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On the large solutions to a class of $k$-Hessian problems

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Abstract: In this paper, we consider the $k$-Hessian problem $S_k(D^2 u) = b(x)f(u)$ in $\Omega$, $u = +\infty$ on $\partial\Omega$, where $\Omega$ is a $C^\infty$-smooth bounded strictly $(k-1)$-convex domain in $\mathbb{R}^N$ with $N \geq 2$, $b \in C^\infty(\Omega)$ is positive in $\Omega$ and may be singular or vanish on $\partial\Omega$, $f \in C[0, \infty) \cap C^1(0, \infty)$ (or $f \in C^1(\mathbb{R})$) is a positive and increasing function. We establish the first expansions (equalities) of $k$-convex solutions to the above problem when $f$ is borderline regularly varying and $\Gamma$-varying at infinity respectively. For the former, we reveal the exact influences of some indexes of $f$ and principal curvatures of $\partial\Omega$ on the first expansion of solutions. For the latter, we find the principal curvatures of $\partial\Omega$ have no influences on the expansions. Our results and methods are quite different from the existing ones (including $k = N$). Moreover, we know the existence of $k$-convex solutions to the above problem (including $k = N$) is still an open problem when $b$ possesses high singularity on $\partial\Omega$ and $f$ satisfies Keller–Osserman type condition. For the radially symmetric case in the ball, we give a positive answer to this open problem, and then we further show the global estimates for all radial large solutions.

Keywords: $k$-Hessian problem; the first expansions; the radial solutions; global estimates

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1 Introduction

This presentation is to investigate the following $k$-Hessian problem

$$S_k(D^2 u) = S_k(\lambda) = b(x)f(u) \text{ in } \Omega, \ u = +\infty \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a $C^\infty$-smooth bounded strictly $(k-1)$-convex domain, $k \in \{1, \ldots, N\}$, $\lambda = (\lambda_1, \cdots, \lambda_N)$ and $\lambda_1, \cdots, \lambda_N$ are the eigenvalues of the Hessian of $u \in C^2(\Omega)$, and

$$S_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} \lambda_{i_1} \cdots \lambda_{i_k}$$

denotes $k$th elementary symmetric function given in [1] and [2]; furthermore, $S_k(\lambda) = 1$ given in [3]. The last condition $u = +\infty$ on $\partial\Omega$ means that $u(x) \to +\infty$ as $d(x) := \text{dist}(x, \partial\Omega) \to 0$ and the solution is called “large solution”, “blow-up solution” or “explosive solution”. The weight $b$ and nonlinearity $f$ satisfy

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(b₁) \( b \in C^\infty(\Omega) \) is positive in \( \Omega \);
(f₁) \( f \in C(0, \infty) \cap C^k(0, \infty) \) is positive and increasing on \((0, \infty)\) with \( f(0) = 0 \) (or \( f_0 \)) \( f \in C^k(\mathbb{R}) \) is positive and increasing on \( \mathbb{R} \);
(f₂) the Keller–Osserman type condition \( \int_0^\infty (f(s))^{-1/(k+1)} ds < \infty, \forall t > 0, F(t) = \int_0^t f(s) ds. \)

We obtain by Definition 1.1 of [3] that \( u \in C^2(\Omega) \) is (strictly) \( k \)-convex if \( S_i(D^2 u) = S_i(\tilde{\lambda})(> 0 \) in \( \Omega \) for \( i = 1, \ldots, k \). Moreover, it follows by Definition 1.2 of [3] that \( \Omega \) is (strictly) \( l \)-convex if

\[
S_i(\tilde{k}(\tilde{x}))(\tilde{\lambda}) \geq 0 \text{ on } \partial \Omega \text{ for } i = 1, \ldots, l \leq N - 1, \text{ with } \tilde{k}(\tilde{x}) = (k_1(\tilde{x}), \ldots, k_{N-1}(\tilde{x})).
\]

where \( k_i(\tilde{x}) \) \( (i = 1, \ldots, N - 1) \) are the principal curvatures of \( \partial \Omega \) at \( \tilde{x} \).

If \( k = 1 \), then problem (1.1) reduces to the semilinear elliptic problem

\[
\Delta u = b(x)f(u) \text{ in } \Omega, \quad u = +\infty \text{ on } \partial \Omega.
\] (1.3)

The study to this problem on the existence, uniqueness and boundary behavior has a long history. In 1916, problem (1.3) first appeared in the work of Bieberbach [4] for \( N = 2 \) in connection with geometric problem of Riemannian surfaces of constant negative curvatures. Then, Rademacher [5], using the idea of Bieberbach, showed the results still hold for \( N > 3 \). Later, Lazer and McKenna [6] extended the above results in bounded domain \( \Omega \subset \mathbb{R}^N \) \( (N \geq 1) \) with a uniform outer sphere condition. Let \( b \equiv 1, f \) satisfy (f₁), Keller [7] and Osserman [8] showed problem (1.3) has solutions if and only if (f₁) holds with \( k = 1 \). For further insights on problem (1.3) and some related problems, we refer the readers to [9]–[19] and the references therein.

Now, let us return to problem (1.1). If \( k = N \), then the problem is Monge–Ampère problem

\[
det(D^2 u) = b(x)f(u) \text{ in } \Omega, \quad u = +\infty \text{ on } \partial \Omega.
\] (1.4)

Problem (1.4) arises from a few geometric problems and was first studied by Cheng and Yau [20], [21] for \( f(u) = \exp(ku) \). Many related works have been considered after Cheng and Yau’s results to problem (1.4). Lazer and McKenna [22] studied the existence, uniqueness and global estimate of strictly convex solutions to problem (1.4), where \( 0 < b \in C^\infty(\Omega), f(u) = u^\gamma \text{ or } f(u) = \exp(u) \). Especially, when \( f(u) = u^\gamma \) with \( \gamma \in (0, N] \), they investigated the nonexistence of solutions. Matero [23], for a class of more general Monge–Ampère problem (i.e., \( b(x)f(u) \) in (1.4) is replaced by \( f(x, u) \)), studied the existence, uniqueness and boundary behavior of strictly convex solutions. Guan and Jian [24] generalized the results of Cheng and Yau [20], [21], in which various existence and nonexistence results were shown for rather general Monge–Ampère equations with gradient terms in bounded (strictly) convex domains. In particular, they also studied the global estimate of strictly convex solutions to the problem in bounded strictly convex domains. Then, the results were extended by Jian [25] to the \( k \)-Hessian problem, where the existence and global estimate of viscosity solutions and the nonexistence of classical solutions were established. In [26], Mohammed studied the existence of strictly convex solutions to problem (1.4) when the following Dirichlet problem

\[
det D^2 u(x) = b(x), \quad x \in \Omega, \quad u|_{\partial \Omega} = 0
\] (1.5)

has a strictly convex solution. In Theorem 1.1 of [27], Caffarelli et al. showed that problem (1.5) admits a convex solution if \( b \in C^\infty(\tilde{\Omega}) \). Cheng and Yau [28] proved that problem (1.5) possesses a strictly convex solution if \( 0 < b(x) < C(d(x))^{\delta-N-1} \) in \( \Omega \) for some constants \( \delta > 0 \) and \( C > 0 \). Mohammed [29] showed that if \( b(x) > C(d(x))^{-N-1} \) in \( \Omega \) with \( C > 0 \), then problem (1.5) has no strictly convex solution. Let \( b \in C^\infty(\tilde{\Omega}) \) be positive in \( \Omega \) and nonnegative on \( \partial \Omega \). Under suitable conditions on \( f \), Cîrstea and Trombetti [30] investigated the existence of strictly convex solutions to problem (1.4) in smooth, bounded, strictly convex domains. And they further gave the boundary behavior and uniqueness of strictly convex solutions by using Karamata regular variation theory. Then, the results were further generalized by Huang [31] to problem (1.1). When \( f(u) = \exp(u) \text{ or } f(u) = u^\gamma, \gamma > N \), and \( b \) satisfies (b₁) and grows like a negative power of \( d(x) \) near boundary, Yang and Chang [32] showed the existence, uniqueness, nonexistence and global estimate of strictly convex solutions to problem (1.4). And when \( \Omega \) is a
ball, they obtained the exact boundary behavior of large solutions. Recently, under the hypotheses $(f_1), (f_2)$ and some additional structural conditions, the existence and boundary behavior of solutions were further studied by Zhang and Du [33] and Zhang [34]–[36] to problem (1.4) and by Ma and Li [37], Wan and Shi [38], Zhang and Feng [39] and Zhang [40] to problem (1.1). Especially, the authors in [37] investigated the existence and boundary behavior of viscosity solutions, and the author in [35] studied the asymptotic behavior of solutions when some indexes (which is relevant to $b$) tend to the corresponding critical values. For other related works, we refer the readers to [3], [41]–[57] and the references therein.

For convenience, we introduce the class of Karamata regularly varying functions as below.

**Definition 1.1.** A positive continuous function $f$ defined on $(a_\infty, \infty)$, for some $a_\infty > 0 (a_0 > 0)$, is called regularly varying at infinity (zero) with index $\rho$, written $f \in RV_\rho (RVZ_\rho)$, if for each $C > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{t \to \infty} f(Ct) = C^\rho \lim_{t \to 0^+} f(t). \quad (1.6)$$

A subclass of $RV_\rho (RVZ_\rho)$ is $NRV_\rho (N RVZ_\rho)$ defined by Proposition A.3 (A.4) in Appendix A.

In this paper, we first establish the first expansion (equality) of $k$-convex solutions near the boundary when $f$ is borderline regularly varying at infinity (i.e., $f \in N RV_k$) and has a lower term. We reveal that the most accurate influences of some indexes of $f$ and principal curvatures of $\partial \Omega$ on the first expansion of solutions. It should be pointed out that the boundary behavior of solutions to problem (1.1) is a hard issue when $f$ is borderline regularly varying at infinity. We analyze briefly the cause of the difficulty by Remark 2.3 (see page 5). Then, we establish the first expansion of solutions to problem (1.1) when $f$ is a $\Gamma$-varying function at infinity (defined on page 5) and $b$ may be vanishing or singular (including borderline singular) on $\partial \Omega$. We find the principal curvatures of $\partial \Omega$ have no influence on the expansions. As we know, $\Gamma$-varying functions is a class of very special rapidly varying functions defined by Appendix A.1 (see Appendix A). By analyzing the first boundary expansions of solutions, we find an interesting fact: Roughly speaking, if $f \in N RV_k (f \in \Gamma)$, then the solutions near the boundary can be expressed by a rapidly (slowly) varying function. Our results and methods are quite different from the existing ones. On the other hand, if $\Omega$ is a ball, $b$ is radially symmetric in $\Omega$ and has high singularity on $\partial \Omega$, $f$ satisfies the following Keller–Osserman type condition

$$\int_1^\infty (F(t))^{-1/(k+1)} dt = \infty. \quad (1.7)$$

we prove problem (1.1) has infinitely many positive $k$-convex radial solutions, which give a positive answer to the open problem (where $\Omega$ is a general $(k - 1)$-convex domain, $b$ satisfies $(b_1)$ and has high singularity on $\partial \Omega$, $f$ satisfies (1.7)) in the case of radial symmetry in the ball. Our result and method are quite different from the ones of Zhang in [35] and Feng and Zhang in [43]. Furthermore, we study the global estimates of all positive radial solutions to this radially symmetric problem.

## 2 Main results

First, we study the influences of some indexes of $f$ and principal curvatures of $\partial \Omega$ on the first expansion of $k$-convex large solutions when $f$ is borderline regularly varying at infinity. To our aim, we show some further hypotheses on $b$ and $f$ as follows:

$(b_2)$ there exist positive constants $b_0, \tilde{\delta}, \delta_0 \ (\tilde{\delta} \leq \delta_0)$ and non-increasing function $\theta \in \Lambda$ such that

$$b(x) = b_0 \theta^{k+1}(d(x)), \ x \in \Omega_{\delta}. \quad (b_2)$$
where \( \Omega_\delta := \{ x \in \Omega : d(x) < \delta \} \), \( A \) (following [11], [12], [30] and [19]) denotes the set of all positive monotonic functions \( \theta \in C^1(0, \delta_0) \cap L^1(0, \delta_0) \) which satisfy

\[
\lim_{t \to 0^+} \frac{d}{dt} \left( \frac{\Theta(t)}{\Theta(t)} \right) := D_{\theta} \in [0, \infty), \quad \Theta(t) = \int_0^t \theta(s)ds.
\]

\((f_3)\) there exist some constant \( t_0 > 0 \) and functions \( f_1, f_2 \) such that \( f(t) := f_1(t) + f_2(t), t \geq t_0 \), where \( f_1 \in C^2([t_0, \infty), g(t) := \frac{f(t)}{f'_1(t)} - k, t \geq t_0 \). And \( g, f_2 \) satisfy the following conditions:

\((S_1)\) \( g(t) \geq 0, t \geq t_0, \) limit \( \lim_{t \to \infty} g(t) = 0, \) limit \( \lim_{t \to \infty} \frac{tg'(t)}{g^2(t)} = 0; \)

\((S_2)\) for any \( \xi > 0, \) limit \( \lim_{t \to \infty} \frac{f(t)}{g(t)f_1(t)} = \xi^k \), and there exists \( C_1 \neq 0 \) such that

\[
\lim_{t \to \infty} \frac{f(t)}{g(t)f_1(t)} = C_1,
\]

or

\((S_3)\) limit \( \lim_{t \to \infty} \frac{f(t)}{g(t)f_1(t)} = 0 \) and there exists \( \mu \leq k \) such that for any \( \xi > 0, \)

\[
\lim_{t \to \infty} \frac{f(t)}{g(t)f_1(t)} = \xi^\mu.
\]

**Theorem 2.1.** Let \( b \) satisfy \((b_1), (b_2), f \) satisfy \((f_1)\) (or \((f_0, \alpha), (f_2), (f_3)\)) and \((S_1)-(S_3)\), then any \( k \)-convex solution \( u \) to problem \((1.1)\) satisfies

\[
u(x) = \exp(\alpha)\psi(\mathcal{B}(x)\Theta(d(x)))(1 + o(1)) \text{ as } d(x) \to 0, \tag{2.1}
\]

where \( \psi \) is uniquely determined by

\[
\int_0^\infty \frac{ds}{s f_1(s)^{1/(k+1)}} = t, \tag{2.2}
\]

\[
\mathcal{A}_0 = \frac{1}{k+1} - \mathcal{C}_2 - \left( \frac{1}{k+1} + \mathcal{K}_g \right)(1 - D_0), \quad \mathcal{C}_2 = \begin{cases} C_1, & \text{if } (S_2) \text{ holds,} \\ 0, & \text{if } (S_3) \text{ holds,} \end{cases} \tag{2.3}
\]

\[
\mathcal{B}(x) = \left( \frac{b_0}{S_{k-1}(\mathcal{E}(x))} \right)^{1/(k+1)}, \tag{2.4}
\]

and

\[
\mathcal{E}(x) = (\mathcal{E}_1(x), \ldots, \mathcal{E}_{N-1}(x)) \text{ with } \mathcal{E}(x) = \frac{\kappa_i(x)}{1 - \kappa_i(x)d(x)}, x \in \Omega \text{ near } \partial \Omega, \tag{2.5}
\]

where \( \kappa_i(x)(i = 1, \ldots, N - 1) \) are principal curvatures of \( \partial \Omega \) at \( \tilde{x} \) defined by \( |x - \tilde{x}| = d(x) \).

**Remark 2.2.** A typical example is \( f(t) = c_1t^\alpha(1 + \alpha \ln t)^\frac{1}{\alpha} + c_2t^\beta(\ln t)^\beta, t \geq t_0, \) where \( c_1 \in \mathbb{R}_+, \alpha \in (0, 1/3(k + 1)), c_2, \beta_1, \beta_2 \in \mathbb{R}. \) By a straightforward calculation, we have

\[
g(t) = \frac{1}{1 + \alpha \ln t}, \quad t \geq t_0, \quad \mathcal{K}_g = -\alpha, \quad \mathcal{C}_2 = \begin{cases} \frac{c_2 \alpha^\frac{1}{\alpha}}{c_1}, & \text{if } \beta_1 = k, \beta_2 = \frac{1}{\alpha} - 1, \\ 0, & \text{if } \beta_1 < k \ (\text{or } \beta_2 < \frac{1}{\alpha} - 1) \end{cases}
\]
Remark 2.3. By Lemma B.3 (ix) (see Appendix B), we see that in Theorem 2.1, any solution of problem (1.1) is expressed near the boundary by the rapidly varying function defined by Definition A.1 (see Appendix A). As we know, the rapidly varying function is extremely sensitive to the microvariations of independent variable. For instance, without loss of generality, in Remark 2.2 for small enough (no matter how small) $\varepsilon > 0$, we have

$$\lim_{t \to 0^+} \frac{\psi((1 - \varepsilon)t)}{\psi(t)} = \infty \quad \text{and} \quad \lim_{t \to 0^+} \frac{\psi((1 + \varepsilon)t)}{\psi(t)} = 0.$$  

This implies that $\psi(t)$ can not be effectively controlled by $\psi((1 \pm \varepsilon)t)$ near zero. Furthermore, the following hold

$$\lim_{t \to 0^+} \lim_{t \to 0^+} \sup \frac{\psi(t)}{\psi((1 - \varepsilon)t)} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \lim_{t \to 0^+} \inf \frac{\psi(t)}{\psi((1 + \varepsilon)t)} = \infty.$$  

So, we see that it is incorrect to describe the asymptotic behavior of $\psi(t)$ at zero by $\psi((1 \mp \varepsilon)t)$.

