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A curvature flow to the $L_p$ Minkowski-type problem of $q$-capacity

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Abstract: This article concerns the $L_p$ Minkowski problem for $q$-capacity. We consider the case $p \geq 1$ and $1 < q < n$ in the smooth category by a kind of curvature flow, which converges smoothly to the solution of a Monge-Ampère type equation. We show the existence of smooth solution to the problem for $p \geq n$. We also provide a proof for the weak solution to the problem when $p \geq 1$, which has been obtained by Zou and Xiong.

Keywords: $q$-capacity, Minkowski-type problem, curvature flow

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1 Introduction

The classical Minkowski problem of convex bodies (i.e., a convex and compact subset of $\mathbb{R}^n$ with nonempty interior), developed by Minkowski, Aleksandrov, Fenchel, and others, asks for necessary and sufficient conditions in order that a given measure arises as the measure generated by a convex body. Minkowski [25] himself solved this problem for the case when the given measure is either discrete or has a continuous density. Aleksandrov [3,4] and Fenchel and Jessen [11] independently solved the problem in 1938 for arbitrary measures: If $\mu$ is not concentrated on any great subsphere of $S^{n-1}$, then $\mu$ is the surface area measure of a convex body if and only if $\int_{S^{n-1}} \xi d\mu(\xi) = 0$.

The $L_p$ Brunn-Minkowski theory is an extension of the classical Brunn-Minkowski theory. The $L_p$ surface area measure, introduced in [23], is a fundamental notion in the $L_p$-theory. $\mathcal{K}^n$ is the class of convex bodies in $\mathbb{R}^n$, and $\mathcal{K}_o^n$ is the class of convex bodies in $\mathbb{R}^n$ containing the origin in their interiors. For each $K \in \mathcal{K}^n$, the support function $h_K : S^{n-1} \to \mathbb{R}$ is defined by

$$h_K(x) = \max \{ \langle x, z \rangle : z \in K \}.$$ 

Let $\nu_K : \partial K \to S^{n-1}$ be the Gauss map of $\partial K$, namely,

$$\nu_K(z) = \{ x \in S^{n-1} : \langle z, x \rangle = h_K(x) \}.$$ 

Here, the Gauss map is defined on the subset $\partial' K$ of those points of $\partial K$ that have a unique outer unit normal. For fixed $p \in \mathbb{R}$, and a convex body $K \in \mathcal{K}_o^n$, the $L_p$ surface area measure $S^p(K, \cdot)$ of $K$ is a Borel measure on $S^{n-1}$ defined, for a Borel set $\eta \subset S^{n-1}$, by
\[ S^{(p)}(K, \eta) = \int_{\xi \in \nu(K(\eta))} (\xi \cdot v_\eta(\xi))^{1-p} \, d\mathcal{H}^{n-1}(\xi). \]

There have been a lot of meaningful results for the $L_p$ Minkowski problem. In [23], Lutwak proved that the solution to the $L_p$ Minkowski problem is unique for $p > 1$ and $p \neq n$ if $\mu$ is even and positive. In [24], Lutwak and Oliker proved the regularity of the solution to this case. When $p = -n$, it is the centroaffine Minkowski problem that was studied by Chou and Wang [7], Lu and Wang [21], Zhu [32], and Li [20]. In [7], the authors also considered the $L_p$ Minkowski problem without the evenness assumption on $\mu$ and proved the existence of the $C^2$ convex solution for the case $p \geq n$ and the weak solution for the case $1 < p < n$. More recently, Guang et al. [14] studied the super-critical case $p < -n$.

Recall that for $1 < q < n$, the $q$-capacity of a bounded convex domain $\Omega$ in $\mathbb{R}^n$ is defined as follows:

\[ C_q(\Omega) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^q \, dx : u \in C_0^\infty(\mathbb{R}^n) \text{ and } u \geq 1 \text{ on } \Omega \right\}. \]

In his celebrated article [17], Jerison solved the Minkowski problem that prescribes the electrostatic (or Newtonian) capacity measure:

\[ C_q(\Omega) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^q \, dx : u \in C_0^\infty(\mathbb{R}^n), u \geq 1 \text{ on } \Omega \right\}. \]

This work demonstrates the variational formula for the electrostatic capacity and reveals a striking similarity with the Minkowski problem for the surface area measure. Regularity was also obtained. The uniqueness was settled by Caffarelli et al. in [5]. Currently, Colesanti et al. [9] extended Jerison’s work to $q$-capacity. Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$ and $1 < q < n$. They established the Hadamard variational formula for $q$-capacity

\[
\frac{dC_q(K + tL)}{dt} \bigg|_{t=0^+} = (q-1) \int_{\mathbb{S}^{n-1}} h_L(x) \, d\mu_q(K, x),
\]

and therefore, the Poincaré $q$-capacity formula

\[ C_q(K) = \frac{q-1}{n-q} \int_{\mathbb{S}^{n-1}} h_K(x) \, d\mu_q(K, x). \]

Here, $\mu_q(K, \cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, called the electrostatic $q$-capacitary measure of $K$, defined by

\[ \mu_q(K, \eta) = \int_{\nu(K(\eta))} |\nabla U|^q \, d\mathcal{H}^{n-1}, \]

for Borel set $\eta \subset \mathbb{S}^{n-1}$, where $\nabla$ is the covariant derivative with respect to an orthonormal frame on $\mathbb{R}^n$, and $U$ is the $q$-equilibrium potential of $K$, which will be introduced in the next section.

**Definition 1.1.** Let $p \in \mathbb{R}$ and $1 < q < n$. Suppose $K$ is a convex body in $\mathbb{R}^n$ with the origin in its interior. The $L_{p,q}$-capacitary measure $\mu_{p,q}(K, \cdot)$ of $K$ is a finite Borel measure on $\mathbb{S}^{n-1}$ defined, for Borel set $\omega \subset \mathbb{S}^{n-1}$, by

\[ \mu_{p,q}(K, \omega) = \int_{\omega} h_K(x)^{1-p} \, d\mu_q(K, x). \]

Consequently, along with the $L_p$ Minkowski problem for volume, there is a parallel $L_p$ Minkowski-type problem for $q$-capacity: Suppose $\mu$ is a finite Borel measure on $\mathbb{S}^{n-1}$, $1 < q < n$, and $p \in \mathbb{R}$. What are the necessary and sufficient conditions on $\mu$ so that $\mu$ is the $L_{p,q}$-capacitary measure $\mu_{p,q}(K, \cdot)$ of a convex body $K$ in $\mathbb{R}^n$? Namely,
\[ \text{d} \mu_{p,q}(K, \cdot) = \text{d} \mu. \]  \hfill (1.2)

If \( \mu \) is absolutely continuous with respect to \( \sigma_{n-1} \) and \( f = \frac{\text{d} \mu}{\text{d} \sigma_{n-1}} \), then (1.2) is reduced to solving a Monge-Ampère type equation:

\[ \det(\nabla^2 h_K + h_K I) = fh_K^{-q} |\nabla U(\nu_K^n)|^{-q} \quad \text{on } S^{n-1}, \] \hfill (1.3)

where \( I \) denotes the identity matrix, and \( \nabla \) is the covariant derivative with respect to an orthonormal frame on \( S^{n-1} \). Throughout this article, we say that \( h \in C^2(S^{n-1}) \) is uniformly convex if the matrix \( \{\nabla^2 h + hI\} \) is positive-definite.

As mentioned earlier, Jerison [17] solved the problem for the classical case \( p = 1 \) and \( q = 2 \). For general measures, the \( L_p \) Minkowski problem for \( q \)-capacity was studied by Colesanti et al. [9] and Akman et al. [1], for \( p = 1 \) and \( 1 \leq q < n \) by Zou and Xiong [33] and for \( p > 1 \) and \( 1 \leq q < n \) by Hong et al. [15]. Moreover, Xiong and Xiong [29] established the continuity of the solutions for \( p > 1 \) and \( 1 < q < n \), and Hong et al. [15] studied the corresponding Orlicz Minkowski problem for \( 1 < q < n \). However, when \( p < 1 \), only discrete results have been obtained by Xiong et al. [28] for \( 0 < p < 1 \) and \( 1 < q < 2 \) and by Xiong and Xiong [30] for the logarithmic case \( p = 0 \) with \( 1 < q < n \). The continuous case for \( p < 1 \) is still open although it is important.

These results mentioned earlier are all for \( 1 < q < n \), while Akman et al. [2] extend the \( q \)-index to \( q \geq n \) and solved the Minkowski problem for \( q \)-capacity. Furthermore, Lu and Xiong [22] developed the \( L_p \) Minkowski problem for \( 0 < p < 1 \) and \( q \geq n \) under the discrete condition. Xiong and Xiong [31] solved the corresponding Orlicz Minkowski problem for \( q > n \), which is an extension of the \( L_p \) Minkowski problem for \( p > 1 \).

The aforementioned results mainly used the variational method. In this article, we consider the case \( p \geq 1 \) and \( 1 \leq q < n \) in the smooth category by a kind of curvature flow.

**Theorem 1.2.** Let \( 1 < q < n \), \( f \in C^\infty(S^{n-1}) \) and \( f > 0 \), then

(i) If \( p > n \), then there is a smooth and uniformly convex body \( \Omega \in \mathcal{K}_n^q \) satisfying (1.3).

(ii) If \( p = n > 2 \), then there is a smooth and uniformly convex body \( \Omega \in \mathcal{K}_n^q \) satisfying (1.3).

Our main result is the smooth solution for \( p \geq n \). For the completeness of the article, we also provide a proof for the weak solution when \( \mu \) is a non-zero finite Borel measure, which is not concentrated on a closed hemisphere (abbreviated as \( \mu \in NCH \)) and \( p \geq 1 \).

**Theorem 1.3.** Let \( 1 < q < n \) and \( \mu \in NCH \), then

(i) If \( p > n \), then there is a convex body \( \Omega \in \mathcal{K}_n^q \) satisfying (1.2).

