\sum_{j=1}^k \frac{1}{(1 + |z - \hat{x}_{k,j}|)^{\frac{N-2m}{2} + \tau}} \right)^{m^*-1} dz \\ &\leq C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2m}} \sum_{j=1}^k \frac{1}{(1 + |z - \hat{x}_{k,j}|)^{\frac{N-2m}{2} + \tau + \frac{4m(\tau-\tau_1)}{N-2m}}} \left(\sum_{j=1}^k \frac{1}{(1 + |z - \hat{x}_{k,j}|)^{\tau_1}} \right)^{\frac{4m}{N-2m}} dz \\ &\leq C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2m}} \sum_{j=1}^k \frac{1}{(1 + |z - \hat{x}_{k,j}|)^{\frac{N-2m}{2} + \tau + \frac{4m(\tau-\tau_1)}{N-2m}}} dz \\ &\leq C \sum_{j=1}^k \frac{1}{(1 + |y - \hat{x}_{k,j}|)^{\frac{N-2m}{2} + \tau + \frac{4m(\tau-\tau_1)}{N-2m}}}, \end{aligned}$$

where $\hat{x}_{k,j} = \mu_k x_{k,j}$ and $\tau_1 \in \left(\frac{N-2m-2}{N-2m}, \tau\right)$. Noting that

$$\frac{N-2m}{2} + \tau + \frac{4m(\tau-\tau_1)}{N-2m} > \frac{N-2m}{2} + \tau,$$

we can choose τ_1 such that $\frac{\frac{N-2m}{2} - \tau}{\frac{4m(\tau-\tau_1)}{N-2m}}$ is not an integer.

Let $\eta = \frac{4m(\tau-\tau_1)}{N-2m}$ and $l = \left\lceil \frac{\frac{N-2m}{2} - \tau}{\frac{4m(\tau-\tau_1)}{N-2m}} \right\rceil$, then we have

$$|\hat{u}_k(y)| \leq C \sum_{j=1}^k \frac{1}{(1 + |y - \hat{x}_{k,j}|)^{\frac{N-2m}{2} + \tau + \eta}}.$$

Continuing this process, we have

$$|\hat{u}_k(y)| \leq C \sum_{j=1}^k \frac{1}{(1 + |y - \hat{x}_{k,j}|)^{\frac{N-2m}{2} + \tau + \eta + \frac{4m(\tau+\eta-\tau_2)}{N-2m}}},$$

where $\tau_2 \in \left(\frac{N-2m-2}{N-2m}, \tau + \eta\right)$. We can choose τ_2 such that $\frac{4m(\tau+\eta-\tau_2)}{N-2m} = \eta$, then we have

$$|\hat{u}_k(y)| \leq C \sum_{j=1}^k \frac{1}{(1 + |y - \hat{x}_{k,j}|)^{\frac{N-2m}{2} + \tau + 2\eta}}.$$

Repeating this process again, we have

$$|\hat{u}_k(y)| \leq C \sum_{j=1}^k \frac{1}{(1 + |y - \hat{x}_{k,j}|)^{\frac{N-2m}{2} + \tau + bl}}.$$

So

$$|\hat{u}_k(y)| \leq C \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2m}} \sum_{j=1}^k \frac{1}{(1+|z-\hat{x}_{k,j}|)^{\frac{N+2m}{2}+\tau+l\eta+\eta}} dz \leq C \sum_{j=1}^k \frac{1}{(1+|y-\hat{x}_{k,j}|)^{N-2m}}.$$

And the desired result follows immediately. $\square$

Lemma 3.2. *There exists a constant $C > 0$ such that for all $y \in \mathbb{R}^N$ and $r = 1, \dots, 2m-1$,*

$$\left| \frac{(\partial)^r u_k}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}}(y) \right| \leq C \sum_{j=1}^k \frac{\mu_k^{\frac{N-2m}{2}+j}}{(1+\mu_k|y-x_{k,j}|)^{N-2m+j}},$$

where $\sum_{s=1}^N i_s = r$.

Proof. Let $\hat{u}_k = \mu^{-\frac{N-2m}{2}} u_k(\mu_k^{-1}y)$, then

$$(-\Delta)^m \hat{u}_k = K(\mu_k^{-1}y) \hat{u}_k^{m^*-1}.$$

We have

$$\hat{u}_k(y) = \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2m}} K(\mu_k^{-1}z) \hat{u}_k^{m^*-1}(z) dz.$$

Hence,

$$\begin{aligned} \left| \frac{(\partial)^r \hat{u}_k}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}}(y) \right| &= C \left| \int_{\mathbb{R}^N} \frac{(\partial)^r}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}} \left(\frac{1}{|z-y|^{N-2m}} \right) K(\mu_k^{-1}z) \hat{u}_k^{m^*-1}(z) dz \right| \\ &\leq C \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2m+j}} |K(\mu_k^{-1}z) \hat{u}_k^{m^*-1}(z)| dz. \end{aligned}$$

Similar to the proof of Lemma 3.1, we can prove

$$\left| \frac{(\partial)^r \hat{u}_k}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}}(y) \right| \leq C \sum_{j=1}^k \frac{1}{(1+|y-\hat{x}_{k,j}|)^{N-2m+j}}.$$

Therefore,

$$\left| \frac{(\partial)^j u_k}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}}(y) \right| \leq C \sum_{j=1}^k \frac{\mu_k^{\frac{N-2m}{2}+j}}{(1+\mu_k|y-x_{k,j}|)^{N-2m+j}}. \quad \square$$

In the following, we shall prove Theorem 1.1 by using contradiction arguments. Suppose that there are $k_n \rightarrow +\infty$, satisfying

$$L_{k_n} \xi_n = 0, \quad (3.2)$$

but $\xi_n \neq 0$. Without loss of generality, we may assume $\|\xi_n\|_* = 1$ and obtain the contradictions through the following steps. Set

$$\hat{\xi}_n(y) = \mu_{k_n}^{-\frac{N-2m}{2}} \xi_n(\mu_{k_n}^{-1}y + x_{k_n,1}). \quad (3.3)$$

Lemma 3.3. *It holds*

$$\hat{\xi}_n \rightarrow b_0 \psi_0 + b_1 \psi_1, \quad (3.4)$$

uniformly in $C^m(B_R(0))$ for any $R > 0$, where b_0 and b_1 are some constants,

$$\psi_0 = \frac{\partial U_{0,\mu}}{\partial \mu} \Big|_{\mu=1}, \quad \psi_i = \frac{\partial U_{0,1}}{\partial y_i}, \quad i = 1, \dots, N.$$

Proof. In view of $|\widehat{\xi}_n| \leq C$, we may assume that $\widehat{\xi}_n \rightarrow \xi$ in $C_{\text{loc}}^m(\mathbb{R}^N)$. Then ξ satisfies

$$(-\Delta)^m \xi = (m^* - 1)U_{0,1}^{m^*-1} \xi, \quad \text{in } \mathbb{R}^N, \quad (3.5)$$

which gives

$$\xi = \sum_{i=0}^N b_i \psi_i. \quad (3.6)$$

Since ξ_n is even in y_i , $i = 2, \dots, N$, it holds $b_i = 0$, $i = 2, \dots, N$. □

We decompose

$$\xi_n(y) = b_{0,n} \mu_{k_n} \sum_{j=1}^{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \sum_{j=1}^{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} + \xi_n^*,$$

where ξ_n^* satisfies

$$\int_{\mathbb{R}^N} U_{x_{k_n,j}, \mu_{k_n}}^{m^*-2} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} \xi_n^* = \int_{\mathbb{R}^N} U_{x_{k_n,j}, \mu_{k_n}}^{m^*-2} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \xi_n^* = 0.$$

It follows from Lemma 2.3 that $b_{0,n}$ and $b_{1,n}$ are bounded.