Next, we show the first expansions of $k$-convex solutions to problem (1.1) when $f$ is $\Gamma$-varying at infinity. For convenience, we introduce $\Gamma$-varying functions as below:

Definition 2.4. A non-decreasing function $f$ defined on $(A_\infty, \infty)$ is $\Gamma$-varying at infinity (written $f \in \Gamma$) if

$$\lim_{t \to \infty} f(t + \lambda \chi(t)) = e^\lambda, \forall \lambda \in \mathbb{R}.$$  

Some typical $\Gamma$-varying functions are

(i) $f(t) = \exp(t^p)$, $p > 0$, $t > 0$, where the auxiliary function $\chi(t) = p^{-1}t^{1-p}$;

(ii) $f(t) = \exp(\ln t)$, $t > 0$, where the auxiliary function is given by

$$\chi(t) = \begin{cases} 1, & \text{if } t \in (0, 1], \\ (\ln t)^{-1}, & \text{if } t \in (1, \infty); \end{cases}$$  

(iii) $f(t) = \exp(\exp(t))$, $t > 0$, where the auxiliary function $\chi(t) = \exp(-t)$.

If $f$ is non-decreasing on $(0, \infty)$, then by Theorem 1.28 in [58] we see that the following statements are equivalent:

(i) $f$ is $\Gamma$-varying at infinity;

(ii) there exist some constant $B_\infty > 0$ and positive function $T \in C^1[B_\infty, \infty)$ with $\lim_{t \to \infty} T'(t) = 0$ such that

$$f(t) \sim \hat{f}(t) := \exp \left( \int_{B_\infty}^t \left| \frac{ds}{F(s)} \right| \right) \text{ as } t \to \infty, \tag{2.6}$$  

where “$f(t) \sim \hat{f}(t)$ as $t \to \infty$” means that $f(t)/\hat{f}(t) \to 1$ as $t \to \infty$;

(iii) $\lim_{t \to \infty} \frac{\int_0^t [f(s)/f(\hat{f}(s))]^2 ds}{\hat{f}(t)^2} = 1$, where $F(t) = \int_0^t f(s)ds$.

Theorem 2.5. Let $b$ satisfy $(b_3)$ and the following condition

$(b_3)$ there exist positive constants $b_1, b_2$ and $\theta \in \Lambda$ with $D_\theta > 0$ such that
\[ b_1 := \liminf_{d(x) \to 0} \frac{b(x)}{d^{k+1}(d(x))} \leq \limsup_{d(x) \to 0} \frac{b(x)}{d^{k+1}(d(x))} =: b_2. \]

If \( f \in \Gamma \) satisfy \((f_1)\) (or \((f_{01})\)), then any \( k \)-convex solution \( u \) to problem (1.1) satisfies
\[ u(x) = \mathcal{L}((d(x)))^{-1}(1 + o(1)) \text{ as } d(x) \to 0, \]
where \( \mathcal{L} \) is defined by
\[ \int_{0}^{\delta_2} \frac{L(s)}{s} \, ds < \infty \quad \text{and} \quad L(t) = c \exp \left( \frac{z(t)}{s} \right) \quad c > 0, \quad z(t) \in C(0, \delta_2] \text{ and } z(t) \to 0 \text{ as } t \to 0^+. \]

Define
\[ \mathcal{L}_1 := \left\{ L \in \mathcal{L} : \int_{0}^{\delta_2} \frac{L(s)}{s} \, ds < \infty \right\} \quad \text{and} \quad \mathcal{L}_2 := \left\{ L \in \mathcal{L} : \int_{0}^{\delta_2} \frac{L(s)}{s} \, ds = \infty \right\}. \]

**Theorem 2.7.** Let \( b \) satisfy \((b_1)\) and the following condition
\[(b_4)\] there exist positive constants \( b_3, b_4 \) and \( L \in \mathcal{L}_1 \) such that
\[ b_3 := \liminf_{d(x) \to 0} \frac{b(x)}{d^{k+1}(d(x))} \leq \limsup_{d(x) \to 0} \frac{b(x)}{d^{k+1}(d(x))} =: b_4. \]

If \( f \in \Gamma \) satisfy \((f_1)\) (or \((f_{01})\)), then any \( k \)-convex solution \( u \) to problem (1.1) satisfies
\[ u(x) = \mathcal{L}((r(d(x)))^{-1})(1 + o(1)) \text{ as } d(x) \to 0, \]
where
\[ r(d(x)) = \left( \int_{0}^{d(x)} \frac{L(s)}{s} \, ds \right)^{k/(k+1)}. \]

\( \mathcal{L} \) and \( h \) are defined by (2.8), (2.9).

**Remark 2.8.** Since \( \Gamma \)-varying functions are rapidly varying at infinity, it is clear that if \( f \in \Gamma \), then \((f_2)\) holds naturally.
When $\Omega$ is a spherical region, we show the existence of positive $k$-convex radial solutions to problem (1.1) as follows. Without loss of generality, we suppose $\Omega$ is a unit ball.

**Theorem 2.9.** Let $\Omega$ be a unit ball, $b \in C(\Omega)$ with $b(x) = \tilde{b}(r)$ and
\[
\int_0^1 \left( \int_0^r \tilde{b}(t) \, dt \right)^{1/k} \, dr = \infty, \tag{2.10}
\]
f satisfy (f$_1$) (or (f$_{01}$)) and (1.7) hold, then problem (1.1) has infinitely many positive $k$-convex radial solutions.

**Remark 2.10.** By the proof of Theorem 2.9, we find that in Theorem 2.9, if $\Omega = \mathbb{R}^N$ and we replace (2.10) by
\[
\int_0^{R_\infty} \left( \int_0^r \tilde{b}(t) \, dt \right)^{1/k} \, dr = \infty \text{ with } R_\infty = \infty, \tag{2.11}
\]
and other conditions still hold, then the entire blow-up problem
\[
S_k(D^2u) = \tilde{b}(|x|)f(u) \text{ in } \mathbb{R}^N, \ u(x) \to \infty \text{ as } |x| \to \infty
\]
has infinitely many positive entire $k$-convex radial solutions.

In fact, by a simple calculation, we see that (2.10) is equivalent to (2.11) with $R_\infty = 1$.

To investigate the global estimates of radial solutions, we extend the class of functions $\mathcal{L}_k$ by the following way. For each $L \in \mathcal{L}_k$, let $\tilde{L} \in C^1([0, \infty))$ be a positive differential extension of $L$. Define
\[
\tilde{\mathcal{L}}_k := \{ \tilde{L} : L \in \mathcal{L}_k \}.
\]
For two functions $h_1$ and $h_2$ defined on $S$, $h_1(x) \asymp h_2(x), x \in S$ means that there exist positive constants $c_1, c_2$ such that $c_1 h_1(x) \leq h_2(x) \leq c_2 h_1(x), x \in S$. Without loss of generality, let $\Omega$ be still a unit ball. We next show the global estimates of positive $k$-convex radial solutions to problem (1.1).

**Theorem 2.11.** Let $\Omega$ be a unit ball in $\mathbb{R}^N, b = \tilde{b}(r)$ satisfy (b$_2$) and the following condition

(b$_2$) there exists $\tilde{L} \in \tilde{\mathcal{L}}_k$ such that $t \mapsto t^{-\gamma} \tilde{L}_k(t)$ ($\gamma \geq k + 1$) is non-increasing on $(0, r_0)(r_0 > 1)$ and the following holds
\[
b(x) = \tilde{b}(r) \asymp (1 - r)^{-\gamma} \tilde{L}_k(1 - r), \quad r < 1 \text{ close to } 1, \tag{2.12}
\]
f satisfy (f$_r$) (or (f$_{0r}$)) and $f(t) \asymp \tilde{f}(t)$ for $t > 0$, where $\tilde{f} \in C([0, \infty)) \cap \text{RVZ}_{p_0} \cap \text{RV}_{p_\infty}$ with $p_0 \leq k$ and $p_\infty < k$.

In particular, if $p_0 = k$, we need to verify
\[
\int_0^1 (\tilde{f}(s))^{-1/(k+1)} \, ds < \infty, \quad \tilde{f}(t) = \int_0^t \tilde{f}(s) \, ds. \tag{2.13}
\]

Then any positive $k$-convex radial solution $u$ to problem (1.1) satisfies
\[
\Phi^{-1}(u(r)) \asymp (\nu(1 - r^2))^{\frac{k}{(k+1)}}, \quad r \in [0, 1], \tag{2.14}
\]
where $\Phi^{-1}$ is the inverse of $\Phi$ which is uniquely determined by
\[
\int_0^r ((k + 1)\tilde{f}(s))^{-1/(k+1)} \, ds = t, \quad \tilde{f}(t) = \int_0^t \tilde{f}(s) \, ds; \tag{2.15}
\]
and
\[ v(t) = \int_{r_0}^{r_0} k \int_{r_s}^{r \gamma L^k(r) dr} ds. \] (2.16)

**Remark 2.12.** In Theorem 2.11, \( p_0 < k \) and \( p_\infty < k \) imply that (2.13) and the following hold
\[ \int_{1}^{\infty} (\tilde{f}(s))^{-1/(k+1)} ds = \infty. \]

**Theorem 2.13.** Let \( \Omega \) be a unit ball in \( \mathbb{R}^N \), \( b(x) = \tilde{b}(r) \) satisfy (\( b_1 \)) and (\( b_2 \)), \( f \) satisfy (\( f_1 \)) (or (\( f_0 \))) and \( f(t) \propto \tilde{f}(t) \) for \( t > 0 \), where \( \tilde{f} \in C[0, \infty) \) and satisfies
\[ \limsup_{t \to \infty} \frac{(\tilde{f}(t))^{k/(k+1)}}{f(t)} < \infty \quad \text{and} \quad \liminf_{t \to \infty} \frac{(\tilde{f}(t))^{k/(k+1)}}{f(t)} > 0. \] (2.17)

Let \( \mathcal{C}_+ \) be an arbitrary positive constant, we have
(i) if \( \gamma > k + 1 \) in (\( b_2 \)), then any positive \( k \)-convex radial solution \( v \) with \( v(0) > \mathcal{C}_+ \) to problem (1.1) satisfies (2.14), where \( \Phi^{-1} \) is the inverse of \( \Phi \) which is uniquely determined by
\[ \Phi^{-1}(\tilde{f}(t)) = \ln t, \quad t \in (0, \infty). \] (2.18)

(ii) if \( \gamma = k + 1 \) in (\( b_2 \)) and \( \limsup_{r \to 0^+} \tilde{L}(r) < \infty \), then any positive \( k \)-convex radial solution \( v \) with \( v(0) > \mathcal{C}_+ \) to problem (1.1) satisfies \( \Phi^{-1}(v(r)) \propto \nu(1 - r^2), \quad r \in [0, 1], \) where \( \Phi^{-1} \) is the inverse of \( \Phi \) which is uniquely determined by (2.18), \( v \) is given by (2.16).

**Corollary 2.14.** Let \( b \) satisfy the hypotheses in Theorem 2.13, \( f \) satisfy (\( f_1 \)) (or (\( f_0 \))) and \( f(t) \propto t^k \) for \( t \geq 1 \), then the results of Theorem 2.13 hold with
\[ \Phi^{-1}(t) = \ln t, \quad t > 1. \]

**Remark 2.15.** The condition (2.13) is unnecessary in Theorem 2.13 and Corollary 2.14.

Obviously, in Theorems 2.11, 2.13 and Corollary 2.14, \( \tilde{b}(r) \) is controlled by a regularly varying function. Now, we consider the global estimate of positive \( k \)-convex radial solutions to problem (1.1) when \( \tilde{b}(r) \) is controlled by a function which is rapidly varying to infinity at zero. For convenience, we now introduce a new class of rapidly varying functions as follows.

Let \( \Lambda_\infty \) denote the set of all positive decreasing functions \( \theta_\infty \in C^1(0, 1) \) which satisfy
\[ \lim_{t \to 0^+} \frac{d}{dt} \left( \frac{\Theta_\infty(t)}{\Theta_\infty(t)} \right) = 0, \quad \Theta_\infty(t) = \int_{t}^{1} \theta_\infty(s) ds. \] (2.19)

**Theorem 2.16.** Let \( \Omega \) be a unit ball in \( \mathbb{R}^N \), \( b(x) = \tilde{b}(r) \) satisfy (\( b_1 \)) and the following condition (\( b_2 \)) there exists \( \theta_\infty \in \Lambda_\infty \) such that \( \tilde{b}(r) \propto \theta_\infty(1 - r^2), \quad r < 1 \) close to 1.

(i) If \( f \) satisfies the hypotheses in Theorem 2.11, then any positive \( k \)-convex radial solution \( v \) to problem (1.1) satisfies
\[ \Phi^{-1}(v(r)) \propto \Theta_\infty(1 - r^2), \quad r \in [0, 1], \] (2.20)
where \( \Phi^{-1} \) is the inverse of \( \Phi \) which is uniquely determined by (2.15).
(ii) If \( f \) satisfies the hypotheses in Theorem 2.13 and \( \mathcal{C}_s \) is an arbitrary positive constant, then any positive \( k \)-convex radial solution \( v \) with \( v(0) > \mathcal{C}_s \) to problem (1.1) satisfies (2.20), where \( \Phi \) is replaced by the solution of (2.18).

**Remark 2.17.** By Lemma B.2 (iv) (see Appendix B), we obtain that \( \theta_\infty \) is rapidly varying to infinity at zero. So, we have \( \theta_\infty(1 - r^2) \) cannot be controlled by \( \theta_\infty(1 - r) \). This implies that the condition \( \tilde{b}(r) \asymp \theta_\infty(1 - r^2) \) cannot be replaced by \( \tilde{b}(r) \asymp \theta_\infty(1 - r) \) in Theorem 2.16.

Let \( \Omega \) (see [33], [39]) denote the set of all the decreasing functions \( q \in C(0, \infty) \) which satisfy

\[
\int_0^1 (Q(t))^{1/k} dt = \infty \quad \text{and} \quad \int_1^\infty (Q(t))^{1/k} dt < \infty, \quad \text{where} \quad Q(t) = \int_t^\infty q(s) ds.
\]

For more general class \( \Omega \), we have the following results.

**Theorem 2.18.** Under the hypotheses in Theorem 2.11, if we replace \((b_3)\) by the condition

\[
\tilde{b}(r) \asymp q(1 - r^2), \quad r < 1 \text{ close to } 1 \text{ for some } q \in \Omega
\]

and other conditions still hold, then the results of Theorem 2.11 hold, where \( v \) is replaced by

\[
v(t) = \int_t^\infty (kQ(s))^{1/k} ds.
\]

**Theorem 2.19.** Under the hypotheses in Theorem 2.13, if we replace \((b_3)\) by (2.21) and other conditions still hold. Then

(i) When

\[
\liminf_{t \to 0^+} \frac{(kQ(t))^{1+1/k}}{q(t) \int_t^\infty (kQ(s))^{1/k} ds} > 0,
\]

the results of (i) in Theorem 2.13 still hold, where \( v \) is replaced by (2.22).

(ii) When

\[
\limsup_{t \to 0^+} \frac{(Q(t))^{k+1/k}}{q(t)} < \infty,
\]

the results of (ii) in Theorem 2.13 still hold, where \( v \) is replaced by (2.22).

### 3 Some preliminary results

In this section, we collect some well-known results which are important to our results on the boundary behaviors of solutions.

**Lemma 3.1.** Let \( \Omega \) be a bounded domain, \( b \) satisfy \((b_1)\), \( f \) satisfy \((f_1)\) (or \((f_{01})\)) and \( u, v \in C(\bar{\Omega}) \cap C^2(\Omega) \) be \( k \)-convex functions with \( u \leq v \) on \( \partial \Omega \). If for any \( x \in \Omega \),

\[
S_k(D^2u(x)) \geq b(x)f(u(x)) \quad \text{and} \quad S_k(D^2v(x)) \leq b(x)f(v(x)),
\]

then \( u \leq v \) in \( \Omega \).
Proof. The proof is similar to the one of Lemma 2.1 of Jian’s paper [25] and we omit it here. □

Lemma 3.2. (Lemma 14.16 of [59]) Let \( \Omega \) be a bounded domain and \( \partial \Omega \in C^m \) for \( m \geq 2 \). Then there exists a positive constant \( \delta_1 \) depending on \( \Omega \) such that \( d \in C^m(\Omega_{\delta_1}) \), where \( \Omega_{\delta_1} \) is given as shown in (b2) (please refer to page 3), i.e., \( \Omega_{\delta_1} := \{ x \in \Omega : d(x) < \delta_1 \} \).

Lemma 3.3. (Lemma 14.17 of [59]) Let \( \Omega \) and \( \delta_1 \) satisfy the condition of Lemma 3.2 and let \( x \in \Omega_{\delta_1}, \bar{x} \in \partial \Omega \) such that \( |x - \bar{x}| = d(x) \). Then, in terms of a principal coordinate system at \( \bar{x} \), we have

\[
D^2d(x) = \text{diag}\left(-\bar{\mathcal{E}}(x), 0\right) \text{ and } \nabla d(x) = (0, \ldots, 0, 1),
\]

where \( \bar{\mathcal{E}}(x) = (\mathcal{E}_1(x), \ldots, \mathcal{E}_{N-1}(x)) \) is given by (2.5).