(ii) If \( p = n \), then there is a convex body \( \Omega \in \mathcal{K}_n^q \) satisfying (1.2).

(iii) If \( 1 \leq p < n \), and \( p \neq n - q \), then there is a convex body \( \Omega \in \mathcal{K}_n^q \) satisfying (1.2). If \( p = n - q \), then there exists a number \( \lambda > 0 \) and a convex body \( \Omega \in \mathcal{K}_n^q \) satisfying (1.2) with \( \mu \) replaced by \( \lambda \mu \).

When \( p \geq n \) and \( \mu \in NCH \), the weak solution \( \Omega \) contains the origin in its interior, which has been shown in [33] by adapting an argument from Hug et al. [16]. In this article, we omit this step.

For \( 1 \leq p \leq n \), our proof is inspired by [6]. Similarly given \( \varepsilon \in (0, 1) \) small enough, we introduce a function on \([0, \infty)\) that satisfies

\[ \tilde{F}_\varepsilon(s) := \begin{cases} \frac{1}{p}s^p, & \text{if } s \geq 2\varepsilon, \\ \frac{s^{\alpha_\varepsilon}}{n + \varepsilon}, & \text{if } 0 \leq s \leq \varepsilon. \end{cases} \] \hfill (1.4)

For \( s \in (\varepsilon, 2\varepsilon) \), we suitably choose \( \tilde{F}_\varepsilon(s) \) so that it is smooth and strictly increasing on \([0, \infty)\). Let \( F_\varepsilon(s) = \tilde{F}_\varepsilon(s) \) be the derivative of \( \tilde{F}_\varepsilon \). Then \( F_\varepsilon \) is a smooth function in \((0, \infty)\) that satisfies
\[ F(s) = \begin{cases} s^{p-1}, & \text{if } s \geq 2\varepsilon, \\ s^{n-1+\varepsilon}, & \text{if } 0 < s \leq \varepsilon, \end{cases} \] (1.5)

and \( F(s) > 0 \) for all \( s > 0 \). To be precise, let us assume
\[ \frac{1}{2} s^{n-1+\varepsilon} \leq F(s) \leq 2s^{p-1}, \quad \text{for } s \in (\varepsilon, 2\varepsilon). \]

Given a positive function \( f \in C^{\infty}(\mathbb{S}^{n-1}) \), and a smooth, closed and uniformly convex hypersurface \( M_0 = \partial \Omega_0 \) with \( \Omega_0 \in \mathcal{K}_n^{\alpha} \). We consider the flow
\[
\begin{aligned}
\frac{\partial X}{\partial t}(x, t) &= -f(v(X, v)F_s((X, v))K(v) + \eta(t), X, \\
X(x, 0) &= X_0(x),
\end{aligned}
\] (1.6)

where \( X_0 \) is the parametrization of \( M_0 \), \( v \), and \( K \) are the unit outer normal and Gauss curvature of \( M_t \) at \( X(x, t) \), respectively, \( U(t, \cdot) \) is the \( q \)-equilibrium potential of \( M_t \), and
\[ \eta(t) = \frac{\int_{\mathbb{S}^{n-1}} f(x)h(x, t)F_s(h(x, t))dx}{\sum_{s=1}^{2} h(x, t)d\mu_q(\Omega_t, x)}. \] (1.7)

Accordingly, consider the following functional
\[ J_s(h(\cdot, t)) = \int_{\mathbb{S}^{n-1}} \tilde{F}_s(h(x, t))f(x)dx - \frac{1}{n-q} \int \eta(t)h(x, t)d\mu_q(\Omega_t, x). \] (1.8)

We will show later that \( J_s(h(\cdot, t)) \) is strictly monotone along the flow (1.6), and \( \frac{d}{dt}J_s(h(\cdot, t)) = 0 \) if and only if \( h(\cdot, t) \) solves the elliptic equation:
\[ \det(\nabla^2 h + hI) = \frac{f(x)F_s(h)}{\eta(t)|\nabla U(\nabla h, t)|^q}. \] (1.9)

For \( p > n \), we only need to take \( \tilde{F}_s(s) = \frac{1}{p}s^p \), \( s > 0 \) in the aforementioned process.

**Theorem 1.4.** Let \( 1 < q < n \), \( p \geq 1 \), and \( K \) and \( L \) be bounded convex domains in \( \mathbb{R}^n \) of class \( C^{1,\alpha}_e \). If they have the same \( L_{p,q} \)-capacitary measure, then
(i) for \( p = 1 \), \( K \) and \( L \) are translates when \( q \neq n-1 \), and homothetic when \( q = n-1 \).
(ii) for \( p > 1 \), \( K = L \).

This article is organized as follows. In Section 2, we introduce some basic properties of the \( q \)-equilibrium potential and show the reason why we choose the functional \( J_s(h(\cdot, t)) \) and \( \eta(t) \). In Section 3, we give some a priori estimates of the flow (1.6), which implies the long time existence and uniqueness of the smooth solution to the flow. In Section 4, we prove the main conclusions (Theorems 1.2 and 1.3). The uniqueness has been proved in [9] for \( p = 1 \) and [33] for \( p > 1 \). For the completeness of our article, we also list their proof briefly in Section 5.

### 2 Preliminaries

We first introduce some basic properties of convex hypersurfaces in \( \mathbb{R}^n \). Let \( \Omega \in \mathcal{K}_n^{\alpha} \) be a convex body, and \( M = \partial \Omega \) be a smooth, closed, and uniformly convex hypersurface in \( \mathbb{R}^n \). Since the support function \( h = h_{\Omega} \) of \( \Omega \) can also be written as follows:
\[ h(x) = \langle x, v_\Omega(x) \rangle, \]
it is apparently that
\[ v_Ω^1(x) = \nabla h + hx = \nabla h. \]
The radial function \( r = r_Ω : S^{n-1} \rightarrow \mathbb{R} \) is defined by
\[ r(\xi) = \max\{\lambda : \lambda \xi \in K\}. \]
Then, if we define \( v_Ω^{-1}(x) = r(\xi)\xi \), it can be seen that
\[ r^2(\xi) = (\nabla h(x))^2 + h(x)^2. \tag{2.1} \]
The principal radii of \( M \) at \( v_Ω^{-1}(x) \) are the eigenvalues of the matrix \( \{b_{ij}\} \), where
\[ b_{ij} = h_{ij} + h\delta_{ij}, \]
in a local coordinates system on sphere \( S^{n-1} \). Then, the Gauss curvature of \( M \) at \( v_Ω^{-1}(x) \) is given by
\[ K = \frac{1}{\det(\nabla^2 h + hI)}. \]
Moreover, by some simple calculation (see [12,27]), we have
\[ x = \frac{r_Ω^2 - \nabla r}{\sqrt{r^2 + |\nabla r|^2}}, \quad h = \frac{r^2}{\sqrt{r^2 + |\nabla r|^2}}, \quad g_{ij} = r^2\delta_{ij} + r\eta_j, \quad h_{ij} = \frac{r^2\delta_{ij} + 2r\eta_j - r\eta_j}{\sqrt{r^2 + |\nabla r|^2}}, \tag{2.2} \]
where \( g_{ij} \) and \( h_{ij} \) are, respectively, the metric and second fundamental forms of \( M \) in terms of the radial function.

The variational structure of the definition of \( q \)-capacity leads naturally to the formulation of the problem:
\[
\begin{cases}
\Delta_q U = 0 & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \\
U = 1 & \text{on } \partial \Omega, \\
U(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty,
\end{cases}
\tag{2.3}
\]
where \( U \) is called the \( q \)-equilibrium potential of \( \Omega \), and \( \Delta_q \) is the \( q \)-Laplace operator defined by
\[ \Delta_q U = \nabla (|\nabla U|^{q-2} \nabla U). \]
For any \( 0 < b < 1, y \in \partial \Omega \), the non-tangential cone is defined as follows:
\[ \Gamma(y) = \{x \in \mathbb{R}^n \setminus \bar{\Omega} : d(x, \partial \Omega) > b|x - y|\}. \]
According to [9], the following property can be obtained.

**Lemma 2.1.** [9] Suppose \( 1 < q < n \), and let \( \Omega \subset \mathbb{R}^n, n \geq 2 \), be a bounded convex domain. Then
\[ \nabla U(y) = \lim_{x \rightarrow y, x \in \Gamma(y)} \nabla U(x), \]
exists for \( \mathcal{H}^{n-1} \) almost all \( y \in \partial \Omega \). Moreover, for \( \mathcal{H}^{n-1} \) almost all \( y \in \partial \Omega \),
\[ |\nabla U(y)| = -|\nabla U(y)|v_Ω(y), \]
and \( |\nabla U| \in L^q(\partial \Omega, \mathcal{H}^{n-1}) \).

Hence, the \( q \)-capacitary measure \( \mu_Ω(Q, \cdot) \) of \( \Omega \) is well defined. By Lewis’ work [19] (see also [8]), we can conclude the following.

**Theorem 2.2.** [19] Suppose \( 1 < q < n \), and let \( \Omega \subset \mathbb{R}^n, n \geq 2 \), be a bounded convex domain. Then there exists a unique weak solution \( U \) to (2.3) satisfying the following:
(1) \( U \in C^\infty(\mathbb{R}^n \setminus \bar{\Omega}) \cap C(\mathbb{R}^n \setminus \bar{\Omega}) \).
(2) \(0 < U < 1\) and \(|\nabla U| \neq 0\) in \(\mathbb{R}^n \backslash \bar{\Omega}\).

(3) \(C_q(\Omega) = \int_{\mathbb{R}^n} |\nabla U|^q dx\).

(4) If \(U\) is defined to be 1 in \(\Omega\), then \(\Omega_t = \{x \in \mathbb{R}^n : U(x) > t\}\) is convex for each \(t \in [0, 1]\) and \(\partial \Omega_t\) is a \(C^{\infty}\) manifold for \(0 < t < 1\).

