Regularity of optimal mapping between hypercubes

Abstract: In this note, we establish the global $C^{3,\alpha}$ regularity for potential functions in optimal transportation between hypercubes in $\mathbb{R}^n$ for $n \geq 3$. When $n = 2$, the result was proved by Jhaveri. The $C^{3,\alpha}$ regularity is also optimal due to a counterexample in the study by Jhaveri.

Keywords: Monge-Ampère equation, optimal transportation, global regularity

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1 Introduction

Let $\Omega, \Omega^*$ be two bounded domains in $\mathbb{R}^n$. Assume that $\rho, \rho^*$ are two positive density functions supported on $\Omega, \Omega^*$, respectively, and satisfy the balance condition $|\rho|_{L^1(\Omega)} = |\rho^*|_{L^1(\Omega^*)}$. The optimal mapping $T : \Omega \to \Omega^*$ is the minimiser of the functional

$$C(s) = \frac{1}{2} \int_{\Omega} |x - s(x)|^2 \rho(x) \, dx$$

among all measure-preserving maps $s : \Omega \to \Omega^*$ such that $s_*\rho = \rho^*$, [15,20,21].

In [2], Brenier obtained the existence and uniqueness of the optimal mapping $T$ that is the gradient of a convex function $u$, which is called potential function and satisfies a natural boundary condition of the Monge-Ampère equation:

$$\det D^2u = \frac{\rho}{\rho^*} = Du \quad \text{in } \Omega, \quad Du(\Omega) = \Omega^*. \quad (1.1)$$

Regularity of the optimal mapping $T$ (equivalently, of the potential function $u$) is a fundamental problem in the theory of optimal transportation. For interior regularities, $C^{1,\alpha}, W^{2,p}$, and $C^{2,\alpha}$ estimates for $u$ have been obtained by Caffarelli [3,4] under appropriate assumptions. For regularity near the boundary, if $\Omega, \Omega^*$ are smooth, uniformly convex, and $\rho, \rho^* > 0$ are smooth, Delanoë [10] and Urbas [19] proved that $u \in C^\infty(\Omega)$. If $\Omega, \Omega^*$ are $C^2$ smooth and uniformly convex, and $\rho, \rho^*$ are Hölder continuous, Caffarelli [6] proved that $D^2u$ are Hölder continuous up to the boundary.

In applications such as in machine learning [9], computer vision [1] and computer graphics [16,18], the domains may fail to be uniformly convex or smooth. When the domains $\Omega, \Omega^*$ are convex and the densities $\rho, \rho^*$ are bounded from zero and infinity, Caffarelli [5] proved that $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. When $\Omega, \Omega^*$
are $C^{1,1}$ and convex, recently in [8], we obtained the regularity $u \in C^{2,\alpha}(\overline{Q})$ if $\rho, \rho^* \in C^{\alpha}$; and $u \in W^{2,p}(\overline{Q})$ for all $p > 1$ if $\rho, \rho^* \in C^0$. In dimension two, for constant densities, Savin and Yu [17] obtained the global $W^{2,p}$ estimate for arbitrary bounded convex domains $\Omega, \Omega^* \subset \mathbb{R}^2$.

In practice, a typical domain in medical image processing like the magnetic resonance imaging and computed tomography images is the hypercube $Q = (0, 1)^n$. Optimal transport between the hypercubes was also used by Caffarelli [7] in proving the FKG type inequalities. An interesting question is whether one can obtain higher regularity for this special case. The techniques used in previous works [6,8,10, 19] do not apply to the case when the domain $\Omega = Q$, due to the loss of regularity of $\partial Q$ at the corners. Very recently, Jhaveri constructed a counterexample showing that there exist smooth densities such that $T \in C^{2,\alpha}(\overline{Q})$ for every $\alpha < 1$ but not $C^{4}(\overline{Q})$, see [12, Theorem 3.7]. Hence, the best possible regularity one can expect is $T \in C^{2,\alpha}(\overline{Q})$.