By Lemma 3.3, we see that

\[
D_i d(x) = \begin{cases} 
0, & \text{if } i \neq N; \\
-\mathcal{E}_i(x), & \text{if } i = j \neq N; \\
0, & \text{if } i = j = N.
\end{cases}
\]

Lemma 3.4. (Corollary 2.3 of [31]) Let \( h \in C^2(0, \delta_1) \) and \( \Omega \) be bounded with \( \partial \Omega \in C^m \) for \( m \geq 2 \). Assume that \( x \in \Omega_{\delta_1} \) and \( \bar{x} \in \partial \Omega \) is the nearest point to \( x \), i.e., \( |x - \bar{x}| = d(x) \), then

\[
S_k(D^2h(d(x))) = (-h'(d(x)))^k S_k(\bar{\mathcal{E}}(x)) + (-h'(d(x)))^{k-1} h''(d(x)) S_{k-1}(\bar{\mathcal{E}}(x)),
\]

where \( \bar{\mathcal{E}} \) is given by (2.5).

Our proofs of Theorems are given in Sections 4–9. Some auxiliary lemmas and preliminaries of Karamata regular (rapid) variation theory are given in Appendix A and B.

4 Proof of Theorem 2.1

In this section, we prove Theorem 2.1.

Proof. Let

\[
\varepsilon \in (0, (|A_0| + 1)/2)
\]

and

\[
A_\pm := A_0 \pm \varepsilon \text{ and } \tau_\pm := \exp(A_\pm), \quad (4.1)
\]

where \( A_0 \) is given by (2.3). It is clear that

\[
\exp\left(-\frac{3(|A_0| + 1)}{2}\right) < \tau_- < \tau_+ < \exp\left(\frac{3(|A_0| + 1)}{2}\right).
\]

Take \( \delta_* \in (0, \min\{ \delta_1, \bar{\delta} \}/2) \) (\( \delta_1 \) is given in Lemma 3.2 and \( \bar{\delta} \) is given in (b2)), \( \sigma \in (0, \delta_* \) and define

\[
D_-^\sigma := \Omega_{2\delta_*} \backslash \bar{\Omega}_\sigma \text{ and } D_+^\sigma := \Omega_{2\delta_* - \sigma}.
\]

Let

\[
u_+(x) := \tau_+ \psi(\mathcal{B}(x) \Theta_+(d(x))), \quad x \in D_-^\sigma \text{ and }
\]

\[
u_-(x) := \tau_- \psi(\mathcal{B}(x) \Theta_+(d(x))), \quad x \in D_+^\sigma,
\]
where $\mathcal{B}$ is given by (2.4) and 
\[ \Theta_{\pm}(d(x)) := \Theta(d(x)) \mp \Theta(\sigma). \]

Since $\Omega$ is a bounded $C^\infty$-smooth domain, for any $m \geq 4$ we have $\Omega$ is a $C^m$-smooth domain. By Lemma 3.3, we can always adjust $\delta_\xi$ (small enough) such that for any $x \in \bar{\Omega}_d$, there hold
\[ \kappa_i(\bar{x}(x)) = \frac{-D_iu(x)}{1 - d(x)D_{ii}u(x)} \text{ for } i = 1, \ldots, N - 1. \]

It follows by Lemma 3.2 (with $m \geq 4$) that $\kappa_i(x) \in C^{m-2}(\bar{\Omega}_d)$. Moreover, the strict $(k - 1)$-convexity of $\bar{\Omega}$ implies that we can further adjust $\delta_\xi$ such that $S_{k-1}(\bar{\Omega}) > 0$ in $\bar{\Omega}_d$. So, we arrive at $\mathcal{B} \in C^{m-2}(\bar{\Omega}_d)$.

A straightforward calculation shows that
\[
D_{ij}u_{\pm}(x) = \tau_\pm \psi''(\mathcal{B}(x)\Theta_{\pm}(d(x)))\left(D_{ij}\mathcal{B}(x)\cdot D_j\mathcal{B}(x) + D_i\mathcal{B}(x)\cdot D_j\mathcal{B}(x)\right) + \tau_\pm \psi'(\mathcal{B}(x)\Theta_{\pm}(d(x)))\frac{\psi'(\mathcal{B}(x)\Theta_{\pm}(d(x)))}{\psi''(\mathcal{B}(x)\Theta_{\pm}(d(x)))};
\]
(i) if $i \neq j$, $i \neq N$ and $j \neq N$, then
\[
D_{ij}u_{\pm}(x) = \tau_\pm \psi''(\mathcal{B}(x)\Theta_{\pm}(d(x)))\left(D_{ij}\mathcal{B}(x)\cdot D_j\mathcal{B}(x) + \tau_\pm \mathcal{B}(x)\Theta_{\pm}(d(x))\right)
\times \left(D_{ij}\mathcal{B}(x)\cdot D_j\mathcal{B}(x)\right) + \tau_\pm \psi'(\mathcal{B}(x)\Theta_{\pm}(d(x)))\left(D_{ij}\mathcal{B}(x)\cdot \mathcal{B}(x)\Theta_{\pm}(d(x))\right);
\]
(ii) if $i = N$ and $j \neq N$, then
\[
D_{ij}u_{\pm}(x) = \tau_\pm \psi''(\mathcal{B}(x)\Theta_{\pm}(d(x)))\left(D_{ij}\mathcal{B}(x)\cdot D_j\mathcal{B}(x) + \tau_\pm \mathcal{B}(x)\Theta_{\pm}(d(x))\right)
\times \left(D_{ij}\mathcal{B}(x)\cdot D_j\mathcal{B}(x)\right) + \tau_\pm \psi'(\mathcal{B}(x)\Theta_{\pm}(d(x)))\left(D_{ij}\mathcal{B}(x)\cdot \mathcal{B}(x)\Theta_{\pm}(d(x))\right);
\]
(iii) if $i \neq N$ and $j = N$, then
\[
D_{ij}u_{\pm}(x) = \tau_\pm \psi''(\mathcal{B}(x)\Theta_{\pm}(d(x)))\left(D_{ij}\mathcal{B}(x)\cdot D_j\mathcal{B}(x) + \tau_\pm \mathcal{B}(x)\Theta_{\pm}(d(x))\right)
\times \left(D_{ij}\mathcal{B}(x)\cdot D_j\mathcal{B}(x)\right) + \tau_\pm \psi'(\mathcal{B}(x)\Theta_{\pm}(d(x)))\left(D_{ij}\mathcal{B}(x)\cdot \mathcal{B}(x)\Theta_{\pm}(d(x))\right).
\]
(iv) If \( i = j = N \), then
\[
D_{ij}u_+(x) = \tau_+ \psi''(\mathcal{B}(x)\Theta_+(d(x))) \left( \Theta_+(d(x))(D_N \mathcal{B}(x))^2 + 2 \mathcal{B}(x)\Theta_+(d(x)) \right) \\
\times (d(x))D_N \mathcal{B}(x) + \mathcal{B}^2(x)\theta^2(d(x))) + \tau_+ \psi'(\mathcal{B}(x)\Theta_+(d(x))) \\
\times (\Theta_+(d(x))D_{NN} \mathcal{B}(x) + 2\theta(d(x))D_N \mathcal{B}(x) + \mathcal{B}(x)\theta''(d(x))) \\
= \psi''(\mathcal{B}(x)\Theta_+(d(x)))\theta^2(d(x)) \left[ \tau_+ (D_N \mathcal{B}(x))^2 \left( \frac{\Theta_+(d(x))}{\theta(d(x))} \right)^2 \\
+ \frac{2\tau_+ \Theta_+(d(x))}{\theta(d(x))} \mathcal{B}(x)D_N \mathcal{B}(x) + \tau_+ \mathcal{B}^2(x) + \tau_+ \right. \\
\times \psi'(\mathcal{B}(x)\Theta_+(d(x))) \left( \Theta_+(d(x)) \frac{\Theta_+(d(x))}{\theta(d(x))} \right)^2 D_{NN} \mathcal{B}(x) \\
\times \mathcal{B}(x) + 2\mathcal{B}(x)\Theta_+(d(x))D_N \mathcal{B}(x) + \mathcal{B}^2(x) \theta_+(d(x)) \frac{\Theta_+(d(x))}{\theta(d(x))} \right].
\]

(v) If \( i \neq j \), then
\[
D_{ij}u_+(x) = \tau_+ \psi''(\mathcal{B}(x)\Theta_+(d(x))) (D_i \mathcal{B}(x))^2 \Theta_+(d(x)) + \tau_+ \psi' \\
\times (\mathcal{B}(x)\Theta_+(d(x))) (\Theta_+(d(x))D_{ii} \mathcal{B}(x) + \mathcal{B}(x)\theta(d(x))D_{ii} \mathcal{B}(x)) \\
= -\psi'(\mathcal{B}(x)\Theta_+(d(x)))\theta(d(x))A_i^+(x),
\]

where
\[
A_i^+(x) = -\tau_+ \frac{\psi''(\mathcal{B}(x)\Theta_+(d(x))) (D_i \mathcal{B}(x))^2 \Theta_+(d(x))}{\psi'(\mathcal{B}(x)\Theta_+(d(x))) \mathcal{B}(x)} \left( \Theta_+(d(x)) \frac{\Theta_+(d(x))}{\theta(d(x))} \right)^2 D_{NN} \mathcal{B}(x) \\
+ \frac{D_{ii} \mathcal{B}(x) + \mathcal{B}(x) \Theta_+(d(x))\theta(d(x))}{\theta(d(x))} \frac{\Theta_+(d(x))}{\theta(d(x))} \frac{\Theta_+(d(x))}{\theta(d(x))} \right].
\]

The condition (b₂) implies that \( \theta \) is a non-increasing function. We conclude by the above (i)–(v), Lemma B.1, Lemma B.3 (i), (ii), (ix)–(xii) (see Appendix B) and the calculation of the sum of principal minors of size \( k \) to the Hessian \( D^2u_+(x) \) that
\[
S_k(D^2u_+(x)) = \psi''(\mathcal{B}(x)\Theta_+(d(x)))\theta^2(d(x)) \left[ \tau_+ (D_N \mathcal{B}(x))^2 \left( \frac{\Theta_+(d(x))}{\theta(d(x))} \right)^2 + 2\tau_+ \Theta_+(d(x)) \mathcal{B}(x) \\
\times D_N \mathcal{B}(x) + \tau_+ \mathcal{B}^2(x) + \tau_+ \psi'(\mathcal{B}(x)\Theta_+(d(x))) \left( \Theta_+(d(x)) \frac{\Theta_+(d(x))}{\theta(d(x))} \right)^2 D_{NN} \mathcal{B}(x) \\
+ 2\mathcal{B}(x) \Theta_+(d(x)) D_N \mathcal{B}(x) + \mathcal{B}^2(x) \theta_+(d(x)) \frac{\Theta_+(d(x))}{\theta(d(x))} \right] \\
\times \left[ (-\psi'(\mathcal{B}(x)\Theta_+(d(x)))k^k-1\theta^{k-1}(d(x))S_{k-1}(A_i^+(x), ..., A_{N-1}^+(x)) \\
+ O\left(-\psi'(\mathcal{B}(x)\Theta_+(d(x)))k^k-2\theta^{k-2}(d(x))\psi''(\mathcal{B}(x)\Theta_+(d(x)))\Theta_+(d(x)) \right) \right] \\
+ O\left((-\psi'(\mathcal{B}(x)\Theta_+(d(x)))k^k\theta^k(d(x))) \right) = (-\psi'(\mathcal{B}(x)\Theta_+(d(x)))k^k-1\psi''(\mathcal{B}(x)\Theta_+(d(x)))\theta^{k+1}(d(x)) \\
\times \left[ \frac{\tau_+ \mathcal{B}^2(x)\psi'(\mathcal{B}(x)\Theta_+(d(x))) \Theta_+(d(x))\theta(d(x))}{\psi'(\mathcal{B}(x)\Theta_+(d(x))) \mathcal{B}(x)\Theta_+(d(x)) \Theta_+(d(x)) g(\mathcal{B}(x)\Theta_+(d(x))) + \tau_+ D_N \mathcal{B}(x)^2 \mathcal{B}(x) \Theta_+(d(x)) \Theta_+(d(x))} \\
\Theta_+(d(x)) \mathcal{B}(x) \Theta_+(d(x)) \end{array} \right].
\[
\times \frac{g'((\mathcal{R}(x)\Theta_z(d(x))))}{\theta(d(x))} D_N \mathcal{R}(x) + \frac{\tau_x g'((\mathcal{R}(x)\Theta_z(d(x))))}{\theta''((\mathcal{R}(x)\Theta_z(d(x)))) (\mathcal{R}(x)\Theta_z^2(d(x)))}
\]
\[
\times \frac{\tau_x g'((\mathcal{R}(x)\Theta_z(d(x))))}{\theta''(d(x))} D_{NN} \mathcal{R}(x) + \frac{2\mathcal{R}(x)\Theta_z(d(x))D_N \mathcal{R}(x)}{g((\mathcal{R}(x)\Theta_z(d(x))))} \left( \frac{\mathcal{R}(x)\Theta_z^2(d(x))}{g((\mathcal{R}(x)\Theta_z(d(x))))} \right) \right]
\]
\[
\times \left[ \frac{t - 1}{2} \frac{\mathcal{R}(x)\Theta_z(d(x))}{\theta''(d(x))} \right] + O \left( \frac{\mathcal{R}(x)\Theta_z(d(x))}{\theta''(d(x))} \right)
\]
\[
= (-\mathcal{R}'((\mathcal{R}(x)\Theta_z(d(x))))^k - \mathcal{R}''((\mathcal{R}(x)\Theta_z(d(x))))^k + 1 + o(g((\mathcal{R}(x)\Theta_z(d(x))))))
\]
\[
\times \left( S_{k-1}(\tilde{\mathcal{R}}(x)) + o(g((\mathcal{R}(x)\Theta_z(d(x)))))) + o(g((\mathcal{R}(x)\Theta_z(d(x)))))) \right)
\]
\[
= (-\mathcal{R}'((\mathcal{R}(x)\Theta_z(d(x))))^k - \mathcal{R}''((\mathcal{R}(x)\Theta_z(d(x))))^k + 1 + o(g((\mathcal{R}(x)\Theta_z(d(x))))))
\]
\[
\times \left[ \frac{\mathcal{R}'((\mathcal{R}(x)\Theta_z(d(x))))}{\mathcal{R}''((\mathcal{R}(x)\Theta_z(d(x))))} \right] S_{k-1}(\tilde{\mathcal{R}}(x)) + S_{k-1}(\tilde{\mathcal{R}}(x))
\]
\[
+ o(g((\mathcal{R}(x)\Theta_z(d(x)))))) = f_1((\mathcal{R}(x)\Theta_z(d(x)))g((\mathcal{R}(x)\Theta_z(d(x))))^k + 1 + o(g((\mathcal{R}(x)\Theta_z(d(x))))))
\]
\[
\times \frac{g((\mathcal{R}(x)\Theta_z(d(x))))}{k + 1} \left( 1 + \frac{f_1((\mathcal{R}(x)\Theta_z(d(x)))}{f_1((\mathcal{R}(x)\Theta_z(d(x))))} \right) S_{k-1}(\tilde{\mathcal{R}}(x)) + o(1)
\]

This implies that
\[
S_k(D^2 u_+(x)) - b(x) f(u_+(x))
\]
\[
= f_1((\mathcal{R}(x)\Theta_z(d(x)))g((\mathcal{R}(x)\Theta_z(d(x))))^k + 1 + o(g((\mathcal{R}(x)\Theta_z(d(x))))))
\]
\[
\times \left[ \frac{\mathcal{R}'((\mathcal{R}(x)\Theta_z(d(x))))}{\mathcal{R}''((\mathcal{R}(x)\Theta_z(d(x))))} \right] S_{k-1}(\tilde{\mathcal{R}}(x)) + \frac{1}{k + 1} (g((\mathcal{R}(x)\Theta_z(d(x)))) - b(x)(f_1(u_+(x)) + f_2(u_+(x)))
\]
\[
\leq f_1((\mathcal{R}(x)\Theta_z(d(x)))g((\mathcal{R}(x)\Theta_z(d(x))))^k + 1 + o(g((\mathcal{R}(x)\Theta_z(d(x))))))
\]
\[
\times \theta^{k+1}(d(x)) r^k_+ \mathcal{R}(x) I_-(\sigma, x),
\]

and
\[
S_k(D^2 u_-(x)) - b(x) f(u_-(x)) \geq f_1((\mathcal{R}(x)\Theta_+(d(x)))g((\mathcal{R}(x)\Theta_+(d(x))))^k + 1 + o(g((\mathcal{R}(x)\Theta_+(d(x))))))
\]
\[
\times \theta^{k+1}(d(x)) r^k_+ \mathcal{R}(x) I_+(\sigma, x),
\]
where

\[
I_\pm(\sigma, x) = S_{k-1}(\tilde{\xi}(x)) - \left( \frac{\psi(\xi(x)\Theta_\pm(d(x)))f_1(\psi(\xi(x)\Theta_\pm(d(x))))}{f_1(\psi(\xi(x)\Theta_\pm(d(x))))g(\psi(\xi(x)\Theta_\pm(d(x))))\Theta_\pm(d(x))} \right)_{-\infty}^x \\
\times \Theta(d(x))\theta'(d(x)) + \frac{(g(\psi(\xi(x)\Theta_\pm(d(x))))^{-1}}{k+1} - \frac{f_1(\tau_k \psi(\xi(x)\Theta_\pm(d(x))))}{\tau_k f_1(\psi(\xi(x)\Theta_\pm(d(x))))} - \frac{f_1(\tau_k \psi(\xi(x)\Theta_\pm(d(x))))}{\tau_k f_1(\psi(\xi(x)\Theta_\pm(d(x))))} + o(1).
\]