In particular, when \(\Omega\) is a ball of radius \(R\), problem (2.3) has a unique solution

\[ u(x) = \left( \frac{R}{|\hat{\nu}|} \right)^{n-q}. \]

By a simple calculation, we can obtain

\[ C_q(B_R) = \omega_n \left( \frac{n - q}{q - 1} \right)^{q-1} R^{n-q}, \]

where \(\omega_n\) denotes the surface area of the unit sphere in \(\mathbb{R}^n\).

We next show that our flow keep the \(q\)-capacity. Multiplying the both sides of the parabolic flows (1.6) by the unit outer normal of \(\partial M_t\), we obtain the following evolution equation:

\[
\begin{aligned}
\frac{\partial h}{\partial t}(x, t) &= - \frac{f(x)h(x, t)E(h)}{|\nabla U(\nabla h, t)|^q} K(x, t) + \eta_t(x)h(x, t) \quad \text{in} \quad S^{n-1} \times (0, T), \\
h(x, 0) &= h_0(x),
\end{aligned}
\]

where \(h_0(x)\) is the support function of the initial hypersurface \(M_0\).

**Lemma 2.3.** Let \(1 < q < n\), \(1 \leq p \leq n\), and \(M_t = X(S^{n-1}, t)\) be a smooth, closed, and uniformly convex solution to the flow (1.6). Suppose that the origin lies in the interior of the convex body \(\Omega_t\) enclosed by \(M_t\) for all \(t \in [0, T]\). Then

\[ C_q(\Omega_t) = C_q(\Omega_0), \quad \forall t \in [0, T). \]

**Proof.** By Hadmard variational formula (1.1), the evolution equation (2.5), and the change of the variation formula

\[ d\mu_q(\Omega_t, x) = \frac{|\nabla U(\nabla h, t)|^q}{K(x, t)} dx, \]

we have

\[
\begin{aligned}
\frac{d}{dt} C_q(\Omega_t) &= (q - 1) \int_{S^{n-1}} h(x, t) d\mu_q(\Omega_t, x) \\
&= (q - 1) \int_{S^{n-1}} \left( - \frac{f(x)h(x, t)E(h)}{|\nabla U(\nabla h, t)|^q} K(x, t) + \eta_t(x)h(x, t) \right) d\mu_q(\Omega_t, x) \\
&= (q - 1) \left( - \int_{S^{n-1}} f(x)h(x, t)E(h) dx + \eta_t(x) \int_{S^{n-1}} h(x, t) d\mu_q(\Omega_t, x) \right) = 0.\]
\]

**Lemma 2.4.** Let \(1 < q < n\), \(1 \leq p \leq n\), then functional \(J_q(h\cdot, t)\) given by (1.8) is non-increasing, namely,

\[ \frac{d}{dt} J_q(h\cdot, t) \leq 0, \quad \forall t \in [0, T). \]

**Proof.** As \(C_q(\Omega_t)\) remains unchanged, we obtain
\[
\frac{d}{dt} J_t(h(\cdot,t)) = \int_{S^{n-1}} f(x) F_t(h) h_t dx \\
= \int_{S^{n-1}} f(x) F_t(h) \left( - \frac{f(x) h(x, t) F_t(h)}{|\nabla U(\nabla h, t)|^q} K(x, t) + \eta_t(t) h(x, t) \right) dx \\
= \left( \int_{S^{n-1}} h \mu_q \right)^{-1} \left[ \left( \int_{S^{n-1}} f F_t(h) h dx \right)^2 - \left( \int_{S^{n-1}} f^2 h^2 F_t(h) h dx \right) \int_{S^{n-1}} h dx \mu_q \right] \\
\leq 0,
\]

where the last inequality is due to the Hölder’s inequality and (2.6). Moreover, the equality holds if and only if
\[
\frac{f^2 h^2 F_t^2(h)}{|\nabla U(\nabla h, t)|^q} K = c^2(t) h^2 \frac{|\nabla U(\nabla h, t)|^q}{K},
\]
that is,
\[
\frac{F_t(h)}{|\nabla U(\nabla h, t)|^q} K = c(t),
\]
for some function \(c(t)\).

Indeed, if (2.7) occurs, by (1.7) and (2.6),
\[
\eta_t(t) = \frac{\int_{S^{n-1},c} c(t) h(x, t) d\mu_q(\Omega_t, x)}{\int_{S^{n-1},h} h(x, t) d\mu_q(\Omega_t, x)} = c(t). \tag{2.8}
\]

\[\square\]

3 A priori estimates

We first show the uniformly lower and upper bounds of the solution to flow (1.6). Let \(T\) be the maximal time such that the non-degenerated, smooth, and uniformly convex solution to the flow (1.6) exists.

**Lemma 3.1.** Let \(1 < q < n, 1 \leq p \leq n, f \in C^\infty(S^{n-1}), f > 0, \) and \(h_0 \in C^\infty(S^{n-1})\) be a positive and uniformly convex function. Let \(M_t = X(S^{n-1}, t)\) be a smooth and uniformly convex solution to the flow (1.6). Then there is a positive constant \(C\) depending only on \(n, p, q, f\) and the initial hypersurface, but independent of \(\varepsilon\) such that
\[
\max_{S^{n-1}} h(\cdot, t) \leq C, \quad \forall t \in [0, T), \tag{3.1}
\]
and
\[
\max_{S^{n-1}} |\nabla h(\cdot, t) | \leq C, \quad \forall t \in [0, T). \tag{3.2}
\]

**Proof.** Let \(\Omega_t\) be the convex body whose support function is \(h(\cdot, t)\). By virtue of Lemma 2.4 and equation (1.8), we have
\[
J_t(X(\cdot, 0)) + (n - q) C_q(\Omega_0) \geq J_t(X(\cdot, t)) + (n - q) C_q(\Omega_t) = \int_{S^{n-1}} F_t(h(x, t)) f(x) dx.
\]

Let \(x_t \in S^{n-1}\) be a unit vector such that \(h(x_t, t) = \max_{S^{n-1}} h(\cdot, t)\). We may assume \(h(x_t, t) > 10\). Then, as \(\varepsilon\) can be chosen small enough, we infer from Lemma 2.6 in [6]:
\[
h(x, t) \geq \frac{1}{2} h(x_t, t) > 2\varepsilon, \quad \forall x \in S^{n-1} \text{ with } x \cdot x_t \geq \frac{1}{2}.
\]
It is clear that $\tilde{E}(s) \geq 0$. Hence,
\[
\int_{S^n} \tilde{E}(h(x, t)) f(x) \, dx \geq \frac{1}{p} \int_{\{x \in S^{n-1} : x \cdot x_i \geq 1/2\}} \left( \frac{1}{2} h(x_i, t) \right)^p f(x) \, dx \\
\geq h(x_i, t)/C_t,
\]
where $C_t$ is a positive constant depending only on $p$ and the bounds of $f$ in $\{x \in S^{n-1} : x \cdot x_i \geq 1/2\}$. Apparently, $h(., t)$ has a uniform upper bound for $t < T$.

By virtue of (2.1) and $\max_{S^{n-1}} h = \max_{S^{n-1}} \tilde{r}$, we have
\[
\max_{S^{n-1}} \nabla |h| \leq \max_{S^{n-1}} r.
\]

Hence, we finish the proof. \qed

In [10], Evans and Gariepy found that the $q$-capacity $C_q$ is increasing with respect to the inclusion of sets and positively homogeneous of order $(n - q)$. That is, if $E \subset F$, then $C_q(E) \leq C_q(F)$ and $C_q(sE) = s^{n-q} C_q(E)$, for $s > 0$. Also, it is rigid invariant, i.e., $C_q(L(E) + x) = C_q(E)$, for $x \in \mathbb{R}^n$ and each affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$. Then we can estimate the uniform bounds of $\eta_t$.

**Lemma 3.2.** Let $1 < q < n$, $1 \leq p \leq n$, $f \in C^\infty(S^{n-1}), f > 0$, and $h_0 \in C^\infty(S^{n-1})$ be a positive and uniformly convex function. Let $M_1 = X(S^{n-1}, t)$ be a smooth and uniformly convex solution to the flow (1.6). Then there is a positive constant $C$ depending only on $n, p, q, f$ and the initial hypersurface, but independent of $\varepsilon$ such that
\[
\frac{1}{C} \leq \eta_t(t) \leq C, \quad \forall t \in [0, T).
\]

**Proof.** Let $x_i \in S^{n-1}$ be a unit vector such that $h(x_i, t) = \max_{S^{n-1}} h(., t)$. By Lemma 3.1, $\Omega_t$ can be enclosed by a sphere of radius $h(x_i, t)$. Clearly, it implies
\[
C_q(\Omega_t) \leq C_q(B_{h(x_i, t)}) = \omega_n \left( \frac{n - q}{q - 1} \right)^{q-1} (h(x_i, t))^{n-q}.
\]
This shows that $\max_{S^{n-1}} h(., t)$ has a lower bound independent of $t$, and we may assume $h(x_i, t) > 10$. Then
\[
h(x, t) \geq \frac{1}{2} h(x_i, t) > 2\varepsilon, \quad \forall x \in S^{n-1} \text{ with } x \cdot x_i \geq 1/2,
\]
for some $\varepsilon$ small enough. Therefore,
\[
\eta_t(t) = \left( \frac{n - q}{q - 1} \right)^{-1} \int_{S^{n-1}} f(x) h(x, t)E_t(h(x, t)) \, dx \\
\geq \frac{1}{C} \int_{\{x \in S^{n-1} : x \cdot x_i \geq 1/2\}} f(x) h(x, t) h^{p-1}(x) \, dx \\
\geq \frac{1}{C} \int_{\{x \in S^{n-1} : x \cdot x_i \geq 1/2\}} \left( \frac{1}{2} h(x_i, t) \right)^p f(x) \, dx \\
\geq h(x_i, t)/C \\
\geq \frac{1}{C'},
\]
where $C$ is a positive constant depending only on $n, p, q, f$, and the initial hypersurface.

