Smooth approximation of twisted Kähler-Einstein metrics

Abstract: In this article, we prove the existence of smooth approximations of twisted Kähler-Einstein metrics using the variational method.

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1 Introduction

Let \((M, \omega_0)\) be a compact Kähler manifold and \(T\) be a closed positive current. Assume that \(c_1(M, T) = 2\pi c(M) - [T]\) is a positive class and \(\omega \in c_1(M, T)\). We say that \(\omega\) is a twisted Kähler-Einstein metric if

\[
\text{Ric} \omega = \omega + T
\]

holds as currents. Twisted Kähler-Einstein metric can be considered as a generalization of Kähler-Einstein metric. The twisted term can be a current in general. If the current is the Dirac measure along a smooth divisor, the metric is the conic Kähler-Einstein metric. The existence of twisted Kähler-Einstein metric is proved in [3,8,16]. The metric \(\omega\) is obtained using the variational method, so there is little information of the metric geometry of \(\omega\). As a first step, we want to study the smooth approximation of metric \(\omega\) as shown in [13,14].

We always assume that \(T\) is a closed positive current with klt singularities. By choosing a smooth \((1,1)\)-form \(\theta\) in the same cohomological class of \(T\), we obtain

\[
T = \theta + \sqrt{-1} \partial \bar{\partial} \psi,
\]

where \(\psi\) is a quasi-psh function such that \(e^{-\psi} \in L^p(M, \omega_0)\) for some \(p > 1\). Then the following holds.

**Theorem 1.1.** Let \(\omega_0\) be a smooth Kähler metric and \(\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi\) be a twisted Kähler-Einstein metric such that \(\varphi\) is bounded. If \(T\) is smooth on an open set \(U\), then \(\varphi\) is smooth on \(U\). Moreover, if \(T\) has analytic singularity and \(\text{Aut}^0(X, T) = 0\), there exists a sequence of smooth metric \(\omega_t\) with Ricci curvature bounded from below such that \(\omega_t\) converges to \(\omega\) smoothly outside the singularity of \(T\).

The smoothness of \(\omega\) on the regular part of \(T\) is proved in Proposition 2.1. This result is essentially proved in [11] (see also Appendix B in [1]). The existence of smooth approximation is proved in Proposition 3.1 using the perturbation method in [14].
2 Regularity of twisted Kähler-Einstein metric

In this section, we prove the smoothness of $\varphi$ in the region where $T$ is smooth.

**Proposition 2.1.** Let $(M, \omega_0)$ be a compact Kähler manifold and $T$ be a closed positive current, $c(T, M) = [\omega_0]$. Assume that there exists a twisted Kähler-Einstein metric $\omega_\varphi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$ with bounded potential. If for the neighborhood $U$ of $x \in M$, $T|_U$ is smooth, then $\omega_\varphi$ is smooth on $U$.

Since $\omega_\varphi$ is a twisted Kähler-Einstein metric, it satisfies

$$ \text{Ric}(\omega_\varphi) = \omega_\varphi + T. $$

For $c(T, M) = [\omega_0]$, there is a smooth function $h$ such that

$$ \omega_0 = \text{Ric}(\omega_0) + \sqrt{-1} \partial \bar{\partial} h - \theta. $$

So we obtain

$$ \text{Ric}(\omega_\varphi) = \text{Ric}(\omega_0) + \sqrt{-1} \partial \bar{\partial}(h + \varphi + \psi), $$

which is equivalent to

$$ (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{h - \varphi - \psi} \omega_0^n $$

by adding a constant to $\varphi$. We only need to prove that $\varphi$ is smooth in the region where $\psi$ is smooth. First, we give the $C^0$-estimate. Since $e^{-\varphi} \in L^p$ and $\varphi$ is bounded, we obtain $f = e^{-h - \varphi - \psi} \in L^p$, so $C^0$-estimate is obtained by Corollary 6.9 in [9].

Next we show the $C^2$-estimate. By Theorem 9.1 in [6], we know the following:

**Theorem 2.2.** Let $\Phi$ be a quasi-psh function on compact Kähler manifold $(M, \omega_0)$ such that for a smooth $(1,1)$ form $\theta$

$$ \sqrt{-1} \partial \bar{\partial} \Phi \geq \theta. $$

Then there exists a decreasing sequence $\Phi_\varepsilon \in C^\infty(M)$ having the following properties:

(i) There exists a constant $C$ such that

$$ \sqrt{-1} \partial \bar{\partial} \Phi_\varepsilon \geq \theta - C \omega_0. $$

(ii) $\lim_{\varepsilon \to 0} \Phi_\varepsilon(x) = \Phi(x)$ for all $x \in M$.