By Lemma B.1 (i), Lemma B.3 (vi)–(viii) (see Appendix B) and (4.1), we obtain

\[
\lim_{(\sigma, x) \to (0, 0)} I_\pm(\sigma, x) = (\mathcal{A}_0 - \ln \tau_k)S_{k-1}(\tilde{k}(x)) = \mp \in S_{k-1}(\tilde{k}(x)),
\]

where \(\tilde{k}(x)\) is given by (1.2). So, we can take small enough constants \(\delta_k \leq \delta_k\) and \(\sigma \in (0, \delta_k)\) such that \(u_+\) and \(u_-\) are, respectively, supersolution and subsolution in \(D_-^\sigma\) and \(D_+^\sigma\), where

\[
D_-^\sigma := \Omega_{\tilde{\delta}_k} \setminus \tilde{\Omega}_\sigma \quad \text{and} \quad D_+^\sigma := \Omega_{\tilde{\delta}_k - \sigma}.
\]

In fact, we can always adjust \(\delta_k\) such that for \(j = 1, \ldots, k\),

\[
S_j(D^2u_\pm(x)) = (-\psi'(\xi(x)\Theta_\pm(d(x))))^{-1}\psi''(\xi(x)\Theta_\pm(d(x)))\theta^j(d(x))\tau_k^j \\
\times \Theta_\pm^j(\tilde{x}(\xi(x)\Theta_\pm(d(x)))) + S_{j-1}(\tilde{\xi}(x)) + S_{j-1}(\tilde{\xi}(x)) + o(g(\psi(\xi(x)\Theta_\pm(d(x)))))) > 0, \quad x \in D_-^\sigma,
\]

i.e., \(u_+\) and \(u_-\) are strictly \(k\)-convex upper and lower solutions of problem (1.1) in \(D_-^\sigma\) and \(D_+^\sigma\), respectively.

Let \(u\) be an arbitrary \(k\)-convex solution to problem (1.1). Now, we prove there exists a large positive constant \(\mathcal{M}\) such that

\[
u(x) \leq u_+(x) + \mathcal{M}, \quad x \in D_-^\sigma \quad \text{and} \quad u_-(x) \leq u(x) + \mathcal{M}, \quad x \in D_+^\sigma.
\]

It is clear that we can choose some large positive constant \(\mathcal{M}\) (independent of \(\sigma\)) such that

\[
u \leq u_+ + \mathcal{M} \quad \text{on} \quad \{x \in \Omega: d(x) = 2\tilde{\delta}_k\}
\]

and

\[
u_- \leq u + \mathcal{M} \quad \text{on} \quad \{x \in \Omega: d(x) = 2\tilde{\delta}_k - \sigma\}.
\]

On the other hand, it is easy to see

\[
u < u_+ = \infty \quad \text{on} \quad \{x \in \Omega: d(x) = \sigma\} \quad \text{and} \quad \nu_- < u = \infty \quad \text{on} \quad \partial\Omega.
\]

Take a small enough constant \(\rho \in (0, \delta_k)\) such that

\[
\sup_{x \in D_-^\sigma} u(x) \leq u_+(x), \quad x \in D_-^\sigma \setminus \tilde{D}_-^\rho \quad \text{and} \quad \sup_{x \in D_+^\sigma} u_-(x) \leq u(x), \quad x \in D_+^\sigma \setminus \tilde{D}_+^\rho.
\]
where
\[ \bar{D}^0 := \Omega_{2\delta_r} \setminus \bar{\Omega}_{(1+\rho)\sigma} \quad \text{and} \quad \bar{D}_+^0 := \Omega_{2\delta_r, -\sigma} \setminus \bar{\Omega}_\sigma. \]

It follows from (f1) (or (f2)) we see that \( u_x + M \) and \( u + M \) are both supersolutions in \( \bar{D}_-^0 \) and \( \bar{D}_+^0 \), respectively. By (4.4)–(4.6) and Lemma 3.1, we obtain
\[ u \leq u_+ + \mathcal{M} \text{ in } \bar{D}_-^0 \quad \text{and} \quad u_- \leq u + \mathcal{M} \text{ in } \bar{D}_+^0. \]

This fact, combined with (4.6), shows that (4.3) holds. Passing to \( \sigma \to 0 \), we have for any \( x \in \Omega_{2\delta_r}, \) there hold
\[ \frac{u(x)}{\psi(\mathcal{B}(x)\Theta(d(x)))} \leq \tau_+ + \frac{\mathcal{M}}{\psi(\mathcal{B}(x)\Theta(d(x)))} \quad \text{and} \quad \frac{u(x)}{\psi(\mathcal{B}(x)\Theta(d(x)))} \geq \tau_- - \frac{\mathcal{M}}{\psi(\mathcal{B}(x)\Theta(d(x)))}. \]

By Lemma B.3 (i) (see Appendix B), we arrive at
\[ \limsup_{\psi(\mathcal{B}(x)\Theta(d(x))) \to 0} \frac{u(x)}{\psi(\mathcal{B}(x)\Theta(d(x)))} \leq \tau_+ \quad \text{and} \quad \liminf_{\psi(\mathcal{B}(x)\Theta(d(x))) \to 0} \frac{u(x)}{\psi(\mathcal{B}(x)\Theta(d(x)))} \geq \tau_- . \]

Letting \( \varepsilon \to 0 \), we obtain (2.1). \( \Box \)

5 Proof of Theorem 2.5

In this section, we prove Theorem 2.5. Since \( \Gamma \)-varying functions are rapidly varying to infinity at infinity, we find by Lemma B.4 (iv) (see Appendix B) that \( \mathcal{L} \) is slowly varying at zero. We know that slowly varying functions are insensitive to the perturbations of independent variable. So, we will prove the first expansion of solutions by using a simpler method than the above.

Proof. Define
\[ M_k = \max_{x \in \Omega} S_{k-1}(\bar{k}(x)) \quad \text{and} \quad m_k = \min_{x \in \Omega} S_{k-1}(\bar{k}(x)), \quad (5.1) \]
where \( \bar{k}(x) \) is given by (1.2). Let
\[ \varepsilon \in (0, \min\{1, b_1/2(C_0 + k + 1)\}), \quad C_0 > \frac{b_1 + b_2}{2} \]
and
\[ \tau_- := \left( \frac{b_1 - (k + 1 + C_0)\varepsilon}{2D_0(k + 1)M_k} \right)^{1/(k+1)} \quad \text{and} \quad \tau_+ := \left( \frac{b_1 + (k + 1 + C_0)\varepsilon}{2D_0(k + 1)m_k} \right)^{1/(k+1)}. \]

It follows that
\[ \left( \frac{b_1}{2D_0(k + 1)M_k} \right)^{1/(k+1)} < \tau_- < \tau_+ < \left( \frac{3b_2}{2D_0(k + 1)m_k} \right)^{1/(k+1)} . \]

As before, for any \( \delta > 0 \), we define
\[ \Omega_\delta := \{ x \in \Omega : 0 < d(x) < \delta \}. \]

Next, we consider the following two cases.

Case (I) \( \theta \) is non-increasing on \((0, \delta_0] \). From (b2), Lemma B.1 (i) and Lemma B.4 (i) and (iv) (see Appendix B), we see that there exists sufficiently small \( \delta_r \) such that for any \( (x, r) \in \Omega_{2\delta_r} \times (0, \delta_r) \), there hold
\[ \frac{m_k}{1 + \varepsilon} < S_{k-1}(\bar{k}(x)) < \frac{M_k}{1 - \varepsilon}; \quad (5.2) \]
\[
\frac{b_1 - \epsilon C_0}{1 - \epsilon} < \frac{b(x)}{\theta^{k+1}(d(x))} < \frac{b_2 + \epsilon C_2}{1 + \epsilon};
\]

(5.3)

\[
\Lambda'(1/(r_\tau \Theta_\tau(d(x)))) > 0,
\quad \Lambda''(1/(r_\tau \Theta_\tau(d(x)))) < 0;
\]

(5.4)

\[
\Lambda^I(x, \Theta_\tau(d(x))) := \frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))} \frac{\tau^k \Lambda^I(1/(r_\tau \Theta_\tau(d(x))))}{\Lambda^I(1/(r_\tau \Theta_\tau(d(x))))(1/(r_\tau \Theta_\tau(d(x))))} - \frac{2\tau^{k+1} \Lambda'(1/(r_\tau \Theta_\tau(d(x))))}{\Lambda''(1/(r_\tau \Theta_\tau(d(x))))(1/(r_\tau \Theta_\tau(d(x))))} > 0 \text{ for } i = 1, \cdots, k;
\]

(5.5)

\[
\Lambda^I(x, \Theta_\tau(d(x))) := \frac{\Lambda'(1/(r_\tau \Theta_\tau(d(x))))}{\Lambda''(1/(r_\tau \Theta_\tau(d(x))))(1/(r_\tau \Theta_\tau(d(x))))} \frac{\Theta(d(x))\theta'(d(x))}{\theta^2(d(x))} \left(1 + \frac{\Lambda^I(1/(r_\tau \Theta_\tau(d(x))))}{\Lambda^I(1/(r_\tau \Theta_\tau(d(x))))(1/(r_\tau \Theta_\tau(d(x))))} \right)
\]

(5.6)

Let \( \sigma \in (0, \delta) \) and define

\[
u_\tau(x) = \Lambda(1/(r_\tau \Theta_\tau(d(x)))) \text{ for } x \in D^\tau_\tau, \text{ with } \Theta_\tau(d(x)) = \Theta(d(x)) + \Theta(\sigma),
\]

where \( D^\tau_\tau \) are defined as (4.2). By Lemma 3.4, (5.2)–(5.7), we have for any \( x \in D^\tau_\tau \), there holds

\[
S_k(D^2 u_\tau(x)) - b(x)f(u_\tau(x)) \leq -\Lambda'(1/(r_\tau \Theta_\tau(d(x)))) - \Lambda''(1/(r_\tau \Theta_\tau(d(x))))(1 - \epsilon)\theta^{k+1}(d(x))
\]

(5.7)
i.e., \( u_- \) is a supersolution of Eq. (1.1) in \( D^- \). In a similar way, we can show that \( u_+ \) is a subsolution of Eq. (1.1) in \( D^+_\). Moreover, \( \mathcal{A}^+(x, \Theta_\tau(d(x))) > 0 \) for \( i = 1, \ldots, k \) imply that \( u_- \) and \( u_+ \) are strictly \( k \)-convex in \( D^- \) and \( D^+_\), respectively.

**Case (II) \( \theta \) is non-decreasing on \( (0, \delta_3) \).** From Lemma B.1 and Lemma B.4 (see Appendix B), we see that there exists sufficiently small \( \delta_2 < \min \{ \delta_0, \delta_1 \} / 2 \) such that for any \( x \in \Omega_{2b_2}, \) (5.2), (5.3) and the following hold

\[
\mathcal{L}'(1/(\tau_\Theta(d(x)))) > 0 \quad \text{and} \quad \mathcal{L}'(1/(\tau_\Theta(d(x)))) < 0,
\]

(5.8)

\[
\mathcal{A}^+(x, \Theta(d(x))) > 0 \quad \text{for} \quad i = 1, \ldots, k, \quad \mathcal{A}^+(x, \Theta(d(x))) < \varepsilon / 2
\]

(5.9)

and

\[
\mathcal{A}_2^+(\Theta(d(x))) < \varepsilon / 2(b_2 + c_0),
\]

(5.10)

where \( \mathcal{A}^+, \mathcal{A}^+_2 \) are defined as shown in (5.5)-(5.7).

Let \( \sigma \in (0, \delta_3) \) and define

\[
u(x) = \mathcal{L}(1/(x_\Theta(d(x)))), \quad x \in D^+_\eta,
\]

where \( d_\eta(x) = d(x) + \sigma \) and \( D^+_\eta \) are defined as (4.2). Combining (5.8)–(5.10) with (5.2), (5.3), we obtain by Lemma 3.4 for any \( x \in D^+ \), there holds

\[
S_\eta(D_\nu u_\nu(x)) - b_\nu f(u_\nu(x))
\]

\[
= -(\mathcal{L}'(1/(\tau_\Theta(d(x))))^{k-1} \mathcal{L}''(1/(\tau_\Theta(d(x))))(\tau_\Theta(d(x)))^{-2k+1}(1 - \varepsilon)^{-1} \theta^{k+1}(d(x))
\]

\[
\times \left[ \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \left( \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \right)^k \right]
\]

\[
\leq -\left(\mathcal{L}'(1/(\tau_\Theta(d(x))))^{k-1} \mathcal{L}''(1/(\tau_\Theta(d(x))))(\tau_\Theta(d(x)))^{-2k+1}(1 - \varepsilon)^{-1} \theta^{k+1}(d(x))
\]

\[
\times \left[ \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \left( \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \right)^k \right]
\]

\[
\leq -\left(\mathcal{L}'(1/(\tau_\Theta(d(x))))^{k-1} \mathcal{L}''(1/(\tau_\Theta(d(x))))(\tau_\Theta(d(x)))^{-2k+1}(1 - \varepsilon)^{-1} \theta^{k+1}(d(x))
\]

\[
\times \left[ \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \left( \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \right)^k \right]
\]

\[
\leq -\left(\mathcal{L}'(1/(\tau_\Theta(d(x))))^{k-1} \mathcal{L}''(1/(\tau_\Theta(d(x))))(\tau_\Theta(d(x)))^{-2k+1}(1 - \varepsilon)^{-1} \theta^{k+1}(d(x))
\]

\[
\times \left[ \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \left( \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \right)^k \right]
\]

\[
\leq -\left(\mathcal{L}'(1/(\tau_\Theta(d(x))))^{k-1} \mathcal{L}''(1/(\tau_\Theta(d(x))))(\tau_\Theta(d(x)))^{-2k+1}(1 - \varepsilon)^{-1} \theta^{k+1}(d(x))
\]

\[
\times \left[ \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \left( \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \right)^k \right]
\]

\[
\leq -\left(\mathcal{L}'(1/(\tau_\Theta(d(x))))^{k-1} \mathcal{L}''(1/(\tau_\Theta(d(x))))(\tau_\Theta(d(x)))^{-2k+1}(1 - \varepsilon)^{-1} \theta^{k+1}(d(x))
\]

\[
\times \left[ \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \left( \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \right)^k \right]
\]

\[
\leq -\left(\mathcal{L}'(1/(\tau_\Theta(d(x))))^{k-1} \mathcal{L}''(1/(\tau_\Theta(d(x))))(\tau_\Theta(d(x)))^{-2k+1}(1 - \varepsilon)^{-1} \theta^{k+1}(d(x))
\]

\[
\times \left[ \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \left( \frac{\mathcal{L}'(1/(\tau_\Theta(d(x))))}{\mathcal{L}''(1/(\tau_\Theta(d(x))))(1/(\tau_\Theta(d(x))))} \right)^k \right]
\]
Proof. Let \( u \) be a supersolution of Eq. (1.1) in \( D_-^\varepsilon \). In a similar way, we can show that \( u_- \) is a supersolution of Eq. (1.1) in \( D_-^\varepsilon \). As before, \( \Psi_i(x, \Theta(d_\varepsilon(x))) > 0 \) for \( i = 1, \cdots, k \) imply that \( u_- \) and \( u_+ \) are strictly \( k \)-convex in \( D_-^\varepsilon \) and \( D_-^\varepsilon \), respectively.

For Cases (I)-(II), let \( u \) be an arbitrary \( k \)-convex solution to problem (1.1). Now, we show there exists a large constant \( \mathcal{M} > 0 \) independent of \( \sigma \) such that

\[
 u(x) \leq u_-(x) + \mathcal{M}, \ x \in D_-^\varepsilon \quad \text{and} \quad u_+(x) \leq u(x) + \mathcal{M}, \ x \in D_-^\varepsilon.
\]

By similar arguments as in the proof of Theorem 2.1, we obtain for any \( x \in \Omega_{\delta \varepsilon} \), the following hold

\[
 \frac{u(x)}{\Sigma(1/(r_\varepsilon \Theta(d_\varepsilon(x)))} \leq 1 + \frac{\mathcal{M}}{\Sigma(1/(r_\varepsilon \Theta(d_\varepsilon(x)))} \quad \text{and} \quad \frac{u(x)}{\Sigma(1/(r_\varepsilon \Theta(d_\varepsilon(x)))} \geq 1 - \frac{\mathcal{M}}{\Sigma(1/(r_\varepsilon \Theta(d_\varepsilon(x)))}.
\]

Consequently, from (2.8), Lemma B.4 (iv) (see Appendix B) and Proposition A.2 (see Appendix A), we have

\[
 \lim_{d_\varepsilon \to 0} \frac{u(x)}{\Sigma(1/(\Theta(d_\varepsilon)))} = 1.
\]

This implies that (2.7) holds. The proof is finished. \( \square \)

6 Proof of Theorem 2.7

In this section, we prove Theorem 2.7.