Lemma 3.4. *It holds*

$$\|\xi_n^*\| \leq C \mu_{k_n}^{-1-\sigma}, \quad (3.7)$$

where $\sigma > 0$ is a small constant.

Proof. Since $L_{k_n} \xi_n = 0$. We can deduce that

$$\begin{aligned} L_{k_n} \xi_n^* &:= (-\Delta)^m \xi_n^* - (m^* - 1)K(|y|)u_{k_n}^{m^*-2} \xi_n^* \\ &= -(m^* - 1)(K(|y|) - 1)u_{k_n}^{m^*-2} \sum_{j=1}^{k_n} \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \\ &\quad - (m^* - 1) \sum_{j=1}^{k_n} \left(u_{k_n}^{m^*-2} - U_{x_{k_n,j}, \mu_{k_n}}^{m^*-2} \right) \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right). \end{aligned}$$

Then we can prove

$$\left\| (K(|y|) - 1)u_{k_n}^{m^*-2} \sum_{j=1}^{k_n} \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \right\|_{**} \leq \left\| (K(|y|) - 1) \left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} \right)^{m^*-1} \right\|_{**} \leq C \mu_{k_n}^{-1-\sigma}. \quad (3.8)$$

In fact, without loss of generality, we may assume $y \in \Omega_1$, then we compute

$$\left| (K(|y|) - 1) \left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} \right)^{m^*-1} \right| \leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{N+2m}} + C \left(\sum_{j=2}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{N-2m}} \right)^{\frac{N+2m}{N-2m}} =: I_1 + I_2.$$

Since $\frac{N-2m}{2} - \tau \frac{N-2m}{N+2m} > 1$, we have

$$\begin{aligned} I_2 &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \left(\frac{k_n}{\mu_{k_n}} \right)^{\frac{N+2m}{2} - \tau} \\ &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \frac{1}{\mu_{k_n}^{\frac{2}{N-2m}(\frac{N+2m}{2} - \tau)}} \\ &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \mu_{k_n}^{-1-\sigma}. \end{aligned}$$

Now, we turn to consider the first term I_1 . We split the slice Ω_1 into two parts, namely,

$$J_1 := \{y \in \Omega_1 \mid |y| - r_0 \geq \delta\}, \quad J_2 := \{y \in \Omega_1 \mid |y| - r_0 \leq \delta\},$$

where $\delta > 0$ is a fixed constant.

For the region J_1 , we have

$$|y - x_{k_n,1}| \geq ||y| - |x_{k_n,1}|| \geq ||y| - r_0| - |r_0 - |x_{k_n,1}|| \geq \frac{1}{2}\delta.$$

Hence,

$$I_1 \leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \frac{1}{\mu_{k_n}^{N+2m-\tau}} \leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \mu_{k_n}^{-1-\sigma}.$$

For the region J_2 , we have

$$|K(|y|) - 1| \leq C||y| - r_0|^2 \leq C(|y| - |x_{k_n,1}| + ||x_{k_n,1}| - r_0|)^2 \leq C(|y| - |x_{k_n,1}|)^2 + \mu_{k_n}^{-2-\sigma}.$$

Thus, for $\sigma > 0$ small such that $\frac{N+2m}{2} - \tau - 1 - \sigma \geq 2m + \frac{1}{2} - \tau - \sigma > 0$, we have

$$\begin{aligned} I_1 &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \frac{||y| - |x_{k_n,1}||^2 + \mu_{k_n}^{-2-\sigma}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} - \tau}} \\ &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} (\mu_{k_n}^{\frac{N+2m}{2} - \tau} + \mu_{k_n}^{-2-\sigma}) \\ &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \mu_{k_n}^{-1-\sigma}. \end{aligned}$$

In the following, for simplicity, without loss of generality, we assume $y \in \Omega_1$.

Case 1. $N \geq 6m$: We have

$$\begin{aligned} &\left| \left(\left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} \right)^{m^*-2} - U_{x_{k_n,1}, \mu_{k_n}}^{m^*-2} \right) \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \right| \\ &\leq C U_{x_{k_n,1}, \mu_{k_n}}^{m^*-1} \sum_{j=2}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} \\ &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \sum_{j=2}^{k_n} \frac{1}{(\mu_{k_n}|y - x_{k_n,j}|)^{\frac{N+2m}{2} - \tau}} \\ &\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \mu_{k_n}^{-1-\sigma}. \end{aligned}$$

Case 2. $N < 6m$: We have

$$\begin{aligned}
 & \left| \left(\left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} \right)^{m^*-2} - U_{x_{k_n,1}, \mu_{k_n}}^{m^*-2} \right) \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \right| \\
 & \leq C \left(U_{x_{k_n,1}, \mu_{k_n}}^{m^*-2} \sum_{j=2}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} + U_{x_{k_n,1}, \mu_{k_n}} \left(\sum_{j=2}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} \right)^{m^*-2} \right) \\
 & \leq C \left(\frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n} |y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \sum_{j=2}^{k_n} \frac{1}{(\mu_{k_n} |y - x_{k_n,j}|)^{\frac{N+2m}{2} - \tau}} \right. \\
 & \quad \left. + \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n} |y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \left(\sum_{j=2}^{k_n} \frac{1}{(\mu_{k_n} |y - x_{k_n,j}|)^{(N-2m) \frac{m+\frac{N}{2}-\tau}{4m}}} \right)^{\frac{4m}{N-2m}} \right) \\
 & \leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n} |y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \mu_{k_n}^{-1-\sigma}.
 \end{aligned}$$

So we have proved

$$\begin{aligned}
 & \left| \left(\left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} \right)^{m^*-2} - U_{x_{k_n,1}, \mu_{k_n}}^{m^*-2} \right) \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \right| \\
 & \leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n} |y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \mu_{k_n}^{-1-\sigma}.
 \end{aligned} \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}
 & \left| \sum_{j=2}^{k_n} \left(\left(\sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-2} - U_{x_{k_n,1}, \mu_{k_n}}^{m^*-2} \right) \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \right| \\
 & \leq C \sum_{j=2}^{k_n} \left(U_{x_{k_n,1}, \mu_{k_n}}^{m^*-2} + \left(\sum_{i=2}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-2} \right) U_{x_{k_n,j}, \mu_{k_n}} \\
 & \leq C \sum_{j=2}^{k_n} \left(U_{x_{k_n,1}, \mu_{k_n}}^{m^*-2} + \frac{C \mu_{k_n}^{2m}}{(1 + \mu_{k_n} |y - x_{k_n,1}|)^{4m - \frac{4m\tau}{N-2m}}} \right) U_{x_{k_n,j}, \mu_{k_n}} \\
 & \leq \frac{C \mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n} |y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau}} \mu_{k_n}^{-1-\sigma}.
 \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we obtained

$$\left| \sum_{j=1}^{k_n} \left(\left(\sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-2} - U_{x_{k_n,j}, \mu_{k_n}}^{m^*-2} \right) \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \right|_{**} \leq \mu_{k_n}^{-1-\sigma}.$$