On the other hand,
where the last inequality is an immediate consequence of Lemma 3.1. □

Lemma 3.3. Let $1 < q < n$, $1 \leq p \leq n$, $f \in C^0(S^{n-1})$, $f > 0$, and $h_0 \in C^0(S^{n-1})$ be a positive and uniformly convex function. Let $M_t = X(S^{n-1}, t)$ be a smooth and uniformly convex solution to the flow (1.6). Then there is a positive constant $C_c$ depending only on $n$, $p$, $q$, $f$, $\varepsilon$, and the initial hypersurface, such that

$$
\min_{S^n} h(\cdot, t) \geq V C_c, \quad \forall t \in [0, T).
$$

(3.6)

Proof. Since we have already obtained the uniform upper bound of $\Omega$, generated by $M_t$, and $M_t$ is smooth. We can infer from Lemma 2.18 in [9] that there exists $C$ depending only on $n$, $q$ and the uniform upper bound of $\Omega$, such that

$$
|\nabla U| \geq C^{-1}, \quad \text{on } \partial \Omega_t \times [0, T).
$$

Let $\min_{S^n} h(\cdot, t) = h(x_t, t)$. If $h(x_t, t) > \varepsilon$, the lower bound is obvious. So we may assume $h(x_t, t) < \varepsilon$. At the point $(x_t, t)$, we have

$$
\nabla h = 0, \quad \nabla^2 h \geq 0.
$$

Consequently, at this point,

$$
h_t \geq -f h^{p+1} |\nabla U|^q h^{-n} + \eta_x h \geq -f h^{p+1} C_q + \eta_x h = f h C_q \left( h^\varepsilon + \frac{\eta_x}{f C_q} \right).
$$

This implies either $h_t(x_t, t) \geq 0$, or

$$
h(x_t, t) \geq \left( \frac{\eta_x}{f C_q} \right)^{\frac{1}{\varepsilon}}.
$$

(3.7)

Therefore,

$$
h(x, t) \geq \min \left\{ \varepsilon, h(\cdot, 0), \left( \frac{\eta_x}{f C_q} \right)^{\frac{1}{\varepsilon}} \right\}.
$$

☐

Now we have obtained the uniform lower and upper bounds of the convex bodies generated by $M_t = X(S^{n-1}, t)$. For every fixed $y \in \partial \Omega_t$, there exists a ball $B$ included in $\partial \Omega_t$ and tangent to $\partial \Omega_t$ at $y$. Let $\bar{U}$ be the $q$-equilibrium potential of $B$. Then we have $U(\cdot, t) \geq \bar{U}(\cdot)$ on $\partial \Omega_t$. By the comparison principle, we obtain $U(\cdot, t) \geq \bar{U}(\cdot)$ in $\mathbb{R}^n \setminus \bar{\Omega}_t$. Since $U(y, t) = \bar{U}(y)$, we obtain that

$$
|\nabla U(y, t)| \leq |\nabla \bar{U}(y)|.
$$

It is easy to conclude that

$$
|\nabla U(\nabla h(x, t), t)| \leq C, \quad \forall (x, t) \in S^{n-1} \times [0, T).
$$

Moreover, by virtue of Schauder’s theory (see, e.g., Lemmas 6.4 and 6.17 in [13]), there is a constant $C$, independent of $t$, satisfying that

$$
|\nabla^k U(\nabla h(x, t), t)| \leq C, \quad \forall (x, t) \in S^{n-1} \times [0, T),
$$

for all integer $k \geq 2$.  


Lemma 3.4. Let \( 1 < q < n, 1 \leq p \leq n, f \in C^\infty(S^{n-1}), f > 0, \) and \( h_0 \in C^\infty(S^{n-1}) \) be a positive and uniformly convex function. Let \( M_t = X(S^{n-1}, t) \) be a smooth and uniformly convex solution to the flow (1.6). Then there is a positive constant \( C_\varepsilon \) depending only on \( n, p, q, f, \varepsilon, \) and the initial hypersurface, such that
\[
K(x, t) \leq C_\varepsilon.
\]

**Proof.** Consider the auxiliary function
\[
W(x, t) = \frac{-\partial_i h(x, t) + \eta_i(t) h(x, t)}{f(x)[h(x, t) - \varepsilon_0]}, \tag{3.8}
\]
where
\[
\varepsilon_0 = \frac{1}{2^{n-1}} \inf_{x \in S^{n-1}} h(x, t)
\]
is a positive constant. It suffices to show
\[
\max_{x \in S^{n-1}} W(x, t) \leq C, \quad \forall t \in [0, T).
\]

For each \( t \in [0, T), \) assume \( W(x_t, t) = \max_{x \in S^{n-1}} W(x, t). \) Take a normal coordinates at \((x_t, t), \) we have at this point that
\[
0 = W_i = \frac{-\partial_j h_i + \eta_i h_j}{f(h - \varepsilon_0)} = \frac{-\partial_i h + \eta_i h}{f(h - \varepsilon_0)^2} h_i, \tag{3.9}
\]
\[
0 \geq W_{ij} = \frac{-\partial_i h_j + \eta_i h_j}{f(h - \varepsilon_0)} + \frac{(\partial_i h - \eta_i h)}{f(h - \varepsilon_0)^2} h_{ij}. \tag{3.10}
\]
Subsequent estimates are all processed at this point. Let \( b_{ij} = h_{ij} + h \delta_{ij}, \) and \( b^{ij} \) be its inverse matrix. Then, \( K = \det(b_{ij}), \) by (3.9) and (3.10), we obtain
\[
\partial_i K = -K b^{ij} \partial_i b_{ij}
\]
\[
= -K b^{ij} (\partial_i h_{ij} + \partial_i h \delta_{ij})
\]
\[
\leq -K b^{ij} (\partial_i h_{ij} - Wf h_{ij} + \partial_i h \delta_{ij})
\]
\[
= -K b^{ij} (\eta_i h_{ij} - Wf h_{ij} - Wf(h - \varepsilon_0) \delta_{ij})
\]
\[
= -K b^{ij} (\eta_i b_{ij} - Wf b_{ij} + Wf \varepsilon_0 \delta_{ij})
\]
\[
= -K \eta_i (n - 1) + K Wf(n - 1) - K Wf \varepsilon_0 \text{tr}(b^{ij}).
\]

Notice that \( W \) can also be written as follows:
\[
W(x, t) = \frac{E_i(h)[\nabla U]^{-q} K}{(h - \varepsilon_0)}, \tag{3.11}
\]
and by virtue of the previous estimate, we can easily obtain
\[
\frac{1}{C_1} W(x, t) \leq K(x, t) \leq C_1 W(x, t), \tag{3.12}
\]
where \( C_1 \) is a positive constant depending only on \( \varepsilon_0, \) the upper and lower bounds of \( h \) on \( S^{n-1}, \) and \( |\nabla U| \) on \( \partial \Omega. \) Combining with the fact that
\[
\frac{1}{n-1} \text{tr}(b^{ij}) \geq \det(b^{ij})^{\frac{1}{n-1}} = K^{\frac{1}{n-1}},
\]
we have
\[
\partial_i K \leq -C_1^{-1} \eta_i (n - 1) + C_1 Wf(n - 1) - C_1^{-1} Wf \varepsilon_0 (n - 1) K^{\frac{1}{n-1}} \leq C_2 W^0 - C_3 W^\frac{3}{n-1}, \tag{3.13}
\]
where $C_2$ and $C_3$ are some positive constants depending only on $n$, $\varepsilon_0$, $C_1$, the upper and lower bounds of $f$ on $\mathbb{S}^{n-1}$, and the constant $C$ in Lemma 3.2.

Recalling that
\[
|\nabla U| = -\nabla U \cdot x, \quad \dot{U} = \frac{\partial U}{\partial t} = |\nabla U|\partial_t h, \quad \forall x \in \mathbb{S}^{n-1},
\] (3.14)
(for more details, one can refer to Lemmas 2.13 and 3.1 in [9]) combining with (3.8) and (3.9), we have
\[
\partial_t|\nabla U| = \partial_t((-\nabla U \cdot x)
= -\nabla^2 U \cdot \nabla(\partial_t h) - \nabla \dot{U} \cdot x
= -\nabla^2 U \cdot (\eta \quad Wf)\nabla h + \nabla(|\nabla U|)\nabla(\partial_t h) \cdot x
= -\nabla^2 U \cdot (\eta \quad Wf)\nabla h + |\nabla U|^{-1}\nabla^2 U \cdot \nabla U(\eta h - Wf(h - \varepsilon_0)) + |\nabla U|(\eta \quad Wf)\nabla h \cdot x.
\]
Then,
\[
|\partial_t h| + |\partial_t(|\nabla U|)| = |\eta h - Wf(h - \varepsilon_0)| + |\partial_t(|\nabla U|)| \leq C_4(1 + W),
\] (3.15)
where $C_4$ is a positive constant depending only on $\varepsilon_0$, the upper bounds of $f$ on $\mathbb{S}^{n-1}$, $|h|$ and $|\nabla h|$ on $\mathbb{S}^{n-1}$, $|\nabla U|$ and $|\nabla^2 U|$ on $\partial \Omega_1$, and the constant $C$ in Lemma 3.2.