Proof. Let

\[
 \varepsilon \in (0, \min\{1, b_3/2(C_1 + k + 1)\}), \ C_1 > \frac{b_3 + b_4}{2}
\]

and

\[
 r_- := \left( \frac{(b_3 - (C_1 + k + 1)\varepsilon)(k + 1)^{k-1}}{k^k M_k} \right)^{1/(k+1)},
\]

\[
 r_+ := \left( \frac{(b_4 + (C_1 + k + 1)\varepsilon)(k + 1)^{k-1}}{k^k m_k} \right)^{1/(k+1)},
\]

where \( M_k \) and \( m_k \) are given by (5.1). So, we have

\[
 \left( \frac{b_3(k + 1)^{k-1}}{2k^k M_k} \right)^{1/(k+1)} < r_- < r_+ < \left( \frac{3b_4(k + 1)^{k-1}}{2k^k m_k} \right)^{1/(k+1)}.
\]

From (h_3), Lemma B.4 and Lemma B.5 (see Appendix B), we see there exist small enough \( \delta_1, \delta_2 \in (0, \min\{ \bar{\delta}_1, \bar{\delta}_2 \}/2) \) (where \( \bar{\delta}_1 \) is given in Remark 2.6) such that for any

\[
 (x, t, r) \in \Omega_{\delta_1} \times (0, \delta_1^2) \times (0, 2\delta_1^2),
\]

Equation (5.2) and the following hold

\[
 \frac{b_3 - \varepsilon C_1}{1 - \varepsilon} < \frac{b(x)}{(d(x))^{-(k+1)} L'(d(x))} < \frac{b_4 + \varepsilon C_1}{1 + \varepsilon}; \quad \text{(6.1)}
\]

\[
 L'(1/r) > 0 \quad \text{and} \quad \frac{\Sigma'(1/r)}{(1/r) L''(1/r)} < 0; \quad \text{(6.2)}
\]
\[ t^{k+1} \mathcal{B} = \left[ -\left( \frac{k}{k+1} \right)^{k+1} L(t) \int_0^{L(t)} \frac{L(t)}{s} ds - \frac{2k+1}{k+1} \left( \frac{k}{k+1} \right)^k \frac{L'(1/r)}{L''(1/r)(1/r)} \int_0^{L(t)} \frac{L(t)}{s} ds \right] \]

\[ - \left( \frac{k}{k+1} \right)^k \frac{L'(1/r)}{L''(1/r)(1/r)} \left( 1 - \frac{L'(t)}{L(t)} \right) > 0; \quad (6.3) \]

\[ \mathfrak{B}_T(x, t, r) = t^{k+1} \left[ \left( \frac{k}{k+1} \right)^k \frac{L'(1/r)}{L''(1/r)(1/r)} S_k(\tilde{B}_t(x)) + \left( \frac{k}{k+1} \right)^k \frac{L'(1/r)}{L''(1/r)(1/r)} \right] \times \int_0^{L(t)} L'(t) \frac{L(t)}{s} ds - \frac{2k+1}{k+1} \left( \frac{k}{k+1} \right)^k \frac{L'(1/r)}{L''(1/r)(1/r)} \int_0^{L(t)} L(t) \frac{L(t)}{s} ds \]

\[ + \left( \frac{k}{k+1} \right)^k \frac{L'(1/r)}{L''(1/r)(1/r)} \frac{L'(1/r)}{L'(1/r)} + M_k \left( \frac{L'}{L''} \right) < \frac{\varepsilon}{2} \quad (6.4) \]

and

\[ \mathfrak{B}_T(r) = \left| \frac{f(\Sigma(1/r))}{h^s(\Sigma(1/r)(1/r)^k+1) \Sigma''(1/r)(1/r)} \left( \frac{h(\Sigma(1/r))}{\Sigma''(1/r)(1/r)} \right)^k + \frac{1}{k+1} \right| \]

\[ < \frac{\varepsilon}{2(b_4 + c_4)} \quad (6.5) \]

Take \( \sigma \in (0, \delta^2) \) with

\[ \sigma < \left( \frac{b_3(k+1)^{k+1}}{2k^k M_k} \right)^{1/(k+1)} \left( \int_0^{L(s)} \frac{L(s)}{s} ds \right)^{k/(k+1)} \]

and let \( u \) be an arbitrary \( k \)-convex solution to problem (1.1). Define

\[ \mathcal{D}^- := \Omega_0 \setminus \Omega^x, \quad \mathcal{D}^+ := \Omega_0 \setminus \Omega^x \]

where

\[ \Omega^x = \left\{ x \in \Omega_{\delta^2}; \tau_+ \left( \int_0^{L(s)} \frac{L(s)}{s} ds \right)^{k/(k+1)} \leq \sigma \right\} \]

and \( \delta^x \in (0, \delta^2) \) is an appropriate constant such that

\[ \mathcal{L} \left[ \tau_+ \left( \int_0^{L(s)} \frac{L(s)}{s} ds \right)^{k/(k+1)} + \sigma \right]^{-1} \leq u(x), \quad x \in \Omega_{\delta^2 - \delta^x}. \]
In fact, by adjusting $\delta_\varepsilon^2$ if necessary, we always assume that

$$\left(\frac{3b_1(k+1)^{k-1}}{2k^k m_k}\right)^{1/(k+1)} \left(\int_0^{d(x)} \frac{L(s)}{s} \, ds\right)^{k/(k+1)} < \delta_\varepsilon^2, \; x \in \Omega_{\delta_\varepsilon}.$$ 

Let

$$u_\pm(x) = \mathcal{L}(1/r_\pm), \; x \in \mathcal{D}_\sigma^\varepsilon,$$

where

$$r_\pm(d(x)) = r_\pm\left(\int_0^{d(x)} \frac{L(s)}{s} \, ds\right)^{k/(k+1)} \mp \sigma.$$

By Lemma 3.4, (5.2) and (6.1)–(6.5), we obtain for any $x \in \mathcal{D}_\sigma^\varepsilon$, there holds

$$S_\varepsilon(D^2 u_\mp(x)) - b(x) f(u_\mp(x))$$

$$= -(\mathcal{L}'(1/r_-) r_-^{2(k+1)} (d(x))^{-k+1} L^k(d(x)) - r_-^{k+1} \left(\frac{k}{k+1}\right)^k \mathcal{L}'(1/r_-) r_-^{k+1} L^k(d(x))$$

$$\times \frac{\mathcal{L}'(1/r_-)}{\mathcal{L}'(1/r_-) - \mathcal{L}(d(x))} \frac{\mathcal{L}(d(x))}{d(x)} \frac{\mathcal{L}'(1/r_-) - \mathcal{L}(d(x))}{d(x)}$$

$$\times \left(\frac{k}{k+1}\right)^k d(x) S_\varepsilon(\mathcal{E}(x)) (1 - \varepsilon)^{-1} \left\{ -r_-^{k+1} \left(\frac{k}{k+1}\right)^k \frac{\mathcal{L}'(1/r_-)}{\mathcal{L}'(1/r_-) - \mathcal{L}(d(x))} \frac{\mathcal{L}(d(x))}{d(x)} \right\}$$

$$\leq -(\mathcal{L}'(1/r_-) r_-^{2(k+1)} (d(x))^{-k+1} L^k(d(x))(1 - \varepsilon)^{-1} \left\{ -r_-^{k+1} \left(\frac{k}{k+1}\right)^k \frac{\mathcal{L}'(1/r_-)}{\mathcal{L}'(1/r_-) - \mathcal{L}(d(x))} \frac{\mathcal{L}(d(x))}{d(x)} \right\}$$

$$\times \left(\frac{k}{k+1}\right)^k d(x) S_\varepsilon(\mathcal{E}(x)) (1 - \varepsilon) + \left[-r_-^{k+1} \left(\frac{k}{k+1}\right)^k \frac{\mathcal{L}(d(x))}{d(x)} \frac{\mathcal{L}'(1/r_-)}{\mathcal{L}'(1/r_-) - \mathcal{L}(d(x))} \frac{\mathcal{L}(d(x))}{d(x)} \right] M_k$$

$$\times \left[ b(x) - \varepsilon C_1 \right] \frac{f(\mathcal{L}(1/r_-))}{\mathcal{L}'(1/r_-) - \mathcal{L}(d(x))} \frac{\mathcal{L}(d(x))}{d(x)}$$
\[ \begin{align*}
&\times \left\{ -\frac{\mathcal{L}'(1/r_\pm)}{\mathcal{L}''(1/r_\pm)(1/r_\pm)} \left[ \frac{d(x)}{r_\pm \pm \sigma} \right] S_j(\bar{\sigma}(x)) \right. \\
&\times \left[ -\left( \frac{k r_\pm}{k+1} \right) \left( \frac{2 \mathcal{L}'(1/r_\pm)}{\mathcal{L}''(1/r_\pm)(1/r_\pm)} + 1 \right) \frac{L(d(x))}{d(x)} \right. \\
&\left. \left. - \frac{1}{k+1} \frac{\mathcal{L}'(1/r_\pm)}{\mathcal{L}''(1/r_\pm)(1/r_\pm)} \frac{r_\pm}{r_\pm \pm \sigma} \frac{L'(d(x))}{d(x)} \right) \right]\ \\
&\times S_{j-1}(\bar{\sigma}(x)) \bigg) > 0 \text{ for } j = 1, \ldots, k.
\end{align*} \]

We see that \( u_- \) and \( u_+ \) are strictly \( k \)-convex in \( \mathcal{D}^+ \) and \( \mathcal{D}^- \), respectively.

The following process is similar to that of Theorem 2.5. So we omit it here. \( \square \)

7 Proof of Theorem 2.9

Proof. Let

\[ \mathcal{N}(r) := \int_0^r \left[ t^{k-N} \int_0^t \left( t^{k-N-1} \mathcal{L}^{N-1}(s) \right) ds \right]^{1/k} dt, \quad r \in [0, 1). \]

For any \( a > 0 \), let

\[ P_a(t) := \int_a^t (F(s))^{-(1/k+1)} ds, \quad t \geq a. \quad (7.1) \]

From (1.7), we get

\[ P_a(\infty) := \lim_{t \to \infty} P_a(t) = \infty \text{ and } P'_a(t) = (F(t))^{-(1/k+1)} > 0, \quad t > a. \]
This implies that \( F_a \) has the increasing inverse function \( F_a^{-1} \) on \([0, \infty)\) with
\[
F_a^{-1}(0) = a \quad \text{and} \quad F_a^{-1}(\infty) := \lim_{t \to \infty} F_a^{-1}(t) = \infty. \tag{7.2}
\]

A straightforward calculation shows that \( v \in C^2(0, 1) \) is a positive radial solution to Eq. (1.1) if and only if \( v \) is the solution of the following initial problem
\[
C_{N-1} \left( \frac{v'(r)}{r} \right)^{k-1} v'' + C_{N-1} \left( \frac{v'(r)}{r} \right)^k = k^{-1} r^{N-k} (v')^k = \bar{b}(r) f(v(r)), \quad r \in [0, 1), \quad v(0) > 0, \quad v'(0) = 0.
\]

So, we investigate the existence of the above problem with the following initial conditions
\[
v(0) = a > 0 \quad \text{and} \quad v'(0) = 0. \tag{7.4}
\]

Problem (7.3) with (7.4) is equivalent to
\[
v(r) = a + \int_0^r \left( t^{N-1} \int_0^t (C_{N-1})^{k-1} ks^{N-1} \bar{b}(s) f(v(s)) ds \right)^{1/k} dt, \quad r \in [0, 1).
\]

To establish the existence of the positive \( k \)-convex solution to this equation, we use successive approximation method. Let \( \{ v_m \}_{m \geq 1} \) be the sequence of positive continuous functions defined by
\[
v_1(r) = a,
\]
\[
v_2(r) = a + \int_0^r \left( t^{N-1} \int_0^t (C_{N-1})^{k-1} ks^{N-1} \bar{b}(s) f(v_1(s)) ds \right)^{1/k} dt,
\]
\[
\ldots
deprecated
\]
\[
v_m(r) = a + \int_0^r \left( t^{N-1} \int_0^t (C_{N-1})^{k-1} ks^{N-1} \bar{b}(s) f(v_{m-1}(s)) ds \right)^{1/k} dt,
\]
\[
\ldots
deprecated
\]

By \((f_1)\) or \((f_{01})\), we obtain
\[
v'_m(r) = \left( r^{N-k} \int_0^r (C_{N-1})^{k-1} ks^{N-1} \bar{b}(s) f(v_{m-1}(s)) ds \right)^{1/k} > 0, \quad r > 0
\]
and
\[
v_m(r) > a + (f(a))^{1/k} N'(r). \tag{7.5}
\]

Hence the sequence \( \{ v_m \}_{m \geq 1} \) is an increasing sequence of positive increasing functions. We note that \( v_m \) satisfies
\[
\left( r^{N-k} (v'_m)^k \right)' = (C_{N-1})^{k-1} kr^{N-1} \bar{b}(r) f(v_{m-1}(r)), \quad m \geq 1.
\]

And by the monotonicity of \( \{ v_k \}_{k \geq 1} \), we have
\[
\left( r^{N-k} (v'_m)^k \right)' \leq (C_{N-1})^{k-1} kr^{N-1} \bar{b}(r) f(v_m(r)), \quad m \geq 1. \tag{7.6}
\]

For an arbitrary \( r_* \in (0, 1) \), let
\[
B_{r_*} := \max_{0 \leq r \leq r_*} (C_{N-1})^{k-1} kr^N \bar{b}(r).
\]
Using this and \( v'_m \geq 0 \), we see that (7.6) yields
\[
k(v'_m)^{k-1}v''_m \leq B_r f(v_m) \text{ on } [0, r_+].
\]
Multiply this by \( v'_m \) and integrate to obtain
\[
(v'_m(r))^{k+1} \leq \frac{k+1}{k} B_r \int_a^{v_m(r)} f(t) dt, \ r \in [0, r_+]. \tag{7.7}
\]
This implies that
\[
F_a(v_m) \leq R_+.
\]
From (7.1), we see that
\[
F_a(v_m) \leq \mathcal{R}_+.
\]
This fact, combined with (7.2), shows that
\[
v_m \leq F_a^{-1}(R_+) \text{ on } [0, r_+]. \tag{8.7}
\]
Thus, \( \{v_m\}_{m \geq 1} \) is uniformly bounded on \([0, r_+]\) for any \( r_+ \in (0, 1) \). Moreover, by (7.7) and (7.8), we see that \( \{v'_m\}_{m \geq 1} \) is uniformly bounded on \([0, r_+]\). By Arzelà-Ascoli’s theorem, we can choose a subsequence of \( \{v_m\}_{m \geq 1} \), still denoted by \( \{v_m\}_{m \geq 1} \), such that \( v_m \rightarrow v \) on \([0, r_+]\). By the arbitrariness of \( r_+ \), we obtain that \( v \) is a positive \( k \)-convex solution to problem (7.3), (7.4). Since (2.10) is equivalent to (2.11) with \( R_\infty = 1 \), we obtain by (7.5) and (2.10) that \( v(r) \rightarrow \infty \) as \( r \rightarrow 1 \). By the arbitrariness of \( a \), we see problem (1.1) possesses infinitely many positive \( k \)-convex radial solutions.

\[\square\]