By the symmetry of $Q$, we can make even extensions for the densities. Hence, by Caffarelli’s interior regularity [3,5], we see that for any positive $\rho, \rho^* \in C^{\alpha}(\overline{Q})$ satisfying $\|\rho\|_{L^1(\Omega)} = \|\rho^*\|_{L^1(\Omega)}$, the optimal mapping $T \in C^{1,\alpha}(\overline{Q})$ [7, Corollary 3]. Moreover, $T$ maps each face of $Q$ to itself correspondingly. In dimension two, by using the partial Legendre transform, Jhaveri [12, Theorem 3.3] proved that if further $\rho, \rho^* \in C^{1,\alpha}(\overline{Q})$, then $T \in C^{2,\alpha}(\overline{Q})$.

In this article, we establish the optimal global $C^{2,\alpha}$ regularity of $T$ in higher dimensions.

**Theorem 1.1.** Assume the positive densities $\rho, \rho^* \in C^{1,\alpha}(\overline{Q})$ for some $\alpha \in (0, 1)$, and satisfy the balance condition $\|\rho\|_{L^1(\Omega)} = \|\rho^*\|_{L^1(\Omega)}$. Then the optimal mapping $T \in C^{2,\alpha}(\overline{Q})$.

We remark that in dimension two, Jhaveri [12] used the partial Legendre transform to change the Monge-Ampère equation (1.1) to a quasi-linear, uniformly elliptic equation due to the fact $u \in C^{2,\alpha}(\overline{Q})$, then he further obtained $u \in C^{3,\alpha}(\overline{Q})$ by an estimate for uniformly elliptic equations. However, this method no longer works when the dimension $n > 2$. In higher dimensions, to obtain $u \in C^{3,\alpha}(\overline{Q})$, we shall adopt the result of [13] for the regularity of solutions along a given direction. The proof of Theorem 1.1 is contained in §2. For the Dirichlet problem of the Monge-Ampère equation in convex polygonal domains in $\mathbb{R}^2$, Le and Savin [14] recently obtained global $C^{2,\alpha}$ estimates of solution $u$ by assuming there exists a globally $C^2$, convex, strict subsolution.

## 2 Proof of Theorem

First, we do the following even reflections around the origin. Let $\bar{Q} = (-1, 1)^n$,

\[
\bar{\rho}(x) = \check{\rho}(x_1, ..., x_n) = \rho(|x_1|, ..., |x_n|), \quad x \in \bar{Q};
\]

\[
\bar{\rho}^*(y) = \check{\rho}^*(y_1, ..., y_n) = \rho^*(|y_1|, ..., |y_n|), \quad y \in \bar{Q}.
\]

(2.1)

If $\rho, \rho^* \in C^{\alpha}(Q)$ for some $\alpha \in (0, 1)$, then $\check{\rho}, \check{\rho}^* \in C^{\alpha}(\bar{Q})$. By the assumption of Theorem 1.1, we further have $\check{\rho}, \check{\rho}^* \in C^{3,\alpha}(\bar{Q})$.

Let $\hat{u}$ be the potential function of optimal transportation from $(\bar{Q}, \bar{\rho})$ to $(\bar{Q}, \bar{\rho}^*)$. By symmetry and the uniqueness of optimal mapping, we see that $D\hat{u} = D\hat{u}$ in $\bar{Q}$. We recall some known regularities as follows:

(i) By Caffarelli’s $C^{1,\alpha}$ regularity [5], we have $\hat{u} \in C^{1,\alpha}(\bar{Q})$ for some $\alpha \in (0, 1)$, provided $\check{\rho}, \check{\rho}^*$ are positive and bounded.

(ii) Furthermore, since $\check{\rho}, \check{\rho}^* \in C^{\beta}(\bar{Q})$ for all $\beta \in (0, 1)$, by the interior regularity [3,4], we have $\hat{u} \in C^{2,\beta}(\bar{Q})$ for all $\beta \in (0, 1)$, and thus, $u \in C^{2,\beta}(B_{1/4}(0) \cap Q)$ for all $\beta \in (0, 1)$.

(iii) By doing the same argument for each corner of $Q$ and using a covering argument, we can obtain $u \in C^{2,\beta}(Q)$ for all $\beta \in (0, 1)$.

Hence, under the assumption of Theorem 1.1 that $\rho, \rho^* \in C^{1,\alpha}(\overline{Q})$, for simplicity, we may write (1.1) as follows:
\[
\det D^2 u = f \quad \text{in } Q,
\]
\[
Du(Q) = 0,
\]
where \( f = \frac{\rho}{\rho^* - Du} \in C^{1,1}(Q) \). To prove Theorem 1.1, it suffices to prove \( u \in C^{1,1}(Q) \).