Moreover, for $N \geq 6m$, we have

$$\left\| \sum_{j=1}^{k_n} \left(\left(\sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-2} - u_{k_n}^{m^*-2} \right) \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \right\|_{**}$$

$$\begin{aligned}
&\leq C \left\| \sum_{j=1}^{k_n} \left(\left(\sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-2} - u_{k_n}^{m^*-2} \right) U_{x_{k_n,j}, \mu_{k_n}} \right\|_{**} \\
&\leq C \left\| \sum_{j=1}^{k_n} \left(\sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-3} \omega_{k_n} U_{x_{k_n,j}, \mu_{k_n}} \right\|_{**} \\
&\leq C \left\| \left(\sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-2} \omega_{k_n} \right\|_{**} \leq C \|\omega_{k_n}\|_* \\
&\leq C \mu_{k_n}^{-1-\sigma}.
\end{aligned}$$

The penultimate term follows the fact that

$$\begin{aligned}
&\left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2}}} \right)^{\frac{4m}{N-2m}} \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau}} \right) \\
&\leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{(N-2m-\eta)\frac{4m}{N-2m} + \frac{N-2m}{2} + \tau - \eta}} \\
&= C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N+2m}{2} + \tau + 2m - \eta\frac{N+2m}{N-2m}}} \\
&\leq C \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N+2m}{2} + \tau}},
\end{aligned}$$

where $\eta = \frac{N-2m-2}{N-2m}$ and the last inequality follows the fact that $2m - \eta\frac{N+2m}{N-2m} > 0$.

For $N < 6m$, we can also prove

$$\begin{aligned}
&\left\| \sum_{j=1}^{k_n} \left(\left(\sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-2} - u_{k_n}^{m^*-2} \right) \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \right\|_{**} \\
&\leq C \left(\left\| \left(\sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-2} \omega_{k_n} \right\|_{**} + \left\| \sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} (\omega_{k_n})^{m^*-2} \right\|_{**} \right) \\
&\leq C (\|\omega_{k_n}\|_* + \|\omega_{k_n}\|_*^{m^*-2}) \\
&\leq C \mu_{k_n}^{-1-\sigma}.
\end{aligned} \tag{3.11}$$

In fact, noting that

$$\begin{aligned}
&\frac{N-2m}{2} - \frac{N-2m-2}{N-2m} \frac{N+2m}{N-2m} + \tau \frac{4m}{N-2m} - \tau \\
&\geq \frac{N-2m}{2} - \frac{N-2m-2}{N-2m} \frac{N+2m}{N-2m} + \frac{N-2m-2}{N-2m} \frac{4m}{N-2m} - \frac{N-2m-2}{N-2m} \\
&= \frac{N-2m}{2} - 2 \frac{N-2m-2}{N-2m} > 0,
\end{aligned}$$

the inequality (3.11) follows from

$$\left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2}}} \right)^{\frac{4m}{N-2m}} \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau}} \right) \leq C \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N+2m}{2} + \tau}},$$

and

$$\begin{aligned} & \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2}}} \right) \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau}} \right)^{\frac{4m}{N-2m}} \\ & \leq C \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{N - \frac{N-2m-2}{N-2m} \frac{N+2m}{N-2m} + \tau \frac{4m}{N-2m}}} \\ & \leq C \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N+2m}{2} + \tau}}. \end{aligned}$$

As a consequence, we have proved

$$\left\| \sum_{j=1}^{k_n} \left(\sum_{i=1}^{k_n} U_{x_{k_n,i}, \mu_{k_n}} \right)^{m^*-2} - u_{k_n}^{m^*-2} \left(b_{0,n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \right) \right\|_{**} \leq C \mu_{k_n}^{-1-\sigma}.$$

On the other hand, from

$$\int_{\mathbb{R}^N} U_{x_{k_n,j}, \mu_{k_n}}^{m^*-2} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} \xi_n^* = \int_{\mathbb{R}^N} U_{x_{k_n,j}, \mu_{k_n}}^{m^*-2} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r} \xi_n^* = 0$$

and Lemma 3.1, we can see that there exists $\rho > 0$, such that

$$\|L_{k_n} \xi_n^*\|_{**} \geq \rho \|\xi_n^*\|_*.$$

Thus, the result follows. $\square$

Lemma 3.5. *There exists a constant $C > 0$ such that for all $y \in \mathbb{R}^N$ and $r = 1, \dots, 2m - 1$,*

$$\left| \frac{(\partial)^r \xi_n^*}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}}(y) \right| \leq C \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2} + j}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau + j}} \mu_{k_n}^{-1-\sigma},$$

where $\sum_{s=1}^N i_s = r$.

Proof. Since

$$L_{k_n} \xi_n^* := (-\Delta)^m \xi_n^* - (m^* - 1)K(|y|)u_{k_n}^{m^*-2} \xi_n^*,$$

we have

$$\begin{aligned} \left| \frac{(\partial)^r \xi_n^*}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}}(z) \right| &= C \int_{\mathbb{R}^N} \frac{(\partial)^r}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}} \left(\frac{1}{|z - y|^{N-2m}} \right) ((m^* - 1)K(|z|)u_{k_n}^{m^*-2} \xi_n^* + L_{k_n} \xi_n^*) dz \\ &\leq C \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2m+j}} |(m^* - 1)K(|z|)u_{k_n}^{m^*-2} \xi_n^* + L_{k_n} \xi_n^*| dz. \end{aligned}$$

By using the fact $\|L_{k_n} \xi_n^*\|_{**} \leq C \mu_{k_n}^{-1-\sigma}$, we obtain

$$\int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2m+j}} |L_{k_n} \xi_n^*| dz \quad (3.12)$$

$$\begin{aligned}
&\leq C\mu_{k_n}^{-1-\sigma} \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2m+j}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|z-x_{k,j}|)^{\frac{N+2m}{2}+\tau}} dz \\
&\leq C \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+j}}{(1+\mu_{k_n}|y-x_{k,j}|)^{\frac{N-2m}{2}+\tau+j}} \mu_{k_n}^{-1-\sigma}.
\end{aligned}$$

Note that $\|\xi_n^*\|_* \leq C\mu_{k_n}^{-1-\sigma}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2m+j}} |(m^*-1)K(|y|)u_{k_n}^{m^*-2}\xi_n^*| dz \\
&\leq C\mu_{k_n}^{-1-\sigma} \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2m+j}} \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|z-x_{k,j}|)^{N-2m}} \right)^{m^*-2} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|z-x_{k,j}|)^{\frac{N-2m}{2}+\tau}} dz \\
&\leq C\mu_{k_n}^{-1-\sigma} \sum_{j=1}^N \int_{\Omega_j} \frac{1}{|z-y|^{N-2m+j}} \frac{\mu_{k_n}^{\frac{N+2m}{2}}}{(1+\mu_{k_n}|z-x_{k,j}|)^{\frac{N+2m}{2}+2m+\tau-\eta(m^*-1)}} dz \quad (3.13) \\
&\leq C\mu_{k_n}^{-1-\sigma} \sum_{j=1}^N \frac{1}{|z-y|^{N-2m+j}} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1+\mu_{k_n}|y-x_{k,j}|)^{\frac{N+2m}{2}+2m+\tau-\eta(m^*-1)}} \\
&\leq C\mu_{k_n}^{-1-\sigma} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+j}}{(1+\mu_{k_n}|y-x_{k,j}|)^{\frac{N-2m}{2}+\tau+j}}.
\end{aligned}$$

Combining (3.12) and (3.13), we obtain

$$\left| \frac{(\partial)^r \xi_n^*}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}}(y) \right| \leq C \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+j}}{(1+\mu_{k_n}|y-x_{k,j}|)^{\frac{N-2m}{2}+\tau+j}} \mu_{k_n}^{-1-\sigma}.$$

□

Lemma 3.6. *There exists a constant $C > 0$ such that for all $y \in \mathbb{R}^N$ and $r = 1, \dots, 2m-1$,*

$$\left| \frac{(\partial)^r \omega_{k_n}}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}}(y) \right| \leq C \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+j}}{(1+\mu_{k_n}|y-x_{k,j}|)^{\frac{N-2m}{2}+\tau+j}} \mu_{k_n}^{-1-\sigma},$$

where $\sum_{s=1}^N i_s = r$.