Now we can estimate $\partial_t W$ with the help of (3.12), (3.13), and (3.15),
\[
\partial_t W = \partial_t \left( \frac{F_3(h)|\nabla U|^{-k}}{(h - \varepsilon_0)} \right)
= \frac{F_3(h)|\nabla U|^{-k} - qF_3(|\nabla U|^{1-k}|\nabla U|K + F_3(|\nabla U|^{-k}|\nabla U|K \partial_t K)}{(h - \varepsilon_0)^2}
\leq C_5|\partial_t h| + |\partial_t(|\nabla U|)|K + C_6\partial_t K
\leq C_7|\partial_t h| + (C_7C_6(1 + W) + C_6(C_7W^2 - C_6W^{p+1}))
\leq C_7|\partial_t h| + (C_7C_6 + C_7W^2 - C_6W^{p+1}),
\] (3.16)
where $C_7$ and $C_8$ are some positive constants depending only on $\varepsilon_0$, the upper and lower bounds of $h$ on $\mathbb{S}^{n-1}$, and $|\nabla U|$ on $\partial \Omega_1$. It is clear that if $W(x_t, t)$ is sufficiently large, we have $\partial_t W < 0$, which implies $W$ has a uniform upper bound.

\[\text{Lemma 3.5. Let } 1 < q < n, 1 \leq p \leq n, f \in C^\infty(\mathbb{S}^{n-1}), f > 0, \text{ and } h_0 \in C^\infty(\mathbb{S}^{n-1}) \text{ be a positive and uniformly convex function. Let } M_t = X(\mathbb{S}^{n-1}, t) \text{ be a smooth and uniformly convex solution to the flow (1.6). Then there is a positive constant } C_q \text{ depending only on } n, p, q, f, \varepsilon, \text{ and the initial hypersurface, such that the principal curvature of } X(\cdot, t) \text{ are bounded from below}
\]
\[
\kappa_i(x, t) \geq \frac{1}{C_q}, \quad \forall(x, t) \in \mathbb{S}^{n-1} \times [0, T),
\]
for $i = 1, \ldots, n - 1$.

\[\text{Proof. Let } b_{ij} = h_j + h^2h_j \text{ as before, } \{b_{ij}\} \text{ be the inverse of the matrix } \{b_{ij}\}, \text{ and } \lambda_{\max}(b_{ij}) \text{ be the maximal eigenvalue of the matrix } \{b_{ij}\}. \text{ Consider the auxiliary function}
\]
\[
w(x, t) = \log \lambda_{\max}(b_{ij}) - A \log h + B|\nabla h|^2, \quad \forall(x, t) \in \mathbb{S}^{n-1} \times [0, T),
\]
where $A$ and $B$ are some large constants to be determined later. It suffices to prove that $w$ has a uniform upper bound, which further implies the conclusion. Fix an arbitrary $T' \in (0, T)$, and assume $w$ attains its maximum at $(x_0, t_0) \in \mathbb{S}^{n-1} \times [0, T']$. By rotating coordinates properly at this point, we can further assume $\{b_{ij}(x_0, t_0)\}$ is diagonal and $\lambda_{\max}(b_{ij}(x_0, t_0)) = b_{11}(x_0, t_0)$. Now the auxiliary function becomes
\[
w(x, t) = \log b_{11} - A \log h + B|\nabla h|^2, \quad \forall(x, t) \in \mathbb{S}^{n-1} \times [0, T').
Hence, we have, at \((x_0, t_0)\),

\[
0 = w_i = b^{11} b_{11;i} - A \frac{h_i}{h} + 2B \sum_k h_k h_{k;i},
\]

(3.17)

\[
0 \geq w_{ij} = b^{11} b_{11;ij} - (b^{11})^2 b_{11;i} b_{11;j} - A \left( \frac{h_{ij}}{h} - \frac{h_i h_j}{h^2} \right) + 2B \sum_k (h_k h_{k;i} + h_k h_{k;j}).
\]

(3.18)

We also have

\[
0 \leq \partial_t w = b^{11} \partial_t b_{11} - A \frac{\partial_t h}{h} + 2B \sum_k h_k \partial_t h_k.
\]

(3.19)

The evolution equation (2.5) can be written as follows:

\[
\log(\eta, h - \partial_t h) = \log K + \log \frac{\rho F}{|\nabla U|^q}.
\]

(3.20)

Set

\[
\phi = \log f + \log h + \log F(h) - q \log|\nabla U|.
\]

By differentiating both sides of equation (3.20), we obtain

\[
\eta, h_k - \partial_t h_k = -b^{1i} b_{1;i, k} + \phi_k,
\]

(3.21)

\[
\eta, h_{11} - \partial_t h_{11} = \frac{(\eta, h_1 - \partial_t h_1)^2}{(\eta, h_1 - \partial_t h_1)^2} - b^{1i} b_{1;i,11} + b^{1i} b^{1i}(b_{1;i})^2 + \phi_{11}.
\]

(3.22)

By the Ricci identity, we have

\[b_{1;i,11} = b_{11;i} - \delta_j b_{11} + \delta_{11} b_{ij} - \delta_{ij} b_{11} + \delta_{ij} b_{11}.
\]

It is direct to calculate

\[b^{11} \partial_t h_{11} + \partial_t \frac{\eta, h_{11} - \eta, h - \eta, h + \partial_t h}{\eta, h - \partial_t h} = b^{11} \partial_t h_{11} - \eta, h_{11} + \eta, b_{11} - \eta, h + \partial_t h
\]

\[= b^{11} \frac{(\eta, h_1 - \partial_t h_1)^2}{(\eta, h_1 - \partial_t h_1)^2} + b^{1i} b^{1i} b_{1;i,11} - b^{1i} b^{1i} b^{1i}(b_{1;i})^2 - b^{1i} b^{1i} + \eta, h_{11} - \partial_t h_{11} - b^{11}
\]

\[\leq b^{1i} b^{1i} b_{1;i,11} - b_{11} + b_{ij} - b^{1i} b^{1i} b^{1i}(b_{1;i})^2 - b^{11} b^{1i} + \frac{\eta, h_{11}}{\eta, h - \partial_t h} - b^{11}
\]

\[= b^{11} b^{1i} b_{1;i,11} - b^{1i} b^{1i} b^{1i}(b_{1;i})^2 - \sum_i b^{1i} + (n - 2 - \phi_{11}) b^{11} + \frac{\eta, h_{11}}{\eta, h - \partial_t h}.
\]

(3.23)

Employing (3.18), and noticing \(b_{1;i,k}\) is symmetric in all indices, we have

\[b^{11} b^{1i} b_{1;i,11} - b^{1i} b^{1i} b^{1i}(b_{1;i})^2 = b^{11}(b^{1i} b_{1;i,11} - (b^{1i})^2(b_{1;i})^2)
\]

\[\leq b^{1i} \left( \frac{h_{ii}}{h} - \frac{(h_i)^2}{h^2} \right) - b^{1i} 2B \sum_k ((h_k)^2 + h_k h_{k;i})
\]

\[= Ab^{1i} h_{ii} - h_{ii} + Ab^{1i} (h_i)^2 \frac{h}{h^2} - 2B b^{1i} (h_k) - 2B b^{1i} (b_{1;i} - h_k h_{k;i})
\]

\[= A \left( n - 1 \right) h - A \sum_i b^{1i} - Ab^{1i} (h_i)^2 \frac{h}{h^2} - 2B b^{1i} (b_{1;i} - h) - 2B b^{1i} h_k h_{k;i}
\]

\[+ 2B b^{1i} (h_k)^2
\]

\[= A \left( n - 1 \right) h - A + 2B h^2 \sum_i b^{1i} - Ab^{1i} (h_i)^2 \frac{h}{h^2} - 2B b^{1i} h_{1;i} + 4B(n - 1) h
\]

\[- 2B b^{1i} h_k h_{k;i} + 2B b^{1i} (h_k)^2.
\]

(3.24)
From (3.21), we further calculate

\[
A \frac{\partial h}{h(\eta_h - \partial h)} = A \frac{\partial h - \eta_h + \eta_h}{h(\eta_h - \partial h)} = -\frac{A \eta_h}{\eta_h - \partial h},
\]

\[
2B \sum_k \frac{h_k \partial_i h_k}{\eta_i h - \partial h} = 2B \sum_k \frac{h_k (\partial_i h_k - \eta_i h_k + \eta_i h_k)}{\eta_i h - \partial h} = 2B \sum_k \left( h_k b^{ii} b_{ii;k} - h_k \phi_k \right) + \frac{2B|\nabla h|}{\eta_i h - \partial h}. \tag{3.25}
\]

Now by dividing (3.19) by $\eta_i h - \partial h$ and using (3.23), (3.24), and (3.25), we have

\[
0 \leq b^{ii} \partial_i h_{ii} + \partial h - A \frac{\partial h}{h(\eta_h - \partial h)} + 2B \sum_k \frac{h_k \partial_i h_k}{\eta_i h - \partial h}
\]

\[
= \frac{A(n-1)}{h} - (A + 2Bh^2) \sum_i b^{ii} - Ab^{ii}(\frac{h_i^2}{h^2}) - 2B \sum_i b^{ii} + 4B(n-1)h
\]

\[
= \frac{A(n-1)}{h} - (A + 2Bh^2 + 1) \sum_i b^{ii} - (A - 2Bh^2) b^{ii}(\frac{h_i^2}{h^2}) - 2B \sum_i b^{ii} + 4B(n-1)h
\]

\[
+ (n - 2 - \phi_{ii}) b^{ii} - \frac{A(n-1)\eta_e - 2B|\nabla h|}{\eta_i h - \partial h} - 2B \sum_k h_k \phi_k.
\]

For any fixed $B$, we choose $A$ large enough such that $A - 2Bh^2 \geq 0$ and $(A - 1)\eta_e - 2B|\nabla h| \geq 0$. Then, we can infer from the aforementioned inequality that

\[
(A - n + 1) \sum_i b^{ii} + 2B \sum_i b^{ii} \leq C_1(A + B) - \phi_{ii} b^{ii} - 2B \sum_k h_k \phi_k, \tag{3.26}
\]

where $C_1$ is a positive constant depending only on $n$ and the upper and lower bounds of $h$ on $\mathbb{S}^{n-1}$.