8 Proofs of Theorems 2.11, 2.13 and 2.16

8.1 Proof of Theorem 2.11

**Proof.** By Proposition A.9 (i) (see Appendix A) and Lemma B.5 (see Appendix B), we obtain
\[
\lim_{t \rightarrow 0^+} \left( \frac{k}{t} \int_0^t s^{-\gamma} \tilde{L}^k(s) ds \right)^{(k+1)/k} \tau^{-\gamma} L^k(t) v(t) = \begin{cases} 
\frac{\gamma - k - 1}{\gamma - 1}, & \text{if } \gamma > k + 1; \\
0, & \text{if } \gamma = k + 1,
\end{cases} \tag{8.1}
\]
where \( \tilde{L} \in \mathcal{L}_2 \) and \( v \) is given by (2.16). Moreover, since \( t \mapsto \tau^{-\gamma} \tilde{L}(t) \) is non-increasing on \((0, r_0)\), by a direct calculation, we obtain
\[
\left( \frac{k}{t} \int_0^t s^{-\gamma} \tilde{L}^k(s) ds \right)^{(k+1)/k} \tau^{-\gamma} L^k(t) v(t) = \left( \frac{k + 1}{k} \int_0^t s^{-\gamma} \tilde{L}^k(s) ds \right)^{(k+1)/k} \tau^{-\gamma} \tilde{L}^k(t) v(t) \leq k + 1. \tag{8.2}
\]
Let
\[
y(r) = 1 - r^2, \quad \omega(r) := \Phi(c(v(y(r)))^{k/(k+1)}, \ r \in [0, 1),
\]
where $\Phi$ is the solution of (2.14) and $c$ is a positive constant to be determined. Then, we have
\[
\lim_{t \to 0^+} \Phi(t) = 0, \quad \lim_{t \to \infty} \Phi(t) = \infty
\]
and
\[
(\Phi'(t))^{k-1} \Phi''(t) = f(\Phi(t)), \quad \frac{\Phi'(t) t^{-\gamma}}{\Phi(t)} = \frac{((k + 1)\tilde{f}(\Phi(t)))^{1/k}}{\tilde{f}(\Phi(t))} \int_0^t ((k + 1)\tilde{f}(s))^{-1/(k+1)} ds \tag{8.3}
\]
By a direct calculation, we obtain
\[
\omega'(r) = \frac{2ck}{k + 1} \Phi'(c(v(y(r)))^{k/(k+1)}) (v(y(r)))^{-\gamma} \left( \int_{y(r)}^{r_s} \frac{r_s}{k} \int_{y(r)}^{r_s} k s^{-\gamma} \tilde{L}^k(s) ds \right)^{1/k} r
\]
and
\[
\omega''(r) = \left( \frac{2ck}{k + 1} \right)^2 \Phi''(c(v(y(r)))^{k/(k+1)}) (v(y(r)))^{-\gamma} \left( \int_{y(r)}^{r_s} \frac{r_s}{k} \int_{y(r)}^{r_s} k s^{-\gamma} \tilde{L}^k(s) ds \right)^{2/k} r^2 - \frac{2zk}{(k + 1)^2} \Phi'(c(v(y(r)))^{k/(k+1)})
\]
\[
\times (v(y(r)))^{-\gamma} \left( \int_{y(r)}^{r_s} \frac{r_s}{k} \int_{y(r)}^{r_s} k s^{-\gamma} \tilde{L}^k(s) ds \right)^{2/k} r^2 + \frac{2zk}{k + 1} \Phi'
\]
\[
\times (c(v(y(r)))^{k/(k+1)}) (v(y(r)))^{-\gamma} \left( \int_{y(r)}^{r_s} \frac{r_s}{k} \int_{y(r)}^{r_s} k s^{-\gamma} \tilde{L}^k(s) ds \right)^{(1-k)/k}
\]
\[
\times (y(r))^{-\gamma} \tilde{L}^k(y(r)) r^2 + \frac{2zk}{k + 1} \Phi'(c(v(y(r)))^{k/(k+1)}) (v(y(r)))^{-\gamma} \left( \int_{y(r)}^{r_s} \frac{r_s}{k} \int_{y(r)}^{r_s} k s^{-\gamma} \tilde{L}^k(s) ds \right)^{1/k}.
\]
So, we further obtain
\[
C_{N-1}^{k-1} \left( \frac{\omega'(r)}{r} \right)^{k-1} \omega''(r) + C_N^{k-1} \left( \frac{\omega'(r)}{r} \right)^k = \left( \frac{2k}{k + 1} \right)^k c^{k+1} \Phi'(c(v(y(r)))^{k/(k+1)}) c^{-1} \Phi''(c(v(y(r)))^{k/(k+1)})(y(r))^{-\gamma} \tilde{L}^k(y(r)) \tilde{g}(c, r)
\]
\[
= \left( \frac{2k}{k + 1} \right)^k c^{k+1} \Phi'(c(v(y(r)))^{k/(k+1)}) (C_{N-1}^{k-1} + C_N^{k-1}) \Phi''(c(v(y(r)))^{k/(k+1)}) C_{N-1}^{k-1} c^{k+1} \Phi'(c(v(y(r)))^{k/(k+1)})(y(r))^{-\gamma} \tilde{L}^k(y(r)) \tilde{g}(c, r), \tag{8.4}
\]
where
\[
\tilde{g}(c, r) := \frac{2C_{N-1}^{k-1} \left( \int_{y(r)}^{r_s} \frac{r_s}{k} \int_{y(r)}^{r_s} k s^{-\gamma} \tilde{L}^k(s) ds \right)^{(k+1)/k}}{(k + 1)v(y(r))(y(r))^{-\gamma} \tilde{L}^k(y(r))} + \frac{\Phi'(c(v(y(r)))^{k/(k+1)}) (C_{N-1}^{k-1} + C_N^{k-1})}{\Phi''(c(v(y(r)))^{k/(k+1)}) c^{k+1} \Phi'(c(v(y(r)))^{k/(k+1)})(y(r))^{-\gamma} \tilde{L}^k(y(r))}
\]
\[
\times \frac{1}{(y(r))^{-\gamma} \tilde{L}^k(y(r))} + \frac{2C_{N-1}^{k-1} \Phi'(c(v(y(r)))^{k/(k+1)}) r^2}{(k + 1)v(y(r))(y(r))^{-\gamma} \tilde{L}^k(y(r))} \left[ 1 - \frac{\int_{y(r)}^{r_s} \frac{r_s}{k} \int_{y(r)}^{r_s} k s^{-\gamma} \tilde{L}^k(s) ds \right)^{(k+1)/k}}{(k + 1)v(y(r))(y(r))^{-\gamma} \tilde{L}^k(y(r))} \right] \tag{8.5}
\]
By (8.3) and Lemma B.6 (iv) (see Appendix B), we see that
\[
\lim_{t \to \infty} \Phi'(t) = \frac{k - p_0}{p_0 + 1} \geq 0 \quad \text{and} \quad \lim_{t \to \infty} \Phi'(t) = \frac{k - p_\infty}{p_\infty + 1} > 0. \tag{8.6}
\]

This, combined with (8.1), (8.2) and the definition of \( y \), implies that there exist constants \( m_1 \) and \( m_2 \) (\( m_1 \) and \( m_2 \) can be chosen to be independent of \( c \)) with
\[
\begin{cases}
  m_1 > 0, & \text{if } p_0 < k \text{ (or } p_0 = k, \ c \geq 1) \\
  m_1 = 0, & \text{if } p_0 = k \text{ and } c < 1
\end{cases}
\]
such that
\[
m_1 < \tilde{y}(c, r) < m_2 \quad \text{for } (c, r) \in (0, \infty) \times [0, 1). \tag{8.7}
\]

On the other hand, it follows from the definition of \( y \) that
\[
y(r) = 1 - r^2 \geq 1 - r \quad \text{and} \quad y(r) = (1 - r)(1 + r) \leq 2(1 - r) \quad \text{for } r \in [0, 1). \tag{8.8}
\]

Since \( t \mapsto t^{-\gamma L}(t) \) is non-increasing on \( (0, \infty) \), we conclude by (8.8) that
\[
(2(1 - r))^{-\gamma L}(2(1 - r)) \leq (y(r))^{-\gamma L}(y(r)) \leq (1 - r)^{-\gamma L}(1 - r).
\]

Since \( L \in NRVZ_0 \) defined in Proposition A.4 (see Appendix A), we can obtain by Proposition A.2 that there exist positive constants \( c_1 \) and \( c_2 \) such that
\[
c_1(1 - r)^{-\gamma L}(1 - r) \leq (y(r))^{-\gamma L}(y(r)) \leq c_2(1 - r)^{-\gamma L}(1 - r) \quad \text{for } r \in [0, 1),
\]
i.e.,
\[
(y(r))^{-\gamma L}(y(r)) \approx (1 - r)^{-\gamma L}(1 - r), \quad r \in [0, 1). \tag{8.9}
\]

On the other hand, by (2.12) we obtain
\[
\tilde{b}(r) \approx (1 - r)^{-\gamma L}(1 - r), \quad r \in [0, 1).
\]

This combined with (8.9) shows that
\[
\tilde{b}(r) \approx (y(r))^{-\gamma L}(y(r)), \quad r \in [0, 1). \tag{8.10}
\]

By (8.7), (8.10) and \( f(u) \approx \tilde{f}(u) \), \( u > 0 \), we can take a small enough constant \( c = c_1 < 1 \) and a large enough constant \( c = c_2 > 1 \) such that
\[
C_{k-1}^{k-1} \left( \frac{\omega_1'(r)}{r} \right)^{k-1} \omega_1''(r) + C_{k-1}^{k-1} \left( \frac{\omega_1'(r)}{r} \right)^{k} \leq \tilde{b}(r)f(\omega_1), \quad r \in [0, 1) \tag{8.11}
\]
and
\[
C_{k-1}^{k-1} \left( \frac{\omega_2'(r)}{r} \right)^{k-1} \omega_2''(r) + C_{k-1}^{k-1} \left( \frac{\omega_2'(r)}{r} \right)^{k} \geq \tilde{b}(r)f(\omega_2), \quad r \in [0, 1), \tag{8.12}
\]
where
\[
\omega_i = \Phi(c_i(v(y(r)))^{k/(k+1)}), \quad r \in [0, 1), \quad i = 1, 2
\]
with
\[
\omega_1(r) < \omega_2(r), \quad r \in [0, 1). \tag{8.13}
\]

Take any \( c \in (\omega_1(0), \omega_2(0)) \) and let \( v_c \) be the unique solution of the following initial value problem
\[
C_{k-1}^{k-1} \left( \frac{v'(r)}{r} \right)^{k-1} v''(r) + C_{k-1}^{k-1} \left( \frac{v'(r)}{r} \right)^{k} = \tilde{b}(r)f(v), \quad v(0) = c, \ v'(0) = 0.
\]
It follows from Lemma 5.1 and Lemma 5.2 of [39] that

\[ \omega_1(r) \leq v_c(r) \leq \omega_2(r), \quad r \in [0, 1) \text{ with } v'_c(r) > 0, \; v''_c(r) > 0, \; r \in (0, 1). \]

Since \( \omega_1(r) \to \infty \) as \( r \to 1 \), we obtain \( v_c(r) \to \infty \) as \( r \to 1 \). Hence \( v_c \) is a \( k \)-convex radial solution of problem (1.1) and satisfies (2.14). In fact, for any \( c > 0 \), we can always take a sufficiently small constant \( c_1 > 0 \) and a sufficiently large constant \( c_2 > 0 \) such that \( c \in (\omega_1(0), \omega_2(0)) \). This implies that (2.14) holds for any positive \( k \)-convex solution \( v_c \) with \( t(0) = c \).

\[ \square \]

8.2 Proof of Theorem 2.13

Proof. Case (I) \( \gamma > k + 1 \). Let

\[ \varphi(t) := \frac{(\tilde{F}(t))^{k/(k+1)}}{f(t)}, \quad t > 0. \]  

(8.14)

It follows from (2.17) that there exist a large positive constant \( T_* \) and two positive constants \( m_* \) and \( m^* \) such that

\[ m_* < \varphi(t) < m^*, \quad t \geq T_. \]

Moreover, by (8.14), we obtain the following equation

\[ \frac{d\tilde{F}(t)}{dt} = \frac{(\tilde{F}(t))^{k/(k+1)}}{\varphi(t)}, \text{ i.e., } \frac{d\tilde{F}(t)}{F_{k/(k+1)}(t)} = \frac{dt}{\varphi(t)}. \]

Integration it from \( T_* \) to \( t > T_* \), we obtain

\[ (\tilde{F}(t))^{1/(k+1)} = \tilde{F}(T_*) + \frac{1}{k+1} \int_{T_*}^{t} \frac{ds}{\varphi(s)}. \]

This fact, combined with (8.15), shows that

\[ (\tilde{F}(t))^{-1/(k+1)} = \left( \tilde{F}(T_*) + \frac{1}{k+1} \int_{T_*}^{t} \frac{ds}{\varphi(s)} \right)^{-1} > \left( \tilde{F}(T_*) + \frac{t - T_*}{m_*(k+1)} \right)^{-1}. \]

So, we have

\[ \int_{T_*}^{\infty} (\tilde{F}(s))^{-1/(k+1)} ds = \infty. \]  

(8.16)

As before, let

\[ \gamma(r) := 1 - r^2, \; \omega(r) := \Phi(c(\nu(y(r)))^{k/(k+1)}), \quad r \in [0, 1), \]

where \( \Phi \) is the solution of (2.18), \( \nu \) is given by (2.16) and \( c \) is a positive constant to be determined. By (8.16) and (2.18) we obtain

\[ \lim_{t \to 0^+} \Phi(t) = \gamma_+ \quad \text{and} \quad \lim_{t \to \infty} \Phi(t) = \infty \]  

(8.17)

and

\[ (\Phi'(t))^{k-1} \Phi''(t) = \tilde{f}(\Phi(t)), \quad \frac{\Phi'(t)}{\Phi''(t)} = (k + 1)^{k/(k+1)} \gamma(\Phi(t)), \quad t > 0. \]

(8.18)

Moreover, by (8.14), we see that

\[ \lim_{t \to 0^+} \frac{\Phi'(t)}{\Phi''(t)} = (k + 1)^{k/(k+1)} \gamma(\gamma_+) = (k + 1)^{k/(k+1)} \frac{\tilde{F}(\gamma_+)^{k/(k+1)}}{f(\gamma_+)} > 0. \]

(8.19)
And by (8.17), we see that there exists a large constant \( t_* > 0 \) such that
\[
\Phi(t) \geq T_*, \quad t \geq t_*.
\]
This together with (8.15) and (8.18) implies that
\[
(k + 1)^{k/(k+1)} m_* < \frac{\Phi'(t)}{\Phi''(t)} < (k + 1)^{k/(k+1)} m_*, \quad t \geq t_*.
\]
(8.20)

On the other hand, by the same calculation as (8.4), we obtain
\[
\frac{\omega''(r)}{r} = \frac{2k}{k+1} c^k \int f(\omega(r)) (y(r))^{-k} L^k(y(r)) c_3(r, r),
\]
where \( c_3(r, r) \) is given by (8.5). Combining (8.1) with (8.19), (8.20), we can also take a sufficiently large constant \( c > 1 \) and some positive constant \( m_5 \) such that
\[
c_3(r, r) > m_5.
\]
Moreover, we can take a sufficiently small constant \( c < 1 \) and some positive constant \( m_4 \) such that
\[
c_3(r, r) < m_4,
\]
where \( m_5 \) and \( m_4 \) can be chosen to be independent of \( c \).

**Case (II)** \( \gamma = k + 1 \) and \( \limsup_{t \to 0^+} \tilde{L}(t) < \infty \).

By Proposition A.9 (i) (see Appendix A), we see that
\[
\limsup_{t \to 0^+} \frac{\int_0^t k s^{-k-1} L^k(s) ds}{t^{-k-1} L^k(t)} = \limsup_{t \to 0^+} \tilde{L}(t) < \infty.
\]
(8.21)

Let
\[
\omega(r) := \Phi(cv(y(r))), \quad y(r) := 1 - r^2, \quad r \in [0, 1),
\]
where \( \Phi \) is the solution of (2.18). A direct calculation shows that
\[
\omega'(r) = 2c \Phi'(cv(y(r))) \left( \int_{y(r)}^{r} k s^{-k-1} L^k(s) ds \right)^{1/k} r
\]
and
\[
\omega''(r) = (2c)^2 \Phi''(cv(y(r))) \left( \int_{y(r)}^{r} k s^{-k-1} L^k(s) ds \right)^{2/k} r^2 + 2c^2 \Phi'(cv(y(r))) (y(r))^{-k-1} L^k(y(r)) r^2
\]
\[
\times \left( \int_{y(r)}^{r} k s^{-k-1} L^k(s) ds \right)^{(1-k)/k} + 2c \Phi'(cv(y(r))) \left( \int_{y(r)}^{r} k s^{-k-1} L^k(s) ds \right)^{1/k}.
\]

So, we have
\[
\frac{\omega'(r)}{r} = \frac{\omega''(r)}{r} + c_N \frac{\omega'(r)}{r} \]
\[
= (2c)^k (\Phi'(cv(y(r))))^{k-1} (\Phi''(cv(y(r))) (y(r))^{-k-1} L^k(y(r))) c_3(c, r)
\]
\[
= (2c)^k \tilde{f}(\omega(r))(y(r))^{-k-1} L^k(y(r)) c_3(c, r),
\]
where

\[
\mathcal{F}(c, r) := \frac{2cC_{N-1}^k \left( \int_{y(r)}^{T} ks^{k-1} l(s) ds \right)^{k+1/2}}{(y(r))^{k-1} L^k(y(r))} + 2C_{N-1}^{k-1} \frac{\Phi'(cv(y(r)))^2}{\Phi''(cv(y(r)))} \left( \frac{\int_{y(r)}^{T} ks^{k-1} l(s) ds}{(y(r))^{k-1} L^k(y(r))} \right)^{k-1/2}.
\]

By (8.19)–(8.21), we obtain that there exist positive constants \( m_5 \) and \( m_6 \) such that

\[
m_5 < \mathcal{F}(c, r) < m_6 \text{ for } (c, r) \in (0, \infty) \times [0, 1].
\]

The rest of the proof is similar to the one of Theorem 2.11 and thus is omitted. \(\square\)

### 8.3 Proof of Theorem 2.16

**Proof** Case (i) When \( f \) satisfies the conditions in Theorem 2.11, let

\[
\omega(r) := \Phi(c\Theta_\infty(y(r))), \quad y(r) := 1 - r^2, \quad r \in [0, 1),
\]

where \( \Phi \) is the solution of (2.15) and \( c \) is a positive constant to be determined. By a direct calculation, we obtain

\[
\omega'(r) = 2c\Phi'(c\Theta_\infty(y(r)))\theta_\infty(y(r))r
\]

and

\[
\omega''(r) = (2c)^2\Phi''(c\Theta_\infty(y(r)))\theta_\infty^2(y(r))r^2 - 2c\Phi'(c\Theta_\infty(y(r)))\theta_\infty'(y(r))r^2 + 2c\Phi'(c\Theta_\infty(y(r))).
\]

So, we have

\[
C_{N-1}^k \left( \frac{\omega(r)}{r} \right)^{k-1} \omega''(r) + C_{N-1}^k \left( \frac{\omega'(r)}{r} \right)^k = (2c)^{k+1}\int \omega(r)\theta_\infty^{2k+1}(y(r))\mathcal{F}(c, r),
\]

where

\[
\mathcal{F}(c, r) := C_{N-1}^{k-1} - \frac{\Phi'(c\Theta_\infty(y(r)))C_{N-1}^{k-1}}{\Phi''(c\Theta_\infty(y(r)))c\Theta_\infty(y(r))} \left( C_{N-1}^{k-1} \frac{\Theta_\infty(y(r))}{\theta_\infty^2(y(r))} \right) + \frac{\Theta_\infty(y(r))\Phi'(c\Theta_\infty(y(r)))C_{N-1}^{k-1}}{2\Phi''(c\Theta_\infty(y(r)))c\Theta_\infty(y(r))}.
\]

By Lemma B.2 (i), (ii) (see Appendix B) and (8.6), we see that there exist constants \( m_7 \) and \( m_8 \) \((m_7 \text{ and } m_8 \text{ can be chosen to be independent of } c)\) with

\[
\begin{align*}
m_7 > 0, & \quad \text{if } p_0 < k \text{ (or } p_0 = k, \ c \geq 1) \\
m_7 = 0, & \quad \text{if } p_0 = k \text{ and } c < 1
\end{align*}
\]

such that

\[
m_7 < \mathcal{F}(c, r) < m_8 \text{ for } (c, r) \in (0, \infty) \times [0, 1).
\]

On the other hand, it follows from (H\(b_0\)) that

\[
\tilde{b}(r) \approx \theta_\infty(y(r)), \quad r \in [0, 1).
\]
This combined with (8.22) implies that we can take a small enough constant \( c = c_1 < 1 \) and a large constant \( c = c_2 \) such that (8.11), (8.12) hold, where

\[ \omega_i(r) = \Phi(c_i \Theta_\infty(y(r))), \ r \in [0, 1), \ i = 1, 2 \]

with (8.13) holds here.