So we have the decreasing sequences of smooth quasi-psh functions $\{\Phi_\varepsilon\}, \{\psi_\varepsilon\}$ converging to $\varphi, \psi$, respectively. Since $\varphi$ is continuous, $[\Phi_\varepsilon]$ converge to $\varphi$ in $C^0$-topology. And since

$$ |e^{-\psi} - e^{-\psi_\varepsilon}| \leq e^{-\psi}, \quad e^{-\varphi} \in L^p, $$

$e^{-\Phi_\varepsilon}$ converges to $e^{-\varphi}$ in $L^p$ norm by dominated convergence theorem. By the result of Yau [15], the equation

$$ (\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi_\varepsilon)^n = e^{h - \varphi - \psi_\varepsilon} \omega_0^n $$

has smooth solution $\varphi_\varepsilon$.

**Proposition 2.3.** Assume $\varphi_\varepsilon$ satisfies

$$ (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon)^n = e^{h - \varphi_\varepsilon - \psi_\varepsilon} \omega_0^n, $$

then $\Delta \varphi_\varepsilon = O(e^{-\psi_\varepsilon})$.

**Proof.** Write $(\Delta, \text{tr})$ and $(\Delta_{\omega_\varepsilon}, \text{tr}_{\omega_\varepsilon})$ as the Laplace operator and trace with respect to $\omega_0, \omega_\varepsilon$, and
\[ \omega_k = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_k. \]

We only need to prove

\[ \text{tr}(\omega_k) \leq A e^{-\psi}. \]

Recall the Laplace inequality for the second-order estimate in [12].

**Lemma 2.4.** If \( \tau \) and \( \tau' \) are two Kähler forms on a complex manifold, then there exists a constant \( B > 0 \) only depending on a lower bound for the holomorphic bisectional curvature of \( \tau \) such that

\[ \Delta_k \log \text{tr}(\tau') \geq - \frac{\text{tr Ric}(\tau')}{\text{tr}(\tau')} - B \text{tr} \tau. \]

It follows that

\[ \Delta_{\omega_k} \log \text{tr}(\omega_k) \geq - \frac{\text{tr Ric}(\omega_k)}{\text{tr}(\omega_k)} - B \text{tr} \omega_k. \]

On the other hand, by applying \( \sqrt{-1} \partial \bar{\partial} \log \) to (4), we obtain

\[ - \text{Ric}(\omega_k) = - \text{Ric}(\omega_0) - \sqrt{-1} \partial \bar{\partial} (h + \bar{\varphi}_k + \varphi_k) \geq - A \omega_0 - \sqrt{-1} \partial \bar{\partial} (\bar{\varphi}_k + \varphi_k), \]

then

\[ \Delta_{\omega_k} \log \text{tr}(\omega_k) \geq - \frac{A n + \Delta(\bar{\varphi}_k + \varphi_k)}{\text{tr}(\omega_k)} - B \text{tr} \omega_k. \quad (5) \]

Since \( \psi_k, \bar{\varphi}_k \) are quasi-psh functions, we have

\[ 0 \leq A \omega_0 + \sqrt{-1} \partial \bar{\partial} (\psi_k + \bar{\varphi}_k) \leq \text{tr} \omega_k(A \omega_0 + \sqrt{-1} \partial \bar{\partial} (\psi_k + \bar{\varphi}_k)) \omega_k \]

\[ \Rightarrow An + \Delta(\psi_k + \bar{\varphi}_k) \leq \left( A \text{tr} \omega_k + \Delta_k(\psi_k + \bar{\varphi}_k) \right) \text{tr} \omega_k \]

\[ \Rightarrow \Delta_{\omega_k}(\psi_k + \bar{\varphi}_k) \geq \frac{An + \Delta(\psi_k + \bar{\varphi}_k)}{\text{tr} \omega_k} - A \text{tr} \omega_k. \quad (6) \]

Actually, constants \( A \) for two inequalities can be chosen as the same. Combining (5) and (6), we obtain

\[ \Delta_{\omega_k}(\log \text{tr}(\omega_k) + \psi_k + \bar{\varphi}_k) \geq - A \text{tr} \omega_0. \]