Let $e^1, e^2, \ldots, e^{n-1}$ be an orthonormal frame on $\mathbb{S}^{n-1}$, and by Gauss formula on $\mathbb{S}^{n-1}$, we deduce that

\[
|\nabla U| = (-\nabla U \cdot x)_i = -\nabla U \cdot e^i - \nabla^2 U (x(h_i \cdot e^i + h_{\alpha})) = -\nabla^2 U \cdot e^i b_{\alpha i}, \tag{3.27}
\]

\[
|\nabla U| = -\nabla^3 U e^k \cdot e^i \cdot x_{b_k b_{\alpha i}} - \nabla^3 U e^j \cdot e^k b_{\alpha i} + \nabla^3 U \cdot e^i b_{\alpha i} = -\nabla^3 U \cdot e^i b_{\alpha i}. \tag{3.28}
\]

Then, due to (3.17),

\[
-2B \sum_k h_k \phi_k = -2B \sum_k h_k \left( \frac{f_i}{f} + \frac{h_k}{h} + \frac{F_i h_k}{F_i} - q \frac{|\nabla U|}{|\nabla U|} \right) - 2B \sum_k \nabla^2 U \cdot e^i b_{\alpha i},
\]

\[
\leq C_2 B - \frac{q \nabla^2 U \cdot e^i}{|\nabla U|} \sum_k h_k \phi_k
\]

\[
\leq C_2 B + q \frac{\nabla^2 U \cdot e^i}{|\nabla U|} \left( |\nabla h| \phi_{ii} - A \frac{h_i}{h} \right)
\]

\[
\leq C_2 B + C_2 A + q \frac{\nabla^2 U \cdot e^i}{|\nabla U|} b_{\alpha i} \phi_{ii},
\]

where $C_2$ is a positive constant depending only on $q, f, h, |\nabla f|$ and $|\nabla h|$ on $\mathbb{S}^{n-1}$, and $|\nabla U|$ and $|\nabla^2 U|$ on $\partial \omega$. $C_3$ is a positive constant depending only on $q, h$ and $|\nabla h|$ on $\mathbb{S}^{n-1}$, $|\nabla U|$ and $|\nabla^2 U|$ on $\partial \omega$. Moreover,
\[-\phi_{11} b^{11} = \left( \frac{f_{11}^2}{f} - \frac{h_{11}^2}{h} - \frac{h_{11}}{h} - \frac{F_{e} h_{11}^2 + F_{h} h_{11}}{F_{e}} + \frac{F_{h} h_{11}}{F_{e}^2} \right) b^{11} - q \frac{\|\nabla U\|_{1}^2 + q \|\nabla U\|_{1} \|\nabla U\|}{\|\nabla U\|} b^{11} + q \frac{\nabla U \cdot e}{|\nabla U|} b^{11} \]

\[
\leq C_{4} (1 + b_{11} + b^{11}) - q \frac{\nabla U \cdot e}{|\nabla U|} b_{11}^{11},
\]

where \(C_{4}\) is a positive constant depending only on \(q\), the upper and lower bounds of \(f\) and \(h\) on \(S^{n-1}\), the upper bounds of \(|\nabla f|\) and \(|\nabla h|\) on \(S^{n-1}\), \(|\nabla U|\), and \(|\nabla^{3} U|\) on \(\partial \Omega\). Consequently, plugging (3.30) and (3.29) into (3.28), we obtain

\[
(A - n + 1) \sum b_{i}^{ii} + 2B \sum b_{i}^{ii} \leq C_{6} (A + B) + C_{6} (1 + b_{11} + b^{11}) + C_{6}B + C_{A}.
\]

This implies

\[
(A - n + 1 - C_{6}) \sum b_{i}^{ii} + (2B - C_{6}) \sum b_{i}^{ii} \leq C_{6} (A + B) + C_{6} + C_{6}B + C_{A}.
\]

Provided \(A\) and \(B\) are suitably large, and we obtain \(b_{ii} \leq C\), then \(b_{11}(x, t_{0})\) is bounded from above. We complete the proof. \(\square\)

Through the aforementioned estimates, we know that equations (1.6) are uniformly parabolic. By the \(C_{0}\) estimate (Lemmas 3.1 and 3.3), the gradient estimate (Lemma 3.1), the \(C_{2}\) estimate (Lemmas 3.4 and 3.5), and the Krylov’s and Nirenberg’s theory \([18,26]\), we obtain the Hölder continuity of \(\nabla h\) and \(h\). Then we can obtain higher order derivative estimates by the regularity theory of the uniformly parabolic equations. Therefore, we obtain the long time existence and the uniqueness of the smooth solution to the normalized flows (1.6).

### 4 Existence of solution

In this section, we complete the proof of Theorems 1.2 and 1.3. Since \(\mu \in NCH\) can be approximated by a family of measures with positive and smooth densities (Lemma 3.7 of [6]), we first prove the result for some smooth and positive function \(f\).

**Proof of Theorem 1.3.** For parts (ii) and (iii):

Step 1: \(d\mu = f dx\) for \(f \in C^{\infty}(S^{n-1})\) and \(f > 0\).

By Lemma 2.4, \(\frac{d}{dt} \mathcal{F}_{\epsilon}(h(\cdot, t)) \leq 0\), for all \(t > 0\). Then

\[
\int_{0}^{t} - \frac{d}{dt} \mathcal{F}_{\epsilon}(h(\cdot, t)) dt = \mathcal{F}_{\epsilon}(h(\cdot, 0)) - \lim_{t \to \infty} \mathcal{F}_{\epsilon}(h(\cdot, t)) \leq \mathcal{F}_{\epsilon}(h(\cdot, 0))
\]

This implies that there exists a subsequence of time \(\{t_{j}\} \to \infty\) such that \(- \frac{d}{dt} \mathcal{F}_{\epsilon}(h(\cdot, t)) \to 0\), as \(t_{j} \to \infty\). By the equality condition (2.7), there exists a convex body \(\Omega_{\epsilon}\) with the support function \(h_{\epsilon}\), the \(q\)-equilibrium potential \(\mathcal{U}_{\epsilon}\), and the Gauss curvature \(\mathcal{K}_{\epsilon}\) satisfying that

\[
\frac{F_{\epsilon}(h_{\epsilon})}{|\nabla U_{\epsilon}|} \mathcal{K}_{\epsilon} = C_{\epsilon},
\]

where
There is a sequence \( \{\epsilon_i\} \rightarrow 0 \) such that \( C_{\epsilon_i} \rightarrow C_0 \) and \( \Omega_{\epsilon_i} \rightarrow \Omega \). Since the lower bound of the support function of \( \Omega_{\epsilon_i} \) may close to zero if \( \epsilon_i \) approaches zero, we need to discuss the dimensions of convex bodies.

Case 1: If \( \dim \Omega \leq n - q \), then \( \mathcal{H}^{n-q}(\Omega) < \infty \). From [10] (for more details, see Proposition 3.5), we have \( C_q(\Omega) = 0 \). It is a contradiction.

Case 2: If \( n - q < \dim \Omega \leq n - 1 \). By Lemma 3.1 and 3.2, \( C_{\epsilon_i} \) and \( F_{\epsilon_i}^{-1}(h_{\epsilon_i}) \) are positively bounded from below. Therefore, by the dominated convergence theorem,

\[
\int f \, dx = \lim_{\epsilon_i \to 0} \int_{S^{n-1}} C_{\epsilon_i} F_{\epsilon_i}^{-1}(h_{\epsilon_i}) \nabla U_{\epsilon_i} \frac{1}{K_{\epsilon_i}} \, dx \geq C_1 \liminf_{\epsilon_i \to 0} \mu_q(\Omega_{\epsilon_i}, S^{n-1}),
\]

where \( C_1 \) is a positive constant depending only on \( q \), and the constants in Lemma 3.1 and 3.2. It follows directly from the combination of equation (13.48) and Propositions 13.5 and 13.6 in [1] that

\[
\liminf_{\epsilon_i \to 0} \mu_q(\Omega_{\epsilon_i}, S^{n-1}) \rightarrow \infty.
\]

This contradicts to the definition of \( f \). As a result, \( \Omega \in \mathcal{K}^n \) is non-degenerated. On the other hand, we have

\[
\int_{\omega} f h_{\epsilon_i}^p \, dx = \int_{\omega} C_{\epsilon_i} h_{\epsilon_i}^p F_{\epsilon_i}^{-1}(h_{\epsilon_i}) \nabla U_{\epsilon_i} \frac{1}{K_{\epsilon_i}} \, dx,
\]

for all Borel set \( \omega \subset S^{n-1} \). Hence, by the dominated convergence theorem,

\[
\int_{\omega} f h_{\epsilon_i}^p \, dx = \int_{\omega} C_{\epsilon_i} h_{\epsilon_i} \nabla U_{\epsilon_i} \frac{1}{K_{\epsilon_i}} \, dx, \quad \forall \omega \subset S^{n-1},
\]

where \( h_{\epsilon_i} \), \( U_{\epsilon_i} \), and \( K_{\epsilon_i} \) are the support function, \( q \)-equilibrium potential, and the Gauss curvature of \( \Omega \), respectively.

Moreover, by Lemma 2.4, \( J_f(h(\cdot, t)) \) is strictly monotone along the flow; thus,

\[
\int_{S^{n-1}} \tilde{F}_{\epsilon_i}(h_{\epsilon_i}) f(x) \, dx = \inf \left\{ \int_{S^{n-1}} \tilde{F}_{\epsilon_i}(h_{\epsilon_i}) f(x) \, dx : C_q(K) = C_q(\Omega_{\epsilon_i}) \right\}.
\]

By the dominated convergence theorem again, we have

\[
\int_{S^{n-1}} h_{\epsilon_i}^p f(x) \, dx = \inf \left\{ \int_{S^{n-1}} h_{\epsilon_i}^p f(x) \, dx : C_q(K) = C_q(\Omega) \right\}. \tag{4.1}
\]

Step 2: \( \mu \in NCH \).