Proof. From (3.2) of [24] and similar to the proof of Lemma 3.5, we can prove

$$\left| \frac{(\partial)^r \omega_{k_n}}{(\partial)_{y_1}^{i_1} \dots (\partial)_{y_N}^{i_N}}(y) \right| \leq C \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+j}}{(1+\mu_{k_n}|y-x_{k,j}|)^{\frac{N-2m}{2}+\tau+j}} \mu_{k_n}^{-1-\sigma}.$$

□

Lemma 3.7. $\hat{\xi}_n \rightarrow 0$ uniformly in $C^m(B_R(0))$ for any $R > 0$.

Proof. The proof consists of the following steps.

Step 1. Recall that

$$\Omega_j = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{(y', 0)}{|y'|}, \frac{x_{k_n,j}}{|x_{k_n,j}|} \right\rangle \geq \cos \frac{\pi}{k_n} \right\}.$$

To prove $b_{1,n} \rightarrow 0$, we apply the identities in Lemmas 2.1 and 2.2 in the domain Ω_1 .

Case 1. m is even: By Lemma 2.1, we have

$$\int_{\mathbb{R}^N} u_{k_n}^{m^*-1} \xi_n \langle \nabla K(|y|), y \rangle = 0. \quad (3.14)$$

Thus,

$$\int_{\Omega_1} u_{k_n}^{m^*-1} \xi_n \langle \nabla K(|y|), y \rangle = 0,$$

which implies that

$$r_{k_n} \int_{\Omega_1} u_{k_n}^{m^*-1} \xi_n \frac{\partial K(|y|)}{\partial y_1} = - \int_{\Omega_1} u_{k_n}^{m^*-1} \xi_n \langle \nabla K(|y|), y - x_{k_n,1} \rangle. \quad (3.15)$$

Case 2. If m is odd: By Lemma 2.2, similar to Case 1, we also get (3.15).

Since $\nabla K(x_{k_n,1}) = O(\|x_{k_n,1} - r_0\|) = O(\mu_{k_n}^{-1-\sigma})$ and

$$\begin{aligned} \int_{\Omega_1} u_{k_n}^{m^*-1} \xi_n &= \int_{(\Omega_1)_{x_{k_n,1}, \mu_{k_n}}} \left(\mu_{k_n}^{-\frac{N-2m}{2}} u_{k_n}(\mu_{k_n}^{-1}y + x_{k_n,1}) \right)^{m^*-1} \widehat{\xi}_n \\ &= \int_{(\Omega_1)_{x_{k_n,1}, \mu_{k_n}}} U_{0,1}^{m^*-1}(b_{0,n}\psi_0 + b_{1,n}\psi_1) + O(\mu_{k_n}^{-1-\sigma}) \\ &= \int_{\mathbb{R}^N} U_{0,1}^{m^*-1}(b_{0,n}\psi_0 + b_{1,n}\psi_1) + O(\mu_{k_n}^{-1-\sigma}) \\ &= O(\mu_{k_n}^{-1-\sigma}), \end{aligned} \quad (3.16)$$

where $(\Omega_1)_{x_{k_n,1}, \mu_{k_n}} = \{y : \mu_{k_n}^{-1}y + x_{k_n,1} \in \Omega_1\}$, we find

$$\begin{aligned} \int_{\Omega_1} u_{k_n}^{m^*-1} \xi_n \frac{\partial K(|y|)}{\partial y_1} &= \int_{\Omega_1} u_{k_n}^{m^*-1} \xi_n \left(\frac{\partial K(|y|)}{\partial y_1} - \frac{\partial K(|x_{k_n,1}|)}{\partial y_1} \right) + O(\mu_{k_n}^{-2-2\sigma}) \\ &= \int_{B_{r_{k_n} \sin \frac{\pi}{k_n}}(x_{k_n,1})} u_{k_n}^{m^*-1} \xi_n \left(\frac{\partial K(|y|)}{\partial y_1} - \frac{\partial K(|x_{k_n,1}|)}{\partial y_1} \right) + O(\mu_{k_n}^{-2-2\sigma}) \\ &= \int_{B_{r_{k_n} \sin \frac{\pi}{k_n}}(x_{k_n,1})} u_{k_n}^{m^*-1} \xi_n \left\langle \nabla \frac{\partial K(|x_{k_n,1}|)}{\partial y_1}, y - x_{k_n,1} \right\rangle \\ &\quad + \frac{1}{2} \left\langle (\nabla)^2 \frac{\partial K(|x_{k_n,1}|)}{\partial y_1} (y - x_{k_n,1}), y - x_{k_n,1} \right\rangle + O(|y - x_{k_n,1}|^3) + O(\mu_{k_n}^{-2-2\sigma}) \\ &= \int_{\mathbb{R}^N} U_{0,1}^{m^*-1}(b_{0,n}\psi_0 + b_{1,n}\psi_1) \left\langle \nabla \frac{\partial K(|x_{k_n,1}|)}{\partial y_1}, \frac{y}{\mu_{k_n}} \right\rangle + \frac{1}{2} \left\langle (\nabla)^2 \frac{\partial K(|x_{k_n,1}|)}{\partial y_1} \frac{y}{\mu_{k_n}}, \frac{y}{\mu_{k_n}} \right\rangle \\ &\quad + O(\mu_{k_n}^{-2-2\sigma}) \\ &= \frac{K''(|x_{k_n,1}|)b_{1,n}}{\mu_{k_n}} \int_{\mathbb{R}^N} U_{0,1}^{m^*-1} \psi_1 y_1 + \frac{\frac{\partial \Delta K(x_{k_n,1})}{\partial y_1} b_{0,n}}{2N\mu_{k_n}^2} \int_{\mathbb{R}^N} U_{0,1}^{m^*-1} \psi_0 |y|^2 + O(\mu_{k_n}^{-2-2\sigma}). \end{aligned} \quad (3.17)$$