We have \( \omega_k = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_k \), hence,

\[ n = \text{tr} \omega_0 - \Delta_k \varphi_k. \]

We deduce from (7) that

\[ \Delta_{\omega_k}(\log \text{tr}(\omega_k) + \psi_k + \bar{\varphi}_k - A_1 \varphi_k) \geq \text{tr} \omega_0 - A_2. \]

(8) on \( M \), with constants \( A_1 \) and \( A_2 \). Set

\[ H = \log \text{tr}(\omega_k) + \psi_k + \bar{\varphi}_k - A_1 \varphi_k. \]

Since \( \omega_k \) is smooth on \( X \), \( H \) achieves its maximum at some \( x_0 \) belongs to smooth part, and (8) yields

\[ \text{tr} \omega_k(\omega_0(x_0)) \leq A_2. \]

On the other hand, a trivial inequality shows that

\[ \text{tr}(\tau') \leq \left( \frac{\tau'}{\tau} \right)^n \text{tr}(\tau)^{n-1} \]

for any two Kähler forms \( \tau, \tau' \). Hence,
\[
\log\text{tr}(\omega_\varepsilon) \leq \log\left(e^{h \psi_\varepsilon - \tilde{\psi}}\right) + (n - 1) \log\text{tr}_{\omega_0}(\omega_0) \leq A_1 + A_4\left(\log\text{tr}_{\omega_0}(\omega_0)\right) - (\psi_\varepsilon + \tilde{\phi}_\varepsilon),
\]
then
\[
H \leq \sup_M H = H(x_0) \leq A_1 + A_4\left(\log\text{tr}_{\omega_0}(\omega_0)\right) - A_1 \varphi_\varepsilon \leq A_0
\]
on \(M\), which means that
\[
\log\text{tr}(\omega_\varepsilon) + \psi_\varepsilon + \tilde{\phi}_\varepsilon - A_1 \varphi_\varepsilon \leq A_0.
\]
For \(\varphi\) is bounded and \(\tilde{\phi}_\varepsilon\) converges in \(C^0\)-topology, we infer
\[
\text{tr}(\omega_\varepsilon) \leq A e^{-\psi_\varepsilon}. \quad \Box
\]
Since we have \(e^{h \psi_\varepsilon - \tilde{\psi}} \rightarrow e^{h \psi - \psi}\) in \(L^p\), it follows that \(\varphi_\varepsilon\) converges as \(\varepsilon \rightarrow 0\) to the solution \(\varphi\) of
\[
(\omega_0 + \sqrt{-1} \delta \delta \psi_\varepsilon)^n = \lambda e^{-h \psi - \varphi} \omega_0^n.
\]
So we know that \(\varphi\) satisfies as well \(|\varphi|_{C^{1,1}} \leq A e^{-\psi}\). Thus, for any neighborhood \(U\) with \(\psi\) is smooth, we have
\[
|\varphi|_{C^{2,0}} \leq C.
\]
By the Evans-Krylov theory, there is some \(a \in (0, 1)\) such that
\[
|\varphi|_{C^{2,0}} \leq C'.
\]
By applying \(\partial_t\) to equation (3), we obtain
\[
a_t \partial_t \varphi + (\partial_t \varphi) = f,
\]
where \(f = -\partial_t(h + \psi)\) is smooth on \(U\). Through Schauder interior estimate and bootstrap argument, we obtain the regularity of \(\varphi\) on \(U\). Proposition 2.1 is proved.

3 Approximate metrics with uniform Ricci lower bound

In this section, we prove the second part of Theorem 1.1 when \(\psi\) has analytic singularity, i.e., \(\psi\) is equal to \(u + \sum_{i=1}^m |f_i|^2\) locally, where \(u\) is a smooth function and \(f_i(1 \leq i \leq m)\) are some analytic functions. It is easy to see that \((e^\psi + \delta)^{-1}\) is a smooth function for any real number \(\delta > 0\) or positive smooth function \(\delta\). So we can perturb equation (3) by
\[
(\omega_0 + \sqrt{-1} \delta \delta \varphi_\varepsilon)^n = \lambda e^{-h \psi}(e^\psi + \delta e^{-K\varphi})^{-1} \omega_0^n. \tag{9}
\]
We will use the variational method to solve (9) as shown in [14].