Let \( f_j \) be a sequence of positive and smooth functions on \( S^{n-1} \) such that the measures \( \mu_j \) (with \( d\mu_j = f_j(x) \, dx \)) weakly converge to \( \mu \in NCH \) as \( j \rightarrow \infty \). From Step 1, we can see that there are \( C_j > 0 \) and \( \Omega_j \in \mathcal{K}^n \) such that

\[
\int_{\omega} f_j h_{\epsilon_i}^p \, dx = \int_{\omega} C_{\epsilon_i} h_{\epsilon_i} \nabla U_j \frac{1}{K_j} \, dx, \quad \forall \omega \subset S^{n-1}.
\]

Since the equality is homogeneous, we can take

\[
C_j = \frac{(q - 1) \int_{S^{n-1}} f_j h_{\epsilon_i}^p \, dx}{(n - q) C_q(B_1)},
\]

where \( B_1 \) is the sphere of radius 1, such that \( C_q(\Omega_j) = C_q(B_1) \).
For each \( j \), there is a \( \capS_j \in -x_j^n \) such that \( \capS_j(x_j) = h_{\max}(x_j) \). Since the segment joining the origin and \( (\max_{x_j^n} h_j) x_j \) is contained in \( \Omega_j \), it follows that for \( x \in S^{n-1} \),
\[
h_j(x) \geq x \cdot x_j h_j(x_j), \quad \forall x \cdot x_j > 0.
\]

Thus, from (4.1), we have
\[
\int_{S^{n-1}} f_j dx \geq \int_{S^{n-1}} h_j f_j dx \geq h_j^p(x_j) \int_{\{x \in S^{n-1} : x \cdot x > 0\}} (x \cdot x_j)^p f_j dx.
\]

Since \( f_j dx \to d\mu \in NCH \), then \( h_j(x_j) \leq C(p, \mu) \), namely, \( \Omega_j \) are bounded from above. Then we also obtain \( \frac{1}{C(p, q, n, \mu)} \geq C_j \leq C(p, q, n, \mu) \). By the Blaschke selection theorem, \( \Omega_j \to \Omega \) in Hausdorff distance. Follow the same argument as Cases 1 and 2 in Step 1, we have \( \Omega \in \mathcal{K}^n \), and \( \Omega \to C > 0 \). Then, \( \Omega \) is the solution for equation (1.2) with \( \mu \) replaced by \( \frac{1}{C} \mu \). If \( p \neq n - q \), we can conclude that \( \Omega^* = \Omega \) satisfies equation (1.2).

**For part (i):**

Step 1: \( d\mu = f dx \) for \( f \in C^\infty(S^{n-1}) \) and \( f > 0 \).

For \( p > n \), we consider the flow
\[
\frac{\partial X}{\partial t}(x, t) = -f(v) < X, \nabla U(X, t)> K v + \eta(t) X,
\]
\[
X(x, 0) = X_0(x),
\]
where
\[
\eta(t) = \int_{S^{n-1}} f(x) h_j^p(x, t) dx \int_{S^{n-1}} h_j(x, t) d\mu_q(\Omega_t, x).
\]

It is direct to take \( \tilde{E}(s) = \frac{1}{p} S^p \), \( \forall s > 0 \) in the proof of part (ii), and our purpose is achieved. Namely, we consider the functional
\[
\mathcal{J}(h(\cdot, t)) = \frac{1}{p} \int_{S^{n-1}} h_j^p(x, t) f(x) dx - \frac{1}{n - q} \int_{S^{n-1}} h(x, t) d\mu_q(\Omega_t, x).
\]

Following the same argument as in Lemmas 2.3 and 2.4, we conclude that
\[
\frac{d}{dt} \mathcal{J}(h(\cdot, t)) \leq 0, \quad \forall t \in [0, T),
\]
and
\[
C_q(\Omega_t) = C_q(\Omega_0), \quad \forall t \in [0, T).
\]
As in Lemmas 3.1–3.5, we obtain the \textit{a priori} estimates:
\[
\frac{1}{C} \leq h(\cdot, t) \leq C, \quad \forall t \in [0, T),
\]
\[
\max_{S^{n-1}} |\nabla h(\cdot, t)| \leq C, \quad \forall t \in [0, T),
\]
\[
\forall C \leq \eta(t) \leq C, \quad \forall t \in [0, T),
\]
\[
C^{-1} \int_{S^{n-1}} (\nabla^2 h + h) (x, t) \leq C I, \quad \forall (x, t) \in S^{n-1} \times [0, T),
\]
where \( C \) is a positive constant depending only on \( n, p, q, f \), and the initial hypersurface. We conclude that the flow exists for all time \( t \geq 0 \), and \( h(\cdot, t) \) remains positive, smooth, and uniformly convex. By the monotonicity of \( \mathcal{J}(h(\cdot, t)) \), we see that there is a smooth convex body \( \Omega \in \mathcal{K}^n \), whose support function satisfy the equation (1.3).

Step 2: \( \mu \in NCH \).

Following the same argument in the proof of parts (ii) and (iii), There is a convex body \( \Omega \in \mathcal{K}^n \) that satisfies equation (1.2).

We complete the proof.
Clearly, part (i) in Theorem 1.2 has been proved in the aforementioned argument. The key point of the rest part is to show the positivity of the solution to (1.3) when \( p = n \), we need the following lemma.

**Lemma 4.1.** Let \( p \geq n > 2, f \in C^\infty(S^{n-1}) \) and \( f > 0 \). Let \( \Omega \in \mathcal{K}_0 \) be a convex body whose boundary is a closed, smooth and uniformly convex hypersurface satisfying (1.3), and \( r \) be the radial function of \( \Omega \). If \( p - n \in (0, 1) \), then there is a constant \( C > 0 \), depending only on \( n, q, f \), and the upper bound of \( r \), but independent of \( p \), such that

\[
\max_{s^1} \left| \frac{\nabla r}{r} \right| \leq C \left( 1 + \max_{s^1} r^{p-n} \right),
\]

(4.3)

**Proof.** Since \( \Omega \) satisfies (1.3), we have

\[
K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{1}{f(x)} h^{1-p}(x)|\nabla U(v^{-1}(x))|^q,
\]

where \( h, U \), and \( K \) are the support function, \( q \)-equilibrium potential, and Gauss curvature of \( \Omega \), respectively. By virtue of (2.2), we have

\[
\det(r^2 \delta_{ij} + 2n r f - r g_{ij}) = r^{2n-2p-2}(r^2 + |
\nabla r|^2)^{\frac{2-p}{2}} |\nabla U(r(\xi)\xi)|^q f^{-1}(v_1(r(\xi)\xi))
\]

Set \( w = -\log r \). Then \( w \) satisfies the following equation:

\[
\det(w_{ij} + w_k w_{kj} + \delta_{ij}) = e^{(p-n)w}(1 + |\nabla w|^2)^{\frac{2-p}{2}} |\nabla U(e^{-w}\xi)|^q f^{-1} \left( \frac{\xi + \nabla w}{\sqrt{1 + |\nabla w|^2}} \right).
\]

(4.4)

Take \( Q^1 = \frac{1}{2} |\nabla w|^2 \), we need to show the upper bound of \( Q \). Suppose that \( x_0 \) is the maximum point of \( Q \). Take a normal coordinates at \( x_0 \), then at \( x_0 \)

\[
0 = Q_k = \sum_i w_i w_{ki},
\]

(4.5)

\[
0 \geq Q_{ij} = \left( \sum_k (w_k x_{kj} + w_k w_{kj}) \right).
\]

(4.6)

By a rotation of the coordinates, we may assume \( w_i = |\nabla w| \) at \( x_0 \). Then

\[
w_{ik}(x_0) = 0, \quad \forall k = 1, \ldots, n-1.
\]

(4.7)

Let

\[
a_{ij} = w_{ij} + w_k w_{kj} + \delta_{ij}.
\]

Furthermore, by (4.7), we have \( a_{ik}(x_0) = 0 \) for all \( k \neq 1 \). Hence, without loss of generality, we can also assume \( \{w_i\} \) is diagonal, and namely,

\[
\{a_{ij}\} = \text{diag}(1 + w_{1}^2, 1 + w_{2}^2, \ldots, 1 + w_{(n-1)(n-1)}).
\]

Let \( \{a^{ij}\} \) be the inverse matrix of \( \{a_{ij}\} \), \( f(\xi, \nabla w) = \frac{\xi + \nabla w}{\sqrt{1 + |\nabla w|^2}} \), and \( a_{ik} = \nabla_k a_{ij} \). Differentiating (4.4), and by (4.7), we obtain, at \( x_0 \),

\[
a^{ij} a_{jk} = (p - n) w_k + \frac{|\nabla U|}{|\nabla U|} - \frac{\tilde{f}}{f} \sum_l \frac{\tilde{f}_l w_{lk}}{f}.
\]

(4.8)

We also have,

\[
|\nabla U(e^{-w}\xi)|_k = |\nabla U|^{-1} |\nabla^2 U e^{-w} (w_k \xi \cdot \nabla U + \xi_k \cdot \nabla U)|.
\]

(4.9)

It follows from (4.5)–(4.9) and the Ricci identity, at \( x_0 \),
\[
0 \geq a^i Q_i = \sum_k a^i (w_k w_{ki} + w_k w_{kj}) \\
= \sum_k a^i w_k (w_{ij} + \delta_{ij} w_k - \delta_{ij} w_j) + \sum_{i=2} a^i w_{ii}^2 \\
= \sum_k a^i w_k (a_{ij} - w_k w_j - w_{kj}) + w_{ii}^2 \sum_i a^i - a^{i1} w_i^2 + \sum_{i=2} a^i w_{ii}^2 \\
\geq (p - n) w_i^2 - C_1 e^{-w_i} (w_i + w_1) - C_2 (w_i + 1) + w_{ii}^2 \sum_i a^i + \sum_{i=2} a^i w_{ii}^2, 
\]

(4.10)

where \(C_1\) is a positive constant depending only on \(q\), the lower bound of \(|\nabla U|\) and the upper bound of \(|\nabla^2 U|\) on \(\partial \Omega\). \(C_2\) is a positive constant depending only on the upper and lower bound of \(f\) and the upper bound of \(|\nabla f|\) on \(S^{n-1}\). Since \(n > 2\), we have

\[
\sum_{i=2} a^i w_{ii}^2 = \sum_{i=2} a^i (a_{ii}^2 - 2a_{ii} + 1) \geq -C_3 + \sum_{i=2} a_{ii} \geq -C_3 + (n - 2) \left( \prod_{i=2} a_{ii} \right)^{\frac{1}{n-2}},
\]

where \(C_3\) is a positive constant depending only on \(n, q\), the lower bound of \(|\nabla U|\) on \(\partial \Omega\) and the upper and lower bound of \(f\) on \(S^{n-1}\). By (4.4) and (4.10), we can further estimate

\[
0 \geq -C_4 (1 + w_i + w_{ii}^2) + C_5 e^{-w_i} w_i^{\frac{p-2}{p-1}},
\]

where \(C_4\) is a positive constant depending only on \(C_1, C_2, C_3,\) and \(C_5\) is a positive constant depending only on \(n, q\), the lower bound of \(|\nabla U|\) on \(\partial \Omega\) and the upper and lower bound of \(f\) on \(S^{n-1}\). As \(p \geq n\), we obtain (4.3).