On the other hand, we have

$$\int_{\Omega_1} u_{k_n}^{m^*-1} \xi_n \langle \nabla K(|y|), y - x_{k_n,1} \rangle \quad (3.18)$$

$$\begin{aligned}
&= \int_{\Omega_1} u_{k_n}^{m^*-1} \xi_n \langle \nabla K(|y|) - \nabla K(|x_{k_n,1}|), y - x_{k_n,1} \rangle + O(\mu_{k_n}^{-2-2\sigma}) \\
&= \int_{B_{r_{k_n} \sin \frac{\pi}{k_n}}(x_{k_n,1})} u_{k_n}^{m^*-1} \xi_n \langle \nabla K(|y|) - \nabla K(|x_{k_n,1}|), y - x_{k_n,1} \rangle + O(\mu_{k_n}^{-2-2\sigma}) \\
&= \int_{B_{r_{k_n} \sin \frac{\pi}{k_n}}(x_{k_n,1})} u_{k_n}^{m^*-1} \xi_n \langle (\nabla)^2 K(|x_{k_n,1}|)(y - x_{k_n,1}), y - x_{k_n,1} \rangle + O(\mu_{k_n}^{-2-2\sigma}) \\
&= \int_{\mathbb{R}^N} U_{0,1}^{m^*-1} (b_{0,n} \psi_0 + b_{1,n} \psi_1) \langle (\nabla)^2 K(|x_{k_n,1}|) \mu_{k_n}^{-1} y, \mu_{k_n}^{-1} y \rangle + O(\mu_{k_n}^{-2-2\sigma}) \\
&= \frac{b_{0,n} \Delta K(|x_{k_n,1}|)}{N \mu_{k_n}^2} \int_{\mathbb{R}^N} U_{0,1}^{m^*-1} \psi_0 |y|^2 + O(\mu_{k_n}^{-2-2\sigma}).
\end{aligned}$$

Therefore, (3.17) and (3.18) give

$$b_{1,n} = - \frac{\int_{\mathbb{R}^N} U_{0,1}^{m^*-1} \psi_0 |y|^2}{\mu_{k_n} K''(|x_{k_n,1}|) \int_{\mathbb{R}^N} U_{0,1}^{m^*-1} \psi_1 y_1} \left(\frac{K''(|x_{k_n,1}|)}{r_{k_n} N} + \frac{\frac{\partial \Delta K(x_{k_n,1})}{\partial y_1}}{2N} \right) b_{0,n} + O(\mu_{k_n}^{-1-2\sigma}). \quad (3.19)$$

Step 2. Next we prove that $b_{0,n} = o(1)$.

Case 1. If m is even: We apply Pohozaev identity (2.15) in Lemma 2.1 to u_{k_n} with $x_0 = x_{k_n,1}$ and $d \sim \mu_{k_n}^{-1+\alpha}$,

where $\max\left\{\frac{2}{N}, \frac{1}{\frac{N-2m}{2} + \tau}\right\} < \alpha < \frac{2}{N-2m}$: we have

$$\begin{aligned}
&\int_{B_d(x_{k_n,1})} u_{k_n}^{m^*-1} \xi \langle \nabla K(|y|), y - x_0 \rangle \\
&= \int_{\partial B_d(x_{k_n,1})} K(|y|) u_{k_n}^{m^*-1} \xi_n \langle v, y - x_0 \rangle - \int_{\partial B_d(x_{k_n,1})} \Delta^{\frac{m}{2}} u_{k_n} \Delta^{\frac{m}{2}} \xi_n \langle v, y - x_0 \rangle \\
&\quad + \sum_{i=1}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(- \frac{\partial \Delta^{m-i} u_{k_n}}{\partial v} \Delta^{i-1} \langle \nabla \xi_n, y - x_0 \rangle + \Delta^{m-i} u_{k_n} \frac{\partial \Delta^{i-1} \langle \nabla \xi_n, y - x_0 \rangle}{\partial v} \right. \\
&\quad \left. - \frac{\partial \Delta^{m-i} \xi_n}{\partial v} \Delta^{i-1} \langle \nabla u_{k_n}, y - x_0 \rangle + \Delta^{m-i} \xi_n \frac{\partial \Delta^{i-1} \langle \nabla u_{k_n}, y - x_0 \rangle}{\partial v} \right) \\
&\quad + \frac{N-2m}{2} \sum_{i=1}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(\Delta^{m-i} u_{k_n} \frac{\partial \Delta^{i-1} \xi_n}{\partial v} + \Delta^{m-i} \xi_n \frac{\partial \Delta^{i-1} u_{k_n}}{\partial v} - \frac{\partial \Delta^{m-i} u_{k_n}}{\partial v} \Delta^{i-1} \xi_n - \frac{\partial \Delta^{m-i} \xi_n}{\partial v} \Delta^{i-1} u_{k_n} \right). \quad (3.20)
\end{aligned}$$

Similar to (3.18), we obtain

$$\text{LHS of (3.20)} = b_{0,n} \frac{\Delta K(x_{k_n,1})}{N \mu_{k_n}^2} \int U_{0,1}^{m^*-1} \psi_0 |y|^2 + O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right) = \tilde{a}_{N,m,K} \frac{b_{0,n}}{\mu_{k_n}^2} + O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right), \quad (3.21)$$

where $\tilde{a}_{N,m,K} = \frac{K''(r_0)}{N} \int U_{0,1}^{m^*-1} \psi_0 |y|^2 = - \frac{(N-2m)K''(r_0)}{2N} C_{N,m}^{m^*} \int \frac{|y|^2 (|y|^2 - 1)}{(1+|y|^2)^{N+1}}$.

In the following, we estimate the right-hand side (RHS) of (3.20).

A direct computation leads to

$$\int_{\partial B_d(x_{k_n,1})} K(|y|) u_{k_n}^{m^*-1} \xi_n \langle y - x_{k_n,1}, v \rangle = O\left(\frac{1}{(\mu_{k_n} d)^N}\right) = O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right). \quad (3.22)$$

Integrating by parts, we have

$$\begin{aligned}
 & - \int_{\partial B_d(x_{k_n,1})} \Delta^{\frac{m}{2}} u_{k_n} \Delta^{\frac{m}{2}} \xi_n \langle v, y - x_0 \rangle + \sum_{i=1}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(- \frac{\partial \Delta^{m-i} u_{k_n}}{\partial v} \Delta^{i-1} \langle \nabla \xi_n, y - x_0 \rangle \right. \\
 & \quad \left. + \Delta^{m-i} u_{k_n} \frac{\partial \Delta^{i-1} \langle \nabla \xi_n, y - x_0 \rangle}{\partial v} - \frac{\partial \Delta^{m-i} \xi_n}{\partial v} \Delta^{i-1} \langle \nabla u_{k_n}, y - x_0 \rangle + \Delta^{m-i} \xi_n \frac{\partial \Delta^{i-1} \langle \nabla u_{k_n}, y - x_0 \rangle}{\partial v} \right) \\
 & \quad + \frac{N-2m}{2} \sum_{i=1}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(\Delta^{m-i} u_{k_n} \frac{\partial \Delta^{i-1} \xi_n}{\partial v} + \Delta^{m-i} \xi_n \frac{\partial \Delta^{i-1} u_{k_n}}{\partial v} - \frac{\partial \Delta^{m-i} u_{k_n}}{\partial v} \Delta^{i-1} \xi_n - \frac{\partial \Delta^{m-i} \xi_n}{\partial v} \Delta^{i-1} u_{k_n} \right) \\
 & = - \int_{B_d(x_{k_n,1})} (-\Delta)^m u_{k_n} \langle \nabla \xi_n, y - x_{k_n,1} \rangle + (-\Delta)^m \xi_n \langle \nabla u_{k_n}, y - x_{k_n,1} \rangle - \frac{N-2m}{2} \int_{B_d(x_{k_n,1})} (-\Delta)^m u_{k_n} \xi_n \\
 & \quad + (-\Delta)^m \xi_n u_{k_n}.
 \end{aligned} \tag{3.23}$$

Define

$$I(u, v) := - \int_{B_d(x_{k_n,1})} (-\Delta)^m u \langle \nabla v, y - x_{k_n,1} \rangle + (-\Delta)^m v \langle \nabla u, y - x_{k_n,1} \rangle - \frac{N-2m}{2} \int_{B_d(x_{k_n,1})} (-\Delta)^m uv + (-\Delta)^m vu.$$