**Proposition 3.1.** Assume \(\text{Aut}^q(M, T) = 1\), and \(\theta + K \omega_0 \geq 0\). Then there are constants \(a, b, \delta_0 > 0\) depending on \((M, \omega_0, \psi)\), such that for \(\delta < \delta_0\) (9) has a smooth solution \(\omega_\delta\) with some \(\lambda \in [a, b]\), which converges to \(\omega_\varphi\) for \(\delta\) approaching 0 outside the singularity of \(\psi\). Moreover, the Ricci curvature of \(\omega_\delta\) is greater than \(1 - K\) uniformly.

As shown in [4], define
\[
\text{PSH}_{\text{full}}(M, \omega_0) = \{\varphi \in \text{PSH}(M, \omega_0) | \lim_{j \rightarrow \infty} \int_{\varphi \leq -j} (\omega_0 + \sqrt{-1} \delta \delta \max\{|\varphi|, -j\})^n = 0\},
\]
and the Monge-Ampère energy on \(\text{PSH}_{\text{full}}(M, \omega_0)\):
\[ E(\varphi) = \frac{1}{(n+1)V} \sum_{i=0}^{n} \int_M \varphi \omega_0^i \wedge \omega_{\varphi}^{n-i}. \]

Set

\[ \mathcal{E}(M, \omega_0) = \{ \varphi \in \text{PSH}_{\text{full}}(M, \omega_0) | E(\varphi) > -\infty \} \]

and

\[ \mathcal{E}_c(M, \omega_0) = \{ \varphi \in \mathcal{E}(M, \omega_0), \sup_{M} \varphi \leq C \quad \text{and} \quad E(\varphi) \geq -C \}, \]

which is weakly compact for each \( C > 0 \).

Then, we define

\[ Q = \{ \varphi \in \mathcal{E}(M, \omega_0) | \int_M h_\delta(e^{-\varphi}) \omega^n_0 = \int_M h_\delta(1) \omega^n_0 \}, \]

where

\[ h_\delta(x) = \int_0^x e^{-h(e^\psi + \delta t^k)} dt. \]

By Lemma 6.4 of [2], we obtain

**Lemma 3.2.** The map

\[ \mathcal{E}(M, \omega_0) \rightarrow L^1(M, \omega_0) : \varphi \rightarrow e^{-\varphi} \]

is continuous. Thus, \( Q \) is a closed subset of \( \mathcal{E}(M, \omega_0) \).

We have the following two functionals on \( \mathcal{H} \):

\[ J(\varphi) = \frac{1}{V} \int_M \varphi \omega_0^n - E(\varphi), \]

\[ F_\delta(\varphi) = -E(\varphi) - \log \left( \int_M h_\delta(e^{-\varphi}) \omega^n_0 \right). \]

It is easy to see that

\[ F_\delta(\varphi) = -E(\varphi) + F_\delta(0), \quad F_\delta(0) = -\log \int_M h_\delta(1) \omega^n_0. \]

For \( \delta < 1 \), \( F_\delta(0) \) is uniformly bounded by a constant depending on \( (M, \omega_0, \psi, h) \).

**Lemma 3.3.** \( J(\varphi) \) is lower semi-continuous on \( Q \).

**Proof.** Actually, by Proposition 2.10 in [4], we know that \( J(\varphi) \) is lsc on \( \mathcal{E}(M, \omega_0) \). Since \( \mathcal{H} \) is closed subset of \( \mathcal{E}(M, \omega_0) \), the lemma is proved. \( \square \)

Now we prove the proposition. Since \( \text{Aut}^0(M, T) = 1 \), by Theorem 2.18 in [3], we know that Ding functional

\[ F_\delta(\varphi) = -E(\varphi) - \log \left( \int_M e^{-h_{-\varphi} - \psi} \omega^n_0 \right) \]
is coercive, i.e., there are some positive constants $A$ and $B$, such that

$$F_0(\varphi) \geq AJ(\varphi) - B.$$ 

Clearly, $F_0 \geq F_0$, so $F_0$ is also coercive. Choose a minimizing sequence $\{\varphi_j\}$ of $F_0$ satisfying:

$$\lim_{j \to \infty} F_0(\varphi_j) = \inf_{\varphi \in \Omega} F_0(\varphi).$$

For $j$ large sufficiently, we have

$$J(\varphi_j) \leq \frac{1}{A}(F_0(\varphi_j) + B) \leq \frac{1}{A}(F_0(0) + B) + 1. \quad (10)$$