The rest of the proof is similar to the one of Theorem 2.11 and thus is omitted.

**Case (II)** When \( f \) satisfies the conditions in Theorem 2.13, let

\[ \omega(r) := \Phi(c \Theta_\infty(y(r))), \ y(r) := 1 - r^2, \ r \in [0, 1), \]

where \( \Phi \) is the solution of (2.18) and \( c \) is a positive constant to be determined. A straightforward calculation shows that

\[
C_{k-1}^N \left( \frac{\omega(r)}{r} \right)^{k-1} \omega''(r) + C_{k-1}^N \left( \frac{\omega'(r)}{r} \right)^k = 2^{k+1} c \omega \theta_{x}^{k+1}(y(r)) c_F(c, y(r)).
\]

Combining (8.19), (8.20) with Lemma B.2 (i), (ii) (see Appendix B), we can take a sufficiently large constant \( c > 1 \) and some positive constant \( m_9 \) such that

\[ c_F(c, y(r)) > m_9. \]

Moreover, we can take a sufficiently small constant \( c < 1 \) and some positive constant \( m_{10} \) such that

\[ c_F(c, y(r)) < m_{10}, \]

where \( m_9 \) and \( m_{10} \) can be chosen to be independent of \( c \).

The proof of the rest of the theorem is same as the above arguments, thus we omit it here. \( \Box \)

### 9 Proofs of Theorems 2.18–2.19

**Proof.** The proofs are quite similar to the ones in Theorems 2.11, 2.13, so we omit them here. \( \Box \)

In section, we introduce some Appendixes.

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### Appendix A

In this appendix, we introduce some basic facts of Karamata regular (rapid) variation theory, which come from Bingham, Goldie and Teugels’ book [60], Geluk and de Hann’s book [58] and Resnick’s book [61]. Some new results on rapidly varying functions, we refer to the paper [62].

**Definition A.1.** A positive continuous function \( f \) defined on \([a_\infty, \infty) ((0, a_0]), \) for some \( a_\infty > 0 (a_0 > 0), \) is called **rapidly varying to infinity at infinity (zero)**, written \( f \in \mathcal{N}_\infty (f \in \mathcal{N}_0^{\infty}) \), if for each \( \varepsilon \in (0, 1) (\varepsilon \in (1, \infty)), \)

...
Theorem 1.1 and Remark 1.2 in [62], we can obtain

\[ \lim_{t \to \infty} f(\psi(t)) = \lim_{t \to 0^+} f(\psi(t)) = 0. \]

A positive continuous function \( f \) is called **general rapidly varying to infinity at infinity (zero)**, written \( f \in \mathcal{G}_\infty^\infty \), if for each \( m > 0 \),

\[ \lim_{t \to \infty} t^{-m} f(t) = \infty \quad (\lim_{t \to 0^+} t^{-m} f(t) = \infty). \]

By Theorem 1.1 and Remark 1.2 in [62], we see that \( \mathcal{H}_\infty^\infty \subset \mathcal{G}_\infty^\infty \), and by the similar argument as the ones of Theorem 1.1 and Remark 1.2 in [62], we can obtain \( \mathcal{H}_\infty^\infty \subset \mathcal{G}_\infty^\infty \).

**Proposition A.2.** (Uniform Convergence Theorem). If \( f \in \text{RV}(Z)_\rho \), then (1.6) holds uniformly for \( \psi \in [c_1, c_2] \) with \( 0 < c_1 < c_2 \).

**Proposition A.3.** (Representation Theorem) A function \( L_{\infty} \) is slowly varying at infinity if and only if it may be written in the form

\[ L_{\infty}(t) = \zeta_{\infty}(t) \exp \left( \int_{a_{\infty}}^t \frac{y_{\infty}(s)}{s} \, ds \right), \quad t \geq a_{\infty}, \]

for some \( a_{\infty} > 0 \), where the functions \( \zeta_{\infty} \) and \( y_{\infty} \) are continuous and for \( t \to \infty \), \( y_{\infty}(t) \to 0 \) and \( \zeta_{\infty}(t) \to c_{\infty} \), with \( c_{\infty} > 0 \). We call that

\[ L_{\infty}(t) = c_{\infty} \exp \left( \int_{a_{\infty}}^t \frac{y_{\infty}(s)}{s} \, ds \right), \quad t \geq a_{\infty}, \]

is **normalized slowly varying at infinity** and \( f(t) = t^\rho L_{\infty}(t) \) is **normalized regularly varying at infinity with index** \( \rho \) (and written \( f \in \text{NRV}_{\rho} \)).

**Proposition A.4.** A function \( L_0 \) is slowly varying at zero if and only if it may be written in the form

\[ L_0(t) = \zeta_0(t) \exp \left( \int_{a_0}^t \frac{y_0(s)}{s} \, ds \right), \quad t \leq a_0, \]

for some \( a_0 > 0 \), where the functions \( \zeta_0 \) and \( y_0 \) are continuous and for \( t \to 0^+ \), \( y_0(t) \to 0 \) and \( \zeta_0(t) \to c_0 \), with \( c_0 > 0 \). We call that

\[ L_0(t) = c_0 \exp \left( \int_{a_0}^t \frac{y_0(s)}{s} \, ds \right), \quad t \leq a_0, \]

is **normalized slowly varying at zero** and \( h(t) = t^\rho L_0(t) \) is **normalized regularly varying at zero with index** \( \rho \) (and written \( h \in \text{NRVZ}_\rho \)).

**Proposition A.5.** A function \( f \in C^1[a_{\infty}, \infty] \) (\( f \in C^1(0, a_0] \)), for some \( a_{\infty} > 0 \) (\( a_0 > 0 \)), belongs to \( \text{NRV}_{\rho} \) \( \text{(NRVZ}_{\rho} \) if and only if

\[ \lim_{s \to \infty} \frac{f'(t)}{f(t)} = \rho \quad \left( \lim_{s \to 0^+} \frac{f'(t)}{f(t)} = \rho \right). \]

**Proposition A.6.** If \( h_1 \in \text{NRV}_{\mu_1} \), \( h_2 \in \text{NRV}_{\mu_2} \) with \( \lim_{t \to \infty} h_1(t) = \infty \), then \( h_1 \circ h_2 \in \text{NRV}_{\mu_1 \mu_2} \).

**Proposition A.7.** If \( h_1 \in \text{NRV}_{\mu_1} \), \( h_2 \in \text{NRV}_{\mu_2} \), then \( h_1 \cdot h_2 \in \text{NRV}_{\mu_1 + \mu_2} \).

**Proposition A.8.** If functions \( L, L_1 \) are slowly varying at zero, then

(i) \( L^\rho \) (for every \( \rho \in \mathbb{R} \), \( c_1 L + c_2 L_1 \) \( (c_1 \geq 0, c_2 \geq 0 \) with \( c_1 + c_2 > 0 \), \( L \circ L_1 \) (if \( L_1(t) \to 0 \) as \( t \to 0^+ \)), are also slowly varying at zero;
(ii) for every $\rho > 0$ and $t \to 0^+$, $\psi L(t) \to 0$ and $t^{-\psi} L(t) \to \infty$;
(iii) for $\rho \in \mathbb{R}$ and $t \to 0^+$, $\ln L(t)/\ln t \to 0$ and $\ln(t^\rho L(t))/\ln t \to \rho$.

**Proposition A.9.** (Asymptotic Behavior) Let $L$ be a slowly varying function at infinity.
(i) $\int_0^\infty \psi L(s) ds \sim (\rho - 1)^{-1} t^{1+\rho} L(t)$, $t \to \infty$, for $\rho < -1$;
(ii) $\int_0^1 \psi L(s) ds \sim (\rho + 1)^{-1} t^{1+\rho} L(t)$, $t \to \infty$, for $\rho > -1$.

**Appendix B**

**Lemma B.1.** Let $\theta \in \Lambda$, then
(i) $\lim_{t \to 0^{-}} \frac{\Theta(0,t)}{\Theta(0)} = 0$ and $\lim_{t \to 0^{+}} \frac{\Theta(0,t)}{\Theta(0)} = 1 - D_\theta$;
(ii) if $\theta$ is non-decreasing, then $D_\theta \in [0,1]$; if $\theta$ is non-increasing, then $D_\theta \in [1,\infty)$;
(iii) if $D_\theta > 0$, then $\theta \in NRVZ_{(1-D_\theta)/D_\theta}$.

**Proof.** (i)–(iii) By a direct calculation, we see that (i)–(iii) hold. □

**Lemma B.2.** Let $\theta_\infty \in \Lambda_\infty$, then
(i) $\theta_\infty'(0,t) < 0$, $t \in (0,1]$ and $\lim_{t \to 0^{+}} \frac{\Theta_\infty(0,t)}{\Theta_\infty(0)} = -1$;
(ii) $\lim_{t \to 0^{-}} \frac{\Theta_\infty(0,t)}{\Theta_\infty(0)} = 0$ and $\lim_{t \to 0^{+}} \frac{\Theta_\infty(0,t)}{\Theta_\infty(0)} = 0$;
(iii) $\lim_{t \to 0^{+}} \frac{\Theta_\infty(0,t)}{\Theta_\infty(0)} = \infty$;
(iv) $\theta_\infty \in \mathcal{R}_0^\infty$, i.e., $\theta_\infty$ is rapidly varying to infinity at zero.

**Proof.**
(i) It follows from the definition of $\Lambda_\infty$ that (i) holds.
(ii) We conclude by (2.19) that there exists function $\theta \in C(0,1]$ satisfying $\theta(t) \to 0$ as $t \to 0^+$ such that

$$
\frac{d}{dt} \left( \frac{\Theta_\infty(t)}{\Theta_\infty(0)} \right) = \theta(t), \; t > 0.
$$

Further more, we have

$$
\frac{\Theta_\infty(t)}{\Theta_\infty(0)} = \int_0^t \theta(s) ds.
$$

This implies that

$$
\lim_{t \to 0^{-}} \frac{\Theta_\infty(t)}{\Theta_\infty(0)} = 0.
$$

By L'Hôpital's rule, we obtain

$$
\lim_{t \to 0^{+}} \frac{\Theta_\infty(t)}{\Theta_\infty(0)} = 0.
$$

(iii) The result follows from (i)-(ii).
(iv) It follows by (ii) that

$$
\lim_{t \to 0^{+}} \frac{1}{\Theta_\infty(t)} \int_0^t \theta_\infty(s) ds = \lim_{t \to 0^{-}} \frac{1}{\Theta_\infty(0)} \int_0^t \theta_\infty(s) ds = 0.
$$

(10.1)
We prove for any $\mu \in (1, \infty)$, $\lim_{t \to 0^+} \frac{\theta_{\infty}(\mu_1 t^\rho)}{\theta_{\infty}(t^\rho)} = 0$. Suppose the contrary, that exist $\mu_s > 1$, $\rho_s > 0$ and some decreasing sequence, denoted by $\{t_n\}_{n=1}^\infty$ with $t_n \in (0, 1)$ and $t_n \to 0$ as $n \to \infty$ such that

$$\frac{\theta_{\infty}(\mu_s t_n)}{\theta_{\infty}(t_n)} > \rho_s.$$ 

This fact, combined with (10.1), shows the contradiction

$$0 = -\lim_{n \to \infty} \frac{\theta_{\infty}(t_n \tau)}{\theta_{\infty}(t_n)} \leq -\lim_{n \to \infty} \frac{\theta_{\infty}(t_n \tau)}{\theta_{\infty}(t_n)} \leq -\rho_s (\mu_s - 1) < 0.$$

\[ \square \]

**Lemma B.3.** Let $f$ satisfy $(f_1)$ (or $(f_{01})$, $(f_2)$, $(f_3)$ and $(S_1)$–$(S_2)$, $\psi$ be the unique solution of (2.2). Then

(i) $\psi'(t) = -(\psi(t)f_1(\psi(t)))^{1/(k+1)}$ and $\lim_{t \to 0^+} \psi(t) = \infty$;
(ii) $(-\psi^r(t))^{k-1}\psi''(t) = \frac{1}{k+1} (f_1(\psi(t)) + \psi(t)f_1'(\psi(t)))$;
(iii) $\lim_{t \to 0^+} g(t)^{-1} \left( \frac{f_1(\psi(t))}{\psi(t)^{k+1}} - 1 \right) = \ln \xi, \xi > 0$;
(iv) $\lim_{t \to 0^+} \frac{f_1(\psi(t))}{\psi(t)^{k+1}} = \mathcal{C}_2$, where $\xi > 0$ and $\mathcal{C}_2$ is given by (2.3);
(v) $\lim_{t \to 0^+} \frac{f_1(\psi(t))}{\psi(t)^{k+1}} = \frac{1}{k+1} + \mathcal{K}_t$;
(vi) $\lim_{t \to 0^+} \frac{f_1(\psi(t))}{\psi(t)^{k+1}} = \frac{1}{k+1} - \ln \xi, \xi > 0$;
(vii) $\lim_{t \to 0^+} \frac{f_1(\psi(t))}{\psi(t)^{k+1}} = \mathcal{C}_2$, where $\xi > 0$ and $\mathcal{C}_2$ is given by (2.3);
(viii) $\lim_{t \to 0^+} \frac{f_1(\psi(t))}{\psi(t)^{k+1}} = \frac{1}{k+1} + \mathcal{K}_t$;
(ix) $\lim_{t \to 0^+} \frac{\psi(t)}{\psi(t)^r} = 0$, $\lim_{t \to 0^+} \frac{\psi(t)}{\psi(t)^r} = 0$ and $\psi', \psi'' \in \mathcal{D}_0$;
(x) $\lim_{t \to 0^+} \frac{t}{\psi(t)^r} = 0$;
(xi) $\lim_{t \to 0^+} \frac{\psi(t)}{\psi(t)^r} = 0$.

**Proof.**

(i–ii) By the definition of $\psi$ and a direct calculation, we see that (i–ii) hold.

(iii) If $\xi = 1$, the result is obvious. If $\xi \neq 1$, by $f \in NRV_k$, we can see that

$$\frac{f_1(\xi t)}{\xi^k f_1(t)} - 1 = \exp \left( \int_t^{\xi t} \frac{g(\tau)}{\tau} d\tau \right) - 1, \quad t \geq t_0. \tag{10.2}$$

It follows by $(f_3)$, $g \in NRV_0$ and Proposition A.2 that

$$\lim_{t \to \infty} g(ts)/s = 0 \quad \text{and} \quad \lim_{t \to \infty} g(ts)/g(t) = 1$$

uniformly with respect to $s \in [c_1, c_2]$, where $c_1$ and $c_2$ ($c_2 > c_1$) are positive constants. Hence we have

$$\lim_{t \to \infty} \int_1^{\xi t} \frac{g(\tau)}{\tau} d\tau = \lim_{t \to \infty} \int_1^{\xi t} \frac{g(ts)}{s} ds = 0.$$
Moreover, by Proposition A.2, we have
\[
\lim_{t \to \infty} \frac{\xi}{g(t)} = \int_{1}^{\infty} \frac{1}{s} \, ds = \ln \xi.
\]  
(10.3)

On the other hand, we see that
\[
\exp(t) - 1 \simeq t \quad \text{as} \quad t \to 0
\]  
(10.4)

and
\[
\lim_{t \to \infty} (g(t))^{-1} \left( \exp \left( \int_{1}^{\infty} \frac{g(t)}{r} \, dr \right) - 1 - \int_{1}^{\infty} \frac{g(t)}{r} \, dr \right) = 0.
\]  
(10.5)

It follows by (10.2)–(10.5) that (iii) holds.