Noting that

$$u_{k_n} = \sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} + w_{k_n}$$

and

$$\xi_n = b_{0,n} \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \sum_{j=1}^{k_n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r_{k_n}} + \xi_n^*,$$

we have

$$\begin{aligned}
 I(u_{k_n}, \xi_n) &= b_{0,n} I \left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}}, \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} \right) + b_{0,n} I \left(w_{k_n}, \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} \right) \\
 & \quad + b_{1,n} I \left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}}, \sum_{j=1}^{k_n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r_{k_n}} \right) + b_{1,n} I \left(w_{k_n}, \sum_{j=1}^{k_n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r_{k_n}} \right) \\
 & \quad + I \left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}}, \xi_n^* \right) + I(w_{k_n}, \xi_n^*).
 \end{aligned} \tag{3.24}$$

We can compute that

$$\begin{aligned}
 & I \left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}}, \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} \right) \\
 &= \int_{B_d(x_{k_n,1})} \left(- \frac{N-2m}{2} \left(U_{x_{k_n,1}, \mu_{k_n}}^{m^*-1} \sum_{j=2}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} + (m^* - 1) U_{x_{k_n,1}, \mu_{k_n}}^{m^*-2} \mu_{k_n} \frac{\partial U_{x_{k_n,1}, \mu_{k_n}}}{\partial \mu_{k_n}} \sum_{j=2}^{k_n} U_{x_{k_n,j}, \mu_{k_n}} \right) \right. \\
 & \quad \left. - U_{x_{k_n,1}, \mu_{k_n}}^{m^*-1} \left\langle \nabla \mu_{k_n} \frac{\partial U_{x_{k_n,1}, \mu_{k_n}}}{\partial \mu_{k_n}}, y - x_{k_n,1} \right\rangle - (m^* - 1) U_{x_{k_n,1}, \mu_{k_n}}^{m^*-2} \mu_{k_n} \frac{\partial U_{x_{k_n,1}, \mu_{k_n}}}{\partial \mu_{k_n}} \langle \nabla U_{x_{k_n,1}, \mu_{k_n}}, y - x_{k_n,1} \rangle \right. \\
 & \quad \left. - N U_{x_{k_n,1}, \mu_{k_n}}^{m^*-1} \mu_{k_n} \frac{\partial U_{x_{k_n,1}, \mu_{k_n}}}{\partial \mu_{k_n}} \right) + O \left(\frac{1}{\mu_{k_n}^{2+\sigma}} \right)
 \end{aligned} \tag{3.25}$$

$$\begin{aligned}
&= -\frac{N-2m}{2}\mu_{k_n}\frac{\partial}{\partial\mu_{k_n}}\int_{B_d(x_{k_n,1})}U_{x_{k_n,1},\mu_{k_n}}^{m^*-1}\sum_{j=2}^{k_n}U_{x_{k_n,j},\mu_{k_n}} \\
&\quad +\mu_{k_n}\int_{\partial B_d(x_{k_n,1})}U_{x_{k_n,1},\mu_{k_n}}^{m^*-1}\frac{\partial U_{x_{k_n,1},\mu_{k_n}}}{\partial\mu_{k_n}}\langle v,y-x_{k_n,1}\rangle+O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right) \\
&= -\frac{N-2m}{2}\mu_{k_n}\int_{B_{\mu_{k_n}d}(0)}U_{0,1}^{m^*-1}\frac{\partial}{\partial\mu_{k_n}}\sum_{j=2}^{k_n}U_{\mu_{k_n}(x_{k_n,j}-x_{k_n,1}),1}+O((\mu_{k_n}d)^{-N})+O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right) \\
&= \frac{(N-2m)^2}{2}C_{N,m}\int_{B_{\mu_{k_n}d}(0)}U_{0,1}^{m^*-1}\sum_{j=2}^{k_n}\frac{1}{(\mu_{k_n}|x_{k_n,j}-x_{k_n,1}|)^{N-2m}}+O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right).
\end{aligned}$$

And from [24], we know that

$$\sum_{j=2}^{k_n}\frac{1}{(\mu_{k_n}|x_{k_n,1}-x_{k_n,j}|)^{N-2m}}=-\frac{K''(r_0)\int|y|^2U_{0,1}^{m^*}}{C_{N,m}N^2\mu_{k_n}^2\int U_{0,1}^{m^*-1}}+O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right).$$

Thus,

$$I\left(\sum_{j=1}^{k_n}U_{x_{k_n,j},\mu_{k_n}},\sum_{j=1}^{k_n}\mu_{k_n}\frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial\mu_{k_n}}\right)=\frac{a_{N,m,K}}{\mu_{k_n}^2}+O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right), \quad (3.26)$$

where

$$a_{N,m,K}=-\frac{N-2m}{2N^2}C_{N,m}^{m^*}\int\frac{|y|^2}{(1+|y|^2)^N}\frac{\int\frac{N|y|^{2-2m}}{(1+|y|^2)^{\frac{N+2m}{2}+1}}K''(r_0)}{\int\frac{1}{(1+|y|^2)^{\frac{N+2m}{2}}}}. \quad (3.27)$$

Similarly, noting that $\mu_{k_n}^{-1}\frac{\partial U_{x_{k_n,1},\mu_{k_n}}}{\partial r_{k_n}}, \left\langle \nabla\mu_{k_n}^{-1}\frac{\partial U_{x_{k_n,1},\mu_{k_n}}}{\partial r_{k_n}}, y-x_{k_n,1}\right\rangle$ is odd with respect to $(y-x_{k_n,1})_1$, we have that

$$\begin{aligned}
&I\left(\sum_{j=1}^{k_n}U_{x_{k_n,j},\mu_{k_n}},\sum_{j=1}^{k_n}\mu_{k_n}^{-1}\frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial r_{k_n}}\right) \\
&= -\int_{B_d(x_{k_n,1})}U_{x_{k_n,1},\mu_{k_n}}^{m^*-1}\left\langle \nabla\mu_{k_n}^{-1}\frac{\partial U_{x_{k_n,1},\mu_{k_n}}}{\partial r_{k_n}}, y-x_{k_n,1}\right\rangle -\int_{B_d(x_{k_n,1})}(m^*-1)U_{x_{k_n,1},\mu_{k_n}}^{m^*-2}\mu_{k_n}^{-1}\frac{\partial U_{x_{k_n,1},\mu_{k_n}}}{\partial r_{k_n}} \\
&\quad \times \langle \nabla U_{x_{k_n,1},\mu_{k_n}}, y-x_{k_n,1}\rangle -N\int_{B_d(x_{k_n,1})}U_{x_{k_n,1},\mu_{k_n}}^{m^*-1}\mu_{k_n}^{-1}\frac{\partial U_{x_{k_n,1},\mu_{k_n}}}{\partial r_{k_n}}+O\left(\frac{1}{\mu_{k_n}^{1+\sigma}}\right) \\
&= O\left(\frac{1}{\mu_{k_n}^{1+\sigma}}\right).
\end{aligned} \quad (3.28)$$