Hence,

$$\frac{1}{V} \int_M \varphi_j \omega_0^S \leq |J(\varphi_j)| + |F_0(\varphi_j)| + |F_0(0)| \leq C(A, B, F_0(0)). \quad (11)$$

So we obtain

$$|\sup(\varphi_j)| \leq C(A, B, F_0(0)). \quad (12)$$

From (10) and (12), we know that $\varphi_j$ lies in a weakly compact subset $E^1(M, \omega_0)$ of $E^1(M, \omega_0)$. Hence, by taking a subsequence of $\{\varphi_j\}$, we can assume that $\varphi_j$ converge to a limit $\varphi_0$ in $E^1(M, \omega_0)$. From Lemma 3.3, we know that the functional $-E(\varphi)$ is lower semi-continuous. Thus, $F_0$ is lower semi-continuous. It follows that $\varphi_0$ is a minimizer of $F_0$. As the proof of Theorem 4.1 in [2], we can show that $\varphi_0$ is a solution of (9) for some $\lambda$.

Then, we give the estimate of $\lambda$. By (11), we know that

$$\int_M |\varphi_j| \omega_0^S \leq C(A, B, F_0(0), V).$$

Hence,

$$|\{e^{-\varphi} \geq C_1\}| = |\{\varphi \leq -\ln C_1\}| \leq \frac{\int_M |\varphi_j| \omega_0^S}{\ln C_1} \leq \frac{C(A, B, F_0(0), V)}{\ln C_1}.$$

So we can choose $C_1 > 0$, such that

$$|\{e^{-\varphi} \geq C_1\}| \leq \frac{V}{4}.$$

And we also can choose $\epsilon > 0$, such that

$$|\{e^\varphi \leq \epsilon\}| \leq \frac{V}{4}.$$

Set

$$N = \{e^{-\varphi} \leq C_1\} \cap \{e^\varphi \geq \epsilon\},$$

then

$$|N| \geq \frac{V}{2}.$$

On $N$, there is a $\delta(M, \omega_0, \psi)$ such that for any $\delta \leq \delta_0$, we have

$$1 \leq e^{-\varphi} \leq C_1$$

and

$$(e^\varphi + \delta e^{-\psi})^{-1} \geq \frac{1}{2} e^{-\varphi}.$$
So we obtain
\[
\int_N e^{-h} \phi \left( e^\phi + \delta e^{-K\phi} \right)^{-1} \omega^n_0 \geq C(M, \omega_0, \psi, h).
\]
Combining with perturbed equation, we obtain
\[
\lambda \leq \frac{V}{C(M, \omega_0, \psi, h)}.
\]
On the other hand, we have
\[
e^{-h} \phi \left( e^\phi + \delta e^{-K\phi} \right)^{-1} \leq h_\delta \left( e^{-\phi} \right).
\]
Hence,
\[
\int_M e^{-h} \phi \left( e^\phi + \delta e^{-K\phi} \right)^{-1} \omega^n_0 \leq \int_M h_\delta \left( e^{-\phi} \right) \omega^n_0 = \int_M h_\delta(1) \omega^n_0.
\]
So we obtain
\[
\lambda \geq \frac{V}{\int_M h_\delta(1) \omega^n_0}.
\]
Next, we establish the regularity of \( \phi_\delta \).

**Lemma 3.4.** For some \( \alpha \in (0, 1) \), \( |\phi_\delta|_{C^\alpha(M, \omega_0)} \leq C \), where \( C \) depends on \( (M, \omega_0, \psi) \).

**Proof.** From above, we know that \( \phi_\delta \in \mathcal{E}^1_C(M, \omega_0) \subset \text{PSh}_{\text{full}} \), where \( \mathcal{E}^1_C(M, \omega_0) \) is a weak compact subset. By Proposition 1.4 of [1], there is \( q > 1 \) and \( |e^{-\phi}|_{L^q} \) is uniformly bounded by constant \( C(q) \). Indeed, the map
\[
\mathcal{E}^1 \to L^q(M, \omega_0) : \phi_\delta \to e^{-\phi_\delta}
\]
is continuous. Since \( e^{-\phi} \in L^p \), so
\[
\left( \left( e^\phi + \delta e^{-K\phi} \right)^{-1} \right)_{L^p} \leq |e^{-\phi}|_{L^q} \leq C(M, \omega_0, \psi, p).
\]
Then for any \( p_0 \in (1, p) \) and some constant independent of \( \delta \) satisfies
\[
|e^{-\phi_\delta} \cdot e^{-h} \cdot \left( e^\phi + \delta e^{-K\phi} \right)^{-1}|_{L^{p_0}} \leq C.
\]
By Theorem 2.1 of [10], we have
\[
|\phi_\delta|_{C^\alpha(M, \omega_0)} \leq C.
\]

**Proposition 3.5.** There exists \( \delta_1 \to 0 \) such that \( \phi_\delta \) converges to \( \phi + c \) in the \( C^0 \)-topology for some constant \( c \).