**Proof of Theorem 1.2.** For \(p > n\), the solution has been shown in Step 1 for the proof of part (i) in Theorem 1.3. It remains to show the smooth solution for \(p = n\).

Let \(\{p_i\}\) be a sequence of indices such that \(p_i > n\) and \(p_i \to n\) as \(i \to \infty\). Assume \(p_i \in [n, n + 1]\). According to Step 1 for the Proof of Part (i) in Theorem 1.3, for each \(p_i\), there is a positive, smooth, and uniformly convex solution \(h_i\) to (1.3) with \(p\) replaced by \(p_i\). Let

\[
h_i = \lambda_i^{-\frac{1}{p_i + q - 1}} h_i, \quad \lambda_i = \left( \frac{C_q(\Omega_i)}{C_q(B_i)} \right)^\frac{n+q}{n},
\]

where \(\Omega_i\) is the convex body generated by \(h_i\). Let \(h_i\) be the support function of \(\Omega_i\), and \(U_i\) be the corresponding \(q\)-equilibrium potential. Consequently, we obtain \(C_q(\Omega_i) = C_q(B_i)\), then

\[
\max_{S^{n-1}} h_i = 1 \quad \text{and} \quad \min_{S^{n-1}} h_i = 1.
\]

(4.11)

And \(h_i\) satisfies the following equation:

\[
\det(\nabla^2 h_i + h_i I) = \lambda_i f h_i^{p_i - 1} |\nabla U_i|^{q}, \quad \text{on} \ S^{n-1}.
\]

(4.12)

For each \(i\), there is a \(x_i \in S^{n-1}\) such that \(h_i(x_i) = \max_{S^{n-1}} h_i\). It follows that for \(x \in S^{n-1}\),

\[
h_i(x) \geq x \cdot x h_i(x), \quad \forall x \cdot x > 0.
\]

Thus, from (4.1), (4.11), and \(p_i \in [n, n + 1]\), we have

\[
\int_{S^{n-1}} f \, dx \geq \int_{S^{n-1}} h_i^{p_i} f \, dx \\
\geq h_i^{p_i}(x_i) \int_{\{x \in S^{n-1}; x \cdot x > 0\}} (x \cdot x_i)^{p_i} f \, dx \\
\geq h_i^{p_i}(x_i) \int_{\{x \in S^{n-1}; x \cdot x > 0\}} (x \cdot x_i)^{n+1} f \, dx.
\]
Since $f > 0$, then $h_i(x) \leq C(n, f)$, namely, there exists a positive constant $C_i$, depending only on $n$ and $f$, but independent of $p_i$, such that
\[
\max_{S^{n-1}} h_i \leq C_i. \tag{4.13}
\]
By virtue of (4.3), we can obtain
\[
\max_{S^{n-1}} \frac{|\nabla h_i|}{r} \leq C_2,
\]
where $C_2$ is a positive constant depending only on the constant in (4.3) and $C_1$. That is to say $C_2$ is also independent of $p_i$. As a result, together with (4.11), we have
\[
\min_{S^{n-1}} h_i \geq VC_i. \tag{4.14}
\]
Hence, we also have the uniform upper and lower bounds for $|\nabla U_i|$, which is also independent of $p_i$. On the other hand, by applying the maximum principle to (4.12), we obtain
\[
\frac{(\min_{S^{n-1}} h_i)^{n-p} (\max_{S^{n-1}} |\nabla U_i|)^q}{\max_{S^{n-1}}} \leq \lambda_i \leq \frac{(\max_{S^{n-1}} h_i)^{n-p} (\max_{S^{n-1}} |\nabla U_i|)^q}{\min_{S^{n-1}}}. \tag{4.15}
\]
Owing to (4.13), (4.14), and (4.15), there is a subsequence $p_i \to n$, such that $\Omega_i \to \Omega \in C^n_+$ in Hausdorff distance and $h_i \to h > 0$, $\lambda_i \to \lambda > 0$. By the weak convergence of the $L_{p,q}$-capacitary measure, $h$ is a weak solution to (1.3) with $f$ replaced by $\lambda f$. Hence, the area measure of $\Omega$ satisfies
\[
VC \leq \frac{dS(\Omega_i, \cdot)}{d\sigma_{n-1}} = \lambda h^{n-1} |\nabla U_i|^q \leq C, \quad \text{on } S^{n-1},
\]
for some positive constants $C$. Follow the same argument as in [6] for the proof of Theorem 1.3, and we see that $\partial \Omega$ is a closed, smooth, and uniformly convex hypersurface. We complete the proof. \hfill \Box

5 Uniqueness of solution

We recall the following:

**Theorem 5.1.** [9] Suppose $1 < q < n$. Let $\Omega_0$ and $\Omega_1$ be bounded convex domains in $\mathbb{R}^n$ of class $C^{2,a}_+$. Then
\[
\left( \frac{q-1}{n-q} \int_{S^{n-1}} h_{\Omega_0}(x) d\mu_q(\Omega_0, x) \right)^{n-q} \geq C_q(\Omega_0)^{n-q} C_q(\Omega_1)
\]
with equality if and only if $\Omega_0$, $\Omega_1$ are homothetic.

**Proof of Theorem 1.4.** By the assumption, assume $h_{K}^{1-p} d\mu_q(K, \cdot) = h_{L}^{1-p} d\mu_q(L, \cdot) = d\mu$. When $p = 1$, namely, $d\mu_q(K, \cdot) = d\mu_q(L, \cdot)$. By Theorem 5.1, we have
\[
C_q(K) = \frac{q-1}{n-q} \int_{S^{n-1}} h_{K} d\mu_q(K, \cdot) = \frac{q-1}{n-q} \int_{S^{n-1}} h_{\Omega_0}(x) d\mu_q(\Omega_0, x) \geq C_q(\Omega_0)^{n-q} C_q(K).
\]
When $q = n - 1$, it implies the equality holds in Theorem 5.1, then $K$ and $L$ are homothetic. When $q \neq n - 1$, we have $C_q(K)^{\frac{n-q}{n-q}} \geq C_q(L)^{\frac{n-q}{n-q}}$. Exchanging the roles of $K$ and $L$, we conclude that $C_q(K) = C_q(L)$.\hfill \Box
Then the equality holds in Theorem 5.1. Therefore $K$ and $L$ are homothetic. Assume $K = sL$. We obtain $C_q(K) = s^{n-q}C_q(L)$, which implies $s = 1$, $K$, $L$ are translates.

When $p > 1$, by virtue of the Jensen’s inequality,
\[
\left( \frac{C_q(L)}{C_q(K)} \right)^{\frac{1}{p}} \geq \left( \frac{1}{C_q(K)} \right)^{\frac{1}{p}} \int_{|h| > 0} h_0^p d\mu_q(K, \cdot) - \int_{|h| > 0} h_R^p d\mu_q(K, \cdot) \geq 1 + \frac{q-1}{C_q(K)} \int_{|h| > 0} h_0 d\mu_q(K, \cdot)
\]
\[
= \frac{q-1}{C_q(K)} \int_{|h| > 0} h_0 d\mu_q(K, \cdot).
\]

Since $\int_{|h| = 0} h_0 d\mu_q(K, \cdot) = \int_{|h| = 0} h_R h_0^{-1} d\mu = 0$, we have, under Theorem 5.1, that
\[
\left( \frac{C_q(L)}{C_q(K)} \right)^{\frac{1}{p}} \geq \frac{1}{C_q(K)} \int_{|h| > 0} h_0 d\mu_q(K, \cdot) \geq \left( \frac{C_q(L)}{C_q(K)} \right)^{\frac{n-q+1}{n}}.
\]
Exchanging the roles of $K$ and $L$, we conclude that $C_q(K) = C_q(L)$, then the equality holds in Theorem 5.1. Therefore $K$ and $L$ are homothetic. If $K = sL$, then $C_q(K) = s^{n-q}C_q(L)$, which implies $s = 1$. If $K = L + y$, for some $y \in \mathbb{R}^n$, then
\[
(h_0(x) + x \cdot y)^{1-p} d\mu_q(L, x) = h_0^{1-p}(x) d\mu_q(L, x), \quad \forall x \in \mathbb{S}^{n-1}.
\]
Note that $\mu \in NCH$, we have
\[
\int_{\{x \in \mathbb{S}^{n-1} : x \cdot y > 0\}} (h_0(x) + x \cdot y)^{1-p} d\mu_q(L, x) > \int_{\{x \in \mathbb{S}^{n-1} : x \cdot y > 0\}} h_0^{1-p}(x) d\mu_q(L, x),
\]
which is a contradiction. As a result, $K = L$.

We complete the proof. □

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A curvature flow to the $L_p$ Minkowski-type problem of $q$-capacity