By the even reflections (2.1), we have
\[
\tilde{f}(x) = f(x_1, \ldots, x_n) = f(|x_1|, \ldots, |x_n|) \quad \text{for } x \in \tilde{Q}.
\]
Similarly, \( \tilde{u} \) satisfies
\[
\det D^2 \tilde{u} = \tilde{f} \quad \text{in } \tilde{Q},
\]
\[
D \tilde{u}(\tilde{Q}) = 0.
\]
As mentioned in (ii), by [3–5], we have
\[
\tilde{u} \in C^{2,\beta}(\tilde{Q}) \quad \text{for all } \beta \in (0, 1).
\]

Note that \( \tilde{f} \in C^{0,1}(\tilde{Q}) \) but is not \( C^1(\tilde{Q}) \) in general. Denote \( x = (x_1, x') \), where \( x' = (x_2, \ldots, x_n) \). From the definition and symmetry of \( \tilde{f} \), the partial derivative \( \partial_1 \tilde{f} = \frac{df}{dx_1} \) is well-defined in \( \{ x \in \tilde{Q} : x_1 \neq 0 \} \). Let’s assign \( \partial_1 \tilde{f} = 0 \) on the interface \( \{ x \in \tilde{Q} : x_1 = 0 \} \) so that \( \partial_1 \tilde{f} \) is defined in \( \tilde{Q} \). Let \( v = \partial_1 \tilde{u} \). By (2.5) and approximation, it is easy to see that \( v \in W^{2, p}(\tilde{Q}) \) for any \( p > 1 \), and \( v \) is a strong solution of
\[
\sum_{i,j=1}^n a_{ij} \partial_j v = \partial_1 \tilde{f} \quad \text{in } \tilde{Q},
\]
where \( \{ a_{ij} \} \) is the cofactor matrix of \( D^2 \tilde{u} \). Let \( B_r = B_r(0) \) be the ball with radius \( r \) and centre at the origin. In \( B_{9/10} \subset \tilde{Q}, \{ a_{ij} \} \) is H"older continuous and uniformly positive definite. In fact, from (2.5) and equation (2.4), there is a positive constant \( \lambda > 0 \) depending on \( n, ||\tilde{u}||_{C^0(B_{10})} \), \( \inf_{\tilde{Q}} \tilde{f} \) such that in \( B_{9/10} \),
\[
\lambda I \leq \{ a_{ij} \} \leq \lambda I
\]
in the sense of matrix, where \( I \) is the \( n \times n \) identity matrix. Equation (2.6) is satisfied almost everywhere in \( \tilde{Q}, [11] \).

Now we recall a useful partial directional regularity result from [13]. We say a function \( h \in L^\infty(\Omega) \) is \( C^\alpha \) in \( x' = (x_2, \ldots, x_n) \) for some \( \alpha \in (0, 1) \), if
\[
|h(x + \tau) - h(x)| \leq C |\tau|^{\alpha} \quad \forall \tau \in \text{span}\{e_2, \ldots, e_n\} \text{such that } x, x + \tau \in \Omega,
\]
where \( C > 0 \) is a constant, and denote
\[
|h|_{C^\alpha(\Omega)} = ||h||_{L^\infty(\Omega)} + \sup_{(x, x') \in \Omega, (y, y') \in \Omega} \frac{|h(x, x') - h(x, y')|}{|x' - y'|^{\alpha}}.
\]

**Lemma 2.1.** (A corollary of [13, Theorem 1.4]) Let \( w \in W^{2, p}(B_r) \) be a strong solution of
\[
\sum_{i,j=1}^n b_{ij} \partial_j w = h,
\]
where the coefficients \( b_{ij} \in C^\beta(B_r) \) for all \( \beta \in (0, 1) \) and satisfy (2.7). Suppose that \( h \) is H"older continuous in \( x' \) and \( ||h||_{C^\alpha(\Omega)} < \infty \) for some \( \alpha \in (0, 1) \). Then \( \partial_1 \partial_j w \) is H"older continuous for all \( i = 1, \ldots, n; \ j = 2, \ldots, n \) and
\[
||\partial_1 \partial_j w||_{C^\alpha(B_{1/2})} \leq C \quad \forall i = 1, \ldots, n; \ j = 2, \ldots, n,
\]
where the constant \( C \) depends on \( n, \lambda, ||h||_{C^\alpha(\Omega)} \), and \( ||w||_{L^\infty(B_1)} \).