**Proof.** By Lemma 3.4, we can choose a subsequence \( \phi_{\delta_i} \), which converges to a continuous function \( \phi_0 \). Moreover, some \( \Lambda \) for \( \phi_0 \) satisfy
\[
(\omega_0 + \sqrt{-1} \delta \phi_{\delta_i} \omega^n_0 = \lambda e^{-h} \phi_{\delta_i} \omega^n_0).
\]
Then, through the unique result of Proposition 8.2 of [5], we know that \( \phi_0 = \phi + c \).
Proposition 3.6. Let \( \varphi \) be a solution of
\[
\omega^n = e^{\psi^- - \psi^+} \omega_0^n,
\]
where \( \omega_0 = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi \) and \( \psi^\pm \) are smooth functions.

Further, we assume that there exists \( C > 0 \) such that:
(i) \( |\varphi| \leq C \);
(ii) \( |\psi^+| \leq C \) and \( \sqrt{-1} \partial \bar{\partial} \psi^+ \geq -C\omega_0 \);
(iii) \( \text{Ric}(\omega_0) \) is bounded from below by \(-C\).

Then there exists a constant \( A > 0 \) depending only on \( C \), such that
\[
\frac{1}{A} \omega_0 \leq \omega_0 \leq A\omega_0.
\]

Choose a sequence of smooth \( \omega_0 - \text{psh} \) functions \( \tilde{\varphi}_j \), which converges to \( \varphi_\delta \) in \( C^0 \) norm.

Lemma 3.7. If \( \varphi_j \) is any solution of
\[
(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_j)^n = \Lambda e^{(K-1)\tilde{\psi}_j - h} (e^{\psi^+ - \psi^-} + \delta)^{-1} \omega_0^n,
\]
then for some \( C = C(M, \delta, |\varphi_\delta|_{C^0}) \),
\[
|\Delta \varphi_j| \leq C.
\]

Proof. First, we observe that for any smooth \( f > 0 \),
\[
\sqrt{-1} \partial \bar{\partial} \log (f + \delta) \geq \frac{f}{(f + \delta)} \sqrt{-1} \partial \bar{\partial} \log f.
\]
Let
\[
u_j = \log \left( e^{\psi^+ - \psi^-} + \delta \right).
\]
Then,
\[
\sqrt{-1} \partial \bar{\partial} \nu_j \geq \frac{e^{\psi^+ - \psi^-}}{e^{\psi^+ - \psi^-} + \delta} \sqrt{-1} \partial \bar{\partial} (\psi^+ - \psi^-) \geq - \frac{e^{\psi^+ - \psi^-}}{e^{\psi^+ - \psi^-} + \delta} (\theta + K\omega_0) \geq -(\theta + K\omega_0).
\]
Since \( \theta \) is smooth, then
\[
\sqrt{-1} \partial \bar{\partial} \nu_j \geq -C\omega_0.
\]
Moreover we know \( \omega_0 - \text{psh} \) function \( \tilde{\varphi}_j \) satisfies
\[
\sqrt{-1} \partial \bar{\partial} (K\tilde{\varphi}_j) \geq -K\omega_0.
\]
The right-hand side of (13) can be written as \( e^{\psi_j^- - \psi_j^+} \), where
\[
\psi_j^+ = K\tilde{\varphi}_j; \quad \psi_j^- = \nu_j + \tilde{\varphi}_j + h.
\]
As mentioned earlier, for some constant \( C > 0 \), we have
\[
\sqrt{-1} \partial \bar{\partial} \psi_j^+ \geq -C\omega_0.
\]
Hence, by Proposition 3.6, we have |\( \Delta \varphi_j | \| \leq C_\delta. \]

It follows from the uniqueness theorem for complex Monge-Ampère equations that \( \varphi_j \) converges to \( \varphi_\delta + c \) for some constant \( c \), so we have
\[
|\varphi_j|_{C^1(M, \omega_0)} \leq C_\delta.
\]
By Evans-Krylov theory, we know that for some $a \in (0, 1)$,
\[ |\varphi_\delta|_{L^a(M, \omega_0)} \leq C_\delta, \]
where $C_\delta$ depends on $\delta$. And higher order estimates are obtained by bootstrap. So $\varphi_\delta$ is a smooth function.