(iv) Since
\[
\lim_{t \to \infty} \frac{f_{2}(\xi t)}{\xi^{k} g(t) f_{1}(t)} = \lim_{t \to \infty} \frac{f_{2}(\xi t)}{\xi^{k} f_{2}(t) t} \quad \lim_{t \to \infty} \frac{f_{2}(t)}{g(t) f_{1}(t)}
\]
we see that if (S₂) holds, then
\[
\lim_{t \to \infty} \frac{f_{2}(\xi t)}{\xi^{k} f_{2}(t)} = 1 \quad \text{and} \quad \lim_{t \to \infty} \frac{f_{2}(t)}{g(t) f_{1}(t)} = C_{1};
\]
if (S₃) holds, then
\[
\lim_{t \to \infty} \frac{f_{2}(\xi t)}{\xi^{k} f_{2}(t)} = \xi^{\mu-k} \quad \text{and} \quad \lim_{t \to \infty} \frac{f_{2}(t)}{g(t) f_{1}(t)} = 0.
\]

(v) By (f₃), (g₁) and L'Hôpital's rule, we obtain
\[
\lim_{t \to \infty} \frac{(t f_{1}(t))^{k/(k+1)}}{g(t) f_{1}(t) \int_{t}^{\infty} (s f_{1}(s))^{-1/(k+1)} \, ds} = \lim_{t \to \infty} \frac{\frac{k}{k+1} g(t) - g'(t) t - \frac{1}{k+1} g(t) \frac{f_{1}'(t)}{f_{1}(t)}}{(g(t))^{2}}
\]
\[
= \lim_{t \to \infty} \frac{k}{k+1} g'(t) t + \frac{1}{k+1} \frac{f_{1}'(t)}{f_{1}(t)} - k
\]
\[
= \frac{1}{k+1} + \mathcal{K}.
\]

(vi)–(viii) The results follow by (f₃) and (iii)–(v).

(ix) By (f₃) and (S₁), we have
\[
\lim_{t \to 0^{+}} \frac{\psi''(t)}{\psi'(t)} = k.
\]  
(10.6)

It follows by (i)–(ii) and (viii) (or (v)) that
\[
\lim_{t \to 0^{+}} \frac{\psi'(t)}{\psi''(t)} = -\lim_{t \to 0^{+}} \frac{(k+1)(\psi)(t) f_{2}(\psi(t))^{k/(k+1)}}{f_{1}(\psi(t)) \left( 1 + \frac{\psi(t) f_{1}(\psi(t))}{f_{1}(\psi(t))} \right) g(\psi(t))} = g(\psi(t)) = 0
\]

and
\[
\lim_{t \to 0^{+}} \frac{\psi(t)}{\psi'(t)} = -\lim_{t \to 0^{+}} \frac{\psi(t) f_{1}(\psi(t))^{k/(k+1)}}{f_{1}(\psi(t)) g(\psi(t))} = 0.
\]

Moreover, by the similar argument as the proof of Lemma B.2 (iv), we have $\psi''$, $\psi' \in \mathcal{R}_{0}^{\infty}$. Furthermore, by the Definition A.1 and the L'Hôpital's rule, we obtain $\psi \in \mathcal{R}_{0}^{\infty}$. 
(x) A straightforward calculation shows that

\[
\lim_{t \to 0^+} \frac{t}{g^2(\psi(t))} = \lim_{s \to 0^+} (2g(\psi(t))g'(\psi(t))\psi'(\psi(t)))^{-1} = \lim_{s \to 0^+} (2g(s)g'(s)(s f_1(s)))^{1/(k+1)}
\]

\[
= \lim_{s \to 0^+} \left( \frac{2g'(s)}{g'(s)(s f_1(s)))^{1/(k+1)} \frac{g'(s)}{s} \right)^{-1} = \lim_{s \to 0^+} \left( \frac{s}{f_1(s)/(s f_1(s)))^{2k+1}} \right)^{1/(k+1)} = 0.
\]

(xi) Since \( \frac{1}{k+1} + \mathcal{K} \not= 0 \) (see (S) on page 4), we obtain by (viii), (x) and (10.6) that

\[
\lim_{t \to 0^+} \frac{t}{\psi'(\psi(t))} = \lim_{t \to 0^+} \frac{f_1(\psi(t))(1 + \frac{\psi(0)f'(\psi(0))}{\psi'(\psi(0))})_{g(\psi(t))}}{(k + 1)(\psi(0)f_1(\psi(t)))^{k/(k+1)}} \frac{t}{g^2(\psi(t))} = -\lim_{t \to 0^+} \left( \frac{1}{k + 1} + K_s \right) \frac{t}{g^2(\psi(t))} = 0.
\]

**Lemma B.4.** Under the hypotheses in Theorem 2.5, we have

(i) \( \lim_{t \to \infty} \frac{t \Sigma'(t)}{h(\Sigma(t))} = \frac{1}{k+1} \).

(ii) \( \tilde{f}^{-1} \in \text{NRV}_0 \) and \( (\tilde{f}^{-1})' \in \text{NRV}_1 \).

(iii) \( \tilde{f} \circ \Sigma \in \text{NRV}_{k+1} \).

(iv) \( \Sigma \in \text{NRV}_0 \) and \( \Sigma' \in \text{NRV}_1 \), i.e., \( \lim_{t \to \infty} \frac{\Sigma'(t)}{\Sigma(t)} = -1 \).

**Proof.**

(i) By (2.8), we have

\[
\frac{t \Sigma'(t)}{h(\Sigma(t))} = \frac{(k + 1)\tilde{f} \circ \Sigma(t)}{k+1}.
\]

This fact, combined with (2.9), implies that (i) holds.

(ii) By the definition of \( \tilde{f} \) in (2.6), we see that the inverse of \( \tilde{f} \) denoted by \( \tilde{f}^{-1} \) is the unique solution of the following integral equation

\[
\exp \left\{ \int_{\tilde{f}^{-1}(t)}^{\tilde{f}^{-1}(0)} \frac{ds}{T(s)} \right\} = t, \quad t > 0.
\]

A simple calculation shows that

\[
\lim_{t \to \infty} \frac{t(\tilde{f}^{-1}(t))'}{\tilde{f}^{-1}(t)} = \lim_{t \to \infty} \frac{T(\tilde{f}^{-1}(t))}{\tilde{f}^{-1}(t)} = \lim_{t \to \infty} T'(t) = 0
\]

and

\[
\lim_{t \to \infty} \frac{(\tilde{f}^{-1}(t))''}{(\tilde{f}^{-1}(t))'} = \lim_{t \to \infty} (T'(\tilde{f}^{-1}(t)) - 1) = -1.
\]

From Proposition A.5, we see that \( \tilde{f}^{-1} \in \text{NRV}_0 \) and \( (\tilde{f}^{-1})' \in \text{NRV}_1 \).

(iii) In (2.8), letting \( \tau = f(s) \), a direct calculation shows that

\[
\int_{\Sigma(t)}^{\infty} \frac{h^{k-1}(s)}{f(s)} ds = \int_{\tilde{f}(\Sigma(t))}^{\infty} \frac{h^{k-1}(\tilde{f}^{-1}(\tau)) \cdot (\tilde{f}^{-1}(\tau))'}{\tau} d\tau = \frac{1}{k+1}, \quad t > 0.
\]
Let
\[ A(t) = \int_{t}^{\infty} \frac{h^{k-1}(\hat{f}^{-1}(\tau)) \cdot (\hat{f}^{-1}(\tau))'}{\tau} d\tau. \]

Since \( h \in NRV_{0} \), \( f^{-1} \in NRV_{0} \) and \( (\hat{f}^{-1})' \in NRV_{-1} \), by Propositions A.6, A.7, we arrive at
\[ (h^{k-1} \circ \hat{f}^{-1}) \cdot (\hat{f}^{-1})' \in NRV_{-1}. \]

So, there exist some constant \( a_{\infty} > 0 \) and \( L_{\infty} \in NRV_{0} \) such that
\[ h^{k-1}(\hat{f}^{-1}(t)) \cdot (\hat{f}^{-1}(t))' = t^{-1}L_{\infty}(t), \quad t > a_{\infty}. \]

It follows by using Proposition A.9 (i) that
\[ \lim_{t \to \infty} \frac{tA'(t)}{A(t)} = -1, \]

i.e., \( A \in NRV_{-1} \). This together with (10.8) implies that \( A \circ \hat{f} \circ \Sigma \in NRV_{-(k+1)} \). Let \( A^{-1} \) denote the inverse of \( A \), then \( A^{-1} \) is the unique solution of the following integral equation
\[ \int_{A^{-1}(t)}^{\infty} \frac{h^{k-1}(\hat{f}^{-1}(\tau)) \cdot (\hat{f}^{-1}(\tau))'}{\tau} d\tau = \int_{A^{-1}(t)}^{\infty} \frac{\tau^{-2}L_{\infty}(\tau)d\tau}{A^{-1}(t)} = t. \]

By using Proposition A.9 (i) and a direct calculation, we obtain
\[ \lim_{t \to \infty} \frac{t(A^{-1}(t))'}{A^{-1}(t)} = -1, \]

i.e., \( A^{-1} \in NRV_{-1} \). It follows by Proposition A.6 that
\[ \hat{f} \circ \Sigma = A^{-1} \circ A \circ \hat{f} \circ \Sigma \in NRV_{k+1}. \]

(iv) It follows by (ii), (iii) and Proposition A.6 that \( \Sigma = \hat{f}^{-1} \circ (\hat{f} \circ \Sigma) \in NRV_{0} \). In view of (2.8), we obtain
\[ \Sigma''(t)h^{k-1}(\Sigma(t)) + (k-1)(\Sigma'(t)^{2})h^{k-2}(\Sigma(t))\Sigma'(\Sigma(t)) \]
\[ = -(k+1)(k+2)t^{-k-2}\hat{f}(\Sigma(t)) + (k+1)t^{-k-2}\hat{f}'(\Sigma(t))'. \]

We conclude by \( h \in NRV_{0} \), \( \Sigma \in NRV_{0} \), (10.7) and (iii) that
\[ \lim_{t \to \infty} \frac{t\Sigma''(t)}{\Sigma'(t)} = -\lim_{t \to \infty} \frac{h'(\Sigma(t))\Sigma(t)}{h(\Sigma(t))} \cdot \frac{\Sigma'(t)}{\Sigma'(t)} = -2k - 2 + \lim_{t \to \infty} \frac{(k+1)t^{-k-2}\hat{f}(\Sigma(t))}{h^{k-1}(\Sigma(t))\Sigma'(t)} \cdot \frac{\hat{f}'(\Sigma(t))t}{\hat{f}(\Sigma(t))} = -1. \]

Lemma B.5. (Lemma 2.3 of [63]) Let \( L \in \Sigma \) (see Remark 2.6), then
\[ \lim_{t \to \infty} \frac{L(t)}{\int_{t}^{\infty} \frac{L(s)}{s} ds} = 0. \]

If we further assume \( L \in \Sigma_{1} \), then
\[ \lim_{t \to \infty} \frac{L(t)}{\int_{0}^{\infty} \frac{L(s)}{s} ds} = 0. \]

Lemma B.6. Let \( \tilde{f} \in C[0, \infty) \cap RV_{p_{0}} \cap RV_{p_{\infty}} \) \( (p_{0} = k \) and \( p_{\infty} < k) \) be positive on \( (0, \infty) \). We need to verify (2.13) if \( p_{0} = k, \tilde{f} \) is given by (2.13). Then
(i) \( \lim_{t \to 0^+} \frac{f(t)}{t^{(k+1)/2}} = \infty; \)

(ii) \( \lim_{t \to 0^+} \frac{\tilde{f}(t)}{\tilde{f}(0)} = \frac{1}{1+p} \) and \( \lim_{t \to \infty} \frac{\tilde{f}(t)}{\tilde{f}(0)} = \frac{1}{1+p_\infty}; \)

(iii) \( \lim_{t \to 0^+} \frac{\tilde{f}(t)}{\tilde{f}(0)} \int_0^t (\tilde{f}(s))^{-1/(k+1)} ds = \begin{cases} 
\infty, & \text{if } \tilde{f} \in \text{RVZ}_k, \\
k + 1, & \text{if } \tilde{f} \in \text{RVZ}_{p_0} \text{ with } p_0 < k 
\end{cases} \)

and

\( \lim_{t \to \infty} \frac{\tilde{f}(t)}{\tilde{f}(0)} \int_0^t (\tilde{f}(s))^{-1/(k+1)} ds = \frac{k+1}{k-p_\infty}. \)

(iv) \( \lim_{t \to 0^+} \frac{(k+1)\tilde{f}(t)^{1/(k+1)}}{\tilde{f}(0)\tilde{f}(0)^{1/(k+1)} + \sum_{n=0}^\infty ds} = \frac{k-p_\infty}{p_\infty+1} \) and \( \lim_{t \to \infty} \frac{(k+1)\tilde{f}(t)^{1/(k+1)}}{\tilde{f}(0)\tilde{f}(0)^{1/(k+1)} + \sum_{n=0}^\infty ds} = \frac{k-p_\infty}{p_\infty+1}. \)

Proof.

(i) We assume that it is false. This implies that there exist a positive constant \( \tilde{c} \) and an increasing sequence \( \{t_n\}_{n=1}^\infty \) of real numbers satisfying \( \lim_{n \to \infty} t_n = 0 \) and \( 2t_{n+1} \leq t_n, n = 1, 2, \ldots \), such that

\( (\tilde{F}(t_n))^{-1/(k+1)} \geq 1/(t_n \tilde{c}). \)

A direct calculation shows that

\[ \infty > \int_0^{t_n} (\tilde{f}(s))^{-1/(k+1)} ds = \sum_{n=1}^{\infty} \int_{t_{n+1}}^{t_n} (\tilde{f}(s))^{-1/(k+1)} ds \]

\[ \geq \sum_{n=1}^{\infty} \int_{t_{n+1}}^{t_n} (\tilde{F}(t_n))^{-1/(k+1)} ds \]

\[ \geq \sum_{n=1}^{\infty} \int_{t_{n+1}}^{t_n} \frac{1}{t_n \tilde{c}} ds = \sum_{n=1}^{\infty} \frac{1}{\tilde{c}} t_n - t_{n+1} \]

\[ \geq \lim_{n \to \infty} \frac{n}{2\tilde{c}} = \infty. \]

This is a contradiction. So, (i) holds.

(ii) Since \( \tilde{F}(t) = \int_0^t \tilde{f}(s) ds \), by the Lebesgue’s dominated convergence theorem we obtain

\[ \lim_{t \to 0^+} \frac{\tilde{F}(t)}{\tilde{f}(0)} = \lim_{t \to 0^+} \frac{\tilde{F}(ts)}{\tilde{f}(t)} ds = \int_0^1 s^{k_0} ds = \frac{1}{1+p_0} \]

and

\[ \lim_{t \to \infty} \frac{\tilde{F}(t)}{\tilde{f}(0)} = \lim_{t \to \infty} \frac{\tilde{F}(ts)}{\tilde{f}(t)} ds = \int_0^1 s^{p_\infty} ds = \frac{1}{1+p_\infty}. \]

(iii) Because of (i), we can apply L’Hôpital’s rule to get

\[ \lim_{t \to 0^+} \frac{\int_0^t (\tilde{f}(s))^{-1/(k+1)} ds}{t(\tilde{F}(t))^{-1/(k+1)}} = \lim_{t \to 0^+} \left( 1 - \frac{\tilde{f}(t)t}{(k+1)\tilde{F}(t)} \right)^{-1} = \begin{cases} 
\infty, & \text{if } \tilde{f} \in \text{RVZ}_k, \\
k + 1, & \text{if } \tilde{f} \in \text{RVZ}_{p_0} \text{ with } p_0 < k. 
\end{cases} \]
On the other hand, since $\tilde{f} \in RV_{p_\infty}$ with $p_\infty < k$, we get

$$
\lim_{t \to \infty} \int_0^t (\tilde{F}(s))^{-1/(k+1)} ds = \infty.
$$

It follows from L'Hôpital's rule, we obtain

$$
\lim_{t \to \infty} \frac{\int_0^t (\tilde{F}(s))^{-1/(k+1)} ds}{t} = \lim_{t \to \infty} \left( \frac{t(\tilde{F}(t))^{-1/(k+1)}}{\int_0^t (\tilde{F}(s))^{-1/(k+1)} ds} \right)^{-1} = \lim_{t \to \infty} \left( 1 - \frac{\tilde{f}(t)t}{(k+1)\tilde{F}(t)} \right)^{-1} = \frac{k + 1}{k - p_\infty}.
$$

(iv) The results follow from (ii), (iii).

References