For the term $I\left(w_{k_n},\sum_{j=1}^{k_n}\mu_{k_n}\frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial\mu_{k_n}}\right)$, integrating by parts, we obtain

$$\begin{aligned}
I\left(w_{k_n},\sum_{j=1}^{k_n}\mu_{k_n}\frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial\mu_{k_n}}\right) &= -\int_{\partial B_d(x_{k_n,1})}\Delta_{\mathbb{H}^N}^mw_{k_n}\Delta_{\mathbb{H}^N}^m\sum_{j=1}^{k_n}\mu_{k_n}\frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial\mu_{k_n}}\langle v,y-x_0\rangle \\
&\quad +\sum_{i=1}^{\frac{m}{2}}\int_{\partial B_d(x_{k_n,1})}\left(-\frac{\partial\Delta^{m-i}w_{k_n}}{\partial v}\Delta^{i-1}\left\langle \nabla\sum_{j=1}^{k_n}\mu_{k_n}\frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial\mu_{k_n}}, y-x_0\right\rangle\right)
\end{aligned}$$

$$\begin{aligned}
& + \Delta^{m-i} w_{k_n} \frac{\left\langle \nabla \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}}, y - x_0 \right\rangle}{\partial v} \\
& - \frac{\partial \Delta^{m-i} \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}}}{\partial v} \Delta^{i-1} \langle \nabla w_{k_n}, y - x_0 \rangle + \Delta^{m-i} \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} \frac{\partial \Delta^{i-1} \langle \nabla w_{k_n}, y - x_0 \rangle}{\partial v} \Bigg) \\
& + \frac{N-2m}{2} \sum_{i=1}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(\Delta^{m-i} w_{k_n} \frac{\partial \Delta^{i-1} \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}}}{\partial v} + \Delta^{m-i} \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} \frac{\partial \Delta^{i-1} w_{k_n}}{\partial v} \right. \\
& \left. - \frac{\partial \Delta^{m-i} w_{k_n}}{\partial v} \Delta^{i-1} \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} - \frac{\partial \Delta^{m-i} \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}}}{\partial v} \Delta^{i-1} w_{k_n} \right).
\end{aligned}$$

As a result,

$$\begin{aligned}
& \left| I \left(w_{k_n}, \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}} \right) \right| \\
& \leq C \mu_{k_n}^{-1-\sigma} \left(\int_{\partial B_d(x_{k_n,1})} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+m}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+m}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+m}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+m}} \right. \\
& + \sum_{i=1}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(m-i)+1}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(i-1)+1}} \right. \\
& + \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(m-i)}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+2}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(i-1)+2}} \Bigg) d \\
& + \sum_{i=2}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(m-i)+1}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(i-1)}} \right. \\
& + \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(m-i)}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(i-1)+1}} \Bigg) \\
& + \sum_{i=1}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(i-1)+1}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(m-i)+1}} \right. \\
& + \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+2}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(i-1)+2}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(m-i)}} \Bigg) d \\
& + \sum_{i=2}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(i-1)+1}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(m-i)}} \right. \\
& + \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(i-1)+1}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(m-i)}} \Bigg) \\
& + \sum_{i=1}^{\frac{m}{2}} \int_{\partial B_d(x_{k_n,1})} \left(\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(m-i)}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(i-1)+1}} \right. \\
& \left. + \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(m-i)}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+1}}{(1 + \mu_{k_n} |y - x_{k_n,j}|)^{N-2m+2(i-1)+1}} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)+1}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(m-i)+1}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{N-2m+2(i-1)}} \\
& + \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(i-1)}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)+1}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{N-2m+2(m-i)+1}} \\
& + \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(i-1)+1}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2}+\tau+2(i-1)+1}} \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}+2(m-i)}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{N-2m+2(m-i)}} \Bigg) \\
& \leq C\mu_{k_n}^{-1-\sigma} \int_{\partial B_d(x_{k_n,1})} \left(\frac{\mu_{k_n}^N d}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N-2m}{2}+\tau+N}} + \frac{\mu_{k_n}^{N-1}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{\frac{N-2m}{2}+\tau+N-1}} \right) \\
& \leq C\mu_{k_n}^{-1-\sigma} (\mu_{k_n} d)^{-\left(\frac{N-2m}{2}+\tau\right)} \\
& \leq C\mu_{k_n}^{-2-\sigma}.
\end{aligned}$$

As a consequence, we obtain

$$I\left(w_{k_n}, \sum_{j=1}^{k_n} \mu_{k_n} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial \mu_{k_n}}\right) = O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right). \quad (3.29)$$

Similarly, we have

$$I\left(w_{k_n}, \sum_{j=1}^{k_n} \mu_{k_n}^{-1} \frac{\partial U_{x_{k_n,j}, \mu_{k_n}}}{\partial r_{k_n}}\right) = O\left(\frac{1}{\mu_{k_n}^{1+\sigma}}\right), \quad (3.30)$$

$$I\left(\sum_{j=1}^{k_n} U_{x_{k_n,j}, \mu_{k_n}}, \xi_n^*\right) = O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right), \quad (3.31)$$

and

$$|I(w_{k_n}, \xi_n^*)| \leq C\mu_{k_n}^{-2-\sigma} \int_{\partial B_d(x_{k_n,1})} \left(\frac{\mu_{k_n}^N d}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{N+2\tau}} + \frac{\mu_{k_n}^{N-1}}{(1 + \mu_{k_n}|y - x_{k_n,1}|)^{2\tau+N-1}} \right) \leq C\mu_{k_n}^{-2-\sigma}. \quad (3.32)$$

Combining (3.26)–(3.32) and $b_{1,n} = O\left(\frac{1}{\mu_{k_n}}\right)$ in Step 1, (3.24) becomes

$$I(u_{k_n}, \xi_n) = \frac{a_{N,m,K}}{\mu_{k_n}^2} b_{0,n} + O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right).$$

Combining (3.22)–(3.24), we have

$$\text{RHS of (3.20)} = \frac{a_{N,m,K}}{\mu_{k_n}^2} b_{0,n} + O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right). \quad (3.33)$$

As a consequence, we have that

$$\tilde{a}_{N,s,K} \frac{b_{0,n}}{\mu_{k_n}^2} + O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right) = a_{N,m,K} \frac{b_{0,n}}{\mu_{k_n}^2} + O\left(\frac{1}{\mu_{k_n}^{2+\sigma}}\right),$$

where $a_{N,m,K}$ is defined in (3.27).

A direct computation shows that $\frac{a_{N,m,K}}{\tilde{a}_{N,s,K}} = \frac{N-2m}{2}$, due to $N > 2m + 2$.

Thus, $b_{0,n} = o(1)$.

Case: m is odd. Applying (2.15) and similar to Case 1, we also have $b_{0,n} = o(1)$.

Noting that $b_{0,n} \rightarrow b_0$, $b_{1,n} \rightarrow b_1$, as $n \rightarrow +\infty$ and Lemma 3.3, the result follows. $\square$

Now, we give the proof of Theorem 1.1.

Proof. With the aid of the above prepared lemmas, it is sufficient to get a contradiction with $\|\xi_n\|_* = 1$.