Now we can calculate the Ricci curvature.

**Proposition 3.8.** Assume $\omega_\delta$ is smooth metric that satisfies (9), then
\[ \text{Ric}(\omega_\delta) \geq (1 - K)\omega_\delta. \]

**Proof.** Write (9) as follows:
\[ (\omega_\delta + \sqrt{-1}\partial\bar{\partial}\varphi_\delta)^n = \lambda e^{e^{(K-1)\varphi_\delta - \delta}(e^{(\theta + K\varphi_\delta \delta}) - 1)\omega_\delta^n}. \]

Then the $\text{Ric}(\omega_\delta)$ is equal to
\[ \sqrt{-1}((-1)(\bar{\partial}\partial\varphi_\delta + \bar{\partial}h + \partial\bar{\partial}\log(e^{\partial\bar{\partial}\varphi_\delta} + \delta)) + \text{Ric}(\omega_0)) \]
\[ \geq (1 - K)\sqrt{-1}\partial\bar{\partial}\varphi_\delta + \frac{e^{\partial\bar{\partial}\varphi_\delta}}{e^{\partial\bar{\partial}\varphi_\delta} + \delta} \sqrt{-1}\partial\bar{\partial}(\psi + K\varphi_\delta) + \omega_0 + \theta \]
\[ \geq \omega_\delta - \frac{\delta K}{e^{\partial\bar{\partial}\varphi_\delta} + \delta} \sqrt{-1}\partial\bar{\partial}\varphi_\delta + \frac{\delta}{e^{\partial\bar{\partial}\varphi_\delta} + \delta} \theta \]
\[ = \omega_\delta - \frac{\delta K}{e^{\partial\bar{\partial}\varphi_\delta} + \delta} \omega_\delta + \frac{\delta(K\omega_\delta + \theta)}{e^{\partial\bar{\partial}\varphi_\delta} + \delta} \]
\[ \geq (1 - K)\omega_\delta. \]

**Lemma 3.9.** There exists $C = C(M, \omega_0, |\varphi_\delta|_{C^2}, |h|_{C^2})$ such that
\[ \frac{1}{C} \omega_0 \leq \omega_\delta \leq C \cdot e^{-\Psi} \omega_0. \]

**Proof.** Since the Ricci curvature of $\omega_\delta$ is bounded below by $(1 - K)\omega_\delta$, by the Chern-Lu inequality, we have
\[ \Delta_\omega \log(\text{tr}_{\omega_\delta} \omega_0) \geq (K - 1) - B \text{tr}_{\omega_\delta} \omega_0, \]
where $B$ is the upper bounded of the bisectional curvature of $\omega_0$. Then we have
\[ \Delta_\omega \left( \log(\text{tr}_{\omega_\delta} \omega_0 - (B + 1)\varphi_\delta) \right) \geq \text{tr}_{\omega_\delta} \omega_0 - n(B - 1) + (K - 1). \]

So by the maximum principle, we obtain
\[ \text{tr}_{\omega_\delta} \omega_0 \leq n(B - 1) - (K - 1) \leq C. \]

Moreover, combined with (9)
\[ \text{tr}_{\omega_\delta} \omega_\delta \leq \text{tr}_{\omega_\delta} \omega_0 \cdot \frac{\omega_\delta^2}{\omega_0^2} \leq C \cdot e^{-\Psi}. \]

Then, we obtain both the upper and lower bound of $\omega_\delta$. \hfill \Box

Now $\omega_\delta$ is a sequence of smooth metrics such that $\text{Ric}(\omega_\delta) \geq (1 - K)\omega_\delta$ and the potential $\varphi_\delta$ converges to $\varphi$ in $C^0$ norm. By Lemma 3.9, we have a uniform $C^2$ estimate of $\varphi_\delta$ outside the singularity of $\psi$. Together with the Evans-Krylov theory, we know that $\varphi_\delta$ converges to $\varphi$ smoothly in the regular part. The proof of Proposition 3.1 is complete.

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References