**Proof of Theorem 1.1.** To apply Lemma 2.1 to equation (2.6), we claim that \( \partial_1 \tilde{f} \) is \( C^\alpha \)-continuous in \( x' \) for the same \( \alpha \in (0, 1) \) as in the assumption of Theorem 1.1. To see this, let \( e' \in \mathbb{R}^{n-1} \) be a unit vector, \( \varepsilon > 0 \) such that \( x = (x_1, x'), x_\varepsilon = (x_1, x' + \varepsilon e') \in \tilde{Q} \). It suffices to show
$$|\partial_1 \tilde{f}(x) - \partial_1 \tilde{f}(x_i)| \leq Ce^a. \quad (2.8)$$

By our definition of $\partial_1 \tilde{f}$, if $x_i = 0$, then (2.8) trivially holds since $\partial_1 \tilde{f}(x) = \partial_1 \tilde{f}(x_i) = 0$. So, by symmetry, we can assume $x_i > 0$.

Define the reflection points

$$\hat{x} = (x_1, |x_2|, ..., |x_n|),$$
$$\hat{x}_e = (x_1, |x_2 + \epsilon e_2|, ..., |x_n + \epsilon e_n|),$$

where $e'$ is expressed as $e' = (e'_1, ..., e'_n)$, so that $\hat{x}, \hat{x}_e \in \overline{Q}$. Since $x_i > 0$, we have

$$\partial_1 \tilde{f}(x) = \partial_1 f(\hat{x}) \quad \text{and} \quad \partial_1 \tilde{f}(x_i) = \partial_1 f(\hat{x}_e).$$

Hence, by the triangle inequality and the fact $f \in C^{1,1}(\overline{Q})$, we can obtain

$$|\partial_1 \tilde{f}(x) - \partial_1 \tilde{f}(x_i)| = |\partial_1 f(\hat{x}) - \partial_1 f(\hat{x}_e)| \leq |\hat{x} - \hat{x}_e|^a \leq |x - x_e|^a = Ce^a,$$

and thus, (2.8) is proved. Moreover, from the aforementioned estimates, we have

$$\|\partial_1 \tilde{f}\|_{C^{a}(\overline{Q})} \leq C \quad (2.9)$$

where the constant $C$ depends only on $f \in C^{1,1}(\overline{Q})$.

Back to equation (2.6), since the coefficients $a_{ij} \in C^\beta(B_{1/10})$ for all $\beta \in (0, 1)$ and satisfy (2.7), by (2.9), we can apply Lemma 2.1 to conclude that

$$|D_{q}^{2}v|^{c_{q}(\beta)}_{C^{1,1}(\overline{Q})} \leq C \quad \text{for } i = 1, ..., n; \quad j = 2, ..., n.$$ 

The same estimate applies around each corner of the hypercube $Q$. Thus, by a covering argument, we have

$$|D_{q}^{2}v|^{c_{q}(\beta)}_{C^{1,1}(\overline{Q})} \leq C \quad \text{for } i = 1, ..., n; \quad j = 2, ..., n.$$ 

Consider the restriction of equation (2.6) in $Q$, 

$$v_{11} = \frac{\partial_1 f - \sum_{i=2}^{n} a_{ij} v_{ij} - \sum_{j=2}^{n} a_{ij} v_{ij}}{a_{11}}.$$ 

Since $\partial_1 f \in C^\alpha(Q), a_{ij} \in C^\alpha(Q)$ and $a_{11} \geq \lambda$, we obtain $|v_{11}|^{c_{q}(\beta)}_{C^{1,1}(\overline{Q})} \leq C$. Therefore, $v \in C^{2,\alpha}(\overline{Q})$. This implies that $u \in C^{2,\alpha}(\overline{Q})$, and thus, $T \in C^{2,\alpha}(\overline{Q})$ is proved.

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References