In fact, we have

$$\xi_n(y) = (m^* - 1) \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2m}} K(z) u_{k_n}^{m^*-2}(z) \xi_n(z) dz \quad (3.34)$$

and

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2m}} K(z) u_{k_n}^{m^*-2}(z) \xi_n(z) dz \right| \\ & \leq C \|\xi_n\|_* \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2m}} K(z) u_{k_n}^{m^*-2}(z) \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|z - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau}} dz \\ & \leq C \|\xi_n\|_* \sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau + \theta}}, \end{aligned}$$

for some $\theta > 0$. So we obtain

$$\frac{|\xi_n(y)|}{\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau}}} \leq C \|\xi_n\|_* \frac{\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau + \theta}}}{\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau}}}.$$

Since $\xi_n \rightarrow 0$ in $B_{R\mu_{k_n}^{-1}}(x_{k_n,j})$ and $\|\xi_n\|_* = 1$, we know that

$$\frac{|\xi_n(y)|}{\sum_{j=1}^{k_n} \frac{\mu_{k_n}^{\frac{N-2m}{2}}}{(1 + \mu_{k_n}|y - x_{k_n,j}|)^{\frac{N-2m}{2} + \tau}}}$$

attains its maximum in $\mathbb{R}^N \setminus \bigcup_{j=1}^{k_n} B_{R\mu_{k_n}^{-1}}(x_{k_n,j})$. Thus,

$$\|\xi_n\|_* \leq o(1) \|\xi_n\|_*.$$

So $\|\xi_n\|_* \rightarrow 0$ as $m \rightarrow +\infty$. This is a contradiction to $\|\xi_n\|_* = 1$. $\square$

Funding information: This work was supported by NSFC (No. 11771235 and 12031015).

Conflict of interest: Authors state no conflict of interest.

References

- [1] J. Wei and S. Yan, *Infinitely many solutions for the prescribed scalar curvature problem on $\mathbb{S}^N$* , J. Funct. Anal. **258** (2010), no. 9, 3048–3081.

- [2] A. Ambrosetti, J. Garcia-Azorero, and I. Peral, *Perturbation of $-\Delta u - u^{\frac{N+2}{N-2}} = 0$, the scalar curvature problem in $\mathbb{R}^N$ and related topic*, J. Funct. Anal. **165** (1999), no. 1, 117–149.
- [3] L. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), no. 3, 271–297.
- [4] Y.Y. Li, *On $-\Delta u = K(x)u^5$ in $\mathbb{R}^3$* , Comm. Pure Appl. Math. **46** (1993), no. 3, 303–340.
- [5] C.-S Lin, *On Liouville theorem and a priori estimates for the scalar curvature equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **27** (1998), no. 1, 107–130.
- [6] C.-S Lin and S.-S Lin, *Positive radial solutions for $\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n$ and related topics*, Appl Anal. **38** (1990), no. 3, 121–159.
- [7] S. Yan, *Concentration of solutions for the scalar curvature equation on $\mathbb{R}^N$* , J. Differ. Equ. **163** (2000), no. 2, 239–264.
- [8] G. Bianchi, *Non-existence and symmetry of solutions to the scalar curvature equation*, Comm Partial Differ. Equ. **21** (1996), no. 1–2, 229–234.
- [9] J. Wei and X. Xu, *Classification of solutions of higher order conformally invariant equations*, Math. Ann. **313** (1999), no. 2, 207–228.
- [10] S. M. Paneitz, *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds (summary)*, SIGMA Symmetry Integrability Geom. Methods Appl. **4** (2008), Paper 036, 3.
- [11] T. P. Branson, *Group representations arising from Lorentz conformal geometry*, J. Funct. Anal. **74** (1987), no. 2, 199–291.
- [12] S.-Y.A. Chang and P. C. Yang, *Partial differential equations related to the Gauss-Bonnet-Chern integrand on 4-manifolds, Conformal, Riemannian and Lagrangian geometry*, Lecture Ser. **27** (2002), 1–30.
- [13] S.-Y.A. Chang and P. C. Yang, *A perturbation result in prescribing scalar curvature on $\mathbb{S}^n$* , Duke Math. J. **64** (1991), no. 1, 27–69.
- [14] T. Bartsch, M. Schneider, and T. Weth, *Multiple solutions of a critical polyharmonic equation*, J. Reine Angew. Math. **571** (2004), 131–143.
- [15] T. Bartsch, T. Weth, and M. Millem, *A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator*, Calc. Var. Partial Differ. Equ. **18** (2003), no. 3, 253–268.
- [16] D. E. Edmunds, D. Fortunato, and E. Jannelli, *Critical exponents, critical dimensions and biharmonic operator*, Arch. Ration. Mech. Anal. **112** (1990), no. 3, 269–289.
- [17] Y. Guo, J. Liu, and Y. Zhang, *Nonexistence of positive solutions for polyharmonic systems in $\mathbb{R}^N$* , Adv. Nonlinear Stud. **7** (2007), no. 3, 381–402.
- [18] Y. Guo, and T. Liu, *Large energy bubble solutions for Schrödinger equation with supercritical growth*, Adv. Nonlinear Stud. **21** (2021), no. 2, 421–445.
- [19] F. Gazzola, H.-C. Grunau, and M. Squassina, *Existence and nonexistence results for critical growth biharmonic elliptic equations*, Calc. Var. Partial Differ. Equ. **18** (2003), no. 2, 117–143.
- [20] H.-C Grunau and G. Sweers, *The maximum principle and positive principle eigenfunctions for polyharmonic equations*, Lecture Notes in Pure and Appl. Math. 194, Dekker, New York 1998, pp. 163–183.
- [21] H.-C Grunau and G. Sweers, *Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions*, Math. Ann. **307** (1997), no. 4, 588–626.
- [22] P. Pucci and J. Serrin, *A general variational identity*, Indiana Univ. Math. J. **35** (1986), no. 3, 681–703.
- [23] P. Pucci and J. Serrin, *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Pures Appl. **69** (1990), no. 1, 55–83.
- [24] Y. Guo and B. Li, *Infinitely many solutions for the prescribed curvature problem of polyharmonic operator*, Calc. Var. Partial Differ. Equ. **46** (2013), no. 3–4, 809–836.
- [25] Y. Guo, M. Musso, S. Peng, and S. Yan, *Non-degeneracy of multi-bubbling solutions for the prescribed scalar curvature equations and applications*, J. Funct. Anal. **279** (2020), no. 6, 29.
- [26] C. A. Swanson, *The best Sobolev constant*, Appl. Anal. **47** (1992), no. 4, 227–239.
- [27] Y. Y. Li, J. Wei, and H. Xu, *Multi-bump solutions of $-\Delta u = K(x)u^{\frac{n+2}{n-2}}$ on lattices in $\mathbb{R}^n$* , J. Reine Angew. Math. **743** (2018), 163–211.

Appendix

Basic estimates

In this section, we give some essential estimates which can be found in [27,1]. Define

$$g_{ij}(\gamma) := \frac{1}{(1 + |y - x_i|)^\alpha (1 + |y - x_j|)^\beta},$$

where $x_i \neq x_j$, $\alpha > 0$, $\beta > 0$ are some constants.

Lemma A.1. *For any constant γ satisfying $0 < \gamma \leq \min\{\alpha, \beta\}$, we have*

$$g_{ij}(\gamma) \leq \frac{C}{|x_i - x_j|^\gamma} \left(\frac{1}{(1 + |y - x_i|)^{\alpha+\beta-\gamma}} + \frac{1}{(1 + |y - x_i|)^{\alpha+\beta-\gamma}} \right).$$

Lemma A.2. *For any constant $\kappa > 0$ with $\kappa \neq N - 2m$, there is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2m}} \frac{1}{(1 + |z|)^{2m+\kappa}} dz \leq \frac{C}{(1 + |y|)^{\min\{\kappa, N-2m\}}}.$$

For $\kappa = N - 2m$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2m}} \frac{1}{(1 + |z|)^N} dz \leq C \frac{\max\{1, \log|y|\}}{(1 + |y|)^{\min\{\kappa, N-2m\}}}.$$